1.(6 pts.) Suppose that $s \geq 10$ and that

$$\begin{array}{rcrcl} (s-2)x_1 & + & 3x_2 & = & 5 \\ x_1 & + & (s-4)x_2 & = & 6 \end{array}$$

Which of the following is the value of x_2 in terms of s?

(a) $\dfrac{-6s+17}{s^2-6s+8}$ (b) $\dfrac{5s-38}{s^2-6s+8}$ (c) $\dfrac{5s-38}{s^2-6s+5}$ (d) $\dfrac{6s-17}{s^2-6s+8}$ (e) $\dfrac{6s-17}{s^2-6s+5}$

2.(6 pts.) Let $\mathbb{P}_k$ denote the vector space of all real polynomials of degree at most k in the indeterminate t. Let

$$V_1 = \{\, p(t) \text{ in } \mathbb{P}_2 \colon p(1) = 0 \,\}, \qquad V_2 = \{\, p(t) \text{ in } \mathbb{P}_2 \colon p(0) = 1 \,\}$$

$$V_3 = \{\, at^2 + bt + c \text{ in } \mathbb{P}_2 \colon a - 2b + 3c = 1 \,\}, \qquad V_4 = \{\, at^2 + bt + c \text{ in } \mathbb{P}_2 \colon a - 2b + 3c = 0 \,\}.$$

Which one of the following statements is true?

(a) V_2 is a subspaces of $\mathbb{P}_2$ but V_1, V_3 and V_4 are not.

(b) V_1, V_2 and V_4 are subspaces of $\mathbb{P}_2$ but V_3 is not.

(c) V_1 and V_4 are subspaces of $\mathbb{P}_2$ but V_2 and V_3 are not.

(d) V_3 and V_4 are subspaces of $\mathbb{P}_2$ but V_1 and V_2 are not.

(e) V_1 and V_2 are subspaces of $\mathbb{P}_2$ but V_3 and V_4 are not.

3.(6 pts.) The transformation $T\colon \mathbb{P}_2 \to \mathbb{R}$ given by $T(\mathbf{p}) = \mathbf{p}(2)$ is a linear transformation. Which of the following sets of polynomials is a basis for the kernel (null space) of T?

(a) $\{\, t^2 - 4, 2 - t \,\}$ (b) $\{t^2 - t - 2, 2t - 3\,\}$ (c) $\{\, t^2 - t - 2, t^2 - 4, 2 - t \,\}$

(d) $\{\, t^2 - t - 2 \,\}$ (e) $\{\, t - 2, t^2 - 4, 2 - t \,\}$

4.(6 pts.) Let $A = \begin{bmatrix} 1 & -2 & 2 & 0 & 3 \\ -1 & 5 & -2 & 1 & -1 \\ 2 & -4 & 4 & 0 & 5 \end{bmatrix}$. Which of the following is equal to the dimension of the column space of A?

(a) 3 (b) 2 (c) 1 (d) 0 (e) 4

5.(6 pts.) Suppose that A is a 10×12 matrix such that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has exactly three free variables. Which one of the following statements is NOT true?

(a) The column space of the transpose matrix A^T is nine-dimensional.

(b) The rank of A is 9.

(c) For all $\mathbf{b}$ in $\mathbb{R}^{10}$, the linear system $A\mathbf{x} = \mathbf{b}$ has a solution.

(d) The null space of the transpose matrix A^T is one-dimensional.

(e) The null space of A is three-dimensional.

6.(6 pts.) Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases of $\mathbb{R}^2$ such that $\mathbf{b}_1 = 3\mathbf{c}_1 - 5\mathbf{c}_2$ and $\mathbf{b}_2 = 2\mathbf{c}_1 - 4\mathbf{c}_2$. Let $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ denote the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. Which of the following is equal to p_{12}?

(a) $-\frac{5}{2}$ (b) 2 (c) -5 (d) -2 (e) 3

7.(6 pts.) Let $A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 2 & 4 \\ -3 & 0 & 8 \end{bmatrix}$. Which of the following are the eigenvalues of A, listed according to their multiplicities?

(a) 0, 0, 2 (b) 2, 2, 8 (c) −3, 2, 3 (d) 2, 5, 5 (e) −1, 2, 3

8.(6 pts.) Suppose that A is a 5×5-matrix with eigenvalues $0, 0, 1, 1, 2$ (listed according to their multiplicity). Suppose that the eigenspace of A corresponding to eigenvalue 0 is 2-dimensional and the eigenspace of A corresponding to the eigenvalue 1 is 1-dimensional. Which one of the following statements must be true?

(a) $A - 2I_5$ is invertible. (b) A has rank 3.

(c) A is diagonalizable. (d) $A - 3I_5$ is not invertible.

(e) The eigenspace of A corresponding to the eigenvalue 2 is 2-dimensional.

9.(6 pts.) Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$. Which of the following is a complex eigenvector of A?

(a) $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3-i \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -2-i \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

10.(14 pts.) The matrix $A = \begin{bmatrix} 2 & 4 & 8 & 12 \\ 1 & 2 & 4 & 6 \\ 3 & 6 & 0 & 5 \\ -1 & -2 & 5 & 4 \end{bmatrix}$ is row equivalent to the matrix $B = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(i) *(4 pts.)* Find a basis for the row space of A.

(ii) *(4 pts.)* Find a basis for the column space of A.

(iii) *(6 pts.)* Find a basis for the null space of A.

11.(14 pts.) Let A be the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(i) *(4 pts.)* Find the eigenvalues of A.

(ii) *(6 pts.)* For each eigenvalue of A, determine a corresponding eigenvector.

(iii) *(4 pts.)* Using just your work in parts (i) and (ii), write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

12.(14 pts.) Consider the transformation $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(\mathbf{p}(t)) = (t+2)\mathbf{p}'(t)$, where $\mathbf{p}'(t)$ is the derivative of $\mathbf{p}(t)$. For example, $T(t^2+t) = (t+2)(2t+1) = 2t^2+5t+2$. The function T is a linear transformation. You need not show this.

(i) *(6 pts.)* Calculate the matrix A of T with respect to the basis $\mathcal{B} = \{\, 1, t, t^2 \,\}$ for $\mathbb{P}_2$.

(ii) *(4 pts.)* Determine the kernel (null space) of T. What is its dimension?

(iii) *(4 pts.)* What is the dimension of the range of T?

Solutions

1.

Cramer's Rule: $\det\begin{vmatrix} s-2 & 3 \\ 1 & s-4 \end{vmatrix} = (s-2)(s-4)-3 = s^2-6s+8-3 = s^2-6s+5.$

For x_2 we need to compute $\det\begin{vmatrix} s-2 & 5 \\ 1 & 6 \end{vmatrix} = 6s-12-5 = 6s-17$ and $x_2 = \dfrac{6s-17}{s^2-6s+5}$.

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2. V_1 is a subspace since closed under addition and scalar multiplication. V_2 is not since $1+1=2$, not 1. V_3 is not since $1+1=2$ and V_4 is.

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3. Using basis $\{1, t, t^2\}$ for $\mathbb{P}_2$, the matrix for T is $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$ This matrix is in reduced row echelon form and variables 2 and 3 are free so basis is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ or $t-2$ and t^2-4.

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4. $\begin{bmatrix} 1 & -2 & 2 & 0 & 3 \\ -1 & 5 & -2 & 1 & -1 \\ 2 & -4 & 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 & 0 & 3 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$. There are 3 pivots so the dimension is 3.

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5. Since $A\colon \mathbb{R}^{12} \to \mathbb{R}^{10}$ and the null space has dimension 3. Hence column space has dimension 9 and is not all of $\mathbb{R}^{10}$. The rank of A is 9, the row space has dimension 9 and the column space of A^T has dimension 9. Hence null space of A^T has dimension 1.

6.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\mathbf{e}_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\mathbf{e}_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \text{ so } P = \begin{bmatrix} 3 & 2 \\ -5 & -4 \end{bmatrix} \text{ so } p_{12} = 2.$$

7.

$\det\begin{vmatrix} 2-\lambda & 0 & 3 \\ -1 & 2-\lambda & 4 \\ -3 & 0 & 8-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ -3 & 8-\lambda \end{vmatrix} = (2-\lambda)\big((2-\lambda)(8-\lambda)+9\big) = (2-\lambda)(16-10\lambda+\lambda^2+9) = (2-\lambda)(25-10\lambda+\lambda^2) = (2-\lambda)(5-\lambda)^2.$

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8. The null space of A has dimension 2. Hence A has rank $5-2=3$. Since 1 is a root of multiplicity 2 but the eigenspace only has dimension 1, the matrix cannot be diagonalized. Since 2 is an eigenvalue $A-2I$ can not be inverted. The eigenspace of 2 is one dimensional. Since 3 is not an eigenvalue $A-3I$ can be inverted.

9.

$\det\begin{vmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 3-4\lambda+\lambda^2+2 = \lambda^2-4\lambda+5$ so eigenvalues are $\lambda = \dfrac{4\pm\sqrt{16-20}}{2} = 2\pm i$.

Null space for eigenvalue $2+i$: $\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$ so eigenvector is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ other eigenvector is $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

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10. The pivot columns are 1, 3 and 4.

A basis for the row space is $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$.

A basis for the column space is $\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 4 \\ 0 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 12 \\ 6 \\ 5 \\ 4 \end{bmatrix}$.

To get a basis for the null space we need reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

11.

$\det\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)$ so the eigenvalues are 2 and 3.

The eigenspace for $\lambda=2$ is the null space of $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ which has basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The eigenspace for $\lambda=3$ is the null space for $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ which has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. If you chose $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

12.

$T(1) = 0$ so column 1 is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$T(t) = t + 2$ so column 2 is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

$T(t^2) = 4t + 2t^2$ so column 3 is $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$.

The matrix is $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

Columns 2 and 3 are pivots. Hence the null space has dimension 1 and the range (or column space) has dimension 2. It remains to determine the null space or kernel. Put $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ into reduced row echelon form.

$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and hence $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or the polynomial 1 or the polynomial $1 + 0t + 0t^2$ is a basis for the null space.
