Exam 2D solutions

Multiple choice.

(1) $[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $c_1b_1+c_2b_2 = x$. So $c_1[2\ 1]^T+c_2[-1\ 1]^T = [4\ 5]^T$ where T denotes transpose. That is, $2c_1 - c_2 = 4$, $c + c_2 = 5$. Adding, $3c_1 = 9$, $c_1 = 3$, so $c_2 = 2$ and $[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (or solve by row-reducing the augmented matrix of the system).

(2) The product of the complex eigenvalues (with their multiplicities) is the determinant, which is $2 \cdot 2 - (-2) \cdot 2 = 8$. Or: the characteristic polynomial is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$ where A is the matrix. This is $\lambda^2 - 4\lambda + 8$ with roots $2 \pm 2i$ (by quadratic formula) and product $(2 + 2i)(2 - 2i) = 2^2 + 2^2 = 8$.

(3) The orthogonal complement of a set A of vectors is the set of all vectors orthogonal to each vector in A; it is always a subspace. Upper half plane is not closed under multiplication by negative scalars. Union of two (distinct) lines is not closed under addition. If a 2×2 matrix has two distinct real eigenvalues, the set of all eigenvectors is a union of two (distinct) lines and so is not closed under addition. Obviously, only one of these is a subspace.

(4) Row-reduce: $\begin{bmatrix} -2 & 4 & 1 & 4 \\ -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 2 & 3 \\ -2 & 4 & 1 & 4 \\ -3 & 6 & -1 & 1 \\ 2 & -4 & 5 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The pivot rows of the last matrix (first, second, third) give the desired basis (note rows are written as column vectors in the answers).

(5) Let $\mathcal{E} = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ be the standard basis. Then $P = P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{E}} = P^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$. Note A is the $\mathcal{E}$-matrix of the linear transformation, so $P^{-1}AP = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ is the $\mathcal{B}$-matrix. This simplifies to $\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -6 & -7 \\ 5 & 6 \end{bmatrix}$.

(6) Start row-reducing: $\begin{bmatrix}1 & 3 & -2 & 5 & 3\\ 0 & 1 & -1 & 2 & 1\\ 2 & 1 & 1 & 2 & 3\end{bmatrix}$ $\begin{bmatrix}1 & 3 & -2 & 5 & 3\\ 0 & 1 & -1 & 2 & 1\\ 0 & 5 & -5 & 8 & 3\end{bmatrix}$ $\begin{bmatrix}1 & 3 & -2 & 5 & 3\\ 0 & 1 & -1 & 2 & 1\\ 0 & 0 & 0 & -2 & 2\end{bmatrix}$

This is in echelon form (though not row-reduced) and has 3 pivot positions; the row-reduced echelon form will have (the same) 3 pivot positions so the rank is 3.

(7) $u \cdot u = \|u\|^2$, and this is the only correct statement.

(8) Let $u = \begin{bmatrix}1\\ 7\end{bmatrix}$, $v = \begin{bmatrix}-4\\ 2\end{bmatrix}$. The projection is $x = \frac{u \cdot v}{v \cdot v} v$. Here, $u \cdot v = 1 \cdot (-4) + 7 \cdot 2 = 10$, $v \cdot v = (-4)^2 + 2^2 = 20$ and $x = \frac{1}{2} v = \begin{bmatrix}-2\\ 1\end{bmatrix}$.

(9) If $\det(A) = 0$, then A is not invertible, A has rank less than n, $\det(AB) = \det(A)\det(B) = 0$, $\lambda = 0$ is a root of the characteristic polynomial $\det(A - \lambda I)$ and hence is an eigenvalue of A, and the columns of A *don't* span $\mathbb{R}^n$.

(10) $P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}7 & -3\\ 5 & -1\end{bmatrix}$ and $P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & -2\\ -5 & 2\end{bmatrix}$. So $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} P_{\mathcal{E}\leftarrow\mathcal{B}} =$ $(P_{\mathcal{E}\leftarrow\mathcal{C}})^{-1} P_{\mathcal{E}\leftarrow\mathcal{B}} = \frac{1}{-8}\begin{bmatrix}2 & 2\\ 5 & 1\end{bmatrix}\begin{bmatrix}7 & -3\\ 5 & -1\end{bmatrix} = -\frac{1}{8}\begin{bmatrix}24 & -8\\ 40 & -16\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -5 & 2\end{bmatrix}$.

Partial credit.

(11)(a) $A = \begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 2\\ 0 & 2 & 1\end{bmatrix}$. The characteristic polynomial is $\det(A - \lambda I)$ which is $\det\begin{bmatrix}3-\lambda & 0 & 0\\ 0 & 1-\lambda & 2\\ 0 & 2 & 1-\lambda\end{bmatrix} = (3-\lambda)\det\begin{bmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{bmatrix}$. This equals $(3-\lambda)((1-\lambda)^2 - 4) = -(\lambda-3)^2(\lambda+1)$. Its roots, the eigenvalues of A, are $\lambda = 3, 3, -1$. Now $A - 3I = \begin{bmatrix}0 & 0 & 0\\ 0 & -2 & 2\\ 0 & 2 & -2\end{bmatrix}$. A column vector $[x, y, z]^T$ is in the nullspace of $A - 3I =$ if x is arbitrary and $y = z$. Hence $[1, 0, 0]^T$ and $[0, 1, 1]^T$ forms a basis for the 3-eigenspace. Similarly, $A - (-1)I = A = \begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 2\\ 0 & 2 & 2\end{bmatrix}$. A vector $[x, y, z]^T$ is in the nullspace of $A + I =$ if $x = 0$ and $y = -z$. Hence $[0, -1, 1]^T$ forms a basis for the -1-eigenspace.

(b) Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ have the eigenvectors as columns and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ have the corresponding eigenvalues as diagonal entries. Then $AP = PD$ and $A = PDP^{-1}$.

(12)(a) Gram-Schmidt produces from a linearly independent set $v_1, \ldots, v_n$ an orthogonal set $w_1, \ldots, w_n$ where $w_1 = v_1$ and

$$w_i = v_i - \frac{v_i \cdot w_1}{w_1 \cdot w_1} w_1 - \ldots - \frac{v_i \cdot w_{i-1}}{w_{i-1} \cdot w_{i-1}} w_{i-1}$$

for $i > 1$. Here,

$$w_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{8} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$w_3 = \begin{bmatrix} 0 \\ -2 \\ -2 \\ 4 \end{bmatrix} - \frac{4}{8} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 3 \end{bmatrix},$$

(13)(a) $u_1 \cdot u_2 = 3 \cdot (-4) + 4 \cdot (-3) + 0 \cdot 0 = 0$

(b)

$$w = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{30}{25} u_1 + \frac{-15}{25} u_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

(c) $z = y - w = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

(d)$\|y\|^2 = 6^2 + (-2)^2 = 49$, $\|w\|^2 = 6^2 + 3^2 + 0^2 = 45$ and $\|z\|^2 = 0^2 + 0^2 + (-2)^2 = 4$. One has $\|y\|^2 = \|z\|^2 + \|w\|^2$ by Pythagoras' theorem, since the difference $z = y - w$ between y and its projection w on the span of u_1 and u_2 is orthogonal to each vector (such as w) in that span.