1.(5pts) Let U be an $m \times n$ matrix with orthonormal columns and let $W = \operatorname{Col} U$. Determine which statement is *not* always true.

(a) $\|U\mathbf{x}\| = 1$ for $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$

(b) $U^T U = I$

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$

(d) $\operatorname{proj}_W \mathbf{y} = UU^T\mathbf{y}$ for $\mathbf{y}$ in $\mathbb{R}^m$

(e) The columns of U form a basis for W.

2.(5pts) Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$ onto the subspace W spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 7 \\ 7 \\ 17 \end{bmatrix}$

3.(5pts) Use the Gram-Schmidt process to find an orthogonal basis for the span of the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

4.(5pts) Classify the differential equation $y' = \dfrac{\cos(x) + y\sin(x)}{\ln(x)}, \quad x > 0.$

(a) *linear* (b) *separable* (c) *exact* (d) *autonomous* (e) *none of the above*

5.(5pts) Solve the differential equation $y' + 2y = 3e^t$ subject to the initial condition $y(1) = 0$.

(a) $y = e^t - e^{3-2t}$ (b) $y = e^{3t} - e^3$ (c) $y = e^{2t} - e^{3-t}$ (d) $y = e^t - 1$ (e) $y = e^t$

6.(5pts) Let $\phi(x)$ be the solution of the differential equation $y' = xy^2$ satisfying $\phi(0) = 2$. Find $\phi(3)$.

(a) $-1/4$ (b) 0 (c) $2/9$ (d) $1/3$ (e) $-2/5$

7.(5pts) Determine the interval where the solution to the initial value problem is guaranteed to exist.

$$\cos(t)y' = y + e^{1/t}, \qquad y(\pi) = 0$$

(a) $(\pi/2, 3\pi/2)$ (b) $(0, \infty)$ (c) $(0, 3\pi/2)$ (d) $(-\pi/2, \pi/2)$ (e) (π, ∞)

8.(5pts) Suppose a certain population of red squirrels satisfies the differential equation

$$\frac{dN}{dt} = -0.05\left(1 - \frac{N}{700}\right)\left(1 - \frac{N}{2000}\right)N$$

If the current population is 600, determine the long-term population of red squirrels predicted by this model.

(a) 0 (b) 700 (c) 2000 (d) $\ln(700)/0.05$ (e) $\ln(2000)/0.05$

9.(5pts) Find the solution to the initial value problem $y'' - 5y' - 6y = 0$, $y(0) = 0$, $y'(0) = 1$.

(a) $y = \frac{1}{7}(e^{6t} - e^{-t})$ (b) $y = e^{-2t} - e^{-3t}$ (c) $y = -\frac{1}{5}(e^{-2t} + e^{-3t})$

(d) $y = \frac{1}{7}(e^t - e^{-6t})$ (e) $y = \frac{1}{5}(e^{3t} + e^{2t})$

10.(5pts) Find a value for r such that $y = t^r$ is a solution of the differential equation $9t^2y'' + 12ty' - 2y = 0$.

(a) $-2/3$ (b) 2 (c) 3 (d) $-1/6$ (e) $3/4$

11.(10pts) Find the least-squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$.

12.(10pts) Gases from the ground below a house seep into its 3000 ft^3 basement at the rate of 2 ft^3 per day. Air leaves the basement at the same rate. The incoming gases contain 5% radon. Assuming the basement is initially clear of radon gas, calculate how many days it will take for the basement to reach the dangerous level of 1% radon gas.

13.(10pts) Solve the initial value problem by first finding an integrating factor.

$$y\,\tan(x)\,dx + (y\,\sec(x) - 1)\,dy = 0, \qquad y(\pi) = 2$$

Solutions

1. Since $\|Uc\mathbf{x}\| = c\|U\mathbf{x}\|$ (a) is not always true.

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2. $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal so

$$\operatorname{proj}_W\left(\begin{bmatrix}-2\\4\\3\end{bmatrix}\right) = \frac{\begin{bmatrix}-2\\4\\3\end{bmatrix}\bullet\begin{bmatrix}1\\1\\-1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\-1\end{bmatrix}}\begin{bmatrix}1\\1\\-1\end{bmatrix} + \frac{\begin{bmatrix}-2\\4\\3\end{bmatrix}\bullet\begin{bmatrix}1\\1\\2\end{bmatrix}}{\begin{bmatrix}1\\1\\2\end{bmatrix}\bullet\begin{bmatrix}1\\1\\2\end{bmatrix}}\begin{bmatrix}1\\1\\2\end{bmatrix} =$$

$$\frac{-1}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} + \frac{8}{6}\begin{bmatrix}1\\1\\2\end{bmatrix} = \frac{1}{6}\left(\begin{bmatrix}-2\\-2\\2\end{bmatrix} + \begin{bmatrix}8\\8\\16\end{bmatrix}\right) = \begin{bmatrix}1\\1\\3\end{bmatrix}$$

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3. $\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\bullet\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

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4. Write it as

$$y' + \frac{-\sin(x)}{\ln(x)} y = \frac{\cos(x)}{\ln(x)}$$

shows that it is linear.

5.
The equation is linear first order so $\int 2dt = 2t$ so $\mu = e^{2t}$ and we need to do $\int 3e^t e^{2t} dt = \int 3e^{3t} dt = e^{3t} + C$ and the general solution is $y = e^{-2t}(e^{3t} + C)$. Since $y(1) = 0$, $0 = e^{-2}(e^3 + C)$ so $C = e^{-3}$ and $y = e^{-2t}(e^{3t} - e^3)$.

6.
The equation is separable so $\frac{dy}{y^2} = xdx$ or $-y-1 = x^2/2 + C$. Since $y(0) = 2$, $-1/2 = 0 + C$ so $-\frac{1}{y} = \frac{x^2 - 1}{2}$ or $y = \frac{2}{1 - x^2}$. $\phi(3) = \frac{2}{1 - 3^2} = \frac{2}{-8} = -1/4$

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7. The equation is linear and the standard form is

$$y' + \frac{-1}{\cos(t)} y = \frac{e^{1/t}}{\cos(t)}, \qquad y(\pi) = 0$$

We are looking for the largest open interval containing π such that $t \neq 0$ and $\cos(t) \neq 0$. That interval is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

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8. The equation is autonomous and the zeros of the function are $N = 0$, $N = 700$ and $N = 2000$. At $N = 600$, $\frac{dN}{dt} < 0$ so N is decreasing. For all N between 0 and 700, $\frac{dN}{dt} < 0$ so N eventually goes to 0.

9.
2nd order linear with constant coefficients so solve $r^2 - 5r - 6 = (r - 6)(r + 1) = 0$ so the roots are $6 and_2$ so the general solution is $y = c_1 e^{-t} + c_2 e^{6t}$ and then $y' = -c_1 e^{-t} + 6c_2 e^{6t}$ so $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = -c_1 + 6c_2$. Hence $c_2 = 1/7$ and $c_1 = -1/7$.

10.
$y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$ so

$$9r(r-1)t^r + 12rt^r - 2t^r = (9r^2 - 9r + 12r - 2)t^r$$

which vanishes if and only if $9r^2 + 3r - 2 = (3r-1)(3r+2) = 0$ Hence two values of r work, $r = 1/3$ and $r = -2/3$.

11.
Solve $A^T A\mathbf{x} = A^T\mathbf{b}$. $A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

so solve

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

By inspection $\hat{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

12.
Let $R(t)$ denote the total amount of radon in the air in ft^3. A cubic foot of incoming gas contains $0.05\,\text{ft}^3$ of radon. A cubic foot of outgoing gas contains $\dfrac{R}{3000}\,\text{ft}^3$ of radon.

Hence $0.1\,\text{ft}^3/\text{day}$ is the amount of incoming radon per day and $\dfrac{2R}{3000}\,\text{ft}^3$ is the amount of outgoing radon per day. Hence

$$\frac{dR}{dt} = 0.1 - \frac{R}{1500}$$

This equation is linear and in standard form is

$$R' + \frac{1}{1500}R = 0.1$$

Hence $\mu = e^{t/1500}$ and $\displaystyle\int 0.1e^{-t/1500}dt = 150e^{-t/1500} + C$ so $R = 150 + Ce^{-t/1500}$.

Since the basement is initially clear of radon $R(0) = 0$ and $C = -150$ and so $R = 150(1 - e^{-t/1500})$.

We are looking for the time t for which $R(t) = 0.01 \cdot 3000 = 30$ so $150(1 - e^{-t/1500}) = 30$ or $1 - e^{-t/1500} = 0.2$ or $0.8 = e^{-t/1500}$ or $-t/1500 = \ln(0.8)$ or

$$t = \frac{-\ln(0.8)}{1500} \quad \text{days}$$

With a calculator this is about 335 days.

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13. $M = y\tan(x)$ and $N = y\sec(x) - 1$ so $M_y = \tan(x)$ and $N_x = y\tan(x)\sec(x)$ so $M_y - N_x = \tan(x) - y\tan(x)\sec(x) = \tan(x)(1 - y\sec(x)) = -\tan(x)N$. Hence $\dfrac{\mu'}{\mu} = -\tan(x)$ or $\ln|\mu| = -\ln|\cos(x)|$ so we may take $\mu = \cos(x)$. The new equation is

$$y\ \cos(x)\tan(x)\,dx + \cos(x)(y\ \sec(x) - 1)\,dy = 0, \qquad y(\pi) = 2$$

or

$$y\ \sin(x)\,dx + (y - \cos(x))\,dy = 0, \qquad y(\pi) = 2$$

$\dfrac{\partial\psi}{\partial x} = y\sin(x)$ so $\psi = -y\cos(x) + g(y)$.

Then $\dfrac{\partial\psi}{\partial y} = -\cos(x) + g'(y) = y - \cos(x)$ so $g'(y) = y$ and $\psi = -y\cos(x) + y^2/2$ Since $y(\pi) = 2$, the value of the level curve is $-2\cos(\pi) + 2^2/2 = 2 + 2 = 4$ so $-y\cos(x) + y^2/2 - = 4$ or

$$y^2 - 2y\cos(x) - 8 = 0$$

You can solve for y using the quadratic formula although you were not asked for this. If you do proceed however,

$$y = \frac{2\cos(x) \pm \sqrt{4\cos^2(x) + 32}}{2}$$

When $x = \pi$, $y = \dfrac{-2 \pm \sqrt{4 + 32}}{2} = \dfrac{-2 \pm 6}{2} = 2 \text{ or } -4$. Since we are looking for the solution with $y(\pi) = 2$, the *unique* solution is

$$y = \frac{2\cos(x) + \sqrt{4\cos^2(x) + 32}}{2}$$
