1. Find the reduced echelon form of the matrix $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ where $\sim$ denotes row equivalence.

---

2. Determine by inspection which one of the following sets of vectors is linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \right\}$ (c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Solution. First vector in (a) is non-zero and second is not a scalar multiple of it; so they are linearly independent.

(b) Not linearly independent; $3\mathbf{v}_2 - 2\mathbf{v}_1 = \mathbf{0}$.
(c) Four vectors in $\mathbb{R}^3$ must be linearly dependent.
(d) Not linearly independent; $\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2$
(e) Not linearly independent; contains the zero vector.

---

3. For which value of h is the vector $\begin{bmatrix} 1 \\ h \\ 2 \end{bmatrix}$ in the span of the vectors $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$?

(a) $h = 0$ (b) $h = 1$ (c) $h = 2$ (d) $h = 3$ (e) $h = 4$

Solution. The first vector is in the span of the other two exactly when the linear system with augmented matrix $\begin{bmatrix} 1 & 2 & | & 1 \\ -3 & 3 & | & h \\ 4 & 2 & | & 2 \end{bmatrix}$ is consistent. Row reduce: $\begin{bmatrix} 1 & 2 & | & 1 \\ -3 & 3 & | & h \\ 4 & 2 & | & 2 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 2 & | & 1 \\ 0 & 9 & | & h+3 \\ 0 & -6 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 1 \\ 0 & 1 & | & (h+3)/9 \\ 0 & -6 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 1/3 - 2h/9 \\ 0 & 1 & | & h/9 + 1/3 \\ 0 & 0 & | & 2h/3 \end{bmatrix}$. The system is consistent exactly when $2h/3 = 0$; that is, when $h = 0$.

---

4. Let A be a 3×5 matrix A and $\mathbf{b}$ in $\mathbb{R}^3$. Which of the following statements about the matrix equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbb{R}^5$, and the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$, *could be true*?

(a) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. (b) $A\mathbf{x} = \mathbf{0}$ is inconsistent.
(c) $A\mathbf{x} = \mathbf{0}$ has exactly two solutions.
(d) $A\mathbf{x} = \mathbf{0}$ has a unique solution and $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
(e) $A\mathbf{x} = \mathbf{b}$ has a unique solution and $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if it is consistent and the reduced echelon form of A has at least one free variable, so (a) could be true. A linear system can have (only) 0, 1 or infinitely many solutions and a homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent (with solution $\mathbf{x} = \mathbf{0}$) so (c) and (b) are false. If $A\mathbf{x} = \mathbf{b}$ is consitent, all its solutions are obtained by adding a solutions of the homogeneous system to a particular solution of $A\mathbf{x} = \mathbf{b}$, so $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have exactly the same number of solutions in that case; (d) and (e) are therefore also false.

---

5. Recall that an $m \times n$ matrix has m rows and n columns. Let $T : \mathbb{R}^6 \to \mathbb{R}^8$ be a linear transformation. What is the size of the standard matrix A for T?

(a) 8×6 (b) 6×6 (c) 8×8 (d) 6×8
(e) There is not enough information to determine the answer.

Solution. Since T sends vectors in $\mathbb{R}^6$ to vectors in $\mathbb{R}^8$, the standard matrix A must be 8×6.

---

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. What is the standard matrix A for T?

(a) $A = \begin{bmatrix} 1 & 6 \\ 2 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 \\ 7 & -2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 2 \\ 6 & -4 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}$
(e) Since we do not know $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, there is not enough information to determine the answer.

Solution. The standard matrix for the linear transformation is the matrix $[T(\mathbf{e_1})\ \ T(\mathbf{e_2}))]$, where $\mathbf{e_1}$ is the first column of the 2×2 identity matrix and $\mathbf{e_2}$ is the second column of the 2×2 identity matrix. By linearity, we have that $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$, thus the standard matrix $A = \begin{bmatrix} 1 & 6 \\ 2 & -4 \end{bmatrix}$.

---

7. Which of the following is a subspace of $\mathbb{R}^3$?

(1) The set of all vectors, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where a, b, c are positive.

(2) The set of all vectors, $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$, where a, b are any numbers.

(3) The set of all vectors, $\begin{bmatrix} a \\ 0 \\ c \end{bmatrix}$, where a, c are any numbers.

(4) The set of all vectors, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where a, b, c are integers.

(5) The set of all vectors, $\begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix}$, where a is a real number.

(a) (3) and (5) (b) Only (3) (c) Only (5) (d) (4) and (5) (e) Only (4)

Solution. (1) is not a subspace since it is not closed under multiplication by scalars. For instance, $-2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$ is not in the set. (2) is not a subspace since the zero vector, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, is not in the set. (4) is not a subspace since it is not closed under multiplication by scalars. For instance, $.5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \\ .5 \end{bmatrix}$ is not in the set. (3) and (5) satisfy all of the propeties of a subspace.

---

8. Which matrix below is invertible?

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 4 & 2 \\ 8 & 0 & 6 & 3 \\ -1 & 2 & 7 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & -1 \\ -4 & -8 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

Solution. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ so all columns are pivots so invertible.

$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ columns 1 and 2 are dependent so not invertible.

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 4 & 2 \\ 8 & 0 & 6 & 3 \\ -1 & 2 & 7 & 1 \end{bmatrix}$ are not square matrices and hence not invertible.

$\begin{bmatrix} 1 & 3 & -1 \\ -4 & -8 & 2 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so not invertible.

---

9. Let A be a 3×2, B be a 3×5 matrix, and C be a 2×5 matrix. Which of the following expressions make sense?

(1) AB

(2) BA

(3) $A + B$

(4) $A^T B + C$

(5) $A(B^T) + C$

(a) (4) (b) (2) and (3) (c) (1) and (4) (d) (5) (e) (1) and (5)

Solution. AB does not make sense since the number of columns of A is not equal to the number of rows of B. For the same reason, BA also does not make sense. $A + B$ does not make sense because we cannot sum matrices of different sizes. $A^T B$ is a 2×5 matrix, which can be added to C, since C is a 2×5 matrix. So $A^T B + C$ makes sense. $A(B^T) + C$ does not make sense since the number of columns of A is not the same as the number of rows of B^T. So $A(B^T) + C$ does not make sense.

---

10. Express the solution set of the homogeneous linear system

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & x_3 & + & x_4 & + & 5x_5 & = & 0 \\ 2x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & & & = & 0 \\ 3x_1 & + & 2x_2 & - & 3x_3 & + & 5x_4 & + & 6x_5 & = & 0 \end{array}$$

in *parametric vector form.*

Solution. Row reduce the coefficient matrix: $\begin{bmatrix} 1 & 1 & -1 & 1 & 5 \\ 2 & 1 & -2 & 4 & 0 \\ 3 & 2 & -3 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 & 5 \\ 0 & -1 & 0 & 2 & -10 \\ 0 & -1 & 0 & 2 & -9 \end{bmatrix} \sim$
$\begin{bmatrix} 1 & 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -2 & 10 \\ 0 & -1 & 0 & 2 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -2 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$
The system is equivalent to

$$\begin{aligned} x_1 \quad - \quad x_3 + 3x_4 &= 0 \\ x_2 \quad - 2x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

The bound variables are x_1, x_2 and x_5 (corresponding to pivot columns) and the free variables x_3, x_4 can take arbitrary values. Rewriting with free variables on the right,

$$\begin{aligned} x_1 &= x_3 - 3x_4 \\ x_2 &= 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

(we include the equation $x_i = x_i$, for $i = 3$ or 4, to indicate that the free variable x_i can take arbitrary values). In parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

or writing $x_3 = r$, $x_4 = s$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

---

11. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Row reduce the augmented matrix $[A \ \ I]$: $\begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & -1 & -6 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$
$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -5 \\ 0 & -1 & 0 & -1 & 1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. So $A^{-1} =$
$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$.

---

12. Find a basis for the column space of $A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ -2 & -2 & 4 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. What is the rank of A?

Solution. Row-reduce $\begin{bmatrix} 1 & 0 & -3 & 1 \\ -2 & -2 & 4 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & -2 & -2 & 2 \\ -1 & -1 & -1 & 1 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & -1 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$. From this we see that the pivot columns of A are the first three columns and thus a basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} \right\}$. Alternatively, since Col$A = \mathbb{R}^3$, any basis of $\mathbb{R}^3$ would do. The rank of A is the dimension of Col$A$, which is 3.
