Multiple Choice

1.(6pts) Let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$. What is the distance from the point $\mathbf{y} = \begin{bmatrix} 1 \\ -11 \\ 19 \end{bmatrix}$ to W?

(a) $\sqrt{114}$ (b) $\sqrt{57}$ (c) $\sqrt{19}$ (d) $\sqrt{228}$ (e) 0

Solution. The closest point of W to $\mathbf{y}$ is the orthogonal projection $\widehat{\mathbf{y}}$ of the vector $\mathbf{y}$ onto W. Note u_1 and u_2 are orthogonal, since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 2 + 1 \cdot (-5) + 1 \cdot 3 = 0$. We calculate

$$\widehat{\mathbf{y}} = \mathrm{proj}_{\mathbf{u}_1}\mathbf{y} + \mathrm{proj}_{\mathbf{u}_2}\mathbf{y} = \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{9}{3}\mathbf{u}_1 + \frac{114}{38}\mathbf{u}_2 = 3\mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} 9 \\ -12 \\ 12 \end{bmatrix}.$$

Hence $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}} = \begin{bmatrix} -8 \\ 1 \\ 7 \end{bmatrix}$. The shortest distance from $\mathbf{y}$ to W is $\|z\| = \sqrt{(-8)^2 + 1^2 + 7^2} = \sqrt{114}$.

2.(6pts) Find a QR factorization of the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 2 & 5 \end{bmatrix}$.

(a) $Q = \begin{bmatrix} 2/3 & 0 \\ -1/3 & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} \end{bmatrix}, R = \begin{bmatrix} 3 & 6 \\ 0 & \sqrt{5} \end{bmatrix}$

(b) $Q = \begin{bmatrix} 2 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 2/3 & 4/\sqrt{41} \\ -1/3 & 0 \\ 2/3 & 5/\sqrt{41} \end{bmatrix}, R = \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{41} \end{bmatrix}$

(d) $Q = \begin{bmatrix} 2/3 & 1/\sqrt{5} \\ -1/3 & 2/\sqrt{5} \\ 2/3 & 0 \end{bmatrix}, R = \begin{bmatrix} 3 & 6 \\ 0 & 2\sqrt{5} \end{bmatrix}$

(e) $Q = \begin{bmatrix} 2/3 & 0 \\ -1/3 & -2/\sqrt{5} \\ 2/3 & -1/\sqrt{5} \end{bmatrix}, R = \begin{bmatrix} 3 & 6 \\ 0 & -\sqrt{5} \end{bmatrix}$

Solution. The answer is (a). In order to find Q, we need to find an orthonormal basis for the column space of A using the Gram-Schmidt process on the column vectors $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{a}_2 = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} - \frac{\mathbf{u}_1 \cdot \mathbf{a}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

We normalize them to get an orthonormal basis:

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \frac{\mathbf{u}_1}{3} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \frac{\mathbf{u}_2}{\sqrt{5}} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

Then $Q = \begin{bmatrix} 2/3 & 0 \\ -1/3 & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} \end{bmatrix}$ and $R = Q^T A = \begin{bmatrix} 3 & 6 \\ 0 & \sqrt{5} \end{bmatrix}$.

3.(6pts) Find the least-squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 5/27 \\ 4/27 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$ (d) $\begin{bmatrix} 10/3 \\ -8/3 \end{bmatrix}$ (e) $\begin{bmatrix} 5/21 \\ -4/21 \end{bmatrix}$

Solution. The answer is NOT HERE. We solve the consistent system $A^TAx = A^Tb$.

$$A^TA = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -4 & 5 \end{bmatrix}, A^Tb = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

The matrix A^TA is invertible with inverse $(A^TA)^{-1} = \frac{1}{54}\begin{bmatrix} 5 & 4 \\ 4 & 14 \end{bmatrix}$.

The solution to the least-squares problem is unique and equal to

$$(A^TA)^{-1}(A^Tb) = \frac{1}{54}\begin{bmatrix} 5 & 4 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \frac{1}{54}\begin{bmatrix} -6 \\ -48 \end{bmatrix} = \begin{bmatrix} -1/9 \\ -8/9 \end{bmatrix}$$

Alternatively, augment the matrix (A^TA) with (A^Tb) and take it to reduced row echelon form to give the same solution $\begin{bmatrix} -1/9 \\ -8/9 \end{bmatrix}$:

$$\begin{bmatrix} 14 & -4 & 2 \\ -4 & 5 & -4 \end{bmatrix} \to \begin{bmatrix} 7 & -2 & 1 \\ -28 & 35 & -28 \end{bmatrix} \to \begin{bmatrix} 7 & -2 & 1 \\ 0 & 27 & -24 \end{bmatrix} \to \begin{bmatrix} 7 & -2 & 1 \\ 0 & 9 & -8 \end{bmatrix} \to$$
$$\begin{bmatrix} 63 & -18 & 9 \\ 0 & 9 & -8 \end{bmatrix} \to \begin{bmatrix} 63 & 0 & -7 \\ 0 & 9 & -8 \end{bmatrix} \to \begin{bmatrix} 9 & 0 & -1 \\ 0 & 9 & -8 \end{bmatrix}$$

4.(6pts) Let

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$$

The eigenvalues of A are

(a) $2 \pm i$ (b) $1 \pm 2i$ (c) $3 \pm 2i$

(d) $1 \pm 3i$ (e) $1 - 2i, 1 + 3i$

Solution. The characteristic polynomial of A is given by

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - (-2)(1) = 5 - \lambda - 3\lambda + \lambda^2$$
$$= \lambda^2 - 4\lambda + 5.$$

Setting this equal to zero, we have the roots $\lambda = \frac{1}{2}(4 \pm \sqrt{16-20}) = \frac{1}{2}(4 \pm i\sqrt{4}) = 2 \pm i$.

5.(6pts) Let $y(x)$ be a solution to the initial value problem

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}, \quad y(0) = 1 \ .$$

Compute $y(1)$.

(a) 1 (b) 0 (c) -1 (d) $1/2$ (e) 2

Solution. The 1st order nonlinear ODE is exact and can be solved like any exact differential equation. In more detail, if $M(x, y) = y - 2x$ and $N(x, y) = x - 2y$, then the differential equation can be written as $M(x, y) + N(x, y)y' = 0$. Observe that $M_y = N_x = 1$, which is the exactness condition. Integrating M with respect to x, we get $\psi(x, y) = yx - x^2 + g(y)$. Then $\psi_y = x + h'(y) = x - 2y$, so $h(y) = -y^2$, and we conclude that the general solution in implicit form is $\psi_(x, y) = yx - x^2 - y^2 = c$. With the initial condition $y(0) = 1$, we find that $c = -1$ and we have the solution $y(x) = \frac{x}{2} + \frac{\sqrt{4-3x^2}}{2}$. Finally, we get $y(1) = 1$.

6.(6pts) Let $y(t)$ be a solution to the initial value problem

$$2y' + y = -2t, \quad y(2) = -3/e \ .$$

Compute $y(0)$.

(a) 1 (b) 0 (c) 4 (d) $3 - e$ (e) $1 + e$

Solution. The ODE is a 1st order linear ODE and can be solved using the method of integrating factors. The standard form is $y' + \dfrac{1}{2}y = -t$. The integrating factor is $\mu = e^{\int \frac{1}{2}dt} = e^{\frac{t}{2}}$. Then $\left(e^{\frac{t}{2}}y\right)' = -te^{\frac{t}{2}}$. $\displaystyle\int te^{\frac{t}{2}}dt$: parts $u = t$, $dv = e^{\frac{t}{2}}dt$, $v = 2e^{\frac{t}{2}}$, $du = dx$. $\displaystyle\int te^{\frac{t}{2}}dt = 2te^{\frac{t}{2}} - \int 2e^{\frac{t}{2}}dt = 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + C$ so the general solution is $y = -(2t - 4) + \dfrac{C}{e^{\frac{t}{2}}} = 4 - 2t + Ce^{-\frac{t}{2}}$. Since $y(2) = -\dfrac{3}{e}$, $4 - 2 \cdot 2 + Ce^{-1} = -\dfrac{3}{e}$, so $C = -3$ and the unique solution to the IVP is $y(t) = 4 - 2t - 3e^{-t/2}$. Hence $y(0) = 4 - 3 = 1$.

7.(6pts) Consider the autonomous differential equation

$$y' = y^3 - 1 \ .$$

How many stable equilibrium solutions does this differential equation have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution. There is a unique cube root of 1 given by 1, so there is one equilibrium solution $y = 1$ (alternatively, we have the factorization $y^3 - 1 = (y-1)(y^2+y+1)$ and the quadratic polynomial has negative discriminant, hence has no real roots). The sign of $y^3 - 1$ is positive if $y > 1$ and negative if $y < 1$, hence the sole equilibrium solution is unstable and the answer is 0.

8.(6pts) Consider the initial value problem

$$(\tan(t))^2 y' + e^t y = \frac{t+2}{t-2}, \quad y(1) = \pi \ .$$

What is the maximal interval over which the IVP has a unique solution as guaranteed by the existence and uniqueness theorem for first-order linear ODEs?

(a) $0 < t < \pi/2$

(b) $0 < t < 2$

(c) $-2 < t < 2$

(d) $-\pi/2 < t < \pi/2$

(e) $2 < t < 3\pi/2$

Solution. We rewrite the ODE as $y' + \frac{e^t}{(\tan(t))^2} y = \frac{t+2}{(t-2)(\tan(t))^2}$ corresponding to the standard form $y' + p(t)y = g(t)$. The functions $p(t)$ and $g(t)$ are continuous on the interval $0 < t < \pi/2$ containing 1 and both have discontinuities at 0 and $\pi/2$ (note that $\pi/2 < 2$). Hence the maximal interval over which the solution $y(t)$ is guaranteed to exist by the theorem on existence and uniqueness of 1st order linear ODES is $0 < t < \pi/2$.

9.(6pts) Consider the differential equation

$$\big(\cos(y)+1\big)+x\sin(y)\frac{dy}{dx}=0\ .$$

Which of the following equations gives the general solution $\psi(x,y)=c$ of this differential equation in implicit form?

(a) $-\frac{1}{x}(\cos(y)+1)=c$

(b) $\sin(x)+(x+y)\cos(y)=c$

(c) $x(\cos(y)+1)=c$

(d) $y(\sin(x)+1)=c$

(e) $xy(\sin(x)+\cos(y))=c$

Solution. $M(x,y)+N(x,y)y'=0$ so $M=\cos(y)+1$ and $N=x\sin(y)$: $M_y=-\sin(y)$ and $N_x=\sin(y)$ so close but no cigar, $N_x-M_y=2\sin(y)$.
$\dfrac{\mu'}{\mu}=-\dfrac{N_x-M_y}{N}=-\dfrac{2}{x}$ so $\mu(x)=x^{-2}$. Now $\dfrac{\cos(y)+1}{x^2}+\dfrac{\sin(y)}{x}y'=0$ is exact. $\dfrac{\partial\phi(x,y)}{\partial y}=\dfrac{\sin(y)}{x}$ so $\phi(x,y)=-\dfrac{\cos(y)}{x}+h(x)$; $\dfrac{\partial\phi(x,y)}{\partial x}=\dfrac{\cos(y)}{x^2}+h'(x)=\dfrac{\cos(y)+1}{x^2}$ so $h'(x)=\dfrac{1}{x^2}$, $h(x)=-x{-}1$. Hence $-\dfrac{\cos(y)}{x}-\dfrac{1}{x}=c$.

Partial Credit

10.(15pts) Diagonalize the matrix A if possible, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, for $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 0 & 3 \end{bmatrix}$.

Solution. We can find the eigenvalues of A as roots of $\det(A - \lambda I) = 0$. Since A is lower triangular, its eigenvalues are its diagonal entries 1, 4, 3 (as can also easily be seen by cofactor expansion). Since A is 3×3 with three distinct real eigenvalues, it is diagonalizable. To explicitly diagonalize it, we first compute a basis for its eigenspace for each eigenvalue.

If $\lambda = 1$, the matrix $(A - 1I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with an eigenvector $[1, 0, 1]^T$.

For $\lambda = 4$, the matrix $(A - 4I) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ with an eigenvector $[0\ 1\ 0]^T$.

For $\lambda = 3$, the matrix $(A - 3I) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with an eigenvector $[0\ 0\ 1]^T$.

Choose $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Then P contains these eigenvectors of A as columns, with their eigenvalues as corresponding diagonal entries of D, so $A = PDP^{-1}$.

11.(15pts) The set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ listed below is a basis for a subspace W of $\mathbf{R}^4$. Use the Gram-Schmidt process to produce an orthogonal basis for W.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Let $\mathbf{v}_1 = \mathbf{v}_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$. To avoid unnecessary fractions in the following calculations,

we replace $\mathbf{v}_2$ by $\mathbf{v}_2' = 2\mathbf{v}_2$ to get $\mathbf{v}_2' = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$,

$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \frac{8}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$. We may replace $\mathbf{v}_3$ by $\mathbf{v}_3' = 2\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. So

$\{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$, where $\mathbf{v}_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2' = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, is an orthogonal basis of W.

Remark: You need to use the vectors in the order in which they were given. If you do not you will get wrong answers when you try to find the QR decomposition of a matrix. It is OK to rescale your vectors as above because when you normalize them to have length one you will get the same vector whether you rescaled or not.

12. (16pts) Consider a tank with a total capacity of 1000 liters which initially contains 100 liters of water mixed together with a chemical substance at a concentration of 0.1 grams per liter. Suppose that water drains out of the tank at a constant rate of 2 liters per hour, while water with a chemical concentration of 1 grams per liter flows into the tank at a constant rate of 4 liters per hour. Let $Q(t)$ denote the amount of chemicals in the water (with units given in grams).

(a) Write down the initial value problem which allows one to compute $Q(t)$ before the tank reaches full capacity.

Solution:
The initial condition is that at time $t = 0$, there are $100 \cdot 0.1 = 10$ grams in the water, so $Q(0) = 10$. A differential equation describing the rate of change $\frac{dQ}{dt}$ is given by $\dfrac{dQ}{dt}$ = rate in − rate out. The rate of chemicals flowing in is given as $1 \cdot 4 = 4$. The rate of chemicals flowing out is given as $\dfrac{2}{100+2t}Q$, where we use that the total amount of water in the tank at time t is given by $100 + (4-2)t = 100 + 2t$ before the tank reaches full capacity. We conclude that the relevant initial value problem is

$$\frac{dQ}{dt} = 4 - \frac{2}{100+2t}Q, \quad Q(0) = 10 \ .$$

(b) Solve the initial value problem to find $Q(t)$.

Solution:
We may solve the first-order linear ODE by the method of integrating factors. The standard form is $Q' + \dfrac{2}{100+2t}Q = 4$. We compute the integrating factor to be $\mu(t) = e^{\int p(t)dt} = e^{\ln|100+2t|+C}$. Since we only want one integrating factor, let's take $100+2t$. Then $\big((100+2t)Q\big)' = 4(100+2t)$ so $(100+2t)Q = 400t + 4t^2 + C$. Since $Q(0) = 10$, $10^3 = C$ so

$$Q(t) = \frac{4t^2 + 400t + 1000}{100 + 2t}$$
