1. Find the eigenvalues of $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(a) 1, 5 (b) 3, 2 (c) 5, 6 (d) -1, 2 (e) There are no eigenvalues

Solution. $\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)(3-\lambda)-(2\cdot 2) = 9-6\lambda+\lambda^2-4 = \lambda^2-6\lambda+5 = (\lambda-1)(\lambda-5)$.

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2. Given bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$ for $\mathbb{R}^2$ which matrix below is the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{D}$ where $\vec{b}_1 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{d}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$.

(a) $\begin{bmatrix} -9 & 2 \\ 7 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 2 & 14 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 \\ 7 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ 1 & 7 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 1 \\ 14 & -2 \end{bmatrix}$

Solution. $\begin{matrix} 4 & 6 & 6 & 2 \\ 5 & 7 & 4 & 3 \end{matrix} \xrightarrow{R_1=R_2-R_1} \begin{matrix} 1 & 1 & -2 & 1 \\ 5 & 7 & 4 & 3 \end{matrix} \xrightarrow{R_2=R_2-5R_1} \begin{matrix} 1 & 1 & -2 & 1 \\ 0 & 2 & 14 & -2 \end{matrix} \xrightarrow{R_2=R_2/2}$
$\begin{matrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 7 & -1 \end{matrix} \xrightarrow{R_1=R_1-R_2} \begin{matrix} 1 & 0 & -9 & 2 \\ 0 & 1 & 7 & -1 \end{matrix}$

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3. The first four Chebyshev polynomials of the first kind are $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$ and $T_3(t) = t(4t^2 - 3)$. Determine the coordinates of $T_3(t)$ with respect to the basis $\mathcal{P} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$.

(a) $\begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 1 \\ 4 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ -3 \\ 4 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \end{bmatrix}$

Solution. We need to write $T_3(t)$ as a linear combination of powers of t. First note $T_3(t) = t(4t^2 - 3) = 4t^3 - 3t$ so $\left[T_3(t)\right]_{\mathcal{P}} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}$

Notice that the indexing (ordering) of the basis is important. Four of the listed answers are the correct answer for some indexing (ordering) of the basis.

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4. Let A be a 7×9 matrix. Suppose the rank of A is 7. What is the dimension of the column space of A^T?

(a) 7 (b) 9 (c) 2 (d) 8 (e) Can not be determined from the given information.

Solution. The dimension of the the column space of A^T is the rank of A^T. The rank of A^T equals the rank of A.

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5. The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 12 & 6 \\ 0 & 3 & 8 & 15 \\ 5 & 12 & -15 & 34 \end{bmatrix}$ has $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ as an eigenvector. What is the eigenvalue?

(a) 30 (b) -12 (c) 20 (d) 6 (e) -6

Solution. We are told $A\vec{x} = \lambda\vec{x}$ for some λ. Hence λ is the first row in $A\vec{x}$ which is $1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 = 30$.

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6. Which of the following matrices represents a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which preserves the volume of a parallelepiped S (i.e. where {volume of S} = {volume of $T(S)$})?

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 4 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution. Under any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ the volume multiplies by the absolute value of the determinant of the matrix. Hence we are looking for matrices with determinant ± 1.

(a), (c) and (d) are upper triangular so the determinant is the product of the diagonal entries: (a) is 1, (c) is 6 and (d) is -4. (b) is lower triangular so the determinant is the product of the diagonal entries: -4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. This last matrix is upper triangular so has determinant -4 so the original determinant is 4. We can also expand $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{vmatrix} = 1 \cdot (4) = 4$.

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7. Let A, B, C be 3×3 matrices with $\det(A) = 4$, $\det(B) = -1$, and $\det(C) = 9$. What is $\det(3A(B^T)^2C^{-1})$?

(a) 12 (b) -12 (c) $\frac{4}{3}$ (d) $-\frac{4}{3}$ (e) 4

Solution. $\det(3A(B^T)^2C^{-1}) = 3^3 \det(A) \det(B)^2 \det(C)^{-1} = 27 \cdot 4 \cdot (-1)^2 \cdot \frac{1}{9} = 12$

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8. Which of the following sets are subspaces of the indicated vector spaces?

I. the set of all 2×2 matrices A such that A^2 is the 0-matrix, inside the vector space of 2×2 matrices, $M_{2\times 2}$

II. the set of all polynomials $p(t)$, where $p(t)$ is of the form $p(t) = at^2 + b$, inside $\mathbb{P}_4$ (the vector space of polynomials of degree 4 or less)

III. the set of all vectors of the form $\begin{bmatrix} s-t \\ 2t+1 \\ 5s \end{bmatrix}$ in $\mathbb{R}^3$

IV. the set of all polynomials $p(t)$, where the derivative $\frac{d}{dt}p(t) = 0$, inside $\mathbb{P}_3$ (the vector space of polynomials of degree 3 or less)

(a) II and IV are subspaces (b) II, III, and IV are subspaces (c) II and III are subspaces
(d) all are subspaces (e) none are subspaces

Solution. I Try closed under addition: $(A+B)^2 = A^2 + AB + BA + B^2$. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Not closed.

II is closed under addition and scalar multiplication and $\vec{0}$ is in II hence subspace.

III Not closed under addition: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

IV The derivative of a sum is the sum of the derivatives and the derivative of a constant multiple of a function is the constant times the derivative. The $\vec{0}$ is the constant function 0 so is in IV. Therefore is a subspace.

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9. Which subsets of $M_{2\times 2}$ (the space of 2×2 matrices with entries from the real numbers) are linearly independent?

I. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \right\}$

II. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \right\}$

III. $\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix} \right\}$

IV. $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \right\}$

(a) II and IV (b) all are linearly independent (c) none are linearly independent
(d) II, III, and IV (e) I, II, and III

Solution. $\dim(M_{2\times 2}) = 2 \cdot 2 = 4$.

(I) has 5 vectors so dependent.

(III) $\begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ so dependent.

(II) independent: $a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ 2b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies $a = b = 0$.

(IV) independent: $a \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+b+c & a+2c \\ 3a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The $(2,1)$ position shows $a = 0$; then the $(1,2)$ position shows $c = 0$ and then the $(1,1)$ position shows $b = 0$.

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10. The transformation $T\colon \mathbb{P}_3 \to \mathbb{R}^3$ given by $T\big(p(t)\big) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix}$ is a linear transformation.

(a) Give a basis for the kernel of T.
(b) What is the dimension of the kernel of T?

Solution. Method 1: $T\big(p(t)\big) = \vec{0}$ if and only if $p(1) = 0$, $p(2) = 0$ and $p(3) = 0$. Hence p must have a factor of $t-1$, another of $t-2$ and a third of $t-3$. Hence $p(t) = c(t-1)(t-2)(t-3)$ so the kernel consists of all scalar multiples of $(t-1)(t-2)(t-3)$ and hence is one dimensional with basis $\{(t-1)(t-2)(t-3)\}$.

Method 2: Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis of $\mathbb{R}^3$. We translate the question to one about a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ by taking coordinates in $\mathbb{P}_3$ with respect to $\mathcal{B}$. The standard basis $\mathcal{B}$ of $\mathbb{P}_3$ translates to the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$. Note

$$T(1) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad T(t) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T(t^2) = \begin{bmatrix}1\\4\\9\end{bmatrix}, \quad T(t^3) = \begin{bmatrix}1\\8\\27\end{bmatrix}$$

so

$$S(e_1) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad S(e_2) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad S(e_3) = \begin{bmatrix}1\\4\\9\end{bmatrix}, \quad S(e_4) = \begin{bmatrix}1\\8\\27\end{bmatrix}$$

The matrix of S is

$$A = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 4 & 8\\1 & 3 & 9 & 27\end{bmatrix}$$

so $S(\vec{x}) = A\vec{x}$ for $\vec{x}$ in $\mathbb{R}^4$. The kernel (null space) of T translates to the kernel (nullspace) of S, which is the null space of the matrix A. We find it by row-reducing A to row-reduced echelon form:

$$A \to \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 7\\0 & 2 & 8 & 26\end{bmatrix} \to \begin{bmatrix}1 & 0 & -2 & -6\\0 & 1 & 3 & 7\\0 & 0 & 2 & 12\end{bmatrix} \to \begin{bmatrix}1 & 0 & -2 & -6\\0 & 1 & 3 & 7\\0 & 0 & 1 & 6\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 6\\0 & 1 & 0 & -11\\0 & 0 & 1 & 6\end{bmatrix}.$$

The only free variable is x_4. We have $x_1 + 6x_4 = 0$, $x_2 - 11x_4 = 0, x_3 + 6x_4 = 0$, $x_4 = x_4$ so the solution is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_4 \begin{bmatrix}-6\\11\\-6\\1\end{bmatrix}$$

The null space of S is 1-dimensional, with basis element given by the transpose of the matrix $[-6\ 11\ \ -6\ 1]$. Translating back to $\mathbb{P}_3$, the null space of T is one-dimensional, with basis element equal to the corresponding polynomial $t^3 - 6t^2 + 11t - 6 = (t-1)(t-2)(t-3)$, in agreement with Method 1.

Remark In solving linear equations by row reduction, you should *always* reduce the corresponding matrix to *row reduced echelon form* before converting back to a system of linear equations and writing the solution in terms of the free variables. While it is in principle possible to stop row reduction once one gets to echelon form and complete the remainder of the solution by "back substitution," it is inefficient to do so and is very likely to cause errors in computation by hand. In future exams, students who don't row reduce all the way to row reduced echelon form (when solving linear systems by row reduction), will lose a significant number of points in such partial credit questions, to discourage bad habits.

11. The eigenvalues of the matrix $B = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ are -2 and 4.

(i) Find a basis for the subspace of eigenvectors of B with eigenvalue 4.

(ii) Find a basis for the subspace of eigenvectors if B with eigenvalue -2.

Solution.

To find the eigenvectors for eigenvalue 4. $\begin{bmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -0.5 \\ 0 & 1 & -0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence there is a basis with one vector $\begin{bmatrix} 0.5 \\ 0.5 \\ 1.0 \end{bmatrix}$ or if you prefer integers $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

To find the eigenvectors for eigenvalue -2. $\begin{bmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Hence a basis for the eigenspace for -2 consisting of two vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

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12. Consider the linear system of equations:

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ -x_1 & & & + & 3x_3 & = & 2 \\ 2x_1 & - & x_2 & & & = & 2 \end{array}$$

Use **Cramer's Rule** to find x_3.

Solution. To compute x_3 by Cramer's rule we need to compute

$$x_3 = \frac{\left|A_3(\vec{b})\right|}{\det(A)}$$

where $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Then $A_3(\vec{b}) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 2 \end{bmatrix}$.

To compute $\det(A)$ first do one step in the row reduction: clear the rest of column 1.

$$\begin{vmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -2 & 6 \\ 0 & 3 & -6 \end{vmatrix}$$

and then do expansion down the first column.

$$\begin{vmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -2 & 6 \\ 0 & 3 & -6 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 6 \\ 3 & -6 \end{vmatrix} = 1 \cdot \big((-2) \cdot (-6) - 3 \cdot 6\big) = 12 - 18 = -6$$

To compute $\det(A_3(\vec{b}))$ first do one step in the row reduction: clear the rest of column 1.

$$\begin{vmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix}$$

and then do expansion down the first column.

$$\begin{vmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix} = 1 \cdot \big((-2) \cdot 0 - 3 \cdot 3\big) = -9$$

Hence $x_3 = \frac{-9}{-6} = \frac{3}{2}$.

Just for completeness, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}$.

Remark Note that to receive full points in partial credit questions you must not only show your working and explain your steps, you should also use the method you are told to use (if a particular method such as Cramer's rule is suggested) otherwise you may receive few or no points.
