Math 20580 Tutorial Week 1 - January 19 2019 Name ____________________

Section ________________

Tutorial Worksheet

1. Find all solutions to the linear system by following the given steps

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 3y + z = 4 \\ y + z = 8 \end{cases}.$$

Step 1. Write down the argumented matrix (coefficient and constants) of the system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

Step 2. Replace the second row of the matrix with the second row subtracted by 2 times the first row. In our notation, we would write $R_2 \mapsto R_2 - 2R_1$ to denote this operation. This gives us the matrix

$$\xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{array}\right]$$

Solution:

$$\xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

Step 3. Keep performing the row operations until you can solve the equation.

Solution:

$$\xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -4 & 8 \end{array}\right]$$

This is in echelon form, and the corresponding linear system is already easy to solve, but we keep going on to row reduced echelon form, where it is even easier:

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -4 & 8 \end{array}\right] \xrightarrow{R_3 \mapsto -\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This is in row reduced echelon form; the corresponding linear system is $x = -12$, $y = 10$, $z = -2$ which is the unique solution of the original system.

2. As a generalization to the first question, can you find the solution to the system

$$\begin{cases} x + 2y + 3z = a \\ 2x + 3y + z = b \\ y + z = c \end{cases}$$

for arbitrary real numbers a, b, and c?

Solution: The same row operations as in 1) work:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 2 & 3 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -1 & -5 & b-2a \\ 0 & 1 & 1 & c \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & 2a-b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & -4 & b+c-2a \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2b-3a \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & -4 & b+c-2a \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto -\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2b-3a \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & 1 & (2a-b-c)/4 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2b-3a \\ 0 & 1 & 0 & (5c+b-2a)/4 \\ 0 & 0 & 1 & (2a-b-c)/4 \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (2a+b-7c)/4 \\ 0 & 1 & 0 & (5c+b-2a)/4 \\ 0 & 0 & 1 & (2a-b-c)/4 \end{array}\right]$$

The unique solution is $x = (2a+b-7c)/4$, $y = (5c+b-2a)/4$, $z = (2a-b-c)/4$.

3. Which matrices below are in echelon form? Which are in reduced echelon form?

$$A = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Echelon form :__________________________.

Reduced echelon form :__________________________.

Solution: It is easy to see that all are in echelon form; leading entries of lower non-zero rows are further to the right, all entries below each leading entry are zero, and any zero rows occur below non-zero rows. B and D are not row-reduced echelon because there is a non-zero entry above the leading entry of their second row. A and C are row-reduced echelon since they (are echelon and) have all leading entries 1 and zeros everywhere above and below each leading entry.

Echelon form : A, B, C, D.

Reduced echelon form : A, C.

4. Which columns of the matrix below are pivot and which are free?

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix}$$

Remark: A *free column* is a column which corresponds to a free variable.

Solution: Row-reduce

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

We pause the row reduction here to make to make a comment about writing row-reductions compactly. In a row reduction, if you have *successive* steps involving adding multiples of a *fixed* row to several other rows, with experience they can be displayed as a single step to avoid unnecessary repeated writing of rows. For example, the last two steps which involved adding multiples of R_1 to R_2 and R_3 successively can be represented in a more compact notation

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

This represents the effect of doing two row operations one after the other. Note the order in which they are done does not matter (doing $\xrightarrow{R_2 \mapsto R_2 - 2R_1}$ then $\xrightarrow{R_3 \mapsto R_3 - 3R_1}$ to the original matrix has the same effect as doing $\xrightarrow{R_3 \mapsto R_3 - 3R_1}$ then $\xrightarrow{R_2 \mapsto R_2 - 2R_1}$). On the other hand, for other sequences of row operations, the order in which they are done usually affects the result, and experience shows that trying to represent such series of row operations in a single notation often causes confusion and errors, and so should be avoided.

In the remainder of the row-reduction, we use the above more compact notation.

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - R_2]{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -3 & -3 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\xrightarrow{R_1 \mapsto R_1 + 3R_3} \begin{bmatrix} 1 & 0 & -3 & 0 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The leading entries of the first, second and third rows are their first, second and fourth entries from the left respectively; these are the pivot entries of the original matrix. The pivot columns are therefore the first, second and fourth columns, while the third and fifth columns are non-pivot columns. If this matrix arose as the augmented matrix of a linear system, the bound variables (and columns) would be the first, second and fourth, while the only free variable (or columns) would be the third (since the fifth and last column does not correspond to a variable, but rather contains the constant terms of the linear equations).

Note that to find the pivot positions etc above, we put the matrix in row-reduced echelon form. This is more work than was strictly necessary to find the pivot positions. It actually is enough to get it the matrix to echelon form and take the pivot positions as the positions of leading entries of the non-zero rows of the echelon form matrix (since further row reduction to row reduced echelon form does not change the positions of the leading entries).

5. Find all solutions to the linear system

$$\begin{cases} x - 3y + z - w = 2 \\ 2x - 6y + 3z - w = 3 \\ 3x - 9y + 5z - w = 4 \end{cases}.$$

Row-reduce the augmented matrix of the linear system:

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 2 & -6 & 3 & -1 & 3 \\ 3 & -9 & 5 & -1 & 4 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 \mapsto R_3 - 2R_2]{R_1 \mapsto R_1 - R_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in row reduced echelon form. The corresponding linear system is

$$\left\{\begin{array}{rrrrrrrrr} x & - & 3y & & & - & 2w & = & 3 \\ & & & & z & + & w & = & -1 \\ & & & & & & 0 & = & 0 \end{array}\right.$$

where the last equation $0 = 0$ can obviously be omitted (note on the other hand if we had ended up with any equations $0 = c$ where c is a non-zero constant, then the original system would be inconsistent i.e. have no solutions).

The pivot positions of the row reduced echelon matrix are the first entry of the first row and third entry of the second row; the pivot columns are the first and third columns, and the corresponding variables, x and z, are bound. The other variables, y and w, are free. Rewrite the last linear system (with $0 = 0$ omitted) so the bound variables are on the left and free variables are on the right:

$$\left\{\begin{array}{rrrrrrrr} x & & = & 3 & + & 3y & + & 2w \\ & z & = & -1 & & & - & w \end{array}\right. .$$

The above is one form of the solution; the free variables y and w can take arbitrary values and the bound variables x and z are then determined by the above equations. Letting the values of y and w be s and t respectively, another form of the solution is

$$\left\{\begin{array}{rcl} x & = & 3 + 3s + 2t \\ y & = & s \\ z & = & -1 - t \\ w & = & t \end{array}\right.$$

where s and t are arbitrary scalars. Although we hadn't covered vectors at the time of the first tutorial, we have by now and note that we can write the solution also in the following *parametric vector form*

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where s and t are arbitrary scalars.