HECKE ALGEBRAS AND

REFLECTIONS IN COXETER GROUPS

by

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ABSTRACT

To any reflection subgroup of a Coxeter system, we associate a canonical set of Coxeter generators. A geometric criterion is given for a set of reflections to be the canonical set of Coxeter generators of some reflection subgroup, and used to classify the reflection subgroups of affine Weyl groups up to isomorphism.

A new proof is given of a theorem of A. Björner and M. Wachs stating that the simplicial complex of an open Bruhat interval is a sphere. By associating a reflection subgroup to a Bruhat interval, it is shown that the Kazhdan-Lusztig polynomials $P_{v,w}$ and $Q_{v,w}$ ($\ell(w) - \ell(v) \leq 4$) depend only on the poset [v,w] and have non-negative coefficients.

We describe a construction which produces, from very general data, mutually inverse elements in the incidence algebra of a locally finite poset. We show how this construction may be used to produce the polynomials $R_{x,y}$ defined by Kazhdan and Lusztig for elements x, y of a Coxeter system.

Four conjectural positivity properties of the structure constants of the generic Hecke algebra of a Coxeter group are described. All four properties are proved by elementary combinatorial arguments in the case of the Coxeter groups which are free products of cyclic groups of order two.

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LIST OF SPECIAL NOTATIONS

$\mathbb{N}$; the natural numbers (including zero)

$\mathbb{Z}$; the integers

$\mathbb{Q}$; the rational numbers

$\mathbb{R}$; the real numbers

$\mathbb{R}^+$; the non–negative real numbers

$\binom{m}{n}$; binomial coefficient

$\mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$; Laurent polynomials (in an indeterminate $q^{\frac{1}{2}}$); with non-negative integral coefficients

$M_{2\times 2}(R)$; 2×2 matrices with entries in a (unital) ring R

A^t; transpose of a matrix A

$E \otimes_R A$, $E \otimes_f A$; the A-module obtained by tensoring the R-module E with the ring A, A being regarded as an R-module via a homomorphism $f\colon R \longrightarrow A$ (A, R being commutative rings)

E^*; dual of a vector space E

$\langle\ ,\ \rangle$; pairing between a vector space and its dual

$\|u\|^2$; $(u \mid u)$ where $(\cdot \mid \cdot)$ is a symmetric bilinear form on a real vector space

$\min(A), \max(A)$; the minimum (maximum) element of a subset A of a poset (when it exists)

$A \setminus B$; the set difference of A and B

$\mathcal{P}(A)$; the power set of a set A

$\#A$ or $\#(A)$; the cardinality of the set A

$\langle A\rangle$; the subgroup of a group G generated by a subset A of G

$\delta_{x,y}$; the Kronecker delta

$\dot{\bigcup}$; the indicated union is of disjoint sets

INTRODUCTION

For any Coxeter system (W, R), Kazhdan and Lusztig define in [KL1] a family $P_{v,w}(v, w \in W)$ of polynomials. These Kazhdan-Lusztig polynomials have deep connections with algebraic groups, and Lie algebras.

For example, if (W, R) is a crystallographic Coxeter system, then ([KL2], [H]) $P_{v,w}$ is the Poincaré series of the stalk (at a point in the Bruhat cell corresponding to v) of the cohomology sheaf of the intersection cohomology complex of the Schubert variety corresponding to w (constructed from an infinite-dimensional group associated to a Kac-Moody Lie algebra with (W, R) as Weyl group).

As another example, if (W, R) is a finite Weyl group, the values of the Kazhdan-Lusztig polynomials at 1 are the multiplicities, as composition factors of Verma modules, of certain irreducible highest-weight modules for the corresponding semisimple complex Lie algebra ([BB],[BK]).

Finally, we mention that the Kazhdan-Lusztig polynomials are used to define the (left,right and two-sided) cells of W and thus give rise to certain cell representations of the generic Hecke algebra of (W, R). In this way, the Kazhdan-Lusztig polynomials enter the representation theory of algebraic groups ([L3]).

Despite these important applications, very little is known about Kazhdan-Lusztig polynomials in general. The polynomials $P_{v,w}$ are defined purely algebraically, but certain properties (non-negativity of their coefficients, Property A of cells) expected to hold for general Coxeter systems have been proved only by exploiting interpretations of the $P_{v,w}$ such as those described above (and then only for crystallographic Coxeter systems).

The major part of this thesis has arisen from an attempt to obtain more detailed information about the Kazhdan-Lusztig polynomials. In virtue of the above-mentioned facts about the $P_{v,w}$, it is to be expected that a better understanding of these polynomials in general would have important applications.

Chapter 0 fixes notation concerning Coxeter groups and Hecke algebras, and recalls the definition of Kazhdan-Lusztig polynomials.

Chapter 1 begins a study of reflection subgroups of Coxeter systems. It is shown that any reflection subgroup of a Coxeter system has a canonical set of Coxeter generators. The results are reformulated in terms of a labelled directed graph naturally associated to a Coxeter system, and we give a characterisation of these "Bruhat graphs" by properties of their "dihedral" subgraphs.

In Chapter 3, we require a slight extension of the usual geometric realisation of a Coxeter system. The necessary properties are given at the beginning of Chapter 2; we also observe that standard properties of the geometric realisation continue to hold "generically".

Chapter 3 gives a criterion for a set of reflections to be the canonical set of generators of a reflection subgroup of a Coxeter system. The condition is that the inner products of distinct elements from the corresponding set of positive roots (in a goemetric realisation of the Coxeter system) should all lie in a certain set. We describe an algorithm which, given a finite set of reflections, produces the canonical generators of the group they generate.

In Chapter 4, the criterion of Chapter 3 is applied to classify reflection subgroups of affine Weyl groups up to isomorphism as Coxeter groups. Subsequent chapters are independent of the results of this chapter.

Chapter 5 contains a new proof of a theorem of A. Björner and M. Wachs ([BW]) stating that the simplicial complex associated to an open Bruhat interval is a sphere; the proof is based on a natural decomposition of the simplicial complex into cells. We then show that the reflection subgroup generated by the ratios $x^{-1}y$ of elements x, y of a closed Bruhat interval is actually generated by the ratios of the elements in some fixed maximal chain.

The first part of Chapter 6 describes a construction which produces, from general data, mutually inverse elements in the incidence algebra of a locally finite poset. Under an additional assumption, one obtains an element of the incidence algebra satisfying the same identity $\sum_y R_{x,y}\bar{R}_{y,z} = \delta_{x,z}$ as the polynomials $R_{x,y}$ defined by Kazhdan and Lusztig in [KL1], and one may define formal analogues of the Kazhdan-Lusztig polynomials in this context. We show how this incidence algebra construction applied to Bruhat order gives rise to the polynomials $R_{x,y}$;

the data required for the construction is obtained from certain special total orderings of the reflections of the Coxeter system.

In Chapter 7, we begin a study of properties of the structure constants of the generic Hecke algebra of a Coxeter system. Four conjectural positivity properties [P1]–[P4] of these structure constants are described:

$$\text{[P1]} \qquad C'_x T_y \in \sum_z \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] T_z \qquad (x, y \in W)$$

$$\text{[P2]} \qquad T^{-1}_{x^{-1}} T_y \in \sum_z \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] C_z \qquad (x, y \in W)$$

$$\text{[P3]} \qquad C'_x C'_y \in \sum_z \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] C'_z \qquad (x, y \in W)$$

$$\text{[P4]} \qquad C'_x C_y \in \sum_z \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] C_z \qquad (x, y \in W)$$

Conjectures [P1] and [P2] generalise the conjectured positivity of the Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials, and [P3] is known to hold for crystallographic Coxeter systems. For finite Coxeter systems, [P1] and [P2] are equivalent and [P3] and [P4] are equivalent.

The remainder of Chapter 7 is devoted to a number of special results concerning the Kazhdan-Lusztig polynomials for arbitrary Coxeter systems. We give a number of equivalent conditions for a Bruhat interval to be isomorphic to an interval in a dihedral group. It is shown that the Kazhdan-Lusztig polynomials $P_{v,w}$ $(\ell(w) - \ell(v) \leq 4)$ depend only on the isomorphism type of the poset $[v, w]$ and have non-negative coefficients.

In the last chapters, we give elementary combinatorial proofs of [P1]–[P4] for universal Coxeter systems. Our technique for showing that the Laurent polynomials arising as structure constants have non-negative coefficients is to construct sets whose cardinalities are these coefficients. The explicit definition of these sets is quite intricate, particularly in the case of [P2].

Chapter 0

PRELIMINARIES

This brief chapter fixes some notation and terminology concerning Coxeter groups and Hecke algebras, and recalls the definition of Kazhdan-Lusztig polynomials.

Let (W, R) be a Coxeter system. We say that (W, R) is crystallographic if for all $r, s \in R$ with $r \neq s$, the order $n_{r,s}$ of rs is either 2,3,4,6 or ∞; (W, R) will be called a universal Coxeter system if $n_{r,s} = \infty$ for all $r, s \in R$ with $r \neq s$. Thus, a universal Coxeter group is isomorphic to a free product of cyclic groups of order 2.

The set $\bigcup_{w \in W} wRw^{-1}$ of reflections of (W, R) will usually be denoted by T, and the length function of (W, R) will be denoted by ℓ or ℓ_W. It will be convenient to let $N\colon W \longrightarrow \mathcal{P}(T)$ denote the function defined by $N(w) = \{\, t \in T \mid \ell(wt) < \ell(w)\ (w \in W)\,\}$; some properties of N are given in Chapter 1.

Let $\mathcal{A}$ be any commutative (associative, unital) ring and q be an element of $\mathcal{A}$. The Hecke algrebra $\mathcal{H}_{q,\mathcal{A}}(W)$ is the (associative, unital) $\mathcal{A}$-algebra generated by generators $T_r (r \in R)$ subject to relations

$$T_r^2 = (q-1)T_r + q \cdot 1$$

$$\overbrace{(T_r T_s T_r \dots)}^{n_{r,s}} = \overbrace{(T_s T_r T_s \dots)}^{n_{r,s}} \ (r, s \in R,\ r \neq s,\ n_{r,s} \neq \infty)$$

As an $\mathcal{A}$-module, $\mathcal{H}_{q,\mathcal{A}}(w)$ is free with $\mathcal{A}$-basis $\{T_w\}_{w \in W}$ and the multiplication is determined by

$$T_r T_w = \begin{cases} T_{rw} & (\ell(rw) > \ell(w)) \\ qT_{rw} + (q-1)T_w & (\ell(rw) < \ell(w)). \end{cases}$$

We will generally be concerned with the case when $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ is the ring of Laurent polynomials with integral coefficients in an indeterminate $q^{\frac{1}{2}}$. For $w \in W$, we then write $\varepsilon_w = (-1)^{\ell(w)}$, $q_w^{\pm\frac{1}{2}} = q^{\pm\ell(w)/2}$ and denote $\mathcal{H}_{q,\mathcal{A}}(W)$ simply by $\mathcal{H}(W)$. Writing $\tilde{T}_w = q_w^{-\frac{1}{2}} T_w$, $\{\tilde{T}_w\}_{w\in W}$ is an $\mathcal{A}$-basis of $\mathcal{H}(W)$ and the multiplication is determined by

$$\tilde{T}_r\tilde{T}_w = \begin{cases} \tilde{T}_{rw} & (\ell(rw) > \ell(w)) \\ \tilde{T}_{rw} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{T}_w & (\ell(rw) < \ell(w)). \end{cases}$$

The elements $\{\tilde{T}_w\}_{w\in W}$ of $\mathcal{H}(W)$ are invertible, and we define $\tilde{R}_{x,w} \in \mathcal{A}$ $(x, w \in W)$ by

$$\tilde{T}_{w^{-1}}^{-1} = \sum_{x\in W} \tilde{R}_{x,w}\tilde{T}_x.$$

In the notation of [KL1], $\tilde{R}_{x,w} = q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \bar{R}_{x,w}$, where $a \longmapsto \bar{a}$ is the ring involution of $\mathcal{A}$ determined by $q^{\frac{1}{2}} \longmapsto q^{-\frac{1}{2}}$. We have the recurrence formula ([KL1])

$$\tilde{R}_{x,w} = \begin{cases} \tilde{R}_{xr,wr} & (xr < x) \\ \tilde{R}_{xr,wr} + \alpha\tilde{R}_{x,wr} & (xr > x) \end{cases} \tag{0.1}$$

if $r \in R$ and $wr < w$, where $\alpha = q^{-\frac{1}{2}} - q^{\frac{1}{2}}$.

The ring involution $a \longmapsto \bar{a}$ of $\mathcal{A}$ extends to a ring involution $h \longmapsto \bar{h}$ of $\mathcal{H}(W)$ defined by $\sum_{w\in W} a_w\tilde{T}_w \longmapsto \sum_{w\in W} \bar{a}_w\tilde{T}_{w^{-1}}^{-1}$.

The following fundamental fact is proved in [KL1]:

<u>Theorem</u> For any $w \in W$, there exists a unique element

$$C_w \in \tilde{T}_w + \sum_{v<w} q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]\tilde{T}_v \qquad \text{such that } \bar{C}_w = C_w. \qquad \square$$

The Kazhdan-Lusztig polynomials $P_{y,w} \in \mathcal{A}$ are defined by

$$C_w = \sum_{y\in W} \varepsilon_y\varepsilon_w q_w^{\frac{1}{2}} q_y^{-1} \bar{P}_{y,w} T_y \ (w \in W).$$

For any $w \in W$, let

$$C'_w = \sum_{y \in W} q_w^{-\frac{1}{2}} P_{y,w} T_y.$$

Then C'_w is the unique element of $\tilde{T}_w + \sum\limits_{v<w} q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]\tilde{T}_v$ such that $\bar{C}'_w = C'_w$.

Finally, we recall that the inverse Kazhdan-Lusztig polynomials $Q_{x,y} \in \mathcal{A}$ (for x and y in W) may be defined by

$$T_y = \sum_{x \in W} \varepsilon_x \varepsilon_y q_x^{\frac{1}{2}} Q_{x,y} C'_x.$$

Chapter 1

REFLECTION SUBGROUPS OF COXETER GROUPS

The main result of this chapter is that any reflection subgroup of a Coxeter system has a canonical set of Coxeter generators (Theorem (1.8)).

The proof uses the function which maps an element of the Coxeter group to the set of reflections in the positive roots made negative by that element (in the standard geometric realisation ([De2])). However, we adopt an abstract approach here and defer more geometric considerations to Chapter 3.

1.1 Let W be a group and R be a set of involutions generating W. Define the corresponding length function $\ell\colon W \longrightarrow \mathbb{N}$ by $\ell(w) = \min\{\, n \in \mathbb{N} \mid w \in R^n \,\}$ $(w \in W)$. If $w = r_1 \ldots r_n$ $(r_i \in R, n = \ell(w))$ then $r_1 \ldots r_n$ is called a reduced expression for w.

The set $T = \bigcup\limits_{w \in W} wRw^{-1}$ is the set of reflections of W. The power set $\mathcal{P}(T)$ of T will be regarded as an abelian group with symmetric difference as addition:

$$A + B = (A \cup B)\backslash(A \cap B) \qquad (A, B \in \mathcal{P}(T)).$$

Note that there exists at most one function $N: W \longrightarrow \mathcal{P}(T)$ satisfying (1.1.1) and (1.1.2) below:

(1.1.1) $N(r) = \{r\} \quad (r \in R)$

(1.1.2) $N(xy) = y^{-1}N(x)y + N(y) \quad (x, y \in W)$.

This is because the values of N are determined on R by (1.1.1) and then on $\langle R \rangle = W$ by the cocycle condition (1.1.2). The following two lemmas, which are implicit in [Bo] (Ch.IV, no 1.4), show that such a function N can only exist if (W, R) is a Coxeter system.

1.2 Lemma. Suppose that $N: W \longrightarrow \mathcal{P}(T)$ satisfies (1.1.1) and (1.1.2). Then

(i) For any $w \in W$, $\#N(w) = \ell(w)$. If $w = r_1 \dots r_n$ $(r_i \in R, n = \ell(w))$ then $N(w) = \{t_1, \dots, t_n\}$ where $t_i = r_n \dots r_{i+1} r_i r_{i+1} \dots r_n$ $(i = 1, \dots, n)$.

(ii) For all $t \in T$, $t \in N(t)$

(iii) If $w \in W$, $N(w) = \{\, t \in T \mid \ell(wt) < \ell(w) \,\}$. For any $t \in T$, $\ell(wt) \neq \ell(w)$.

Proof

(i) Suppose that $w = r_1 \dots r_n$ where for $i = 1, \dots, n$ $r_i \in R$, and $n = \ell(w)$. Let $t_i = r_n \dots r_{i+1} r_i r_{i+1} \dots r_n$. If $t_i = t_j$ $(i > j)$, then

$$\begin{aligned} w &= r_1 \dots r_{j-1} r_{j+1} \dots r_n t_j \\ &= r_1 \dots r_{j-1} r_{j+1} \dots r_n t_i \\ &= r_1 \dots r_{j-1} r_{j+1} \dots r_{i-1} r_{i+1} \dots r_n \in W^{n-2} \end{aligned}$$

contrary to $\ell(w) = n$. Hence the t_i are all distinct. By repeated application of (1.1.2),

$$\begin{aligned} N(w) &= N(r_n) + (r_n N(r_{n-1}) r_n) + \dots + (r_n \dots r_2 N(r_1) r_2 \dots r_n) \\ &= \{t_n\} + \{t_{n-1}\} + \dots + \{t_1\} \\ &= \{t_1, \dots, t_n\} \end{aligned}$$

so $\#N(w) = n = \ell(w)$.

(ii) Let $t \in T$ and write $t = r_1 \dots r_{n-1} r_n r_{n-1} \dots r_1$ with n minimal. Define $s_1, \dots, s_{2n-1} \in R$ by

$$(s_1, \dots, s_{n-1}, s_n, s_{n+1}, \dots, s_{2n-1}) = (r_1, \dots, r_{n-1}, r_n r_{n-1}, \dots, r_1)$$

and let $t_i = s_{2n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{2n-1}$ $(i = 1, \dots, 2n-1)$. If $1 \le i \le n$, then

$$tt_i t = (r_1 \dots r_n \dots r_1)(r_1 \dots r_i \dots r_1)(r_1 \dots r_n \dots r_1) = t_{2n-i}$$

and so $t_{2n-i} = t$ if and only if $t_i = t$. But $t_i \ne t$ for $i \le n-1$ by the assumed minimality of n. This shows that $t_j = t$ if and only if $j = n$ $(1 \le j \le 2n-1)$. As in (i), $N(t) = \{t_{2n-1}\} + \dots + \{t_1\}$, so $t \in N(t)$ as claimed.

(iii) Write $w = r_1 \dots r_n$ with $n = \ell(w)$. Then $N(w) = \{t_1, \dots, t_n\}$ where

$$t_i = r_n \dots r_{i+1} r_i r_{i+1} \dots r_n \ (i = 1, \dots, n).$$

Now
$$wt_i = r_1 \dots r_{i-1} r_{i+1} \dots r_n \in W^{n-1}$$

and so $\ell(wt_i) \le n-1 < \ell(w)$. Hence if $t \in N(w)$ then $\ell(wt) < \ell(w)$.

Now suppose that $t \in T$ and $t \notin N(w)$. Then $t \notin t^{-1}N(w)t$ but $t \in N(t)$, so $t \in t^{-1}N(w)t + N(t) = N(wt)$. By what has just been proved, $\ell(w) = \ell(\,(wt)t) < \ell(wt)$. □

1.3 Lemma. The following are equivalent:

(i) (W, R) is a Coxeter system

(ii) There exists a function $N\colon W \longrightarrow \mathcal{P}(T)$ satisfying (1.1.1) and (1.1.2).

<u>Proof</u> The implication (i) implies (ii) is well-known, but can be proved as follows. Suppose that (i) holds. By ([Bo] Ch IV , no 1.4) there is a representation of W as a group of permutations of $T \times \{1, -1\}$ such that $r(t, n) = (rtr, (-1)^{\delta_{r,t}} n)$ $(t \in T, n \in \{1, -1\}, r \in R)$. For $w \in W$, let $N(w) = \{\, t \in T \mid w(t, n) = (wtw^{-1}, -n)$ for $n = \pm 1 \,\}$. The above formula for $r(t, n)$ shows that (1.1.1) holds and that $N(ry) = \{y^{-1}ry\} + N(y)$ $(r \in R, y \in W)$; then (1.1.2) follows by induction on $\ell(x)$.

Now suppose that $N\colon W \longrightarrow \mathcal{P}(T)$ satisfies (1.1.1) and (1.1.2). To prove (i), it suffices to show that (W, R) satisfies the "exchange condition" (1.3.1) below ([Bo] Ch IV, no 1.6):

(1.3.1) if $w \in W$, $r \in R$ and $\ell(wr) \leq \ell(w)$, then for any reduced expression $w = r_1 \dots r_n$, there exists $i \in \{1, \dots, n\}$ such that $r_i \dots r_n = r_{i+1} \dots r_n r$.

In fact, the "strong exchange condition" (1.3.2) holds:

(1.3.2) if $r_i \in R$ $(i = 1, \dots, n)$, $t \in T$ and $\ell(r_1 \dots r_n t) \leq \ell(r_1 \dots r_n)$ then there exists $i \in \{1, \dots, n\}$ such that $r_i \dots r_n = r_{i+1} \dots r_n t$.

For let $t_i = r_n \dots r_{i+1} r_i r_{i+1} \dots r_n$ $(i = 1, \dots, n)$. If $\ell(r_1 \dots r_n t) \leq \ell(r_1 \dots r_n)$, then (1.2) (iii) implies that $t \in N(r_1 \dots r_n)$. But by (1.1.1) and (1.1.2) we have $N(r_1 \dots r_n) = \{t_n\} + \dots + \{t_1\}$. Hence $t = t_i$ for some i, and so $r_i \dots r_n = r_{i+1} \dots r_n t$ as required. $\square$

In (1.4)–(1.27), (W, R) denotes a Coxeter system. We maintain the conventions of (1.1) and let $N\colon W \longrightarrow \mathcal{P}(T)$ be the function determined by (1.1.1) and (1.1.2).

The following simple lemma will often prove useful; it is equivalent to ([Sp], Prop. 1).

1.4 Lemma. Let $t = r_1 \dots r_{2n+1} \in T$ $(r_i \in R)$ with $\ell(t) = 2n + 1$. Then $t = r_1 \dots r_n r_{n+1} r_n \dots r_1$.

Proof Let $x = r_n \dots r_1$ and $y = r_{n+2} \dots r_{2n+1}$. Then $\ell(x) = \ell(y) = n < n+1 = \ell(r_{n+1}x) = \ell(r_{n+1}y)$ and $r_{n+1}yt = x$. By (1.3.2), $t = r_{2n+1} \dots r_i \dots r_{2n+1}$ for some $i \in \{n+1, \dots, 2n+1\}$, and so $x = r_{n+1} \dots r_{i-1} r_{i+1} \dots r_{2n+1}$. Since $\ell(x) = n$, this is a reduced expression for x. Since $\ell(r_{n+1}x) > \ell(x)$, it follows that $i = n+1$ and so $x = y$. Hence $t = x^{-1} r_{n+1} y = x^{-1} r_{n+1} x$ as required. $\square$

Taking $n \geq 1$ in (1.4) immediately gives the following

1.5 Corollary. If $t \in T \backslash R$, there exists $r \in R$ with $\ell(rtr) = \ell(t) - 2$. $\square$

1.6 For any subgroup W' of W, let

(1.6.1) $S(W') = \{\, t \in T \mid N(t) \cap W' = \{t\} \,\}$.

Note that $S(W') \subseteq W'$. If W' is a reflection subgroup of W, it will be shown in (1.8) that $S(W')$ is a set of Coxeter generators for W'. The proof will use (1.3) and the following

1.7 Lemma. Let W' be a subgroup of W.

(i) If $r \in R \setminus W'$ then $S(rW'r) = rS(W')r$.

(ii) If $t \in W' \cap T$ then there exists $m \in \mathbb{N}$ and $t_0, \ldots, t_m \in S(W')$ such that $t = t_m \ldots t_1 t_0 t_1 \ldots t_m$.

(iii) For $w \in W$ let $N'(w) = N(w) \cap W'$. Then if $x \in W$ and $y \in W'$, $N'(xy) = y^{-1}N'(x)y + N'(y)$.

<u>Proof</u>

(i) Let $t \in S(W')$. Then

$$\begin{aligned} N(rtr) \cap rW'r &= (\{rtrtr\} + rN(t)r + \{r\}) \cap rW'r \\ &= r[(\{trt\} + N(t) + \{r\}) \cap W']r \\ &= r[N(t) \cap W']r \qquad \text{since } r \notin W' \text{ and } trt \notin tW't = W' \\ &= \{rtr\} \end{aligned}$$

Hence $rtr \in S(rW'r)$. This proves that $rS(W')r \subseteq S(rW'r)$. But $r \in R \backslash rW'r$, so also $rS(rW'r)r \subseteq S(W')$ and (i) is proved.

(ii) The proof will be by induction on $\ell(t)$. If $\ell(t) = 1$, then $t \in W' \cap R$ so $N(t) \cap W' = \{t\} \cap W' = \{t\}$ and it is sufficient to take $m = 0$ and $t_0 = t \in S(W')$.

Suppose now that $\ell(t) > 1$ and that (ii) holds for all subgroups W'' of W and reflections $t'' \in W'' \cap T$ with $\ell(t'') < \ell(t)$. By (1.5), there exists some $r \in R$ such that $\ell(rtr) < \ell(t)$. Let $W'' = rW'r$ and $t'' = rtr$. By the inductive assumption there exist $m \in \mathbb{N}$ and $t_0, \ldots,\ t_m \in S(W'')$ such that $t'' = t_m \ldots t_1 t_0 t_1 \ldots t_m$. There are now two cases to consider.

Case 1. $r \in W''$

Then $W' = rW''r = W''$ and $N(r) \cap W'' = \{r\} \cap W'' = \{r\}$. Let $t_{m+1} = r$; then $t_i \in S(W'') = S(W')$ $(i = 0, 1, \ldots, m+1)$ and $t = rt''r = rt_m \ldots t_1 t_0 t_1 \ldots t_m r = t_{m+1} \ldots t_1 t_0 t_1 \ldots t_{m+1}$.

Case 2. $r \notin W''$

Let $t'_i = rt_ir$ $(i = 0, \ldots, m)$. Then $t'_i \in rS(W'')r = S(rW''r)$ by (i) and $t = rt''r = rt_m \ldots t_1t_0t_1 \ldots t_mr = t'_m \ldots t'_1t'_0t'_1 \ldots t'_m$.

This completes the proof of (ii).

(iii) If $x \in W$ and $y \in W'$, then

$$\begin{aligned} N'(xy) &= [y^{-1}N(x)y + N(y)] \cap W' \\ &= (y^{-1}N(x)y \cap y^{-1}W'y) + (N(y) \cap W') \quad \text{noting } W' = y^{-1}W'y \\ &= y^{-1}(N(x) \cap W')y + (N(y) \cap W') \\ &= y^{-1}N'(x)y + N'(y) \end{aligned}$$

as required. $\square$

Recall that a subgroup W' of W is said to be a reflection subgroup of W if it is generated by the reflections it contains, i.e. if $W' = \langle W' \cap T \rangle$. We may now prove the main result of this chapter.

1.8 Theorem. Let W' be a reflection subgroup of W', and set $R' = S(W')$. Then

(i) (W', R') is a Coxeter system

(ii) $W' \cap T = \bigcup\limits_{w \in W'} wR'w^{-1}$

(iii) For $w \in W'$, $N(w) \cap W' = \{\, t \in W' \cap T \mid \ell'(wt) < \ell'(w) \,\}$ where $\ell' \colon W' \longrightarrow \mathbb{N}$ is the length function of (W', R').

Proof Let $W'' = \langle R' \rangle$ and $T' = \bigcup\limits_{w \in W''} wR'w^{-1}$. Since $R' \subseteq T \cap W'$ it follows that $W'' \subseteq W'$ and $T' \subseteq T \cap W'$. By (1.7) (ii), $T \cap W' \subseteq T'$, hence $T' = T \cap W'$ and so $W' = \langle W' \cap T \rangle = \langle T' \rangle \subseteq \langle R' \rangle \subseteq W'$. This shows that $W' = \langle R' \rangle = W''$ and also that (ii) holds, since $T \cap W' = T' = \bigcup\limits_{w \in W'} wR'w^{-1}$.

Now $T' \subseteq T$ and so $\mathcal{P}(T')$ is a subgroup of $\mathcal{P}(T)$. Define $N' \colon W' \longrightarrow \mathcal{P}(T')$ by $N'(w) = N(w) \cap W'$. For $r' \in R' = S(W')$, we have $N'(r') = \{r'\}$ by definition of $S(W')$. Also, if $x', y' \in W'$, (1.7) (iii) implies that $N'(x'y') = y'^{-1}N'(x')y' + N'(y')$. Since R' consists of involutions and $W' = \langle R' \rangle$, Lemma (1.3) shows that (W', R') is a Coxeter system, and Lemma (1.2) (iii) then proves (iii). $\square$

If W' is a reflection subgroup of W, then $R' = S(W')$ will be called its set of canonical generators and (W', R') will be said to be a reflection subsystem of (W, R); we then write $(W', R') \leq (W, R)$.

Some simple properties of reflection subsystems are given below.

1.9 Corollary.

(i) Suppose that $(W', R') \leq (W, R)$, and that W'' is a subgroup of W' and $R'' \subseteq W''$. Then $(W'', R'') \leq (W', R')$ if and only if $(W'', R'') \leq (W, R)$.

(ii) If $(W', R') \leq (W, R)$, $(W'', R'') \leq (W, R)$ and W'' is conjugate to W' then the Coxeter systems (W', R') and (W'', R'') are isomorphic.

Proof (i) Let $T' = T \cap W'$ and $N'(w) = N(w) \cap W'$ $(w \in W)$. Note that $N(w) \cap W'' = N'(w) \cap W''$, for any $w \in W$.

Suppose firstly that $(W'', R'') \leq (W, R)$. Then $W'' = \langle W'' \cap T \rangle = \langle W'' \cap W' \cap T \rangle = \langle W'' \cap T' \rangle$ and $R'' = \{\, t \in T \mid N(t) \cap W'' = \{t\} \,\} = \{\, t \in T \mid N'(t) \cap W'' = \{t\} \,\}$, the last set being the set of canonical generators of W'' as a reflection subgroup of (W', R'). Hence $(W'', R'') \leq (W', R')$.

Conversely, suppose that $(W'', R'') \leq (W', R')$. Then $W'' = \langle W'' \cap T' \rangle \subseteq \langle W'' \cap T \rangle \subseteq W''$ so $W'' = \langle W'' \cap T \rangle$. Here $R'' = \{\, t \in T \mid N'(t) \cap W'' = \{t\} \,\} = \{\, t \in T \mid N(t) \cap W'' = \{t\} \,\} = S(W')$ so $(W'', R'') \leq (W, R)$.

(ii) It suffices to prove this when W' and W'' are conjugate by a simple reflection, say $W' = rW''r$ with $r \in R$. If $r \in W''$ then $W' = W''$ and $R' = S(W') = S(W'') = R''$, so $(W', R') = (W'', R'')$. If $r \notin W''$ then $R'' = rR'r$ by (1.7)(i); the map $x \longmapsto rxr$ is an isomorphism of groups $W' \longrightarrow W''$ which restricts to a bijection $R' \longrightarrow R''$ i.e. an isomorphism of Coxeter systems. $\square$

1.10 Remark. There is a standard partial order on W, the Bruhat order; this will be discussed in Chapter 5. If $(W', R') \leq (W, R)$, then the partial order induced on W' as a subset of W is not generally the Bruhat order of (W', R'). We now define a directed graph intimately related to the Bruhat order of (W, R), and prove that the graph associated to any reflection subsystem is a "full" subgraph of this graph.

1.11 Definition. The Bruhat graph $\Gamma_{(W,R)}$ is the directed graph with vertex set W and edge set $E_{(W,R)} = \{\, (x, y) \in W \times W \mid x^{-1}y \in T,\ \ell(x) < \ell(y) \,\}$.

For any subset X of W, there is a corresponding subgraph Γ_X with vertex set X and edge set $E_{(W,R)} \cap (X \times X)$.

1.12 Remark. There is a partial order $\leq$ on W such that $x \leq y$ if and only if there exists a sequence $x_0, x_1, \ldots, x_n$ of elements of W such that $x_0 = x, x_n = y$ and $(x_{i-1}, x_i) \in E_{(W,R)}$ for $i = 1, \ldots, n$. This partial order is the Bruhat order.

1.13 Proposition. Suppose that $(W', R') \leq (W, R)$. Then

(i) $\Gamma_{(W',R')} = \Gamma_{W'}$

(ii) Let xW' be any left coset of W' in W. Then xW' contains a unique element x_0 of minimal length. The map $\theta\colon W' \longrightarrow xW'$ defined by $w \longmapsto x_0 w$ is an isomorphism of directed graphs $\Gamma_{W'} \longrightarrow \Gamma_{xW'}$ For all $w \in W'$,

(1.13.1) $N(w) \cap W' = N(\theta(w)\,) \cap W'$.

Proof Let ℓ' denote the length function on (W', R') and define $N'(w) = N(w) \cap W'$ $(w \in W)$. Let $T' = T \cap W'$.

(i) Both $\Gamma_{(W',R')}$ and $\Gamma_{W'}$ are directed graphs with vertex set W'. Hence it remains to check that they have the same edge set, i.e. that $E_{(W',R')} = E_{(W,R)} \cap (W' \times W')$. Now $E_{(W,R)} \cap (W' \times W')$

$$\begin{aligned}
&= \{\, (x, y) \in W' \times W' \mid x^{-1}y \in T,\ \ell(x) \leq \ell(y) \,\} \\
&= \{\, (x, y) \in W' \times W' \mid x^{-1}y \notin N(x), x^{-1}y \in T' \,\} \text{ by (1.2)(iii)} \\
&= \{\, (x, y) \in W' \times W' \mid x^{-1}y \notin N'(x), x^{-1}y \in T' \,\} \\
&= \{\, (x, y) \in W' \times W' \mid x^{-1}y \in T', \ell'(x) \leq \ell'(y) \,\} \text{ by (1.8)(iii)} \\
&= E_{(W',R')} \text{ by definition, so (i) is proved.}
\end{aligned}$$

(ii) Choose some $x_0 \in xW'$ with $\ell(x_0)$ minimal. Then for any $t \in T'$, $\ell(x_0 t) \geq \ell(x)$. This shows that $N'(x_0) = N(x_0) \cap W' = \emptyset$. It follows from (1.7) (iii) that $N'(w) = N'(\theta(w))$ for all $w \in W'$; hence (1.13.1) is proved. Now θ is certainly a bijection; to prove that it is an isomorphism of directed graphs, it remains to check that if $y, z \in W'$, then (y, z) is an edge of $\Gamma_{W'}$ if and only if $(\theta(y), \theta(z))$ is an edge of $\Gamma_{xW'} = \Gamma_{x_0 W'}$. Fix $y, z, \in W'$.

Now $(y, z) \in E_{(W,R)} \cap (W' \times W')$ if and only if $y^{-1}z \in T$ and $\ell(y) \leq \ell(z)$. This holds if and only if $y^{-1}z \in T'$ and $y^{-1}z \notin N'(y) = N'(\theta(y))$ by (1.13.1). But $y^{-1}z = (x_0 y)^{-1}(x_0 z) = \theta(y)^{-1}\theta(z)$, so we see that $(y, z) \in E_{(W,R)} \cap (W' \times W')$ if and only if $\theta(y)^{-1}\theta(z) \in T'$ and $\theta(y)^{-1}\theta(z) \notin N'(\theta(y))$ i.e., if and only if $\theta(y)^{-1}\theta(z) \in T$ and $\ell(\theta(y)) \leq \ell(\theta(z))$. This last condition is precisely the condition that $(\theta(y), \theta(z))$ be an edge of $\Gamma_{xW'}$.

To complete the proof of (ii), it remains to check that $\ell(x_0) < \ell(x_0 w)$ for all $w \in W' \backslash \{1\}$. Let $w \in W' \backslash \{1\}$ and write $w = r_1 \ldots r_n$ ($r_i \in R'$, $n = \ell'(w)$). Let $w_i = r_1 \ldots r_i$ ($i = 0, 1, \ldots, n$). Then for $i = 1, \ldots, n$, (w_{i-1}, w_i) is an edge of $\Gamma_{(W',R')}$, hence an edge of $\Gamma_{W'}$ (by (i)) and, by what has just been proved, $(x_0 w_{i-1}, x_0 w_i)$ is an edge of $\Gamma_{xW'}$. In particular, $\ell(x_0) = \ell(x_0 w_0) < \ell(x_0 w_1) < \ldots < \ell(x_0 w_n) = \ell(x_0 w)$, so $\ell(x_0) < \ell(x_0 w)$.

1.14 Let $J \subseteq R$, $W_J = \langle J \rangle$ and $W^J = \{ w \in W \mid \ell(wr) \geq \ell(w)$ for all $r \in J .\}$. Then (W_J, J) is a reflection subsystem of (W, R). For such parabolic reflection subsystems, (1.13) is related to the following well-known facts ([De 1]).

(1.14.1) If $w \in W$, there exist unique $x \in W^J$ and $y \in W_J$ with $w = xy$

(1.14.2) $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in W^J$ and $y \in W_J$. □

We now give a number of simple facts concerning dihedral reflection subgroups of (W, R). These facts are part of the basis of an algorithm, to be presented in Chapter 3, for computing the canonical generators of a (finitely generated) reflection subgroup.

1.15 Lemma. Let (W, R) be a Coxeter system and $T = \bigcup_{w \in W} wRw^{-1}$. Suppose that there exist $t, t' \in T$ $(t \neq t')$ with $W = \langle t, t' \rangle$.

Then $\#(R) = 2$.

Proof Let $R' = \{t, t'\}$; then R' consists of 2 involutions and generates W, so (W, R') is a Coxeter system ([Bo] Ch IV, no 1.2). Let ℓ, ℓ' be the length functions on (W, R) and (W, R') respectively. Since $\ell(t), \ell(t')$ are odd, it follows that for all $w \in W$, $\ell(w)$ is odd if and only if $\ell'(w)$ is odd. Now every element $w \in W$ with $\ell'(w)$ odd is a conjugate of t or t', and hence is conjugate to an element of R (since $t, t' \in T$). This shows that $T = \{\, w \in W \mid \ell(w) \text{ is odd}\,\}$.

Suppose that $\#(R) \geq 3$; let $r, s, t \in R$ be distinct. Then $\ell(rst) = 3$, hence $rst \in T$. But by (1.4) this implies that $rst = rsr$, contrary to $t \neq r$.

Hence $\#(R) \leq 2$. Since $t, t' \in T$ and $t \neq t'$, we must have $\#(R) \geq 2$. □

By ([Bo] Ch IV, §1, Ex 8), the assumption that $t, t' \in T$ in the hypotheses of (1.15) is essential.

1.16 Lemma. Let (W, R) be a Coxeter system, $T = \bigcup_{w \in W} wRw^{-1}$ and $t, t' \in T$ with $t \neq t'$. Let $W' = \langle t, t' \rangle$.

Then $S(W') \subseteq \bigcup_{w \in W'} w\{t, t'\}w^{-1}$ and $\#S(W') = 2$.

Proof Let $R' = S(W')$ and ℓ' be the length function on the Coxeter system (W', R'). Then $t, t' \in W' \cap T = \bigcup_{w \in W'} wR'w^{-1}$. By (1.15), $\#S(W') = 2$. Write $S(W') = \{t_1, t_2\}$. As in the proof of (1.15), $\bigcup_{w \in W'} wR'w^{-1} = \{\, w \in W \mid \ell'(w) \text{is odd}\,\} = \bigcup_{w \in W'} w\{t, t'\}w^{-1}$, and in particular, $t_1, t_2 \in \bigcup_{w \in W'} w\{t, t'\}w^{-1}$. □

1.17 Lemma. Let W' be a dihedral reflection subgroup of the Coxeter system (W, R) (i.e.$\#S(W') = 2$). Write $S(W') = \{t_1, t_2\}$. Then for any $t, t' \in W' \cap T$ with $t' \neq t$ and $S(W') \neq \{t, t'\}$,

$$\ell(t_1) + \ell(t_2) < \ell(t) + \ell(t')\,.$$

Proof Write $R' = S(W')$. Suppose without loss of generality that $t \notin \{t_1, t_2\}$ and that $t' \neq t_2$. One may choose "paths" $(x_0, \ldots, x_n)$ $(n \geq 0, x_0 = t_1,\ x_n =$

t' , $(x_{i-1}, x_i) \in E_{(W',R')}$ $(i = 1, \ldots, n)$) and $(y_0, \ldots, y_m)$ ($m \geq 1$, $y_0 = t_2$, $y_m = t$, $(y_{i-1}, y_i) \in E_{(W',R')}$ $(i = 1, \ldots, m)$) in $\Gamma_{(W',R')}$ from t_1 to t', and from t_2 to t respectively (since (W', R') is dihedral). By (1.13), $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_m)$ are paths in $\Gamma_{(W,R)}$; hence $\ell(x_0) < \ldots < \ell(x_n)$ and $\ell(y_0) < \ldots < \ell(y_m)$. Therefore, $\ell(t_1) \leq \ell(t')$ and, since $m \geq 1$, $\ell(t_2) < \ell(t)$. □

The remainder of this chapter is devoted to describing an edge-labelling of the Bruhat graphs and giving a graph-theoretic characterisation of these edge-labelled Bruhat graphs. First, though, we need some terminology concerning edge-labelled directed graphs in general.

1.18 For any set ω and $E \subseteq \omega \times \omega$, define E_α $(\alpha \in \omega)$ by $E_\alpha = [(\{\alpha\} \times \omega) \cup (\omega \times \{\alpha\})] \cap E$.

We will say that $\Gamma = (\omega, E, S, f)$ is a directed graph edge-labelled by S (via f) if ω is a set, $E \subseteq \omega \times \omega$ is an antisymmetric set of directed edges, and $f: E \longrightarrow S$ is a function into a set S such that for all $\alpha \in \omega$, $f\,|_{E_\alpha}: E_\alpha \longrightarrow S$ is a bijection.

Note that if these conditions are satisfied, then each connected component of the underlying undirected graph may also be naturally regarded as a directed graph edge-labelled by S. We will say that $\alpha \in \omega$ is a source if $(\omega \times \{\alpha\}) \cap E = \emptyset$.

For any $S_0 \subseteq S$, we put $E_{S_0} = f^{-1}(S_0)$ and let $f_{S_0}: E_{S_0} \longrightarrow S_0$ denote the restriction of f to E_{S_0}. Then $\Gamma(S_0) = (\omega, E_{S_0}, S_0, f_{S_0})$ is a directed graph edge-labelled by S_0.

It will also be convenient to define $\bar{S}_0 \subseteq S$ by $\bar{S}_0 = \{\, f(\alpha, \beta) \mid (\alpha, \beta) \in E; \alpha, \beta$ in the same connected component of $\Gamma(S_0) \,\}$.

Two different notions of isomorphism of edge-labelled directed graphs $\Gamma = (\omega, E, S, f)$ and $\Gamma' = (\omega', E', S', f')$ will be required. Firstly, Γ and Γ' will be said to be strongly isomorphic if

(i) $S = S'$

and there exists a bijection $\theta: \omega \longrightarrow \omega'$ such that

(ii) $(\alpha, \beta) \in E$ if and only if $(\theta(\alpha), \theta(\beta)) \in E'$ $(\alpha, \beta \in \omega)$

(iii) $f'(\theta(\alpha), \theta(\beta)) = f(\alpha, \beta)$ $((\alpha, \beta) \in E)$.

Secondly, Γ and Γ' will be said to be isomorphic if there exists a bijection $\rho\colon S' \longrightarrow S$ such that Γ and $(\omega', E', S', \rho f')$ are strongly isomorphic.

1.19 For the remainder of this chapter, the Bruhat graph $\Gamma_{(W,R)}$ will be regarded as the edge-labelled directed graph

$$\Gamma_{(W,R)} = (W, E_{(W,R)}, T, f)$$

where $f\colon E_{(W,R)} \longrightarrow T$ is defined by $f(x,y) = x^{-1}y$ ($(x,y) \in E_{(W,R)}$).

We now illustrate some of the notions of (1.18) with these edge-labelled Bruhat graphs.

Firstly, for any $T' \subseteq T$, we have $\bar{T}' = T \cap \langle T' \rangle$. The vertex sets of the connected components of $\Gamma_{(W,R)}(T')$ are the cosets $x\langle T' \rangle$ $(x \in W)$.

Suppose now that $T' = W' \cap T$ for some reflection subgroup W' of W, and set $R' = S(W')$. Then the connected components of $\Gamma_{(W,R)}(T')$ are the graphs $\Gamma_{xW'}$ associated to cosets xW', with labelling induced from $\Gamma_{(W,R)}$. Now if (y,z) and (xy, xz) are both edges of $\Gamma_{(W,R)}$ they receive the same label $y^{-1}z$. It therefore follows from (1.13) that

(1.19.1) the connected components of $\Gamma_{(W,R)}(T')$ are pairwise strongly isomorphic, and the connected component containing $1 \in W$ is $\Gamma_{(W',R')}$. $\square$

For later use, we mention now the simple

1.20 Lemma. If $(x,y) \in E_{(W,R)}$, $r \in R$ and $y \neq xr$ then $(xr, yr) \in E_{(W,R)}$.

Proof Suppose $(x,y) \in E_{(W,R)}$; let $x^{-1}y = t$. Then $t \in N(y)$, so $rtr \in rN(y)r + \{r\} = N(yr)$. Hence $(yr(rtr), yr) = (xr, yr) \in E_{(W,R)}$. $\square$

We now state a characterisation of edge-labelled Bruhat graphs; the result was suggested by ([G], (4.1)).

1.21 Theorem. Let $\Gamma = (\omega, E, L, f)$ be a connected edge-labelled directed graph. Then Γ is isomorphic to $\Gamma_{(W,R)}$ for some Coxeter system (W,R) if and only if (1.21.1)–(1.21.3) below hold:

(1.21.1) Γ has a source 1

(1.21.2) For each $x \in \omega$, the set $\{\, n \in \mathbb{N} \mid \exists (x_0, \ldots, x_n) \in \omega^{n+1}\ (x_{i-1}, x_i) \in E\ (i = 1, \ldots, n),\ x_0 = 1,\ x_n = x \,\}$ is bounded.

(1.21.3) For each $s, t \in L$ $(s \neq t)$ the connected components of $\Gamma(\overline{\{s,t\}})$ are pairwise strongly isomorphic and each is isomorphic to the edge-labelled Bruhat graph of some dihedral Coxeter system. $\square$

The proof of (1.21) will occupy the rest of the chapter.

1.22 Suppose that Γ is isomorphic to $\Gamma_{(W,R)}$; then we may assume $\Gamma = \Gamma_{(W,R)}$ without loss of generality. Then (1.21.1) holds, and for $x \in \omega = W$, the set in (1.21.2) is bounded above by $\ell(x)$. For $s, t \in L$ with $s \neq t$, $\overline{\{s,t\}} = \langle s, t \rangle \cap T$ and $\#S(\langle s, t \rangle) = 2$ by (1.16), so (1.21.3) follows from (1.19.1).

1.23 Henceforward, we assume that (1.21.1)–(1.21.3) hold. Without loss of generality, replace Γ by an isomorphic edge-labelled directed graph so that if $(1, \alpha) \in E$, then $f(1, \alpha) = \alpha \in L$ (this may be done since $L = \{\, f(1, \alpha) \mid (1, \alpha) \in E \,\}$).

For any $s, t \in L$ with $s \neq t$, (1.21.3) implies that there exists $\{t_1, t_2\} \subseteq \overline{\{s,t\}}$ uniquely determined by the conditions $t_1 \neq t_2$ and $(x, t_i) \in E \Rightarrow x = 1$ (x a vertex of the connected component of $\Gamma(\overline{\{s,t\}})$ containing 1); t_1, t_2 correspond to the Coxeter generators, regarded as labels, of the edge-labelled Bruhat graphs of a dihedral Coxeter system. Throughout the proof, t_1 and t_2 will be called the special labels of $\Gamma(\overline{\{s,t\}})$.

Let $\bar{E} = \{\, \{x, y\} \mid (x, y) \in E \,\}$ be the edge set of the underlying undirected graph of Γ; this undirected graph is edge-labelled by the function $\bar{f} \colon \bar{E} \to L$ defined by $\bar{f}(\{x, y\}) = f(x, y)$ $(\,(x, y) \in E)$. If $\tau = (x_0, \ldots, x_n) \in \omega^{n+1}$ and $(x_{i-1}, x_i) \in E$ (respectively $\bar{E}$) for each $i = 1, \ldots, n$, then τ is said to be a path (respectively, undirected path) from x_0 to x_n, of length n. Note that if τ is a path, and $x_0 = 1$, then $x_i \neq 1$ for all $i > 0$ (by (1.21.2)).

For any $t \in L$, let $\pi(t)$ denote the unique permutation of ω such that the orbits of $\pi(t)$ are the sets $\{x, y\} \in \bar{E}$ such that $\bar{f}(\{x, y\}) = t$. We write the action of $\pi(t)$ as $x \longmapsto x\pi(t)$ i.e. on the right. According to our definition of an edge-labelling

(1.23.2) if $t, t' \in L,\ x \in \omega$ and $x\pi(t) = x\pi(t')$, then $t = t'$.

Note that for $s, t \in L$, condition (iii) implies that

(1.23.3) there exists $t' \in \overline{\{s,t\}}$ such that $\pi(s)\pi(t)\pi(s) = \pi(t')$, (this statement being true for the dihedral Bruhat graphs).

For any $t \in L$, we define $\ell'(t)$ to be the maximum of the set $\{ n \in \mathbb{N} \mid \exists (x_0, \ldots, x_n) \in \omega^{n+1}\ (x_{i-1}, x_i) \in E\ (i = 1, \ldots, n),\ x_0 = 1,\ x_n = t \}$ (the set is bounded by (1.21.2) and non-empty since $(1, t) \in E$). We now let $L' = \{ t \in L \mid \ell'(t) = 1 \}$. The crucial step in the proof is the following

1.24 Lemma. If $t \in L$ and $\ell'(t) > 1$, there exist $t_1, t_2 \in L$ with $\ell'(t_i) < \ell'(t)$ $(i = 1, 2)$ and

$$\pi(t) = \overbrace{\pi(t_1)\pi(t_2)\pi(t_1)\ldots\pi(t_1)}^{2m+1} \text{ for some } m \in \mathbb{N}\ (m \geq 1).$$

<u>Proof</u> By assumption, there is a path $(x_0, \ldots, x_n)$ with $x_0 = 1,\ x_n = t$ and $n \geq 2$. Let $t' = f(x_{n-1}, x_n)$ (note $t' \neq t$) and let t_1, t_2 be the special labels of $\Gamma(\overline{\{t, t'\}})$. Since $(x_{n-1}, x_n) \in E$ and $(1, x_n) \in E$ $(1 \neq x_{n-1})$, we have $t \notin \{t_1, t_2\}$. Interchanging t_1 and t_2 if necessary, it follows from simple properties of the dihedral groups that there exists a path $(y_0, \ldots, y_{2m+1})$ $(m \geq 1)$ from 1 to t with

$$f(y_{i-1}, y_i) = \begin{cases} t_1 & (i \text{ odd}) \\ t_2 & (i \text{ even}) \end{cases} \quad (i = 1, \ldots, 2m+1).$$

Also there exists a path $(z_0, \ldots, z_{2p+1})$ from 1 to t with $f(z_0, z_1) = t_2$, and $p \geq 1$. Then $y_1 = t_1$ and $z_1 = t_2$, so it follows that $\ell'(t) \geq \ell'(t_1) + 2m$ and $\ell'(t) \geq \ell'(t_2) + 2p$. In particular, $\ell'(t_1) < \ell'(t)$ and $\ell'(t_2) < \ell'(t)$.

Further, $1\,\overbrace{\pi(t_1)\pi(t_2)\ldots\pi(t_1)}^{2m+1} = t = 1\pi(t)$ and so (1.23.2) and (1.23.3) imply $\overbrace{\pi(t_1)\pi(t_2)\ldots\pi(t_1)}^{2m+1} = \pi(t)$. □

The following result follows immediately by induction on $\ell'(t)$.

1.25 Corollary. If $t \in L$, then there exist $r_1, \ldots, r_m \in L'$ $(m \in \mathbb{N},\ m \geq 1)$ such that $\pi(t) = \pi(r_1) \ldots \pi(r_m) \ldots \pi(r_1)$. $\square$

1.26 Note that $\pi(t)$ $(t \in L)$ is an involution. Let $W = \langle \pi(t) \mid t \in L \rangle$ and $R = \{\, \pi(r) \mid r \in L' \,\}$. Also, let $T = \bigcup_{w \in W} wRw^{-1}$. By (1.23.3) and(1.25), $W = \langle R \rangle$ and the map $L \to W$ given by $t \longmapsto \pi(t)$ $(t \in L)$ induces a bijection $\rho\colon L \to T$.

We make use of (1.3) to show that (W, R) is a Coxeter system.

Define a map $N\colon W \to \mathcal{P}(T)$ by

$$N(w) = \{\, \pi(t) \mid t \in L,\ (1w\pi(t), 1w) \in E \,\} \quad (w \in W).$$

Let $t \in L,\ r \in L',\ w \in \omega$ with $t \neq r$. We claim that

(1.26.1) if $(w\pi(t), w) \in E$ then $(w\pi(t)\pi(r), w\pi(r)\,) \in E$.

To see this, note first that r is a special label of $\Gamma(\overline{\{r,t\}})$; for if r were not a special label, there would exist $x \in \omega,\ x \neq 1$ with $(x, r) \in E$. Letting $t' = f(x, r)$, we could argue as in the proof of (1.24) that r was not a special label of $\Gamma(\overline{\{r,t'\}})$ and so there would exist a path $(1, x_1, x_2, r)$, contrary to $\ell'(r) = 1$. Hence r is a special label of $\Gamma(\overline{\{r,t\}})$ and (1.26.1) follows by applying (1.20) to the connected component of $\Gamma(\overline{\{r,t\}})$ containing x (making use of (1.21.3)).

Note that the preceeding paragraph shows that

(1.26.2) $N(\pi(r)\,) = \pi(r)$ $(r \in L')$.

Regarding $\mathcal{P}(T)$ as an abelian group under symmetric difference, (1.26.1) shows that $N(w\pi(r)\,) = \pi(r)N(w)\pi(r) + \{\pi(r)\}$ $(r \in L',\ w \in W)$. It follows by induction on the length of $\ell(y)$ of y in (W, R) that

(1.26.3) $N(wy) = y^{-1}N(w)y + N(y)$ $(w, y \in W)$ and so (W, R) is a Coxeter system as claimed. Further,

(1.26.4) $N(w) = \{\, t \in T \mid \ell(wt) < \ell(w) \,\}$.

1.27 To finish the proof, it will be shown that the map $\theta\colon W \to \omega$ given by $w \longmapsto 1w$ is a strong isomorphism $\Gamma_{(W,R)} \to (\omega, E, T, \rho f)$.

Firstly, θ is injective since if $(1)v = (1)w$ $(v, w \in W)$ then $N(v) = N(w)$ and so $v = w$. The assumption that Γ is connected gives the surjectivity of θ as follows; if $(x_0, \dots, x_n)$ is any undirected path with $x_0 = 1$, then, letting $t_i = f(x_{i-1}, x_i)$ $(i = 1, \dots, n)$, we have $x_n = \theta(\pi(t_1) \dots \pi(t_n))$.

Now the edges of $\Gamma_{(W,R)}$ are the pairs $(w\pi(t), w)$ with $\ell(w\pi(t)) < \ell(w)$ $(w \in W,\ t \in L)$ and the edges of Γ are the $(1w\pi(t), 1w)$ with $\pi(t) \in N(w)$ $(w \in W, t \in L)$. Hence $(x, y) \in E_{(W,R)}$ if and only if $(\theta(x), \theta(y)) \in E$. Moreover, if $(x, y) \in E_{(W,R)}$, say $(x, y) = (w\pi(t), w)$, then

$$\begin{aligned}
\rho f(\theta x, \theta y) &= \rho f(1w\pi(t), 1w) \\
&= \rho(t) \quad \text{by definition of the permutation } \pi(t) \\
&= \pi(t) \\
&= x^{-1}y
\end{aligned}$$

which is the label (x, y) receives as an edge of $E_{(W,R)}$. Hence θ is a strong isomorphism as claimed. This completes the proof of (1.21). $\square$

Chapter 2

"GENERIC" ROOT SYSTEMS

Chapter 3 contains a criterion for a set of reflections in a Coxeter system to be the canonical set of generators of some reflection subgroup; the condition is that the inner products of the corresponding positive roots in the standard geometric realisation ([De2]) should lie in a certain set.

To determine this set, one needs to know precisely what inner products can occur between simple roots in a geometric realisation of the Coxeter group if the resulting root system is to partition into positive and negative roots as usual. The question is answered by Lemma (2.4) which also summarises all the properties of geometric realisations of Coxeter groups needed for Chapter 3.

As well as the root systems arising from these geometric realisations, one has, in the case of crystallographic Coxeter systems, also the various systems of real roots of corresponding Kac-Moody Lie algebras ([K]). It is of some interest to see in what generality the standard properties of root systems hold. Thus, after proving (2.4), we indicate how the usual arguments may be modified to prove properties of "generic" root systems.

Many of these properties are proved by reducing to the rank two case, and we begin with some 2×2-matrix computations.

2.1 Let $\mathcal{A}$ be a commutative IR-algebra, γ be an element of IR and $q^{1/2},\ X$ be units of $\mathcal{A}$. Define $A, B \in M_{2\times 2}(\mathcal{A})$ by

$$A = \begin{pmatrix} -1 & 2\gamma q^{1/2}X \\ 0 & q \end{pmatrix} \qquad B = \begin{pmatrix} q & 0 \\ 2\gamma q^{1/2}X^{-1} & -1 \end{pmatrix}.$$

It follows by induction on $n \in \mathbb{N}$ that

$$B(AB)^n = \begin{pmatrix} q^{n+1} p_{2n+1} & -q^{n+\frac{1}{2}} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n+2} X^{-1} & -q^n p_{2n+1} \end{pmatrix} \text{ and} \tag{2.1.1}$$

$$(AB)^n = \begin{pmatrix} q^n p_{2n+1} & -q^{n-\frac{1}{2}} p_{2n} X \\ q^{n+\frac{1}{2}} p_{2n} X^{-1} & -q^n p_{2n-1} \end{pmatrix} \tag{2.1.2}$$

where $p_n \in \mathbb{R}$ ($n \in \{-1\} \cup \mathbb{N}$) are defined recursively by

(2.1.3) $p_{-1} = -1,\ p_0 = 0,\ p_{n+1} = 2\gamma p_n - p_{n-1}$ $(n \in \mathbb{N})$.

Now the solution of the recurrence equation (2.1.3) is

$$p_n = \begin{cases} n & (\gamma = 1) \\ (-1)^{n+1} n & (\gamma = -1) \\ \frac{1}{2\sqrt{\gamma^2-1}} \left[\left(\gamma + \sqrt{\gamma^2-1}\right)^n - \left(\gamma - \sqrt{\gamma^2-1}\right)^n \right] & (|\gamma| > 1) \\ \frac{\sin n\theta}{\sin\theta} \qquad (\cos\theta = \gamma) & (|\gamma| < 1). \end{cases} \tag{2.1.4}$$

Here are some properties of the p_n and the matrices A, B. Part (i) of the lemma is particularly important for our applications.

2.2 Lemma.

(i) Conditions (a) and (b) below are equivalent

(a) $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$

(b) $\gamma \in \{ \cos\frac{\pi}{m} \mid m \in \mathbb{N}, m \geq 2 \} \cup [1, \infty)$

(ii) If $\gamma \geq 1$ then for all $n \in \mathbb{N}$, $p_{n+1} > p_n \geq 0$ and $\frac{p_{n+2}}{p_{n+1}} > \frac{p_{n+3}}{p_{n+2}}$.

(iii) If $\gamma = \cos\frac{\pi}{m}$ ($m \in \mathbb{N},\ m \geq 2$) then

$$0 = p_0 < p_1 < \ldots < p_{\lfloor \frac{m}{2} \rfloor} = p_{\lfloor \frac{m+1}{2} \rfloor},\ p_{\lfloor \frac{m+1}{2} \rfloor} > \ldots > p_{m-1} > p_m = 0.$$

Also $\frac{p_2}{p_1} > \frac{p_3}{p_2} > \ldots > \frac{p_m}{p_{m-1}} = 0$ and $\overbrace{(\ldots BAB)}^{m} = \overbrace{(\ldots ABA)}^{m}$.

(iv) If $q = 1$, the matrix AB has order

$$\begin{cases} n & (\gamma = \cos\frac{k\pi}{n}\ (k, n \in \mathbb{N},\ 0 < k < n,\ \gcd(n,k) = 1)) \\ \infty & \text{(otherwise).} \end{cases}$$

Proof (i) Now $p_0 = 0,\ p_1 = 1,\ p_2 = 2\gamma$. Assume that (a) holds. Then $\gamma \geq 0$. If $0 \leq \gamma < 1$, choose θ so that $0 < \theta \leq \frac{\pi}{2}$ and $\cos\theta = \gamma$. Let m be the largest integer such that $0 < \theta < 2\theta < \ldots < m\theta \leq \pi$. If $m\theta < \pi$ then $\pi < (m+1)\theta < 2\pi$ and hence $p_m = \frac{\sin m\theta}{\sin\theta} > 0,\ p_{m+1} = \frac{\sin(m+1)\theta}{\sin\theta} < 0$ contrary to (a). Hence $m\theta = \pi,\ m \geq 2$ and $\gamma = \cos\theta = \cos\frac{\pi}{m}$. This shows that (a) implies (b). Conversely, if (b) holds then it follows from (2.14) that (a) holds.

The first claim in (ii) holds by induction on n, noting that $p_{n+2} - p_{n+1} = p_{n+1} - p_n + (2\gamma - 2)p_{n+1}$. The first claim in (iii) follows from (2.1.4). By induction on $n \in \mathbb{N}$, one has $p_n^2 - p_{n-1}p_{n+1} = 1$; the claims in (ii), (iii) concerning the ratios $\frac{p_{n+1}}{p_n}$ follow readily from this. The remaining assertions of the lemma follow from (2.1.4), (2.1.1), (2.1.2) and analogues for $A(BA)^n,\ (BA)^n$. $\square$

2.3 Let V be a vector space over $\mathbb{R}$ equipped with a symmetric bilinear form $(\cdot|\cdot)$. For non-isotropic $\alpha \in V$, let $r_\alpha\colon V \longrightarrow V$ be the corresponding reflection, defined by

(2.3.1) $r_\alpha(v) = v - 2[(v \mid \alpha)/(\alpha \mid \alpha)]\alpha \qquad (v \in V)$

Let Π be a linearly independent subset of V such that $(\alpha \mid \alpha) = 1$ $(\alpha \in \Pi)$. Let $R = \{\, r_\alpha \mid \alpha \in \Pi \,\},\ W = \langle R \rangle,\ \Phi = W\Pi,\ \Phi^+ = \{\, \sum_{\alpha \in \Pi} m_\alpha \alpha \in \Phi \mid m_\alpha \geq 0$ for all $\alpha \,\},\ \Phi^- = -\Phi^+$ and $T = \bigcup_{w \in W} wRw^{-1}$.

2.4 Lemma.

(i) The conditions (a), (b) below are equivalent

(a) $\Phi = \Phi^+ \cup \Phi^-$

(b) For all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$,

$$(\alpha \mid \beta) \in (-\infty, -1] \cup \{ -\cos \frac{\pi}{m} \mid m \in \mathbb{N},\ m \geq 2 \}$$

(ii) Suppose that conditions (a) (b) of (i) above hold. For $w \in W$, let $\ell(w) = \min\{ n \in \mathbb{N} \mid w \in R^n \}$, $\bar{N}(w) = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \}$ and $N(w) = \{ t \in T \mid \ell(wt) < \ell(w) \}$. Then

(a) The map $i: \alpha \mapsto r_\alpha$ is a bijection $i: \Phi^+ \to T$ and $i\bar{N}(w) = N(w)$ $(w \in W)$.

(b) (W, R) is a Coxeter system and if $\alpha, \beta \in \Pi,\ \alpha \neq \beta$,

$$\operatorname{ord}(r_\alpha r_\beta) = \begin{cases} m & ((\alpha \mid \beta) = -\cos \frac{\pi}{m} \quad (m \in \mathbb{N},\ m \geq 2)) \\ \\ \infty & ((\alpha \mid \beta) \leq -1) \end{cases}$$

(c) If $w r_\alpha w^{-1} = r_\beta$ $(w \in W, \alpha, \beta \in \Phi)$ then $w(\alpha) \in \{\beta, -\beta\}$.

<u>Proof</u> (i) Suppose $\alpha, \beta \in \Pi$ $(\alpha \neq \beta)$ and let $r = r_\alpha,\ s = r_\beta$. In (2.1), take $\gamma = -(\alpha \mid \beta)$, $\mathcal{A} = \mathbb{R}$, $q^{\frac{1}{2}} = X = 1$. Then A and B are the matrices representing the action of r and s (respectively) on the $\langle r, s\rangle$-invariant subspace $\mathbb{R}\alpha + \mathbb{R}\beta$ (with respect to the ordered basis α, β). By (2.1.1) and (2.1.2), the condition that $\langle r, s\rangle\alpha \cup \langle r, s\rangle\beta \subseteq \Phi^+ \cup \Phi^-$ is that $p_n p_{n+1} \geq 0$ for all $n \in \mathbb{N}$, so (a) implies (b) by Lemma (2.2) (i). The implication $(b) \implies (a)$ follows by a standard argument (cf. (2.6.3)).

(ii) All these facts follow from (i) (a) as in [Ste], [De2]. We recall the argument here. First, W is a group of isometries of V; hence $\|w\alpha\|^2 = \|\alpha\|^2$ for any $\alpha \in V,\ w \in W$, and if α is non-isotropic, $w r_\alpha w^{-1} = r_{w(\alpha)}$. This shows that $\|\alpha\|^2 = 1$ and $r_\alpha \in T$ for all $\alpha \in \Phi$. If $\alpha, \beta \in \Phi$ and $r_\alpha = r_\beta$, then $\alpha = k\beta$ for some $k \in \mathbb{R}$, and we must have $k \in \{1, -1\}$ since $\|\alpha\|^2 = \|\beta\|^2 = 1$. These remarks prove (c), and that $i: \Phi^+ \longrightarrow T$ is a bijection. Now (i) (a) implies that for $\alpha \in \Pi,\ r_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$. From this, one may check that $i\bar{N}: W \longrightarrow \mathcal{P}(T)$ satisfies (1.1.1) and (1.1.2), hence by (1.2) and (1.3), (W, R) is a Coxeter system and $i\bar{N} = N$. The claim about $\operatorname{ord}(r_\alpha r_\beta)$ follows from (2.2) (iv) (cf. (2.7)). $\square$

When the conditions (a) and (b) (i) hold, we will say that we have a geometric realisation of (W, R) on V with simple roots Π. As usual, Φ will be called the root system and the elements of $\Phi^+(\Phi^-)$ are called positive (respectively, negative) roots.

2.5 We now describe certain "generic" root systems corresponding to the root systems in (2.4). Every root will turn out to be either positive or negative, the positive roots being linear combinations of simple roots, the coefficients themselves all being Laurent polynomials with non-negative coefficients.

Let (W, R) be a Coxeter system with Coxeter matrix $(m_{r,s})_{r,s\in R}$ (thus, $m_{r,s} = \operatorname{ord}(rs) \in \mathbb{N} \cup \{\infty\}$). In (2.6)–(2.10) $(a_{r,s})_{r,s\in R}$ denotes a fixed real $(R \times R)$ symmetric matrix satisfying

(i) $a_{r,r} = 2$ $(r \in R)$

(ii) $a_{r,s} = -2\cos\frac{\pi}{m_{r,s}}$ $\quad (m_{r,s} \neq \infty \quad r, s \in R,\ r \neq s)$

(iii) $a_{r,s} \leq -2 \quad (m_{r,s} = \infty,\ r, s \in R)$
and $n\colon \{\,(r,s) \in R \times R \mid r \neq s\,\} \longrightarrow \{1, -1\}$ is a fixed function satisfying $n(r,s) = -n(s,r)$ $(r, s \in R,\ r \neq s)$.

2.6 Let $X_{r,s}$ $(r, s \in R,\ r \neq s)$ and $q^{\frac{1}{2}}$ be indeterminates such that $X_{r,s} = X_{s,r}$ $(r, s \in R,\ r \neq s)$, $\mathcal{A} = \mathbb{R}[q^{\pm 1/2}, X_{r,s}^{\pm 1}]$ be the algebra of Laurent polynomials in $q^{1/2}$ and the $X_{r,s}$, and K be the quotient field of $\mathcal{A}$. Let V be a K space on basis $\{e_r\}_{r\in R}$.

For $r \in R$, define $X_r \in \operatorname{End}_K(V)$ by

$$\begin{cases} X_r(e_r) = -e_r \\ X_r(e_s) = qe_s - a_{r,s}q^{1/2}X_{r,s}^{n(r,s)}e_r \quad (s \in R,\ s \neq r) \end{cases}$$

It is well-known that there is a representation $\rho\colon \mathcal{H} \longrightarrow \operatorname{End}_K(V)$ of the Hecke algebra $\mathcal{H} = \mathcal{H}_{q,K}(W)$ such that $\rho(T_r) = X_r$; for finite W, this is esssentially the "reflection representation" defined in [CIK]. To check that ρ exists here, note firstly that $X_r^2 = (q-1)X_r + qId$. One also needs to see that

(2.6.1) $\overbrace{(\ldots X_s X_r X_s)}^{m} e_t = \overbrace{(\ldots X_r X_s X_r)}^{m} e_t$
when $m = m_{r,s}$ is finite $(r, s, t \in R,\ r \neq s)$. Now taking $\gamma = -a_{r,s}/2$ and $X = X_{r,s}^{n(r,s)}$ in (2.1), one sees that A and B are the matrices representing the

action of X_r and X_s on the subspace spanned by the basis vectors e_r and e_s, so (2.6.1) follows from (2.2) (iii) if $t \in \{r, s\}$. If $t \notin \{r, s\}$, then one may choose $k, l \in K$ such that $X_r(e_t + ke_r + le_s) = X_s(e_t + ke_r + le_s) = q(e_t + ke_r + le_s)$ (since the matrix

$$\begin{pmatrix} 1+q & q^{1/2} a_{r,s} X_{r,s}^{n(r,s)} \\ q^{1/2} a_{r,s} X_{r,s}^{-n(r,s)} & 1+q \end{pmatrix}$$

is non-singular) and (2.6.1) follows since both sides are equal to $q^m(e_t + ke_r + le_s) - \overbrace{(\dots X_s X_r X_s)}^{m}(ke_r + le_s)$.

Let $\mathcal{A}^+ = \mathbb{R}^+[q^{\pm 1/2}, X_{r,s}^{\pm 1}]$ be the set of Laurent polynomials with non-negative coefficients. Let $r, s \in R$ with $r \neq s$. Interpreting the matrices A and B as above, (2.1.1) (2.1.2) and (2.2) (ii), (iii) show that

(2.6.2) $\rho\,\overbrace{(\dots T_s T_r T_s)}^{k}\, e_r \in \mathcal{A}^+ e_r + \mathcal{A}^+ e_s \ (0 \leq k < m_{r,s})$.

We now have

(2.6.3) If $w \in W,\ r \in R$ and $\ell(wr) > \ell(w)$, then $\rho(T_w)e_r \in \sum_{t \in R} \mathcal{A}^+ e_t$.

This may be proved by the following standard argument ([De2]). The result holds if $w = 1$, so suppose $w \neq 1$ and argue by induction on $\ell(w)$. Choose $s \in R$ so that $\ell(ws) < \ell(w)$; note $s \neq r$. Use (1.14.1) to write $w = w'w''$ where $w'' \in \langle r, s \rangle$ and $\ell(w'r) = \ell(w's) = \ell(w) + 1$. By (1.14.2), $\ell(w) = \ell(w') + \ell(w'')$. Now $w'' = \overbrace{(\dots srs)}^{k}$ where $1 \leq k < m_{r,s}$ (since $\ell(wr) > \ell(w)$). By (2.6.2), $\rho(T_{w''})e_r \in \mathcal{A}^+ e_r + \mathcal{A}^+ e_s$, and by induction, $\rho(T_{w'})e_r,\ \rho(T_{w''})e_r$ are both in $\sum_{t \in R} \mathcal{A}^+ e_t$. Therefore $\rho(T_w)e_r = \rho(T_{w'})\rho(T_{w''})e_r \in \sum_{t \in R} \mathcal{A}^+ e_t$ as claimed.

2.7 Let $X_{r,s}$ $(r, s \in R,\ r \neq s)$ be indeterminates as in (2.6), $\mathcal{R} = \mathbb{R}[X_{r,s}^{\pm 1}],\ L$ be the quotient field of $\mathcal{R}$ and E be an L-vector space on a basis $\{e_r\}_{r \in R} = \Pi$. Define elements $e_r^\vee \in E^*$ $(r \in R)$ of the dual E^* of E by $\langle e_r, e_r^\vee \rangle = 2$ and

$$\langle e_s, e_r^\vee \rangle = a_{r,s} X_{r,s}^{n(r,s)} \qquad (s \in R,\ s \neq r).$$

By specialising the reflection representation ρ of (2.6) to $q = 1$, one sees that there is a W-action on E such that

(2.7.1) $r(h) = h - \langle h, e_r^\vee \rangle e_r \qquad (r \in R,\ h \in E).$

The contragredient representation of W on E^* is given by

(2.7.2) $r(h^\vee) = h^\vee - \langle e_r, h^\vee \rangle e_r^\vee \qquad (r \in R,\ h^\vee \in E^*).$

Let $\mathcal{R}^+ = \mathbb{R}^+[X_{r,s}^{\pm 1}]$ be the set of Laurent polynomials in $\mathcal{R}$ with non-negative coefficients, $\Phi = W\Pi$ and $\Phi^+ = \{ \sum_{\alpha\in\Pi} m_\alpha \alpha \in \Phi \mid m_\alpha \in \mathcal{R}^+$ for all $\alpha \in \Pi \}$, $\Phi^- = -\Phi^+$. By specialising (2.6.3) to $q = 1$, it follows that

(2.7.3) $\Phi = \Phi^+ \cup \Phi^-$.

The following result is an analogue of (2.4) (c).

2.8 Lemma. If $w, w' \in W$ and $r, r' \in R$ satisfy $wrw^{-1} = w'r'w'^{-1}$, then $w(e_r) = kw'(e_{r'})$ where $\pm k \in \langle X_{r,s}^{\pm 1} \rangle$ (the subgroup of the group of units of $\mathcal{R}$ generated by the $X_{r,s}$).

<u>Proof</u> There is no loss of generality in assuming that $w' = 1$. Let E' be the $\mathcal{R}$-span in E of the elements of Π; thus, E' is a free $\mathcal{R}$-module with $\mathcal{R}$-basis Π, and E' is W-invariant. Write $w(e_r) = \sum_{s\in R} p_s e_s$ ($p_s \in \mathcal{R}, s \in R$), and note that each $p_s \in \pm\mathcal{R}^+$.

Regard $\mathbb{R}$ as an $\mathcal{R}$-module by means of the $\mathbb{R}$-algebra homomorphism $f\colon \mathcal{R} \longrightarrow \mathbb{R}$ with $f(X_{r'',s''}) = 1$ ($r'', s'' \in R$). Now the induced W-action on $E' \otimes_{\mathcal{R}} \mathbb{R}$, given by $w \cdot v = (w \otimes 1)$ (v) ($v \in E' \otimes_{\mathcal{R}} \mathbb{R}$) is a geometric realisation of (W, R) with simple roots $e_t \otimes 1$ ($t \in R$). Since $wrw^{-1} = r'$, it follows from (2.4) (c) that $w \cdot (e_r \otimes 1) = \pm(e_{r'} \otimes 1)$. But $w \cdot (e_r \otimes 1) = \sum_{s\in R} f(p_s)\ (e_s \otimes 1)$. Since $\pm p_s \in \mathcal{R}^+$, we have $f(p_s) = 0$ iff $p_s = 0$. Hence $p_s = 0$ unless $s = r'$. This proves that $w(e_r) = ke_{r'}$ where $k \in \mathcal{R}$ and $f(k) \in \{1, -1\}$. Similarly, one has $w^{-1}(e_{r'}) = k'e_r$ for some $k' \in \mathcal{R}$. Now one sees that $e_r = w^{-1}w(e_r) = k'ke_r$ so k is actually a unit. Since also $f(k) \in \{1, -1\}$, it follows that $\pm k \in \langle X_{r,s} \rangle$ as claimed. $\square$

Lemma 2.8 shows that for each $t \in T$, there is a corresponding element of Φ^+, well defined up to multiplication by monomials in the $X_{r,s}^{\pm 1}$.

As well as the root system Φ, one has a dual root system $\Phi^\vee = W\Pi^\vee$ in E^*, where $\Pi^\vee = \{e_r^\vee\}_{r \in R}$.

The next proposition implies that the element of $\mathcal{R}$ obtained by pairing a root with a dual root is either in $\mathcal{R}^+$, or in $-\mathcal{R}^+$.

2.9 Proposition. Suppose $w \in W$, $r, s \in R$ and $\ell(wr) > \ell(w)$. Write $wrw^{-1} = t$. Then exactly one of (i)–(iv) below holds

(i) $t = s$

(ii) $st = ts,\ \ell(st) = \ell(t) + 1$ and $\langle w(e_r),\ e_s^\vee \rangle = 0$

(iii) $\ell(sts) = \ell(t) + 2$ and $-\langle w(e_r),\ e_s^\vee \rangle \in \mathcal{R}^+ \setminus \{0\}$

(iv) $\ell(sts) = \ell(t) - 2$ and $\langle w(e_r),\ e_s^\vee \rangle \in \mathcal{R}^+ \setminus \{0\}$.

Proof The possibilities (i)–(iv) are clearly mutually exclusive. Note that to prove the result, it suffices to prove that the same holds with (w, r) replaced by any other pair $(w', r') \in W \times R$ with $w'r'w'^{-1} = t$ and $\ell(w'r') > \ell(w')$ (for then $w'(e_{r'}) = kw(e_r)$ for some $k \in \langle X_{r,s}^{\pm 1} \rangle$, by (2.8)).

Suppose that (i) does not hold, but that $st = ts$. Then $\ell(st) > \ell(t)$, else we would have $\ell(sts) = \ell(t) - 2$ by Lemma (1.4). Now by (2.8) $w(e_r) - \langle w(e_r),\ e_s^\vee \rangle e_s = sw(e_r) = kw(e_r)$ for some $k \in \langle \pm X_{r,s}^{\pm 1} \rangle$. Write $w(e_r) = \sum\limits_{r' \in R'} \alpha_{r'} e_{r'}$. Now if $k \neq 1$, it follows that $\alpha_{r'} = 0$ for $r' \neq s$. Hence $w(e_r) = \alpha_s e_s$. Specialising to the geometric realisation as in the proof of (2.8), we have $w(e_r \otimes 1) = f(\alpha_s)(e_s \otimes 1)$, hence $wrw^{-1} = s$ contrary to the assumption that $t \neq s$. Hence $k = 1$ and so $\langle w(e_r),\ e_s^\vee \rangle = 0$ as required for (ii).

Now assume $\ell(sts) = \ell(t) + 2$. First we prove that (iii) holds in the special case when there exist $r', s' \in R$ such that $r, w, s, \in \langle r', s' \rangle$. Let $m = m_{r', s'}$.

In (2.1), take $\mathcal{A} = \mathcal{R}$, $q^{1/2} = 1$, $\gamma = -a_{r',s'}/2$ and $X = X_{r',s'}^{n(r',s')}$. Then for $n \in \mathbb{N}$

$$\begin{aligned} s'(r's')^n e_r' &= p_{2n+1} e_{r'} + p_{2n+2} X^{-1} e_{s'} \\ (r's')^n e_{r'} &= p_{2n+1} e_{r'} + p_{2n} X^{-1} e_{s'} \end{aligned}$$

Since $\langle e_{r'}, e^\vee_{s'}\rangle = -2\gamma X^{-1}$, $\langle e_{s'}, e^\vee_{r'}\rangle = -2\gamma X$ we have

$$(2.9.1)\quad \begin{cases} \langle s'(r's')^n e_{r'}, e^\vee_{r'}\rangle = 2p_{2n+1} - 2\gamma p_{2n+2} = p_{2n+1} - p_{2n+3} \\ \langle (r's')^n e_{r'}, e^\vee_{s'}\rangle = -2\gamma X^{-1} p_{2n+1} + 2p_{2n} X^{-1} = X^{-1}(p_{2n} - p_{2n+2}) \end{cases}$$

Now we are assuming that $r, w, s, \in \langle r', s'\rangle$ and that $\ell(wr) > \ell(w)$, $\ell(sts) = \ell(t) + 2$ where $t = wrw^{-1}$. By the remark at the beginning of the proof and symmetry we only need consider the cases

$$\begin{aligned} &w = s'(r's')^n,\ r = r',\ s = r'\ (5 \leq (4n+3) + 2 \leq m) \\ &w = (r's')^n,\ r = r',\ s = s'\ (3 \leq (4n+1) + 2 \leq m) \end{aligned}$$

and the result is true in these cases by (2.9.1) and (2.2) (ii), (iii).

Now we may deal with the general case of (iii); We proceed by induction on $\ell(t)$. Let $\ell(t) = 2k+1$, and choose $s' \in R$ with $\ell(s't) < \ell(t)$. Write $t = xy$ with $x \in \langle s', s\rangle$ (note $s \neq s'$) and $\ell(sy) = \ell(s'y) = \ell(y) + 1$ (using 1.14.1). Now make use of Lemma (1.4). If $\ell(x) \geq k+1$, we have $t \in \langle s', s\rangle$ and the remark at the beginning reduces us to the special case considered earlier. Hence we may assume $\ell(x) \leq k$, and write $y = y'x^{-1}$ where $\ell(y) = \ell(y') + \ell(x)$ (still using (1.4)). Write $y' = zr'z^{-1}$ ($z \in W$, $r' \in R$, $\ell(y') = 2\ell(z) + 1$). By the remark at the beginning, it will suffice to prove that

$$-\langle xz(e_{r'}), e^\vee_s\rangle \in \mathcal{R}^+ \setminus \{0\}$$

or, equivalently, $-\langle z(e_{r'}),\ x^{-1}(e^\vee_s)\rangle \in \mathcal{R}^+ \setminus \{0\}$ (since it is the case that $t = xy'x^{-1} = xzr'(xz)^{-1}$ and $\ell(t) = 2\ell(xz) + 1$).

Now since $\ell(sts) = \ell(t) + 2$, we must have $x^{-1} = \overbrace{(\ldots s'ss')}^{p}$ where $1 \leq p < m$ ($m = m_{s,s'}$). Also, $\ell(sy') = \ell(s'y') = \ell(y') + 1$; hence $\ell(sy's) \geq \ell(y')$ and $\ell(s'y's') \geq \ell(y')$, at least one of these two inequalities being strict because $\ell(xy'x^{-1}) = 2\ell(x) + \ell(y')$. Moreover, if $\ell(x) = p = m-1$, then sx is the longest element of $\langle s, s'\rangle$ and so both of these inequalities above are strict.

Now in (2.1) let $\mathcal{A} = \mathcal{R}, q^{1/2} = 1,\ X = X^{n(s',s)}_{s,s'},\ \gamma = -a_{s,s'}/2$. We have

$$\begin{aligned} s'e^\vee_s &= e^\vee_s - a_{s,s'}X^{-1}e^\vee_{s'} \qquad & s'e^\vee_{s'} &= -e^\vee_{s'} \\ se^\vee_{s'} &= e^\vee_{s'} - a_{s,s'}Xe^\vee_s \qquad & se^\vee_s &= -e^\vee_s \end{aligned}$$

and so by (2.1.1), (2.1.2), for $n \in \mathbb{N}$

$$
\begin{cases} s'(ss')^n e_s^\vee = p_{2n+1} e_s^\vee + p_{2n+2} X^{-1} e_{s'}^\vee \\ (ss)^n e_s^\vee = p_{2n+1} e_s^\vee + p_{2n} X^{-1} e_{s'}^\vee \end{cases} \tag{2.9.2}
$$

even though $e_s^\vee$ and $e_{s'}^\vee$ may be linearly dependent.

Now we consider two cases.

Case 1. $p+1 < m$.

In this case, (2.9.2) and (2.2) (ii), (iii) show that $x^{-1} e_s^\vee \in (\mathcal{R}^+ \setminus \{0\}) e_s^\vee + (\mathcal{R}^+ \setminus \{0\}) e_{s'}^\vee$. By (iii) and the inductive assumption, we have that

$$
-\langle z(e_{r'}), e_s^\vee) \rangle \qquad \text{and} \qquad -\langle z(e_{r'}), e_{s'}^\vee \rangle
$$

are both in $\mathcal{R}^+$, and at least one is non-zero. Hence $-\langle z(e_{r'}), x^{-1}(e_s^\vee) \rangle \in \mathcal{R}^+ \setminus \{0\}$ as required

Case 2. $p+1 = m$

Here (2.9.2) and (2.2) (iii) give $x^{-1} e_s^\vee \in (\mathcal{R}^+ \setminus \{0\}) e_s^\vee \cup (\mathcal{R}^+ \setminus \{0\}) e_{s'}^\vee$ (depending on the parity of m). By the inductive assumption, we have that both $-\langle z(e_{r'}), e_s^\vee \rangle$ and $-\langle z(e_{r'}), e_{s'}^\vee \rangle$ are in $\mathcal{R}^+ \setminus \{0\}$ so $-\langle xz(e_{r'}), e_{s'}^\vee \rangle \in \mathcal{R}^+ \setminus \{0\}$ here also.

Finally, consider the case (iv) $\ell(sts) = \ell(t) - 2$. In this case, applying (iii) with t replaced by sts gives $-\langle sw(e_r), e_s^\vee \rangle \in \mathcal{R}^+ \setminus \{0\}$ i.e. $\langle w(e_r), e_s^\vee \rangle \in \mathcal{R}^+ \setminus \{0\}$. (One must note that $\ell(sw) < \ell(swr)$ but since $\ell(wr) < \ell(w)$, if this failed it would follow that $sw = wr$ and $s = t$, contrary to assumption). $\square$

We conclude this chapter with some observations about the specialisations of the representation given by (2.7.1).

2.10 Let E' be the $\mathcal{R}$-submodule of E spanned by Π; thus, Π is an $\mathcal{R}$-basis of E'. Let $x_{r,s}$ $(r, s \in R,\ r \neq s)$ be any family of elements of $\mathbb{R}^+ \setminus \{0\}$ with $x_{r,s} = x_{s,r}$ and let $g\colon \mathcal{R} \longrightarrow \mathbb{R}$ be the $\mathbb{R}$-algebra homomorphism with $g(X_{r,s}) = x_{r,s}$ $(r, s \in R,\ r \neq s)$. Then $E' \otimes_g \mathbb{R}$ is a faithful W-module with action given by $w.(v) = (w \otimes 1)v$ $(w \in W,\ v \in E' \otimes_g \mathbb{R})$.

For suppose $(w\otimes 1)v = v$ for all $v \in E' \otimes_g \mathbb{R}$. Write $w(e_r) = \sum_{s\in R} \beta_{r,s} e_s$ and note that $\pm\beta_{r,s} \in \mathcal{R}^+$. Then $e_r \otimes 1 = (w \otimes 1)(e_r \otimes 1) = \sum_{s\in R} g(\beta_{r,s})(e_s \otimes 1)$. Since the $x_{r,s}$ are positive, it follows that $\beta_{r,s} = 0$ for $s \neq r$ and $\beta_{r,r} \in \mathcal{R}^+$. Now specialise to the geometric realisation of W via the homomorphism $f{:}\,\mathcal{R} \longrightarrow \mathbb{R}$ considered in the proof of (2.8). We now have $(w\otimes 1)(e_r \otimes 1) = f(\beta_{r,r})(e_r \otimes 1)$ where $f(\beta_{r,r}) \in \mathbb{R}^+$. Hence w keeps all the simple roots, and hence all the positive roots, positive and so $w = 1$. Hence $E' \otimes_g \mathbb{R}$ is a faithful W-module.

Suppose now that $g'{:}\,\mathcal{R} \longrightarrow \mathbb{R}$ is another homomorphism with $g'(X_{r,s}) = y_{r,s} \in \mathbb{R}^+ \setminus \{0\}$. If $\theta{:}\, E' \otimes_g \mathbb{R} \longrightarrow E' \otimes_{g'} \mathbb{R}$ were an isomorphism of W-modules, we would necessarily have $\theta(e_r \otimes_g 1) \in \mathbb{R}(e_r \otimes_{g'} 1)$; this is because, for any $r \in \mathbb{R}$, $E' \otimes_g \mathbb{R}$ is the direct sum of the 1-eigenspace of r and the (-1)-eigenspace, the latter having $(e_r \otimes_g 1)$ as the basis, and θ would preserve this decomposition. Now assume that R is finite.

Let X denote the matrix with entries $X_{r,r} = 2$ $(r \in R)$; $X_{r,s} = a_{r,s} x_{r,s}^{n(r,s)} (r, s \in R,\ r \neq s)$ and let Y be defined similarly using $y_{r,s}$ in place of $x_{r,s}$. The above comments imply that $E' \otimes_g \mathbb{R}$ and $E' \otimes_{g'} \mathbb{R}$ are isomorphic W-modules iff there exists an invertible diagonal matrix Λ such that $\Lambda X \Lambda^{-1} = Y$.

Thus, if the Coxeter graph of (W, R) is a forest (i.e. all its connected components are trees) all these specialisations are isomorphic, but if the Coxeter graph contains a cycle there are uncountably many non-isomorphic specialisations.

Chapter 3

CANONICAL GENERATORS OF REFLECTION SUBGROUPS

In Chapter 1 it was shown that every reflection subgroup of a Coxeter system has a canonical set of Coxeter generators. In this chapter, our main result is a criterion for a set of reflections to be the canonical Coxeter generators of the reflection subgroup they generate. The condition is that the inner products of distinct elements of the corresponding set of positive roots (in a geometric realisation of the Coxeter system) should all lie in the set $\{ -\cos\frac{\pi}{m} \mid m \in \mathbb{N},\ m \geq 2 \} \cup (-\infty, -1]$.

As an application of this criterion, we describe the canonical generators of reflection subgroups of Weyl groups of type A, B and D. Another application is given in Chapter 4.

We begin by fixing some notation, and then translate some of the results from Chapter 1 into the language of root systems.

3.1 Let (W, R) be a Coxeter system. We may assume that W is a group of isometries of a vector space V as in (2.3), and that R is the set of reflections determined by a set Π of simple roots satisfying the condition (b) of Lemma (2.4) (i). We adopt without change all the notation and terminology of (2.3) and (2.4), and also use the following notation for a reflection subgroup W' of W.

The set of canonical generators of W' will be denoted by $S(W')$ as in Chapter 1. Recall that

(3.1.1) $S(W') = \{ t \in T \mid N(t) \cap W' = \{t\}\}$

The corresponding set of positive roots is denoted $\Delta(W')$. Thus,

(3.1.2) $\Delta(W') = i^{-1}S(W') = \{ \alpha \in \Phi^+ \mid r_\alpha \in S(W') \}$.

We also set

(3.1.3) $\Phi(W') = \{\, \alpha \in \Phi \mid r_\alpha \in W' \}$

(3.1.4) $\Phi^+(W') = \Phi(W') \cap \Phi^+,\ \ \Phi^-(W') = \Phi(W') \cap \Phi^- = -\Phi^+(W')$ and let $\ell_{W'} \colon W' \longmapsto \mathbb{N}$ denote the length function of $(W', S(W'))$.

By Lemma (2.4) (ii) (a),

(3.1.5) $\Delta(W') = \{\, \alpha \in \Phi^+ \mid \bar{N}(r_\alpha) \cap \Phi(W') = \{\alpha\} \,\}$.

In our present notation, theorem (1.8) gives the following three facts:

(3.1.6) $(W', S(W'))$ is a Coxeter system

(3.1.7) $\Phi(W') = W'\Delta(W')$

(3.1.8) $\bar{N}(w) \cap \Phi(W') = \{\, \alpha \in \Phi^+(W') \mid \ell_{W'}(wr_\alpha) < \ell_{W'}(w) \,\}$ $(w \in W')$

We will also need

(3.1.9) if $\alpha \in \Pi$ and $r_\alpha \notin W'$, then $\Delta(r_\alpha W' r_\alpha) = r_\alpha \Delta(W')$.

This follows from Lemma (1.7) (i), noting that $r_\alpha \Delta(W') \subseteq \Phi^+$ since $\alpha \notin \Delta(W')$.

In (3.2)–(3.4), W' denotes a fixed reflection subgroup of W and we write ℓ' for $\ell_{W'}$. The following two lemmas are directed toward the computation of the inner products $(\alpha \mid \beta)$ $(\alpha, \beta \in \Delta(W'))$ in (3.4).

3.2 Lemma. Let $\alpha, \beta \in \Delta(W')$ with $\alpha \neq \beta$ and $\mathrm{ord}(r_\alpha r_\beta) = n$. Then for $0 \leq m < n$, $\overbrace{(\ldots r_\beta r_\alpha r_\beta)}^{m}\,\alpha \in \Phi^+$ and $\overbrace{(\ldots r_\alpha r_\beta r_\alpha)}^{m}\,\beta \in \Phi^+$.

Proof Note that we have $\{r_\alpha, r_\beta\} \subseteq S(W')$ and that ℓ' is the length function of $(W', S(W'))$. Therefore, for $0 \leq m < n$,

$$\ell'(\overbrace{(\ldots r_\beta r_\alpha r_\beta)}^{m}\, r_\alpha) = m + 1 > m = \ell'\,\overbrace{(\ldots r_\beta r_\alpha r_\beta)}^{m}.$$

Write $\overbrace{(\ldots r_\beta r_\alpha r_\beta)}^{m} = w$. Then $\alpha \notin \{\, \gamma \in \Phi^+(W') \mid \ell'(wr_\gamma) < \ell'(w) \,\}$. By (3.1.8), $\alpha \notin \bar{N}(w) \cap \Phi(W')$. But $\alpha \in \Delta(W') \subseteq \Phi^+(W')$, so $\alpha \notin \bar{N}(w)$. Since $\alpha \in \Phi^+$, we have $w(\alpha) \in \Phi^+$ by definition of $\bar{N}$. The other fact is proved similarly. □

3.3 Lemma. Let $\alpha, \beta \in \Delta(W')$ with $\alpha \neq \beta$ and $\mathrm{ord}(r_\alpha r_\beta) = n$. Write

$$\overbrace{(\dots r_\beta r_\alpha r_\beta)}^{m}\,\alpha = c_m\alpha + d_m\beta$$
$$\overbrace{(\dots r_\alpha r_\beta r_\alpha)}^{m}\,\beta = c'_m\alpha + d'_m\beta \qquad (0 \leq m < n)$$

Then $d_m \geq 0,\ d'_m \geq 0,\ c_m \geq 0,\ c'_m \geq 0$ for $0 \leq m < n$.

Proof By symmetry, it will suffice to prove that $d_m \geq 0,\ d'_m \geq 0$. The proof of this will be by induction on $\ell(r_\alpha)$.

Suppose first that $\ell(r_\alpha) = 1$. Then $\alpha \in \Pi$. Write $\beta = \sum_{\gamma\in\Pi} a_\gamma\gamma$ where $a_\gamma \in \mathbb{R}$ $(\gamma \in \Pi)$. Since $\beta \in \Delta(W') \subseteq \Phi^+$, we have $a_\gamma \geq 0$ for all $\gamma \in \Pi$. In fact, $a_{\gamma_0} > 0$ for some $\gamma_0 \in \Pi\backslash\{\alpha\}$, since otherwise we would have $\beta \in \mathbb{R}\alpha$ and so $\beta = \alpha$ (because $\|\beta\|^2 = \|\alpha\|^2 = 1$ and $\alpha, \beta \in \Phi^+$).

Now for $0 \leq m < n$, Lemma (3.2) gives $\overbrace{(\dots r_\beta r_\alpha r_\beta)}^{m}\,\alpha = c_m\alpha + \sum_{\gamma\in\Pi} d_m a_\gamma\gamma \in \Phi^+$. The coefficient of γ_0 in this is $d_m a_{\gamma_0} \geq 0$. Since $a_{\gamma_0} > 0$, it follows that $d_m \geq 0$. Similarly, $d'_m \geq 0$.

Suppose inductively now that the result is true for reflection subgroups W'' of W and $\alpha', \beta' \in \Delta(W'')$ with $\alpha' \neq \beta'$ and $\ell(r_{\alpha'}) < \ell(r_\alpha)$ where $\ell(r_\alpha) \geq 3$. By (1.5), there exists $s \in R$ with $\ell(sr_\alpha s) = \ell(r_\alpha) - 2$. Then $\ell(r_\alpha s) < \ell(r_\alpha)$, so $s \in N(r_\alpha)$. But since $\alpha \in \Delta(W')$, $N(r_\alpha) \cap W' = \{r_\alpha\}$. Since $s \neq r_\alpha$, this shows that $s \notin W'$. Let $W'' = sW's$. By (3.1.9), it follows that $\Delta(W'') = s\Delta(W')$ and therefore $s\alpha, s\beta \in \Delta(W'')$.

Now $r_{s\alpha} = sr_\alpha s,\ r_{s\beta} = sr_\beta s$ and hence $\mathrm{ord}(r_{s\alpha}r_{s\beta}) = \mathrm{ord}(r_\alpha r_\beta) = n$. Since $\ell(r_{s\alpha}) = \ell(sr_\alpha s) = \ell(r_\alpha) - 2$, the inductive assumption gives

$$\overbrace{(\dots r_{s\beta} r_{s\alpha} r_{s\beta})}^{m}(s\alpha) = c_m(s\alpha) + d_m(s\beta)$$
$$\overbrace{(\dots r_{s\alpha} r_{s\beta} r_{s\alpha})}^{m}(s\beta) = c'_m(s\alpha) + d'_m(s\beta)$$

where $d_m, d'_m \geq 0$ for $0 \leq m < n$. Since $r_{s\beta} = sr_\beta s$ and $r_{s\alpha} = sr_\alpha s$, the result follows on applying s to both sides of these equations. $\square$

The following result is the first half of our criterion for a set of reflections to be the set of canonical generators of some reflection subgroup.

3.4 Corollary. For any $\alpha, \beta \in \Delta(W')$ with $\alpha \neq \beta$, let $n_{\alpha,\beta} = \mathrm{ord}(r_\alpha r_\beta)$. Then

$$\begin{cases} (\alpha \mid \beta) = -\cos\frac{\pi}{n_{\alpha,\beta}} & (n_{\alpha,\beta} \in \{2,3,4,\ldots\}) \\ (\alpha \mid \beta) \leq -1 & (n_{\alpha,\beta} = \infty) \end{cases}$$

<u>Proof</u> Let $\Psi = \langle r_\alpha, r_\beta \rangle \Pi'$ where $\Pi' = \{\alpha, \beta\}$, and set $\Psi^+ = \{c\alpha + d\beta \in \Psi \mid c \geq 0, d \geq 0\}$. Now the elements of Ψ are $\pm\overbrace{(\ldots r_\alpha r_\beta r_\alpha)}^{m}\beta$, $\pm\overbrace{(\ldots r_\beta r_\alpha r_\beta)}^{m}\alpha$ $(0 \leq m < n_{\alpha,\beta})$. By Lemma (3.3), $\Psi = \Psi^+ \cup -\Psi^+$. The conclusion of the corollary therefore holds by Lemma (2.4) (i), (ii) (b). □

In (3.5)–(3.7), Γ denotes a fixed subset of Φ^+ such that $(\alpha \mid \beta) \in (-\infty, -1] \cup \{-\cos\frac{\pi}{n} \mid n \in \mathbb{N}, n \geq 2\}$ for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, and $W' = \langle r_\alpha \mid \alpha \in \Gamma \rangle$ denotes the reflection subgroup generated by the reflections in the elements of Γ. The following sections will show that $\Gamma = \Delta(W')$.

3.5 Let Γ' be a subset of Γ.

Let U be a vector space over $\mathbb{R}$ on a basis $\Pi' = \{e_\alpha\}_{\alpha \in \Gamma'}$ and define a symmetric bilinear form $(\cdot \mid \cdot)$ on U by setting $(e_\alpha \mid e_\beta) = (\alpha \mid \beta)$ $(\alpha, \beta \in \Gamma')$. Note that $\|e_\alpha\|^2 = \|\alpha\|^2 = 1$ for all $\alpha \in \Gamma'$.

For non-isotropic $u \in U$, let $s_u : U \longrightarrow U$ denote the corresponding reflection, defined by

$$s_u(v) = v - 2[(v \mid u)/(u \mid u)]u \qquad (v \in U).$$

Let $R' = \{s_u \mid u \in \Pi'\}$, $W'' = \langle R' \rangle$, $\Psi = W''\Pi'$, $\Psi^+ = \{\sum_{\alpha \in \Gamma'} c_\alpha e_\alpha \in \Psi \mid c_\alpha \geq 0 \text{ for all } \alpha \in \Gamma'\}$, and $\Psi^- = -\Psi^+$. Since Π' satisfies the condition (b) of Lemma (2.4) (i), it follows that (W'', R') is a Coxeter system realised geometrically on U with Π' as its set of simple roots, and Ψ^+ is the corresponding set of positive roots.

Note that, by (2.4) (ii) (b), for $\alpha, \beta \in \Gamma'$ with $\alpha \neq \beta$ we have

$$\begin{aligned}
\text{ord }(s_{e_\alpha} s_{e_\beta}) &= \begin{cases} n & (e_\alpha \mid e_\beta) = -\cos\frac{\pi}{n}\ (n \in \mathbb{N},\ n \geq 2) \\ \infty & (e_\alpha \mid e_\beta) \leq -1 \end{cases} \\
&= \begin{cases} n & (\alpha \mid \beta) = -\cos\frac{\pi}{n}\ (n \in \mathbb{N},\ n \geq 2) \\ \infty & (\alpha \mid \beta) \leq -1 \end{cases} \\
&= \text{ord }(r_\alpha r_\beta).
\end{aligned}$$

Since (W'', R') is a Coxeter system, this implies that there exists a homomorphism $\theta: W'' \longrightarrow W'$ such that $\theta(s_{e_\alpha}) = r_\alpha$ $(\alpha \in \Gamma')$.

Let $L: U \longrightarrow V$ be the $\mathbb{R}$-linear map such that $L(e_\alpha) = \alpha$ $(\alpha \in \Gamma')$. We now claim that

(3.5.1) $L(w''u) = \theta(w'')L(u)$ $(w'' \in W'', u \in U)$.

To prove (3.5.1), first notice that if $\alpha, \beta \in \Gamma'$, then

$$\begin{aligned}
L(s_{e_\alpha}(e_\beta)) &= L(e_\beta - 2(e_\beta \mid e_\alpha)e_\alpha) \\
&= \beta - 2(\beta \mid \alpha)\alpha \qquad\qquad \text{since } (e_\beta \mid e_\alpha) = (\beta \mid \alpha) \\
&= r_\alpha(\beta) \\
&= \theta(s_{e_\alpha})L(e_\beta).
\end{aligned}$$

By linearity, this gives $L(s_{e_\alpha}(u)) = \theta(s_{e_\alpha})L(u)$ $(\alpha \in \Gamma', u \in U)$. Since $W'' = \langle s_{e_\alpha} \mid \alpha \in \Gamma' \rangle$ and θ is a homomorphism, the claim (3.5.1) follows by induction on the length of w'' in (W'', R').

We will need to apply the results in (3.5) twice. The first application is to the proof of

3.6 Lemma. With the above notation, $\Delta(W') \subseteq \Gamma$.

Proof Take $\Gamma' = \Gamma$ in (3.5). Since $\theta(R') = \{ r_\alpha \mid \alpha \in \Gamma' \}$ and $W' = \langle r_\alpha \mid \alpha \in \Gamma' \rangle$, it follows that θ is a surjective.

Let $\gamma \in \Delta(W')$. Choose $x \in W''$ with $\theta(x) = r_\gamma \in W'$. Let $s_{e_{\alpha_1}} \dots s_{e_{\alpha_n}}$ $(\alpha_i \in \Gamma')$ be a reduced expression for x in (W'', R'). Note $n \geq 1$. Now $\ell''(xs_{e_{\alpha_n}}) < \ell''(x)$, where ℓ'' is the length function on (W'', R'). Applying (2.4) (i) (a) to the geometric realisation of (W'', R') on U, we have $x(e_{\alpha_n}) \in \Psi^-$, say $x(e_{\alpha_n}) = -\sum_{\alpha\in\Gamma'} c_\alpha e_\alpha$ where $c_\alpha \geq 0$ for all $\alpha \in \Gamma'$. Hence

$$\begin{aligned} r_\gamma(\alpha_n) = \theta(x)L(e_{\alpha_n}) &= L(xe_{\alpha_n}) \qquad\qquad \text{by (3.5.1)} \\ &= L\left(-\sum_{\alpha\in\Gamma'} c_\alpha e_\alpha\right) \\ &= -\sum_{\alpha\in\Gamma'} c_\alpha \alpha. \end{aligned}$$

But $r_\gamma(\alpha_n) \in \Phi$ and each $\alpha \in \Gamma'$ is a non-negative linear combination of elements of Π, so $r_\gamma(\alpha_n) \in \Phi^-$. Since $\alpha_n \in \Gamma' \subseteq \Phi^+(W')$, it follows that $\alpha_n \in \bar{N}(r_\gamma) \cap \Phi(W')$, and so by (3.1.5), $\gamma = \alpha_n \in \Gamma' = \Gamma$. Since $\gamma \in \Delta(W')$ was arbitrary, $\Delta(W') \subseteq \Gamma$ as wanted. □

We are now able to complete the proof of

3.7 Proposition. Let $\Gamma \subseteq \Phi^+$ be such that

(3.7.1) $(\alpha \mid \beta) \in (-\infty, -1] \cup \{-\cos\frac{\pi}{n} \mid n \in \mathbb{N},\ n \geq 2\}$ for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, and let $W' = \langle r_\alpha \mid \alpha \in \Gamma\rangle$. Then $\Gamma = \Delta(W')$.

Proof Take $\Gamma' = \Delta(W')$ in (3.5); this is possible by (3.6). Since $(W', S(W'))$ is a Coxeter system, we know that $\theta\colon (W'', R') \longrightarrow (W', S(W'))$ is an isomorphism of Coxeter systems. In particular,

(3.7.2) $\ell'(\theta(w'')) = \ell''(w'')$ $(w'' \in W'')$ where ℓ'' is the length function of (W'', R').

Let $\gamma \in \Gamma$. Then $r_\gamma \in W' \cap T$; by Theorem (1.8), there exist $\alpha_p, \ldots, \alpha_0 \in \Delta(W')$ such that $r_\alpha = r_{\alpha_p} \ldots r_{\alpha_1} r_{\alpha_0} r_{\alpha_1} \ldots r_{\alpha_p}$. Since $r_{\alpha_p} \ldots r_{\alpha_1} r_{\alpha_0} r_{\alpha_1} \ldots r_{\alpha_p} = r_{\alpha_p} \ldots r_{\alpha_1} r_{\alpha_0} r_{\alpha_0} r_{\alpha_0} r_{\alpha_1} \ldots r_{\alpha_p}$, there is no loss of generality in assuming that $\ell'(r_{\alpha_p} \ldots r_{\alpha_1} r_{\alpha_0}) > \ell'(r_{\alpha_p} \ldots, r_{\alpha_1})$. By (3.7.2), this gives $\ell''(s_{e_{\alpha_p}} \ldots s_{e_{\alpha_1}} s_{e_{\alpha_0}}) > \ell''(s_{e_{\alpha_p}} \ldots s_{e_{\alpha_1}})$ and so by (2.4) (ii) (a), $\beta = s_{e_{\alpha_p}} \ldots s_{e_{\alpha_1}}(e_{\alpha_0}) \in \Psi^+$. Write $\beta = \sum_{\delta\in\Gamma'} a_\delta e_\delta$ where $a_\delta \geq 0$ for all $\delta \in \Gamma'$. We now have $L(\beta) = r_{\alpha_p} \ldots r_{\alpha_1}(\alpha_0) = \sum_{\delta\in\Gamma'} a_\delta \delta \in \Phi^+$, since $\Gamma' \subseteq \Phi^+$ and $L(\beta) \in \Phi$. Write $\beta' = L(\beta)$. Then $r_{\beta'} =$

$r_{\alpha_p} \dots r_{\alpha_1} r_{\alpha_0} r_{\alpha_1} \dots r_{\alpha_p} = r_\gamma$. Since $\gamma, \beta' \in \Phi^+$ and $r_\gamma = r_{\beta'}$, it follows that $\gamma = \beta' = \sum_{\delta \in \Gamma'} a_\delta \delta$.

Now suppose that $\gamma \notin \Delta(W')$. Then by (3.7.1), $(\gamma \mid \delta) \leq 0$ for all $\delta \in \Gamma' = \Delta(W')$, and so

$$1 = (\gamma \mid \gamma) = \sum_{\delta \in \Gamma'} a_\delta (\delta \mid \gamma) \leq 0.$$

This contradiction shows that the assumption $\gamma \notin \Delta(W')$ is false. Hence if $\gamma \in \Gamma$, then $\gamma \in \Delta(W')$ i.e. $\Gamma \subseteq \Delta(W')$. The reverse inclusion is true by (3.6), so the proposition has been proved. $\square$

3.8 We now combine (3.4) and (3.7) to characterise the sets of reflections which arise as canonical generators of reflection subgroups. It is desirable to formulate the result so that it applies directly to classical root systems, so we now allow roots to have different lengths.

Specifically, let V be a real vector space equipped with a symmetric bilinear form $(\cdot \mid \cdot)$ and Π be a linearly independent set of non-isotropic vectors of V. For non-isotropic $\alpha \in V$, let r_α denote the corresponding reflection. Set $R = \{ r_\alpha \mid \alpha \in \Pi \}$, $W = \langle R \rangle$, $\Phi = W\Pi$ and $\Phi^+ = \{ \sum_{\alpha \in \Pi} c_\alpha \alpha \in \Phi \mid c_\alpha \geq 0$ for all $\alpha \in \Pi \}$. We make the following assumptions (cf. [Ste]) concerning this situation:

(i) $(\alpha \mid \alpha) > 0$ for all $\alpha \in \Pi$ (and hence for all $\alpha \in \Phi$)

(ii) If $\alpha \in \Phi$ and $k \in \mathbb{R}$, then $k\alpha \in \Phi$ iff $k \in \{1, -1\}$

(iii) $\Phi = \Phi^+ \cup (-\Phi^+)$.

For $\alpha \in \Phi$, write $\|\alpha\| = (\alpha \mid \alpha)^{\frac{1}{2}}$.

Then (W, R) is a Coxeter system which is realised geometrically on V, with $\Pi' = \{ \|\alpha\|^{-1}\alpha \mid \alpha \in \Pi \}$ as the set of simple roots; for $R = \{ r_\beta \mid \beta \in \Pi' \}$ and $\Phi' = W\Pi' = \{ \|\alpha\|^{-1} w\alpha \mid w \in W, \alpha \in \Pi \} = \{ \|\alpha\|^{-1}\alpha \mid \alpha \in \Phi \}$, so every element of $W\Pi'$ is a combination of elements of Π' with coefficients all of the same sign, whence the result by (2.4) (i). The corresponding set of positive roots is $\Phi'^+ = \{ \|\alpha\|^{-1}\alpha \mid \alpha \in \Phi^+ \}$. By (3.4) and (3.7), we have

3.9 Theorem. Let Γ be a subset of Φ^+. Then $\{r_\alpha \mid \alpha \in \Gamma\}$ is the canonical set of generators $S(W')$ of some reflection subgroup W' of the Coxeter system (W, R) iff

(3.9.1) $\frac{(\alpha|\beta)}{\|\alpha\|\,\|\beta\|} \in \{-\cos\frac{\pi}{n} \mid n \in \mathbb{N}, n \geq 2\} \cup (-\infty, -1]$ for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. (Of course, if these conditions hold then $W' = \langle r_\alpha \mid \alpha \in \Gamma\rangle$). □

Note that condition (iii) of (3.8) is equivalent to the validity of (3.9.1) with $\Gamma = \Pi$ (by (2.4) (i) and the discussion in (3.8)).

3.10 Corollary. Let (W, R) be a Coxeter system, $T = \bigcup_{w \in W} wRw^{-1}$ its set of reflections, and $N(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$ $(w \in W)$. Then for any subset T' of T, (i) and (ii) below are equivalent:

(i) $N(t) \cap \langle T' \rangle = \{t\}$ for all $t \in T'$

(ii) $N(t) \cap \langle t', t \rangle = \{t\}$ for all $t, t' \in T'$.

Proof There is no loss of generality in assuming that (W, R) is a group of isometries of a real vector space V as in (3.1); we also use the rest of the notation there. Now (i) obviously implies (ii). Assume that condition (ii) holds, and let $\Gamma = i^{-1}T' = \{\alpha \in \Phi^+ \mid r_\alpha \in T'\}$, $W' = \langle T' \rangle = \langle r_\alpha \mid \alpha \in \Gamma\rangle$. Let $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Then $N(r_\alpha) \cap \langle r_\alpha, r_\beta \rangle = \{r_\alpha\}$ and $N(r_\beta) \cap \langle r_\alpha, r_\beta \rangle = \{r_\beta\}$, so $\{r_\alpha, r_\beta\} = S(\langle r_\alpha, r_\beta \rangle)$. By (3.9), $\frac{(\alpha|\beta)}{\|\alpha\|\,\|\beta\|} \in (-\infty, -1] \cup \{-\cos\frac{\pi}{n} \mid n \in \mathbb{N}, n \geq 2\}$. It now follows by (3.9) applied to Γ that $T' = S(W')$, and (i) holds by definition of $S(W')$. □

It would be interesting to have a direct proof of (3.10) from the definition of a Coxeter system, in the style of the arguments of Chapter 1.

In (3.15) we will describe a procedure for calculating the set of canonical generators of a (finitely generated) reflection subgroup of a Coxeter system (W, R). As a preliminary, we investigate the canonical generators of a dihedral reflection subgroup.

In (3.11)–(3.13), (W, R) denotes a Coxeter system realised geometrically as a group of isometries of a real vector space V as in (3.1), with simple roots Π and positive roots Φ^+.

3.11 Lemma. Let $\alpha, \beta \in \Phi^+$ with $\alpha \neq \beta$. Then

$$\operatorname{ord}(r_\alpha r_\beta) = \begin{cases} n & (\alpha \mid \beta) = -\cos\frac{k\pi}{n} \; (k, n \in \mathbb{N}, 0 < k < n,\ \gcd(n,k) = 1) \\ \infty & \text{otherwise} \end{cases}$$

<u>Proof</u> This follows from (2.2) (iv) on noting that if $-1 < (\alpha \mid \beta) < 1$, then the restriction of $(\cdot \mid \cdot)$ to $U = \mathbb{R}\alpha + \mathbb{R}\beta$ is positive definite, so V is the direct sum of U and the orthogonal complement of U. $\square$

3.12 Remark. For any $\alpha, \beta \in \Phi$,

$$(\alpha \mid \beta) \in (-\infty, -1] \cup \{\cos\frac{k\pi}{n} \mid k, n \in \mathbb{N}\ n \neq 0\} \cup [1, \infty)$$

(by considering the canonical generators of $\langle r_\alpha, r_\beta \rangle$, this reduces to checking the claim for the geometric realisations of a dihedral group). $\square$

Our next result explicitly describes the canonical generators of a dihedral reflection subgroup containing a simple reflection of (W, R).

3.13 Lemma. Let $\alpha \in \Pi$ and $\beta \in \Phi^+$, with $\alpha \neq \beta$. Let $W' = \langle r_\alpha, r_\beta \rangle$.

(i) If $r_\alpha r_\beta$ has infinite order, then

$$S(W') = \begin{cases} \{r_\alpha, r_\beta\} & (r_\alpha \notin N(r_\beta)) \\ \{r_\alpha, r_\alpha r_\beta r_\alpha\} & (r_\alpha \in N(r_\beta)). \end{cases}$$

(ii) Suppose $(\alpha \mid \beta) = -\cos\frac{k\pi}{n}$ $(k, n \in \mathbb{N},\ 0 < k < n,\ \gcd(n,k) = 1)$. For $p \in \mathbb{N},\ p \geq 1$ set $t_p = s_1 \ldots s_{p-1} s_p s_{p-1} \ldots s_1$ where

$$s_p = \begin{cases} r_\beta & (p \text{ odd}) \\ r_\alpha & (p \text{ even}). \end{cases}$$

Choose $m_1, m_2 \in \mathbb{N}$ satisfying $m_1 k \equiv 1$ (mod n), $m_2 k \equiv -1$ (mod n). Then

$$S(W') = \begin{cases} \{r_\alpha, r_\beta\} & (n = 2) \\ \{r_\alpha, t_{m_2}\} & (n \neq 2, \quad r_\alpha \in N(t_{m_1})) \\ \{r_\alpha, t_{m_1}\} & (n \neq 2, \quad r_\alpha \notin N(t_{m_1})). \end{cases}$$

<u>Proof</u> Note that by (1.16), $\#S(W') = 2$. Also, $r_\alpha \in S(W')$.

(i) Let $R' = \{r_\alpha, r_\beta\}$. Then (W', R') is an infinite dihedral Coxeter system, so any set of two reflections of (W', R') (i.e. elements of $\bigcup_{w \in W'} wR'w^{-1}$) which generate W' is conjugate to R' in W'. In particular, $S(W')$ is conjugate to R' (note $S(W') \subseteq \bigcup_{w \in W'} wR'w^{-1}$ by (1.16)) and contains r_α, so either $S(w') = \{r_\alpha, r_\beta\}$ or $S(W') = \{r_\alpha, r_\alpha r_\beta r_\alpha\}$. If $r_\alpha \in N(r_\beta)$, we must have $S(W') = \{r_\alpha, r_\alpha r_\beta r_\alpha\}$. On the other hand, if $S(W') = \{r_\alpha, r_\alpha r_\beta r_\alpha\}$ then $r_\beta = r_\alpha(r_\alpha r_\beta r_\alpha) r_\alpha$ is a reduced expression for r_β in $(W', S(W'))$, and so $r_\alpha \in N(r_\beta)$ by (1.8) (iii).

(ii) Write $\Delta(W') = \{\alpha, \gamma\}$. Since $(\alpha \mid \beta) = -\cos\frac{k\pi}{n}$ $(0 < k < n,\ \gcd(n, k) = 1)$, $\#(W') = 2n$ by (3.11). Using (3.9) and (3.11) again, we must have $(\alpha \mid \gamma) = -\cos\frac{\pi}{n}$. The restriction of $(\cdot \mid \cdot)$ to $U = \mathbb{R}\alpha + \mathbb{R}\gamma$ is positive definite, and we regard U as a two-dimensional Euclidean space. Let $\Phi' = \{\, \delta \in \Phi \mid r_\delta \in W' \,\}$; since every element of $W' \cap T$ is conjugate in W' to r_α or r_γ, Φ' consists of the $2n$ unit vectors which form with $(-\alpha)$ a directed angle of the form $\frac{m\pi}{n}$ $(0 \leq m < 2n)$ (the root system of a dihedral group of order $2n$, in the standard geometric realisation).

Let

$$\alpha_j = \begin{cases} \beta & (j \text{ odd}) \\ \alpha & (j \text{ even}) \end{cases} \qquad \text{and } \gamma_j = r_{\alpha_1} \ldots r_{\alpha_j}\ (\alpha_{j_1})\ (j \in \mathbb{N}),$$

and assume that the plane U is oriented so that $\beta \in U$ makes a directed angle of $\frac{k\pi}{n}$ with $(-\alpha)$. Then γ_j makes a directed angle of $\frac{jk\pi}{n}$ with $(-\alpha)$. Now there are exactly two roots in Φ' which make a directed angle of $\pm\frac{\pi}{n}$ with $-\alpha$; one is γ, and we denote the other by γ'. Note that $r_\alpha(\gamma) = -\gamma'$, so $r_{\gamma'} = r_\alpha r_\gamma r_\alpha$.

Now γ_{m_1} (respectively, γ_{m_2}) makes a directed angle with $-\alpha$ of the form $(j + \frac{1}{n})\pi$ $(j \in \mathbb{Z})$ (respectively, $(j - \frac{1}{n})\pi$ $(j \in \mathbb{Z})$). Hence $\{r_\gamma, r_{\gamma'}\} = \{r_{\gamma_{m_1}}, r_{\gamma_{m_2}}\} = \{t_{m_1}, t_{m_2}\}$; in particular, $S(W')$ equals either $\{r_\alpha, t_{m_1}\}$ or $\{r_\alpha, t_{m_2}\}$.

If $n = 2$, then $\Phi(W') = \{\pm\alpha, \pm\beta\}$ so $S(W') = \{r_\alpha, r_\beta\}$. Suppose $n \geq 3$. If $r_\alpha \in N(t_{m_1})$, we must have $S(W') = \{r_\alpha, t_{m_2}\}$. On the other hand, if $S(W') = \{r_\alpha, t_{m_2}\}$ then $t_{m_1} = r_\alpha t_{m_2} r_\alpha$ is a reduced expression for t_{m_1} in $(W', S(W'))$ and so $r_\alpha \in N(t_{m_1})$ by (1.8) (iii). □

3.14 Remark. For any reflection subgroup W' of a Coxeter system (W, R), we have $W' \cap R = S(W') \cap R$. It follows from (1.7.1) that for any $r \in R$,

$$S(rW'r) = \begin{cases} rS(W')r & (r \notin S(W')) \\ S(W') & (r \in S(W')). \end{cases}$$

Thus, if one knows $S(W')$ one may determine $S(W'')$ for any conjugate W'' of W'. Taken in conjunction with (3.13), this gives a procedure for calculating $S(W')$ when W' is any dihedral reflection subgroup of (W, R). □

3.15 Let (W, R) be a Coxeter system, T its set of reflections and T' be a finite subset of T. Following is an algorithm for determining $S(W')$ where $W' =< T >$.

Set $T_0 = T'$, and define $T_1, T_2, \ldots$ as follows:

If $S(\langle t, t' \rangle) = \{t, t'\}$ for all $t, t' \in T_i$ $(t \neq t')$ set $T_{i+1} = T_i$. Otherwise, choose $t, t' \in T_i$ with $t \neq t'$ and $S(\langle t, t' \rangle) \neq \{t, t'\}$, and define $T_{i+1} = (T_i \backslash \{t, t'\}) \cup S(\langle t, t' \rangle)$.

We claim that there exists some $i \in \mathbb{N}$ with $T_i = T_{i+1}$, and that $S(W') = T_i$.

For if $T_i \neq T_{i+1}$, then (1.16) and (1.17) show that $\sum\limits_{t \in T_{i+1}} \ell(t) < \sum\limits_{t \in T_i} \ell(t)$; hence $T_i = T_{i+1}$ for some $i \in \mathbb{N}$. By (3.10), it follows that $T_i = S(\langle T_i \rangle)$ and the claim follows on noting that $\langle T_i \rangle = \langle T_{i-1} \rangle = \ldots = \langle T_0 \rangle = W'$.

Note that by (1.16), we have $\#(T_{j+1}) \leq \#(T_j)$ $(j \in \mathbb{N})$ and in addition $T_{j+1} \subseteq \bigcup\limits_{w \in W'} wT_jw^{-1} (j \in \mathbb{N})$. Therefore,

(3.15.1) $\#(S(W')) \leq \#(T')$ and

(3.15.2) $S(W') \subseteq \bigcup\limits_{w \in W'} wT'w^{-1}$

3.16 Corollary. For any subset T' of the reflections T of a Coxeter system (W,R), the following hold:

(i) $\#(S(\langle T'\rangle)) \leq \#(T')$

(ii) $\langle T'\rangle \cap T = \bigcup_{w\in\langle T'\rangle} wT'w^{-1}$

Proof If $\#(T')$ is infinite, (i) follows by a standard cardinality argument, and the case $\#(T')$ finite in (i) is just (3.15.1). To prove (ii), it suffices to show that $S(W') \subseteq \bigcup_{w\in\langle T'\rangle} wT'w^{-1}$ (by (1.8)). Since each element of W' lies inside a subgroup generated by a finite subset of T', this follows from (3.15.2). $\square$

A reflection subgroup W' of (W,R) is said to be a dihedral reflection subgroup if $\#(S(W')) = 2$, or, equivalently, if W' is generated by two (distinct) reflections ((1.16)). At one stage, we will need a result on the existence of maximal dihedral reflection subgroups of (W,R). This is given in (3.18) after a preliminary

3.17 Lemma. Let (W,R) be a Coxeter system realised geometrically on a real vector space V with positive roots Φ^+, and let U be a two dimensional subspace of V. Then there do not exist $\alpha,\beta,\gamma \in \Phi^+ \cap U$ with $(\alpha \mid \beta) \leq 0,\ (\alpha \mid \gamma) \leq 0$ and $(\beta \mid \gamma) \leq 0$.

Proof Suppose such α,β,γ existed. There is no non-trivial relation of linear dependence between α,β,γ with non-negative coefficients so we may write, say, $\gamma = a\alpha + b\beta$ where $a \geq 0,\ b \geq 0$. Then $1 = (\gamma \mid \gamma) = a(\alpha \mid \gamma) + b(\beta \mid \gamma) \leq 0$, a contradiction. $\square$

Lemma (3.17) shows that a reflection subgroup of a dihedral Coxeter system is of rank at most two.

3.18 Corollary. Let W' be a dihedral reflection subgroup of a Coxeter system (W,R). Then there is a dihedral reflection subgroup W_1 of (W,R) with the following property:

if W_2 is any dihedral reflection subgroup of (W,R) such that $W' \subseteq W_2$ then $W_2 \subseteq W_1$.

Proof Assume that (W, R) is realised geometrically on V as in (3.17), and write $\Delta(W') = \{\alpha, \beta\}$. Let $\Phi_1 = \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$ and $W_1 = \langle r_\gamma \mid \gamma \in \Phi_1 \rangle$. By (3.16) (ii), $\Phi(W_1) \subseteq W_1\Phi_1 \subseteq \mathbb{R}\alpha + \mathbb{R}\beta$, and so (3.17) and (3.9) show that $\#S(W_1) \leq 2$; therefore, W_1 is a dihedral reflection subgroup of (W, R). Suppose W_2 is a dihedral reflection subgroup containing W', and let $\Delta(W_2) = \{\gamma, \delta\}$. Using (3.1.7), we have $\{\alpha, \beta\} \in \Phi(W_2) = W_2\Delta(W_2) \subseteq \mathbb{R}\gamma + \mathbb{R}\delta$, hence $\Delta(W_2) \subseteq \Phi_1$ and $W_2 \subseteq W_1$ as desired. □

Corollary (3.18) asserts that every dihedral reflection subgroup is contained in a unique maximal dihedral reflection subgroup. We conclude the chapter with some examples concerning reflection subgroups of "universal" Coxeter systems, and of finite Coxeter systems of types A, B, D.

3.19 Example. Let (W, R) be a Coxeter system such that for $r, s \in R$ with $r \neq s$, rs has infinite order (a universal Coxeter system). Then the product of any two reflections is of infinite order (this is easily checked when one of the reflections is simple, and the general case reduces to this). It follows that every reflection subsystem of (W, R) is a universal Coxeter system.

Consider now the case $\#(R) = 3$, say $R = \{r, s, t\}$. We realise (W, R) geometrically in the standard way; let V be a 3-dimensional vector space over $\mathbb{R}$ with basis $\Pi = \{\alpha, \beta, \gamma\}$, equipped with a bilinear form such that $(\delta \mid \delta) = 1$, $(\delta \mid \varepsilon) = -1$ for any $\delta, \varepsilon \in \Pi$ with $\delta \neq \varepsilon$, and identify r, s, t with the reflections in α, β, γ respectively. One may check that for any $k \in \mathbb{Z}$, $(rs)^k\gamma = \gamma + 2k(2k-1)\beta + 2k(2k+1)\alpha$. Let $\Gamma = \{\, (rs)^k\gamma \mid k \in \mathbb{Z}\,\}$, $R' = \{\, r_\delta \mid \delta \in \Gamma\,\}$ and $W' = \langle R' \rangle$. For any $j, k \in \mathbb{Z}$ with $j \neq k$, we have $((rs)^k\gamma \mid (rs)^j\gamma) = (\gamma \mid (rs)^{|j-k|}\gamma) = 1 - 8|k-j|^2 \leq -1$. Thus, (W', R') is a reflection subsystem of (W, R).

This example is in marked contrast to the case of finite Coxeter systems If (W, R) is a Coxeter system with $\#(W)$ finite, and (W', R) is a reflection subsystem, then $\#(R') \leq\#(R)$ (this follows from our (3.9) and [Bo] Ch V, no 4.8 and 3.5). □

3.20 Example.

(i) Let $l \geq 3$ be an integer and V be a real vector space with basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$, equipped with the bilinear form $(\cdot \mid \cdot)$ determined by $(\varepsilon_i \mid \varepsilon_j) = \delta_{ij}$ $(i, j = 1, \ldots, l)$. The finite Coxeter group (W, R) of type B_l has a standard realisation on V as in (3.2); this is described in ([Bo], page 252). The positive roots Φ^+ are ε_i $(1 \leq i \leq l)$, $\varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq l)$.

We will shortly describe the sets of positive roots corresponding to the canonical generators of all the reflection subgroups of (W, R). First, we describe some standard sets of positive roots satisfying (3.9.1).

Let $J \subseteq \{1, \ldots, l\}$, say $J = \{i_1, \ldots, i_j\}$ where $1 \leq i_1 < i_2 < \ldots < i_j \leq l$. Define

$$\begin{aligned} A(J) &= \{\, \varepsilon_{i_{k+1}} - \varepsilon_{i_k} \mid 1 \leq k \leq j-1 \,\} \\ B(J) &= A(J) \cup \{\varepsilon_{i_j}\} \qquad\qquad (j = \#(J) \geq 1) \\ D(J) &= A(J) \cup \{\varepsilon_{i_{j-1}} + \varepsilon_{i_j}\} \qquad (j = \#(J) \geq 2) \end{aligned}$$

Also, if $J' = \{k_1, \ldots, k_m\}$ $(1 \leq k_1 < \ldots < k_m \leq l)$ is another subset of $\{1, \ldots, l\}$ such that $J \cap J' = \emptyset$, we set

$$A(J, J') = A(J) \cup A(J') \cup \{\varepsilon_{i_j} + \varepsilon_{k_m}\} \ (j, m \geq 1).$$

Given a set X of roots, it will be convenient to define its support $\mathrm{supp}(X)$ to be the smallest subset of $\{\varepsilon_1, \ldots, \varepsilon_l\}$ the linear span of which contains X (thus $\mathrm{supp}(A(J)) = \mathrm{supp}(B(J)) = \mathrm{supp}(D(J)) = \{\, \varepsilon_j \mid j \varepsilon J \,\}$ and $\mathrm{supp}(A(J, J')) = \{\, \varepsilon_i \mid i \varepsilon J \cup J' \,\}$).

It is straightforward to check from (3.9) that a set of roots corresponds to the set of canonical generators of some reflection subgroup of (W, R) iff it is a union of disjointly supported sets of roots each of one of the types $A(J)$ $(\#(J) \geq 2)$, $B(J)$ $(\#(J) \geq 1)$, $D(J)$ $(\#(J) \geq 2)$ or $A(J, J')$ $(J \cap J' = \emptyset,\ \#(J) \geq 1,\ \#(J') \geq 1)$. The expression as a union of such sets is unique except for order (and the notational ambiguity $A(J, J') = A(J', J)$).

(ii) The root system of type D_l is a subsystem of that of type B_l; the positive roots are precisely the roots $\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq l)$. In this case the reflection subgroups of the Coxeter group of type D_l correspond to the unions of disjointly supported sets of roots each of one of the types $A(J)$ $(\#(J) \geq 2)$, $D(J)$ $(\#(J) \geq 2)$ or $A(J, J')$ $(J \cap J' = \emptyset,\ \#(J) \geq 1,\ \#(J') \geq 1)$.

(iii) The root system of type A_{l-1} is a subsystem of that of type D_l; the positive roots are precisely the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$. Here, reflection subgroups of the Coxeter group of type A_{l-1} correspond to unions of disjointly supported sets of roots each of type $A(J)$ $(\#(J) \geq 2)$.

Chapter 4

REFLECTION SUBGROUPS OF AFFINE WEYL GROUPS

The reflection subgroups of finite Weyl groups are known, up to isomorphism, by an algorithm attributed to Borel and Siebenthal, and independently to Dynkin ([BS], [D]; see also [Co]). This algorithm is described in [Ca].

The purpose of this chapter is to prove a result which determines all the reflection subsystems of an affine Weyl group, up to isomorphism as Coxeter systems, from the isomorphism types of reflection subsystems of the corresponding finite Weyl group. The proof is an application of the results of Chapter 3. For our geometric realisation of the affine Weyl group, we use (essentially) the natural representation on the Cartan subalgebra of the corresponding Kac-Moody Lie algebra; this has the advantage that the resulting root system relates in a particularly simple way to the root system of the corresponding finite Weyl group ([K]).

In the proof, we will make use of the classifications of affine Weyl groups and finite root systems ([Bo]) and generalised Cartan matrices of finite and affine types ([K]).

We begin by introducing some notation and terminology that will be required for the statement of the main result, and its proof.

4.1 If $A_1, \ldots, A_n$ are Coxeter systems, their direct product will be denoted by $A_1 \times \ldots \times A_n$; for instance, $(W_1, R_1) \times (W_2, R_2) = (W_1 \times W_2, (R_1 \times \{1\}) \cup (\{1\} \times R_2))$.

For an irreducible root system of type X (either A_l $(l \geq 1)$, B_l $(l \geq 3)$, C_l $(l \geq 2), D_l$ $(l \geq 4)$, E_6, E_7, E_8, F_4 or G_2) let $W(X)$ denote the corresponding finite Coxeter system (of type X) and $\widetilde{W}(X)$ denote the Coxeter system of type $\widetilde{X}$ ([Bo] Ch VI, 4.1 and 4.3; we write $\widetilde{C}_2$ instead of $\widetilde{B}_2$). Note that for

$l \geq 3$, $W(B_l) \cong W(C_l)$ but $\widetilde{W}(B_l) \not\cong \widetilde{W}(C_l)$ (where $\cong$ denotes isomorphism of Coxeter systems).

The following result was conjectured by Coxeter ([Co]):

4.2 Theorem. Let X be an irreducible root system (of one of the above types $A_l, \ldots, G_2$).

(i) If $W(X)$ has a reflection subsystem isomorphic to $W(X_1) \times \ldots \times W(X_n)$, then for any i $(0 \leq i \leq n)$, $\widetilde{W}(X)$ has a reflection subsystem isomorphic to

$$\widetilde{W}(X_1) \times \ldots \times \widetilde{W}(X_i) \times W(X_{i+1}) \times \ldots \times W(X_n)$$

(here the X_i are root systems of the above types $A_l, \ldots, G_2$).

(ii) Every reflection subsystem of $\widetilde{W}(X)$ is isomorphic to one of those described in (i). $\square$

In [Co], it is shown that (i) holds, and that if $\widetilde{W}(X)$ has a reflection subsystem of type $\widetilde{W}(Y)$, then $W(X)$ has a reflection subsystem of type $W(Y)$.A proof of Theorem (4.2) will be given in (4.3)–(4.14). We begin by recalling some facts about sets of vectors with negative inner products.

4.3 Let V be a Euclidean space i.e. a finite dimensional real vector space equipped with a symmetric, positive definite bilinear form $(\cdot|\cdot)$. We will say that a non-empty set $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ of non-zero elements of V is indecomposable if Γ cannot be expressed as a disjoint union $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are non-empty mutually orthogonal subsets of V. Any finite set of non-zero elements of V may be partitioned into non-empty indecomposable mutually orthogonal subsets (its indecomposable components). If Γ is indecomposable and satisfies $(\alpha_i \mid \alpha_j) \leq 0$ $(i \neq j)$, then the matrix $\left(\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\right)_{i,j=1,\ldots,n} = M(\Gamma)$ satisfies the properties (m1)–(m3) of ([K], §4.0) and so is, in the terminology there, either of finite, affine or indefinite type.

The following lemma recalls the implications of these cases.

4.4 Lemma. Let Γ be as immediately above. Then

(i) $M(\Gamma)$ is not of indefinite type

(ii) If $M(\Gamma)$ is of finite type, Γ is linearly independent

(iii) If $M(\Gamma)$ is of affine type, the subspace $\sum_{i=1}^{n} \mathbb{R}\alpha_i$ has dimension $(n-1)$. There exist $c_1 > 0, \ldots, c_n > 0$ such that $\sum_{i=1}^{n} c_i\alpha_i = 0$. Set $c = (c_1, \ldots, c_n)^t$ and $x = (x_1, \ldots, x_n)^t$ $(x_i \in \mathbb{R},\ i = 1, \ldots, n)$. Then $M(\Gamma)x = 0$ iff $x = \lambda c$ for some $\lambda \in \mathbb{R}$.

Proof The matrix $M'(\Gamma) = (\alpha_i \mid \alpha_j)_{i,j=1,\ldots,n}$ is of the same type as $M(\Gamma)$. Since for any $x_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, $\sum_{i,j=1}^{n} (\alpha_i \mid \alpha_j)x_ix_j = \| \sum_{i=1}^{n} x_i\alpha_i\|^2 \geq 0$, $M'(\Gamma)$ is symmetric and positive. The result now follows from ([Bo] Ch. V, no 3.6) and ([K], §4.5). □

Lemma (4.4) has the following simple consequence.

4.5 Corollary. Let Γ' be a finite non-empty set of non-zero elements of V such that $(\alpha \mid \beta) \leq 0$ for all $\alpha, \beta \in \Gamma'$ with $\alpha \neq \beta$. Then either Γ' is linearly independent or there exist non-negative real numbers c_α $(\alpha \in \Gamma')$, not all zero, such that $\sum_{\alpha\in\Gamma'} c_\alpha\alpha = 0$.

Proof Let $\Gamma' = \bigcup_{i=1}^{n} \Gamma_i$ be the decomposition of Γ' into its indecomposable components Γ_i. Since the Γ_i are mutually orthogonal, Γ' is linearly independent iff each Γ_i is independent. The result follows by (4.4). □

4.6 Let (W, R) be a finite Coxeter system realised as a group of isometries of a real vector space V as in (3.8); in addition to the assumptions there, we now require that V be finite dimensional, and that the form $(\cdot|\cdot)$ be positive definite. If W' is a reflection subgroup of W, let $\Psi = \{\, \alpha \in \Phi \mid r_\alpha \in W' \,\}$. One knows that Ψ contains a simple system Γ i.e. a linearly independent subset Γ of Ψ such that each element of Ψ is a linear combination of elements of Γ with coefficients all of the same sign, and that, setting $R' = \{\, r_\alpha \mid \alpha \in \Gamma \,\}$, (W', R') is a Coxeter system. For our purposes, we need to know that the Coxeter system (W, R) contains a reflection subsystem (in the sense of (1.8)) isomorphic to (W', R'). Since all simple systems in Ψ are conjugate under the action of W', the following Lemma shows that $(W', R') \cong (W', S(W'))$.

4.7 Lemma. Let $S(W')$ be the canonical generators of W', and $\Delta(W') = \{\, \alpha \in \Phi^+ \mid r_\alpha \in S(W') \,\}$. Then $\Delta(W')$ is a simple system in Ψ.

Proof Since $\Delta(W') \subseteq \Phi^+$, there do not exist non-negative scalars c_α, not all 0, with $\sum_{\alpha \in \Delta(W')} c_\alpha \alpha = 0$. By (4.5), $\Delta(W')$ is linearly independent. By theorem (3.9), and the immediately following remark, $\Psi = \Psi^+ \cup (-\Psi^+)$ where

$$\Psi^+ = \Big\{ \sum_{\alpha \in \Delta(W')} c_\alpha \alpha \in \Psi \mid c_\alpha \geq 0 \text{ for all } \alpha \in \Delta(W') \Big\}. \qquad \square$$

4.8 Recall the definitions of a generalised Cartan matrix and its Dynkin diagram ([K]). The Dynkin diagrams of all the indecomposable generalised Cartan matrices of affine type are given in ([K], pages 48–49) and we adopt the notation there. We assume that the vertices of each diagram are given a standard indexing by the integers $0, 1, \ldots, l$ with 0 designating the leftmost vertex; the corresponding Cartan matrix $A = (a_{ij})_{i,j=0,\ldots,l}$ is reconstructed as in ([K], §4.7). For $i = 0, 1, \ldots, l+1$ we let a_i be the number in the Dynkin diagram adjacent to the i-th vertex, and $a_i^\vee$ the number adjacent to the i-th vertex of the dual diagram (obtained by reversing the directions of all arrows, keeping the same indexing of the vertices). Thus, $A(a_0, \ldots, a_n)^t = 0$.

4.9 Let X be one of $A_l (l \geq 1), \ldots, G_2$. Consider the affine Cartan matrix $(a_{ij})_{i,j=0,\ldots,l}$ of type $X^{(1)}$ and define $a_i, a_i^\vee$ as in (4.8).

Let V be the real vector space on basis $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ and define a bilinear form $(\cdot|\cdot)$ on V by

(4.9.1) $(\alpha_i \mid \alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ $(i, j = 0, \ldots, l)$. It follows from ([K], §6.2) that

(4.9.2) the form $(\cdot|\cdot)$ is symmetric and positive semidefinite, and that, adopting the notation of (3.8), the assumptions (3.8) (i), (ii), (iii) hold.

Now let $V_0 = \sum_{i=1}^{l} \mathbb{R}\alpha_i$, $\Phi_0 = \Phi \cap V_0$, $\Phi_0^+ = \Phi^+ \cap V_0$ and $\Pi_0 = \Pi \cap V_0$. We collect together some additional facts from ([K], §6.2 and 6.3) as

4.10 Lemma.

(i) The Coxeter system (W,R) is isomorphic to $\widetilde{W}(X)$.

(ii) The restriction of $(\cdot|\cdot)$ to V_0 is positive definite, and Φ_0 is a root system of type X in V_0 with Π_0 as a set of simple roots and Φ_0^+ the corresponding set of positive roots.

(iii) Let $\delta = \sum_{i=0}^{l} a_i\alpha_i$. Then $(\delta \mid \alpha_i) = 0$ $(i = 0,\dots,l)$, $(\delta \mid \delta) = 0$.

(iv) The root system Φ_0 determines Φ as follows:

(4.10.1) $\Phi^+ = \Phi_0^+ \cup \{\, \alpha + n\delta \mid \alpha \in \Phi_0, n \in \mathbb{N}, n \geq 1, \}$. □

For any $\alpha, \beta \in \Phi$, we define

(4.10.2) $c_{\alpha,\beta} = \frac{(\alpha|\beta)}{\|\alpha\|\,\|\beta\|}$, $n_{\alpha,\beta} = \frac{2(\alpha|\beta)}{(\alpha|\alpha)}$.

With the aid of (4.10.1), we immediately reduce the classification of reflection subsystems of $\widetilde{W}(X)$ to a question concerning sets of vectors in the root system Φ_0.

4.11 Lemma. There exists a reflection subsystem of $\widetilde{W}(X)$ with Coxeter matrix $(m_{ij})_{i.j\in I}$ iff there exists a family $\{\beta_i\}_{(i\in I)}$ of elements of Φ_0 such that

(4.11.1) $c_{\beta_i,\beta_j} = -\cos\frac{\pi}{m_{ij}}$ $(i,j \in I)$ $(\frac{\pi}{\infty} = 0$ by convention). □

<u>Proof</u> Note that if (4.11.1) holds, then $\beta_i \neq \beta_j$ for $i, j \in I$ with $i \neq j$.

Since $\{\delta, \alpha_1, \dots, \alpha_l\}$ is a basis of V, we may define a linear map $p\colon V \longrightarrow V_0$ satisfying $p(\delta) = 0$, $p(\alpha_i) = \alpha_i$ $(i = 1,\dots,n)$. We will also let $q\colon V_0 \longrightarrow V$ denote the map $v \longmapsto v + \delta$ $(v \in V_0)$.

Now by (4.10.1),

(4.11.2) $p(\Phi^+) = \Phi_0$; $q(\Phi_0) \subseteq \Phi^+$,

and by (4.10) (iii), we have

(4.11.3) $c_{\alpha,\beta} = c_{p(\alpha),p(\beta)}$ $(\alpha, \beta \in \Phi^+)$; $c_{\alpha,\beta} = c_{q(\alpha),q(\beta)}$ $(\alpha, \beta \in \Phi_0)$.

Observe that the Cauchy-Schwarz inequality on V_0 together with the first part of (4.11.3) implies

(4.11.4) $\mid c_{\alpha,\beta} \mid \leq 1$ $(\alpha, \beta \in \Phi^+)$.

Now apply (3.9) to show that (W, R) has a reflection subsystem with Coxeter matrix $(m_{ij})_{i,j\in I}$ iff there exists a family $\{\gamma_i\}_{i\in I}$ of elements of Φ^+ satisfying

$$c_{\gamma_i,\gamma_j} = -\cos\frac{\pi}{m_{ij}} \quad (i, j \in I).$$

Making use of (4.11.2) and (4.11.3), we see that this is equivalent to the existence of $\{\beta_i\}_{i\in I}$ satisfying (4.11.1). □

4.12 Note that since Φ_0 is a root system of type X, it contains roots of at most two different lengths, and if $\alpha \in \Phi_0$, then $\pm 2\alpha \notin \Phi_0$.

From the second fact and the calculations on ([Bo]), pg 148) it follows that for $\alpha, \beta \in \Phi_0$ with $\alpha \neq \beta$,

(4.12.1) $c_{\alpha,\beta} \in \{ -\cos\frac{\pi}{m} \mid m \in \mathbb{N} \cup \{\infty\}, m \geq 2 \}$ iff $n_{\alpha,\beta} \leq 0$.

Thus, given a family $\{\beta_i\}_{i\in I}$ of elements of Φ_0 satisfying (4.11.1), it follows that the matrix $A = (n_{\beta_i,\beta_j})_{i,j\in I}$ is a generalised Cartan matrix. By (4.4), each connected component of the Dynkin diagram of A is of finite or affine type. The Coxeter graph of the Coxeter matrix $(m_{ij})_{i,j\in I}$ satisfying (4.11.1) is obtained from the Dynkin diagram of A by replacing each connected component of type $A_1^{(1)}$ or $A_2^{(2)}$ by the Coxeter graph of type $\widetilde{A}_1$, and removing all arrows from any multiple bonds on the other components.

4.13 We can now prove (4.2) (ii). Suppose that $\Gamma \subseteq \Phi_0$ and that $n_{\alpha,\beta} \leq 0$ for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Let $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ be the decomposition of Γ into indecomposable mutually orthogonal sets Γ_i. For any non-zero $\alpha \in V_0$, we denote the reflection in α by $s_\alpha \colon V_0 \longrightarrow V_0$.

For each i such that $M(\Gamma_i)$ (defined as in (4.3)) is of finite type, say Y_i', we set $Y_i = Z_i = Y_i'$, $\Gamma_i' = \Gamma_i$, (noting Γ_i' is linearly independent by 4.4 (ii)) $R_i' =$

$\{ s_\beta \mid \beta \in \Gamma_i' \}$ and $W_i' = \langle R_i' \rangle$, $\Phi_i' = W_i' \Gamma_i'$. Then Φ_i' is a root system of type Y_i (in the linear span of Γ_i'), Γ_i' is a simple system in Φ_i' and $(W_i', R_i') \cong W(Y_i)$.

On the other hand, if $M(\Gamma_i)$ is of affine type Y_i', we may write $\Gamma_i' = \{\beta_0, \ldots, \beta_m\}$ so that the matrix $M(\Gamma)$ is the Cartan matrix of type Y_i' determined by the standard labelling of its Dynkin diagram; we let b_j denote the number adjacent to the j-th vertex of that Dynkin diagram. By (4.4) (iii),

(4.13.1) $\sum_{i=0}^{m} b_i \beta_i = 0$, $\{\beta_1, \ldots, \beta_m\}$ is linearly independent.

Now Y_i' cannot be $A_2^{(2)}$ (else $2\beta_0 + \beta_1 = 0$) or $A_{2l}^2 (l \geq 2)$ (since then $\|\beta_l\| = \sqrt{2}\|\beta_{l-1}\| = 2\|\beta_0\|$, assuming the vertices are labelled $0, 1, \ldots, l-1, l$ from left to right).

Let $\Gamma_i' = \{\beta_1, \ldots, \beta_m\}$, $R_i' = \{ s_\beta \mid \beta \in \Gamma_i' \}$, $W_i' = \langle R_i' \rangle$, $\Phi_i' = W_i' \Gamma_i'$. Since Γ_i' is linearly independent, Φ_i' is a root system (in the linear span of Γ_i') and Γ_i' is a simple system in Φ_i'. The Dynkin diagram of Φ_i' is obtained by deleting vertex 0 of the Dynkin diagram of Y_i' ie Φ_i' is of type Z_i, where

$$Z_i = \begin{cases} Z & (Y_i' = Z^{(1)}) \\ C_l & (Y_i' = A_{2l-1}^{(2)} \ (l \geq 3)) \\ B_l & (Y_i' = D_{l+1}^{(2)} \ (l \geq 2)) \\ F_4 & (Y_i' = E_6^{(2)}) \\ G_2 & (Y_i' = D_4^{(3)}). \end{cases}$$

We also define

$$Y_i = \begin{cases} Z & (Y_i' = Z^{(1)}) \\ B_l & (Y_i' = A_{2l-1}^{(2)} (l \geq 3)) \\ C_l & (Y_i' = D_{l+1}^{(2)} (l \geq 2)) \\ F_4 & (Y_i' = E_6^{(2)}) \\ G_2 & (Y_i' = D_4^{(3)}) \end{cases}$$

and note that $W(Y_i) \cong W(Z_i)$. By (4.11) and (4.12), $\widetilde{W}(X)$ has a reflection subsystem isomorphic to $P_1 \times \ldots \times P_n$, where

$$P_i = \begin{cases} W(Y_i) & (Y_i' \text{ of finite type}) \\ \widetilde{W}(Y_i) & (Y_i' \text{ of affine type}) \end{cases}$$

and every reflection subsystem is isomorphic to one arising in this way from some set Γ satisfying $n_{\alpha,\beta} \leq 0$ for $\alpha, \beta \in \Gamma, \alpha \neq \beta$.

Let $R_0 = \{s_{\alpha_1}, \ldots, s_{\alpha_l}\}$, $W_0 = \langle R_0 \rangle$. Then by (4.10) (ii), $(W_0, R_0) \cong W(X)$. To prove (4.2) (i), it will suffice to show (W_0, R_0) has a reflection subsystem isomorphic to $W(Y_1) \times \ldots \times W(Y_n)$.

Let $\Gamma' = \bigcup_{i=1}^{n} \Gamma_i'$; this is linearly independent since the Γ_i' are independent and mutually orthogonal. Let $R' = \{ r_\beta \mid \beta \in \Gamma' \} = \bigcup_{i=1}^{n} R_i'$, $W' = \langle R' \rangle$, $\Phi' = W'\Gamma'$. Since the Γ_i' are mutually orthogonal, $\Phi' = \dot{\bigcup}_{i=1}^{n} \Phi_i'$ and Γ' is a simple system in Φ'. By (4.6), (W_0, R_0) has a reflection subsystem isomorphic to (W', R'). But

$$\begin{aligned} (W', R') &\cong (W_1', R_1') \times \ldots \times (W_n', R_n') \text{ by orthogonality of the } \Gamma_i \\ &\cong W(Z_1) \times \ldots \times W(Z_n) \text{ since } \Phi_i' \text{ is of type } Z_i \\ &\cong W(Y_1) \times \ldots \times W(Y_n) \text{ since } W(Y_i) \cong W(Z_i). \end{aligned}$$

4.14 To complete the proof of (4.2), it suffices to prove that if $W(X)$ has a reflection subsystem (W', R') isomorphic to $W(X_1) \times \ldots \times W(X_n)$, then $W(\widetilde{X})$ has a reflection subsystem isomorphic to $\widetilde{W}(X_1) \times \ldots \times \widetilde{W}(X_n)$.

Identify $W(X)$ with (W_0, R_0) where $R_0 = \{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$, $W_0 = \langle R_0 \rangle$. Let $\Pi' = \{ \alpha \in \Phi_0^+ \mid r_\alpha \in R' \}$ and $\Phi' = W'\Pi'$. Note Π' is linearly independent by (4.7). By (3.9) and (4.11.4)

(4.14.1) $c_{\alpha,\beta} \in \{ -\cos \frac{\pi}{m} \mid m \in \mathbb{N} \cup \{\infty\},\ m \geq 2 \}$ for all $\alpha, \beta \in \Pi',\ \alpha \neq \beta$.

Let $\Pi' = \bigcup_{i=1}^{m} \Pi_i'$ be the decomposition of Π' into mutually orthogonal indecomposable subsets Π_i'. Let $R_i = \{ r_\alpha \mid \alpha \in \Pi_i' \}$, $W_i = \langle R_i \rangle$, $\Phi_i = W_i \Pi_i$. The decomposition of Π' into indecomposable components corresponds to the decomposition of (W', R') into irreducible components, so $m = n$ and it may be assumed that $(W_i, R_i) \cong W(X_i)$.

Suppose that Φ_i is a root system of type Y_i (necessarily one of $A_l, \ldots, G_2$). Since $W(Y_i) \cong (W_i, R_i) \cong W(X_i)$, we have either $Y_i = X_i$ or ($Y_i = B_{l_i},\ X_i = C_{l_i}\ (l_i \geq 3)$) or ($Y_i = C_{l_i},\ X_{l_i} = B_{l_i}\ (l_i \geq 3)$). Let $T_k = (a_{ij}^{(k)})_{i.j=0,\ldots,l_k}$ be the affine Cartan matrix of type

$$\begin{cases} X_k^{(1)} & (X_k = Y_k) \\ D_{l_k+1}^{(2)} & (Y_k = B_{l_k}, X_k = C_{l_k}, l_k \geq 3) \\ A_{2l_k-1}^{(2)} & (Y_k = C_{l_k}, X_k = B_{l_k}, l_k \geq 3) \end{cases}$$

($k = 1, \ldots, n$), and $a_i^{(k)}$ be the number associated to the i-th vertex of the corresponding Dynkin diagram.

We may write $\Pi_k' = \{\alpha_1^{(k)}, \ldots, \alpha_{l_k}^{(k)}\}$ ordered in such a way that $\frac{2(a_i^{(k)}|a_j^{(k)})}{(a_i^{(k)}|a_i^{(k)})} = a_{ij}^{(k)}$ $(i, j = 1, \ldots, l_k)$, and define

$$\alpha_0^{(k)} = -\sum_{i=1}^{l_k} a_i^{(k)} \alpha_i^{(k)},\ \Pi_k = \{\alpha_0^{(k)}\} \cup \Pi_k' \,.$$

One may check from the descriptions of the root systems in ([Bo], pages 250–275) that in each case, $\alpha_0^{(k)} \in \Phi_k$ (in fact, except for the cases where T_k is of type $D_{l_k+1}^{(2)}$ or $A_{2l_k-1}^{(2)}, \alpha_0^{(k)}$ is the negative of the highest root of Φ_k corresponding to the simple system Π_k') and that the matrix $M(\Pi_k)$ is equal to T_k. Since Π_k is orthogonal to Π_j if $j \neq k$, (4.12) shows that $\widetilde{W}(X)$ has a reflection subsystem the Coxeter graph of which has as connected components the Coxeter graphs of type $\widetilde{X}_1, \ldots, \widetilde{X}_n$. This completes the proof of (4.2). $\square$

4.15 Remark We will now describe an algorithm for computing the isomorphism types of reflection subgroups of affine Weyl groups, in terms of operations with their Coxeter graphs. For this purpose, it is convenient to agree that the Coxeter graph of type B_l $(l \geq 3)$ is also of type C_l. We define two types of operations on Coxeter graphs Γ all of whose connected components are of type $A_l, \ldots, G_2, \widetilde{A}_l, \ldots, \widetilde{G}_2$.

By a deletion we will mean the removal of one or more vertices, together with all edges incident with them, from one connected component of Γ.

By a completion, we will mean the replacement of a connected component of Γ which is a Coxeter graph of finite type X with one of type $\widetilde{X}$ ($X = A_l, \ldots, G_2$).

Also, define a move to consist of a completion of a component of finite type, followed by a deletion applied to the newly introduced component of affine type. If X is of finite type $A_l, \ldots, G_2$, then the Coxeter graphs of the reflection subsystems of $W(X)$ are precisely those obtained by beginning with a Coxeter graph of type X and applying a finite number of moves ([Ca]).

It follows from (4.2) that the Coxeter graphs of the reflection subsystems of $\widetilde{W}(X)$ are precisely those obtained by beginning with the Coxeter graph of type X and applying any finite sequence of completions and deletions. Also, the reflection subsystems of $W(X)$ are those whose Coxeter graphs arise in this way and have all their connected components of finite type.

For example, from the Coxeter graph of type $\widetilde{F}_4$ one can obtain by successive deletions and completions the Coxeter graphs of type B_4, $\widetilde{B}_4$, D_4,$\widetilde{D}_4$, $A_1 \times A_1 \times A_1 \times A_1$.Completing in turn each component of type A_1, one sees that $\widetilde{W}(F_4)$ (of rank 5) contains a reflection subsystem of type $\widetilde{W}(A_1) \times \widetilde{W}(A_1) \times \widetilde{W}(A_1) \times \widetilde{W}(A_1)$ (of rank 8).

Chapter 5

THE SIMPLICIAL COMPLEX OF A BRUHAT INTERVAL

To any Coxeter system (W, R), there is associated a standard partial order on W, the Bruhat order ([De1]). This chapter begins by recalling some characterisations of Bruhat order; we then prove two special results that will be used later.

The main result of the chapter is that the simplicial complex of an open Bruhat interval is a sphere; this is a consequence of the lexicographical shellability of Bruhat order proved in [BW], but the proof here depends on a study of a natural decomposition of the simplicial complex into cells.

Finally, it is shown that the subgroup generated by the ratios $x^{-1}y$ of all pairs of elements x, y in some closed Bruhat interval is actually generated by the ratios of the pairs of elements in some fixed maximal chain.

Throughout this chapter, (W, R) denotes a Coxeter system, $\ell\colon W \longrightarrow \mathbb{N}$ is the corresponding length function, and $T = \bigcup_{w \in W} wRw^{-1}$. Recall the definition of the Bruhat graph $\Gamma_{(W,R)}$ (see (1.11)).

5.1 Definition. The Bruhat order $\leq$ on W is the partial order such that $x \leq y$ iff there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of W such that for each $i \in \{1, \ldots, n\}$, $(x_{i-1}, x_i) \in E_{(W,R)}$. In other words, $x \leq y$ iff there exists a path in $\Gamma_{(W,R)}$ from x to y $(x, y \in W)$.

5.2 Remark. Let $\mathcal{P}(W)$ denote the monoid of subsets of W under the product $A \cdot B = \{\, ab \mid a \in A,\ b \in B \,\}$ $(A, B \in \mathcal{P}(W))$. If $r, s \in R$ and $n_{r,s} = \operatorname{ord}(rs)$ is finite, then

$$\overbrace{(\ldots \{1, r\} \cdot \{1, s\} \cdot \{1, r\})}^{n_{r,s}} = \langle r, s \rangle = \overbrace{(\ldots \{1, s\} \cdot \{1, r\} \cdot \{1, s\})}^{n_{r,s}}.$$

It follows by ([Bo] Ch IV, no. 1.5) that there is a function $f\colon W \longrightarrow \mathcal{P}(W)$ such that $f(w) = \{1, r_1\} \cdot \ldots \cdot \{1, r_n\}$ whenever $w = r_1 \ldots r_n$ $(r_i \in R)$ and $n = \ell(w)$. The first of the alternative characterisations of Bruhat order below shows that $f(w) = \{\, v \in W \mid v \le w \,\}$.

5.3 Proposition.

(i) Let $v, w \in W$ and write $w = r_1 \ldots r_n$ $(r_i \in R,\ n = \ell(w))$. Then $v \le w$ iff there exist integers $i_1, \ldots, i_m$ with $1 \le i_1 < \ldots < i_m \le n$ and $w = r_{i_1} \ldots r_{i_m}$ (where $\le$ denotes Bruhat order on W).

(ii) Bruhat order is the unique partial order $\le$ on W such that $1 \le w$ for all $w \in W$ and the following "Z-property" holds:

if $x, y \in W,\ r \in R$ and $\ell(rx) < \ell(x),\ \ell(ry) < \ell(y)$ then conditions (a)–(c) below are equivalent.

$$\text{(a) } x \le y \qquad \text{(b) } rx \le y \qquad \text{(c) } rx \le ry$$

(iii) Bruhat order is the unique partial order $\le$ on W such that

(a) If $x \le y$, then $\ell(x) \le \ell(y)$. If $x \le y$ and $\ell(x) = \ell(y)$ then $x = y$.

(b) If $A \subseteq W$ has a maximum (minimum) element then for any $r \in R,\ \{1, r\}A$ has a maximum (respectively, minimum) element.

Proof Parts (i) and (ii) are proved in [De 1], so we only prove (iii). First, let $\le$ denote Bruhat order. Then (iii) (a) follows from the Definition (5.1). Also, the Z-property implies that

(5.3.1) if $\emptyset \ne A \subseteq W$ and $a = \max\,(A)$, then

$$\max\,(\{1, r\}A) = \begin{cases} a & (\ell(ra) < \ell(a)) \\ ra & (\ell(ra) > \ell(a)), \end{cases}$$

and the statement about the minimum is proved similarly.

Conversely, suppose now that $\le$ is a partial order on W satisfying (iii) (a), (b). We show that $\le$ satisfies the conditions of (ii). Let $w \in W$ and write $w = r_1 \ldots r_n$ $(r_i \in R)$. By repeated application of (iii) (b), the set $B =$

$\{1, r_n\} \dots \{1, r_1\} w$ has a minimal element, say x. Since $1 \in B$, we have $x \le 1$, hence $x = 1$ by (iii) (a). Since $w \in B$, $1 \le w$. Hence $1 \le w$ for all $w \in W$.

Now take $x \in W$. The set $\{1, r\}\{x\} = \{x, rx\}$ has a minimum element by (iii) (b); using (iii) (a), we have

$$\begin{cases} x \le rx & (\ell(rx) > \ell(x)) \\ rx \le x & (\ell(rx) < \ell(x)). \end{cases}$$

To check the Z-property, take $x, y \in W$ and $r \in R$ with $\ell(rx) < \ell(x)$ and $\ell(ry) < \ell(y)$. Then $rx \le x$ and $ry \le y$. Hence if $x \le y$, then $rx \le x \le y$. On the other hand, suppose that $rx \le y$. Since $rx \ne y$, (iii) (a) implies that $\ell(rx) \le \ell(y) - 1$, hence $\ell(x) \le \ell(y)$. The set $\{rx, y\}$ has a maximum element y, so $\{1, r\}\{rx, y\} = \{rx, ry, x, y\}$ has a maximum element which must be either x or y (since $rx < x$ and $ry < y$). In particular, either $x \le y$ or $y \le x$. But $\ell(x) \le \ell(y)$, so if the latter holds, (iii) (a) shows that $x = y$. Hence $x \le y$ in either case. Thus, $x \le y$ iff $rx \le y$. Similarly, $rx \le y$ iff $rx \le ry$. Hence (iii) is proved. □

Henceforward, the Bruhat order on W will always be denoted by $\le$. For any $x, y \in W$, we define

$$(x, y) = \{\, w \in W \mid x < y < w \,\},\ (x, y] = \{\, w \in W \mid x < y \le w \,\} \text{ etc.}$$

5.4 Corollary. Let $A \subseteq W$ have a maximum (minimum) element. Then for any $w \in W$, $[1, w]A$ has a maximum (respectively, minimum) element.

<u>Proof</u> If $w = r_1 \dots r_n$ ($n = \ell(w)$), then by (5.2), $[1, w] = \{1, r_1\} \dots \{1, r_n\}$. The result follows by repeated use of (5.3) (iii) (b). □

Following is an interesting special property of Bruhat intervals of the form $[1, w]$.

5.5 Proposition. For any $w \in W$,

$$\#\{\, r \in R \mid r \le w \,\} \ \le \#\{\, x \in W \mid x \le w,\ \ell(x) = \ell(w) - 1 \,\}.$$

<u>Proof</u> For $w \in W$, let $c(w) = \#\{\, x \in W \mid x \le w,\ \ell(x) = \ell(w) - 1 \,\}$ and $a(w) = \#\{\, r \in R \mid r \le w \,\}$, $b(w) = c(w) - a(w)$. We have to prove that

$b(w) \geq 0$; this is the case $y = 1$ of
(5.5.1) if $w = vy$ where $v \in W^J$ and $y \in W_J$, then $b(w) \geq b(y)$.
Here $J \subseteq R$ and W^J, W_J are as in (1.14). We prove (5.5.1) by induction on $\ell(v)$. If $\ell(v) = 0$ then (5.5.1) is trivial, so assume $\ell(v) \geq 1$ and (5.5.1) holds for v of smaller length. There is no loss of generality in assuming that $J = \{\, r \in R \mid r \leq y \,\}$ ([Bo] Ch IV, no 1.8). Choose $s \in R$ with $\ell(vs) < \ell(v)$, and write $v = v'v''$ where $v'' \in W_{J\cup\{s\}}$ and $v' \in W^{J\cup\{s\}}$. Since $\ell(vs) < \ell(v)$, it follows that $v \neq v'$. But $\ell(v) = \ell(v') + \ell(v'')$, so $\ell(v') < \ell(v)$. By induction, $b(w) = b(v'(v''y)) \geq b(v''y)$, noting that $v''y \in W_{J\cup\{s\}}$. Also, $a(v''y) \leq \#(J \cup \{s\} \leq a(y) + 1$.

Choose $r \in J \cup \{s\}$ so that $\ell(rv'') < \ell(v'')$; this is possible since $v'' \in W_{J\cup\{s\}}$ and $v'' \neq 1$. Then $v = v'v''$ where $\ell(v) = \ell(v') + \ell(v'') = \ell(v'r) + \ell(rv')$ and it follows that $v'' \in W^J$, $rv'' \in W^J$. In particular,

(5.5.2) $v''W_J \neq rv''W_J$ (by (1.14.1)

Let $y_1, \ldots, y_m$ ($m = c(y)$) be the distinct elements of $\{\, x \in W \mid x \leq y,\ \ell(x) = \ell(y) - 1 \,\}$, and consider the elements $v''y_1, \ldots, v''y_m, rv''y$ of $[1, v''y]$. Since $v'' \in W^J$, $rv'' \in W^J$ we have $\ell(v''y_i) = \ell(v'') + \ell(y_i) = \ell(v'') + \ell(y) - 1 = \ell(v''y) - 1$ and $\ell(rv''y) = \ell(rv'') + \ell(y) = \ell(v'') + \ell(y) - 1 = \ell(v''y) - 1$. Making use of (5.5.2), we see that these elements are all distinct. This proves that $c(v''y) = \#\{x \in W \mid x \leq v''y,\ \ell(x) = \ell(v''y) - 1\,\} \geq m + 1$. Hence $b(w) \geq b(v''y) = c(v''y) - a(v''y) \geq (c(y) + 1) - (a(y) + 1) = b(y)$ as required. □

We now turn to the consideration of simplicial complexes of Bruhat intervals. For the statement and proof of our result, we need quite a number of notions and facts from elementary combinatorial topology and it is convenient to collect these all together at the outset.

5.6 In this chapter, the word complex will mean finite, abstract, simplicial complex. Thus, a complex K is a (finite) set of finite sets such that if $A \in K$ and $B \subseteq A$ then $B \in K$; if $A \in K$ then A is said to be a simplex of K, of dimension $\dim A = (\#(A) - 1)$. Note that by our conventions, any non-empty complex contains the empty simplex $\emptyset$, and $\emptyset$ has dimension (-1). A subcomplex of a complex K is defined to be a subset L of K which is itself a complex. An n-simplex ($n \in \mathbb{N} \cup \{-1\}$) is a set of cardinality $(n + 1)$.

Let K be a complex and A be a simplex. We define the complement, star and link of A in K to be the complexes

$$\begin{aligned}\mathcal{C}(A,K) &= \{\, B \in K \mid B \not\supseteq A \,\}\\ \mathrm{st}(A,K) &= \{C \in K \mid A \cup C \in K\,\}\\ \mathrm{lk}(A,K) &= \{\, C \in \mathrm{st}(A,K) \mid A \cap C = \emptyset\}.\end{aligned}$$

For any $L \subseteq K$, its closure is defined to be the complex

$$\bar{L} = \{\, B \in K \mid \text{ there exists } C \in L \text{ with } B \subseteq C\}.$$

If M is any other complex with $K \cap M \subseteq \{\emptyset\}$, define the join of K and M to be

$$KM = \{\, A \cup B \mid A \in K,\ B \in M\,\}.$$

A complex K is defined to be n-homogeneous if for all $A \in K$ there exists $B \in K$ with $A \subseteq B$ and $\dim B = n$. If K is n-homogeneous, its boundary $\overset{\bullet}{K}$ is by definition $\bar{L}$, where L is the set of $(n-1)$-simplexes of K contained in an odd number of n-simplexes of K, and $\overset{\circ}{K} = K \setminus \overset{\bullet}{K}$ ($\overset{\circ}{K}$ is not a complex in general).

5.7 Let K be a complex, $A \neq \emptyset$ a simplex and $\{a\}$ a 0-simplex not in K. Let $P = \mathrm{lk}\ (A,K),\ Q = \mathcal{C}(A,K)$, and note

(5.7.1) $K = AP \cup Q,\ AP \cap Q = \overset{\bullet}{A}P$

Let $L = a\overset{\bullet}{A}P \cup Q$; here, a denotes the complex $\{\,\{a\,\},\emptyset\}$. (One may regard L as a "subdivision" of K.) Then we write $K \xrightarrow{(A,a)} L$. If $(K_\lambda)_{\lambda\in\Lambda}$ is a family of subcomplexes of K and $K_\lambda \xrightarrow{(A,a)} L_\lambda$ $(\lambda \in \Lambda)$ then we will write

$$(K\colon (K_\lambda)) \xrightarrow{(A,a)} (L\colon (L_\lambda)) \qquad \text{and} \qquad (L\colon (L_\lambda)) \xrightarrow{(A,a)^{-1}} (K\colon (K_\lambda)).$$

Often, one writes just $\longrightarrow$ instead of $\xrightarrow{(A,a)^{\pm 1}}$. The equivalence relation on such pairs $(K\colon (K_\lambda)_{\lambda\in\Lambda})$ generated by the relation $\longrightarrow$ will be denoted $\sim$. (In fact, if K and L are complexes, then $K \sim L$ iff K and L have simplicially homeomorphic subdivisions, but this won't be needed here.) Note the following simple properties of $\longrightarrow$:

(5.7.2) If $K_\lambda \overset{(A,a)}{\longrightarrow} L_\lambda$ $(\lambda \in \Lambda)$ then $\cup_\lambda K_\lambda \overset{(A,a)}{\longrightarrow} \cup_\lambda L_\lambda$ and $\cap_\lambda K_\lambda \overset{(A,a)}{\longrightarrow} \cap_\lambda L_\lambda$

(5.7.3) If $K \overset{(A,a)}{\longrightarrow} L$ and $K' \overset{(A,a)}{\longrightarrow} L'$ then $K \subseteq K'$ iff $L \subseteq L'$

(5.7.4) If $K \overset{(A,a)^\varepsilon}{\longrightarrow} L$ and $K' \overset{(A,a)^\varepsilon}{\longrightarrow} L$ then $K = K'$ $(\varepsilon \in \{-1, 1\})$.

From [A], we have

(5.7.5) Suppose $K \overset{(A,a)}{\longrightarrow} L$. Then K is n-homogeneous iff L is n-homogeneous. If K is homogeneous, then $\overset{\bullet}{K} \overset{(A,a)}{\longrightarrow} \overset{\bullet}{L}$.

If K, L are homogeneous complexes, the "elementary move" $K \overset{(A,a)^\varepsilon}{\longrightarrow} L$ is called internal if ($\varepsilon = 1$ and $A \notin \overset{\bullet}{K}$) or ($\varepsilon = -1$ and $A \notin \overset{\bullet}{L}$).

5.8 Definition. For $n \in \{1\} \cup \mathbb{N}$, let $\mathbf{B}^n$ denote a complex consisting of an n-simplex and its subsets, and $\mathbf{S}^n = \overset{\bullet}{\mathbf{B}}{}^{n+1}$. A complex K is called a (combinatorial) n-ball if $K \sim \mathbf{B}^n$ and a (combinatorial) n-sphere if $K \sim \mathbf{S}^n$. We set $\overset{\bullet}{\mathbf{B}}{}^{-1} = \mathbf{S}^{-2} = \emptyset$. The following proposition lists some properties of balls and spheres ([A]).

5.9 Proposition.

(i) Let B be an n-ball $(n \in \{-1\} \cup \mathbb{N})$. Then B is n-homogeneous and $\overset{\bullet}{B}$ is an $(n-1)$-sphere. If $\{a\} \notin B$, then aB is an $(n+1)$-ball and $(aB)^\bullet = B \cup a\overset{\bullet}{B}$. If $A \neq \emptyset$ is a p-simplex of B, then $\mathrm{lk}(A, B)$ is a $(n-p-1)$-sphere if $A \notin \overset{\bullet}{B}$ or an $(n-p-1)$-ball if $A \in B^\bullet$.

(ii) Let S be an n-ball $(n \in \mathbb{N})$. Then S is n-homogeneous and $\overset{\bullet}{S} = \emptyset$. If K is a non-empty n-homogeneous subcomplex of S and $\overset{\bullet}{K} = \emptyset$, then $K = S$. If $\{a\} \notin S$ then aS is an $(n+1)$-ball and $(aS)^\bullet = S$. If $A \neq \emptyset$ is a p-simplex of S, then $\mathrm{lk}(A, S)$ is a $(n-p-1)$-sphere.

(iii) If E_1, E_2 are two n-balls such that $E_1 \cap E_2 = \overset{\bullet}{E}_1 \cap \overset{\bullet}{E}_2 = F$ is an $(n-1)$-ball, then $E_1 \cup E_2$ is an n-ball and $(E_1 \cup E_2)^\bullet = (\overset{\bullet}{E}_1 \cup \overset{\bullet}{E}_2) \setminus \overset{\circ}{F}$. □

The following technical result ([A]) will be used repeatedly

5.10 Theorem. Let B be an n-ball, and J be a complex such that $B \cap J \subseteq \dot{B}$ and $\{a\}$ be a 0-simplex not in $J \cup \dot{B}$. Then there is a sequence of elementary internal moves $B = K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_m = a\dot{B}$ such that $K_i \cap J \subseteq \dot{K}_i = \dot{B}$ for $i = 0, 1, \ldots, m$. Moreover, if in the above $K_i \stackrel{(A,a)^\varepsilon}{\longrightarrow} K_{i+1}$ then also $J' \stackrel{(A,a)^\varepsilon}{\longrightarrow} J'$ for any subcomplex J' of J. $\square$

This completes our inventory or standard facts from combinatorial topology.

5.11 If P is any finite poset, $\sum P$ denotes the complex whose simplexes are the totally ordered subsets (also called chains) of P. For any $x, y \in P$, we define $(x, y) = \{\, z \in P \mid x < z < y \,\}$, $[x, y) = \{\, z \in P \mid x \leq z < y \,\}$ etc. The poset P is said to satisfy the Jordan-Dedekind chain condition if for any $x, y \in W$ with $x \leq y$, $\sum[x, y]$ is an n-homogeneous complex for some $n \in \mathbb{N}$; we then write $n = \ell(x, y)$.

In (5.12)–(5.14), X denotes a finite poset, with a maximum element w, satisfying the Jordan-Dedekind chain condition. We define below a notion of an "X-celled complex"; this is a combinatorial analogue of a CW-complex with cells indexed by X and inclusion relations amongst the closed cells described by the partial order on X.

5.12 Definition. An X-celled complex is a pair $(K\colon (e_x)_{x\in X})$ where K is a complex and the e_x $(x \in X)$ are subcomplexes of K satisfying

(i) e_x is a $[\ell(x, w) - 1]$-ball $(X \in X)$

(ii) $\dot{e}_x = \bigcup_{z>x} e_z \qquad (x \in X)$

(iii) $K = \dot{\bigcup}_x \overset{\circ}{e}_x$

(the unions being taken over all elements of X satisfying any indicated condition).

We need two general lemmas concerning X-celled complexes.

5.13 Lemma. Let $(K\colon(e_x))$ be an X-celled complex and suppose that

$$(K\colon(e_x)) \sim (L\colon(f_x)).$$

Then $(L\colon(f_x))$ is an X-celled complex.

<u>Proof</u> It will suffice to prove this in the case where the complexes differ by a single elementary move $(K\colon(e_x)) \stackrel{(A,a)^\varepsilon}{\longrightarrow} (L\colon(f_x))$. Since e_x is a $[\ell(x,w)-1]$-ball and $e_x \stackrel{(A,a)^\varepsilon}{\longrightarrow} f_x$, it follows that f_x is a $[\ell(x,w)-1]$-ball and $\overset{\bullet}{e}_x \stackrel{(A,a)^\varepsilon}{\longrightarrow} \overset{\bullet}{f}$. Since $\cup_{z>x} e_z \stackrel{(A,a)^\varepsilon}{\longrightarrow} \cup_{z>x} f_z$ and $\overset{\bullet}{e}_x = \cup_{z>x} e_z$, this shows that $\overset{\bullet}{f}_x = \cup_{z>x} f_z$ $(x \in X)$.

Suppose that $x, z \in X$ and $x \neq z$. Then $\overset{\circ}{e}_z \cap \overset{\circ}{e}_x = \emptyset$. That is, $(e_z \setminus \cup_{t>z} e_t) \cap (e_x \setminus \cup_{t>x} e_x) = \emptyset$, or, equivalently, $e_z \cap e_x \subseteq \bigcup\limits_{\substack{t>z \\ \text{or } t>x}} e_t$. This implies that $f_z \cup f_x \subseteq \bigcup\limits_{\substack{t>z \\ \text{or } t>x}} f_t$ and so $\overset{\circ}{f}_z \cap \overset{\circ}{f}_x = \emptyset$.

Finally, $K \stackrel{(A,a)^\varepsilon}{\longrightarrow} L$, $\cup_x e_x \stackrel{(A,a)^\varepsilon}{\longrightarrow} \cup_x f_x$ and $K = \cup_x e_x$ imply that $L = \cup_x f_x$.

□

Together with (5.13), the result below shows that the existence of an X-celled complex is a very restrictive condition on X.

5.14 Lemma. Let $(K\colon(e_x))$ be an X-celled complex. Then $(K\colon(e_x)) \sim (L\colon(f_x))$ where $f_x = \sum[x,w]$ $(x \in W)$ and $L = \cup_x f_x$.
<u>Proof</u> It will be convenient to abbreviate $\ell(x,w)$ by $\ell'(x)$ $(x \in X)$ (recall w is the maximum element of X).

For $m \in \mathbb{N}$, let $X_m = \{\, x \in X \mid \ell(x,w) \leq m \,\}$ and let $\mathcal{K}_{(m)}$ be the set of pairs $(L\colon(f_x)_{x \in X_m})$ where L is a complex with no vertex in $X \setminus X_m$ and the f_x are subcomplexes such that $L = \bigcup\limits_{x \in X_m} f_x$.

If $A = (L\colon(f_x)_{x \in X_m}) \in \mathcal{K}_{(m)}$, define $\hat{A} = (\hat{L}\colon(\hat{f}_x)_{x \in X})$ by setting

$$\hat{f}_x = \begin{cases} f_x & (x \in X_m) \\ \bigcup\limits_{\substack{z>x \\ \ell'(z)=m}} \sum[x,z] f_z & (x \in X \setminus X_m) \end{cases}$$

and $\hat{L} = \bigcup_{x \in X} \hat{f}_x$. We make the following claim:

(5.14.1) if $L_1, L_2 \in \mathcal{K}_{(m)}$ and $L_1 \sim L_2$ then $\hat{L}_1 \sim \hat{L}_2$.

To prove this claim, it will suffice to show that if $L_1 \xrightarrow{(A,a)^\varepsilon} L_2$ where A is a simplex with no vertex in $X \setminus X_m$ and $a \notin X \setminus X_m$ then $\hat{L}_1 \xrightarrow{(A,a)^\varepsilon} \hat{L}_2$.

Write $L_1 = (M\colon (g_x)_{x \in X_m})$ and $L_2 = (N\colon (h_x)_{x \in X_m})$. Note that $\{a\} \notin \hat{M}$ otherwise $\{a\} \in M$ and the elementary move $\xrightarrow{(A,a)}$ couldn't be applied to M, by its definition). It will be sufficient to prove that $\hat{g}_x \xrightarrow{(A,a)} \hat{h}_x$, for then we also have $\hat{M} = \bigcup_{x \in X} \hat{g}_x \longrightarrow \bigcup_{x \in X} \hat{h}_x = \hat{N}$.

Now if $x \in X_m$, then $\hat{g}_x = g_x \xrightarrow{(A,a)} h_x = \hat{h}_x$. If $x \in X \setminus X_m$, then

$$\mathcal{C}(A, \hat{g}_x) = \bigcup_{\substack{z > x \\ \ell'(z) = m}} \sum [x,z) \mathcal{C}(A, g_z)$$

$$\mathrm{st}(A, \hat{g}_x) = \bigcup_{\substack{z > x \\ \ell'(z) = m}} \sum [x,z) \mathrm{st}(A, g_z)$$

and

$$\mathrm{lk}(A, \hat{g}_x) = \bigcup_{\substack{z > x \\ \ell'(z) = m}} \sum [x,z) \mathrm{lk}(A, g_z).$$

Hence

$$\begin{aligned}
\hat{g}_x &\xrightarrow{(A,a)} \mathcal{C}(A, \hat{g}_x) \cup a\,\dot{A}\,\mathrm{lk}(A, \hat{g}_x) \\
&= \bigcup_{\substack{z > x \\ \ell'(z) = m}} \sum [x,z)\ \left(\mathcal{C}(A, g_z) \cup a\,\dot{A}\,\mathrm{lk}(A, g_z)\right) \\
&= \bigcup_{\substack{z > x \\ \ell'(z) = m}} \sum [x,z)\ h_z \\
&= \hat{h}_x
\end{aligned}$$

and so (5.14.1) is proved.

Now we complete the proof of the lemma. Note that if $P \overset{(\{b\},a)}{\longrightarrow} P'$ then P' does not have b as a vertex. Hence there is no loss of generality in assuming that K has no vertex in X. We will show by downward induction on m that $A_\infty \sim \hat{A}_m$, where $A_\infty = (K\colon (e_x)_{x\in X})$ and $A_m = \left(\bigcup_{x\in X_m} e_x\colon (e_x)_{x\in X_m}\right) \in \mathcal{K}_{(m)}$. If $m \geq \max\{\ell(x,w) \mid x \in X\}$ then $\hat{A}_m = A_m = A_\infty$. Suppose inductively that $m \geq 1$ and that $A_\infty \sim \hat{A}_m$.

Choose any $y_0 \in X$ with $\ell'(y_0) = m$. If $y \in X_m$ and $y \neq y_0$, then $\overset{\circ}{e}_{y_0} \cap e_y = \overset{\circ}{e}_{y_0} \cap (\cup_{z\geq y} \overset{\circ}{e}_z) = \emptyset$ since $\overset{\circ}{e}_{y_0} \cap \overset{\circ}{e}_z = \emptyset$ unless $z = y_0$, and $y \not\leq y_0$. Now make use of (5.10) applied to the ball $B = e_{y_0}$ with $a = y_0$ and $J = \bigcup_{\substack{y\in x_m \\ y\neq y_0}} e_y$ to conclude that

$$A_m \sim \left(\bigcup_{x\in X_m} e'_x\colon (e'_x)_{x\in X_m}\right) \quad \text{where} \quad e'_x = \begin{cases} e_x & (x \neq y_0) \\ y_0\, \overset{\bullet}{e}_{y_0} & (x = y_0). \end{cases}$$

Repeating this for each y_0 with $\ell'(y_0) = m$, we have

$$A_m \sim B = \left(\bigcup_{x\in X_m} g_x\colon (g_x)_{x\in X_m}\right) \quad \text{where} \quad g_x = \begin{cases} e_x & (\ell'(x) < m) \\ x\, \overset{\bullet}{e}_x & (\ell'(x) = m). \end{cases}$$

Now if $\ell'(x) = m$ and $y > x$, there exists z with $\ell'(z) = m-1$ and $y \geq z > x$, hence $e_y \subseteq e_z$. Thus,

$$\overset{\bullet}{e}_x = \bigcup_{y>x} e_y = \bigcup_{\substack{z>x \\ \ell'(z)=m-1}} e_z.$$

Therefore, $\hat{B} = \left(\bigcup_{x\in X} g'_x\colon (g'_x)_{x\in X}\right)$ where

$$\begin{aligned} g'_x &= g_x = e_x & (\ell'(x) < m) \text{ and} \\ g'_x &= g_x = \bigcup_{\substack{z>x \\ \ell'(z)=m-1}} \sum [x,z) e_z & (\ell'(x) = m). \end{aligned}$$

If $\ell'(x) > m$, then

$$g'_x = \bigcup_{\substack{z:z>x\\ \ell'(z)=m}} \sum[x,z)g_z = \bigcup_{\substack{z:z>x\\ \ell'(z)=m}} \sum[x,z) \bigcup_{\substack{y:y>z\\ \ell'(x)=m-1}} ze_y$$
$$= \bigcup_{\substack{y:y>x\\ \ell'(y)=m-1}} \Big[\bigcup_{\substack{z:y>z>x\\ \ell'(z)=m}} \sum[x,z)z\Big] e_y$$
$$= \bigcup_{\substack{y:y>x\\ \ell'(y)=m-1}} \sum[x,y)e_y$$

noting that $\sum[x,y) = \bigcup_{\substack{z:y>z>x\\ \ell'(z)=m}} \sum[x,z)z$ since the complex on the right is clearly a subcomplex of that on the left, and every simplex on the left is contained in a maximal simplex, and this maximal simplex has a vertex z with $y > z > x$, $\ell'(z) = m$ and so lies in $\sum[x,z)z$.

The above computations prove that $\hat{B} = \hat{A}_{m-1}$ (by definition of the latter). Using (5.14.1), this gives

$$A_\infty \sim \hat{A}_m \sim \hat{B} = \hat{A}_{m-1}.$$

Finally, $A_\infty \sim \hat{A}_0 = (L\colon (f_x)_{x\in X})$ where for $x \in X$, $f_x = \sum[x,w)e_w = \sum[x,w)$, and $L = \bigcup_{x\in X} f_x$. □

5.15 We are now able to give our main construction. Here X denotes a finite poset, with a minimum element v and a maximum element w, satisfying the Jordan-Dedekind chain condition; for $x \in X$, we abbreviate $\ell(x,w)$ by $\ell'(x)$.

Assume that the group $\{1,r\}$ has an action on the set X satisfying the following "Z-property" (compare (5.3) (ii)):

(i) For all $x \in X$, $|\ell'(rx) - \ell'(x)| = 1$

(ii) For all $x, y \in X$ with $\ell'(rx) > \ell'(x)$ and $\ell'(ry) > \ell'(y)$ the conditions (a)–(c) below are equivalent:
(a) $x \leq y$ (b) $rx \leq y$ (c) $rx \leq ry$

(note that ℓ' increases from top to bottom on$X$).

The following result gives a construction of a $[v, w]$-celled complex from a $[v, rw]$-celled complex, under the above hypotheses.

5.16 Theorem. Suppose that $(K\colon (e_y)_{y\in Y})$ is a Y-celled complex, where $Y = [v, rw]$. Let $\{a\}$ be a 0-simplex not in L. Let $L = aK$ and define subcomplexes f_x $(x \in X)$ of L as follows. If $x \in [v, rw]$ and $rx > x$, let $f_x = ae_x$ and

$$f_{rx} = \begin{cases} e_x & rx \notin [v, rw] \\ ae_{rx} \cup e_x & rx \in [v, rw] \end{cases}$$

Then $(L\colon (f_x)_{x\in X})$ is an X-celled complex.

Proof Note that if $x \leq y$, then $\ell(x, y) = \ell'(x) - \ell'(y)$.

By applying (5.15) (ii) with $x = y$, it follows that for $x \in X$ and $r \in R$, $\ell'(rx) > \ell'(x)$ iff $rx < x$, and $\ell'(rx) < \ell'(x)$ iff $rx > x$. Note (5.15) (ii) shows now that if $y \in X$, then either $ry > y$ and $y \leq rw$, or $ry < y$ and $ry \leq rw$; this proves that f_y has been defined for all $y \in X$. In fact, let $X_* = \{\, x \in Y \mid rx > x \,\}$. Then

(5.16.1) $X = X_* \dot{\bigcup}\, rX_*$ and $Y = X_* \dot{\bigcup} \{rx \mid x \in X_*,\ rx \in Y\,\}$.

The proof that $(L\colon (f_x))$ is an X-celled complex is given in (5.17), (5.18) and (5.19), corresponding to the three parts of Definition (5.12) to be checked.

5.17 Proof that f_x is a $[\ell(x, w) - 1]$-ball $(x \in X)$.

If $x \in [v, rw]$, $rx > x$ and $rx \notin [v, rw]$, then e_x is a $[\ell(x, rw) - 1]$-ball, and so $f_{rx} = e_x$ is an $[\ell(rx, w) - 1]$-ball and $f_x = ae_x$ is a $[\ell(x, w) - 1]$-ball.

Suppose now that $x \in [v, rw]$, $rx > x$, $rx \in [v, rw)$. Then e_x is a $[\ell(x, rw) - 1]$-ball and e_{rx} is a $[\ell(rx, rw) - 1]$-ball. Hence $f_x = ae_x$ is a $[\ell(x, w) - 1]$-ball. Also, ae_{rx} is a $[\ell(rx, w) - 1]$-ball which we now denote by g_x. Note that $e_x \supseteq \dot{e}_x = \bigcup_{\substack{y\in Y\\ y>x}} e_y \supseteq e_{rx}$. Hence

$$\begin{aligned} e_x \cap g_x &= e_x \cap ae_{rx} = e_x \cap e_{rx} = e_{rx} \qquad \text{and} \\ \dot{e}_x \cap \dot{g}_x &= \dot{e}_x \cap (a\,\dot{e}_{rx} \cup e_{rx}) \\ &= (\dot{e}_x \cap a\,\dot{e}_{rx}) \cup (\dot{e}_x \cap e_{rx}) \\ &= (\dot{e}_x \cap \dot{e}_{rx}) \cup e_{rx} \\ &= e_{rx}. \end{aligned}$$

Since e_x and g_x are both $[\ell(rx,w)-1]$-balls and e_{rx} is a $[\ell(rx,w)-2]$-ball, (5.9) (iii) shows that $f_{rx} = e_x \cup g_x$ is a $[\ell(rx,w)-1]$-ball. For future reference, note

(5.17.1) $\overset{\bullet}{f}_{rx} = [(ae_{rx})^{\bullet} \cup \overset{\bullet}{e}_x] \setminus \overset{\circ}{e}_{rx}$

5.18 Proof that $\overset{\bullet}{f}_x = \bigcup_{z>x} f_z$ $(x \in X)$.

Recall that $X_* = \{x \in Y \mid rx > x\}$. Suppose that $x \in X_*$ and $rx \notin Y$. If $z \in X^*$ and $rz \in Y$, we cannot have $z \geq x$ or $rz \geq x$ for these would imply $rw \geq rx$ i.e. $rx \in Y$ contrary to assumption. Hence $z \geq rx$ and $rz \geq rx$ are also impossible (since $rx > x$).

If $x, z \in X_*$, $rz \notin Y$ and $z \neq x$, then $z > x \iff rz > x \iff rz > rx$. Therefore, for $x \in X_*$ with $rx \notin Y$, (5.16.1) and the above comments give

$$\begin{aligned}
\overset{\bullet}{f}_x = (ae_x)^{\bullet} = a\overset{\bullet}{e}_x \cup e_x &= \Big(a \bigcup_{\substack{y\in Y\\ y>x}} e_y\Big) \cup e_x \\
&= \bigcup_{\substack{z\in X_*\\ rz\notin Y\\ z>x}} (ae_z \cup e_z) \cup e_x \\
&= \bigcup_{\substack{z\in X_*\\ rz\notin Y\\ z>x}} (f_z \cup f_{rz}) \cup f_{rx} = \bigcup_{\substack{y\in X\\ y>x}} f_y
\end{aligned}$$

and

$$\overset{\bullet}{f}_{rx} = \overset{\bullet}{e}_x = \bigcup_{\substack{y\in Y\\ y>x}} e_y = \bigcup_{\substack{z\in X_*\\ rz\notin Y\\ z>x}} e_z = \bigcup_{\substack{z\in X_*\\ rz\notin Y\\ z>x}} f_{rz} = \bigcup_{\substack{y\in X\\ y>rx}} f_y.$$

Now consider the case when $x \in X_*$ and $rx \in Y$. Note that for any $z \in X_* \setminus \{x\}$, $rz > x$ if and only if $z > x$. Hence

$$\begin{aligned}
\bigcup_{\substack{z\in X\\ z>x}} f_z &= \bigcup_{\substack{z\in X_*\\ z>x}} (f_z \cup f_{rz}) \cup f_{rx} \\
&= \bigcup_{\substack{z\in X_*\\ rz\notin Y\\ z>x}} (ae_z \cup e_z) \cup \bigcup_{\substack{z\in X_*\\ rz\in Y\\ z>x}} (ae_z \cup ae_{rz} \cup e_z) \cup (ae_{rx} \cup e_x)
\end{aligned}$$

$$= \bigcup_{\substack{z\in X_* \\ rz\notin Y \\ z>x}} ae_z \cup \bigcup_{\substack{z\in X_* \\ rz\in Y \\ z>x}} (ae_z \cup ae_{rz} \cup (ae_{rx} \cup e_x)$$

$$= \bigcup_{\substack{y\in Y \\ y>x}} ae_y \cup e_x = a\overset{\bullet}{e}_x \cup e_x = (ae_x)^\bullet = f_x^\bullet.$$

Also, since if $rz > z$ then $rz > rx$ if and only if $z > x$, it follows that

$$\begin{aligned}
\bigcup_{\substack{z\in X \\ z>rx}} f_z &= \bigcup_{\substack{z\in X_* \\ z>x}} f_{rz} \cup \bigcup_{\substack{z\in X_* \\ z>rx}} f_z \\
&= \bigcup_{\substack{z\in X_* \\ z>x \\ rz\notin Y}} e_z \cup \bigcup_{\substack{z\in X_* \\ z>x \\ rz\in Y}} (ae_{rz} \cup e_z) \cup \bigcup_{\substack{z\in X_* \\ z>rx \\ rz\notin Y}} ae_z \cup \bigcup_{\substack{z\in X_* \\ z>rx \\ rz\in Y}} ae_z \\
&= \bigcup_{\substack{y\in Y \\ y>rx}} ae_y \cup \bigcup_{\substack{y\in Y \\ y>x \\ y\neq rx}} e_y \\
&= \Big[\Big(\Big(\bigcup_{\substack{y\in Y \\ y>rx}} ae_y\Big) \cup e_{rx}\Big) \cup \bigcup_{\substack{y\in Y \\ y>x}} e_y\Big] \setminus \Big(e_{rx} \setminus \bigcup_{\substack{y\in Y \\ y>rx}} e_y\Big) \\
&= [(a\overset{\bullet}{e}_{rx} \cup e_{rx}) \cup \overset{\bullet}{e}_x] \setminus \overset{\circ}{e}_{rx} \\
&= [(ae_{rx})^\bullet \cup \overset{\bullet}{e}_x] \setminus \overset{\circ}{e}_{rx} \\
&= \overset{\bullet}{f}_{rx} \text{ by (5.17.1)}
\end{aligned}$$

5.19 Proof that $aK = \dot{\bigcup}_{x\in X} \overset{\circ}{f}_x$.

Let $x \in X_*$. Then $\overset{\circ}{f}_x = \{\{a\} \cup A \mid A \in \overset{\circ}{e}_x\}$, and

$$\overset{\circ}{f}_{rx} = \begin{cases} \overset{\circ}{e}_x & (rx \notin Y) \\ \{\{a\} \cup A \mid A \in \overset{\circ}{e}_{rx}\} \cup \overset{\circ}{e}_{rx} \cup \overset{\circ}{e}_x & (rx \in Y) \end{cases}$$

(for if $rx \in Y$, then

$$\begin{aligned}
\overset{\circ}{f}_{rx} &= [ae_{rx} \cup e_x] \setminus [((ae_{rx})^\bullet \cup \overset{\bullet}{e}_x) \setminus e^\circ_{rx}] \quad \text{by (5.17.1)} \\
&= [(ae_{rx} \cup e_x) \setminus ((ae_{rx})^\bullet \cup \overset{\bullet}{e}_x)] \cup \overset{\circ}{e}_{rx} \quad \text{since } \overset{\circ}{e}_{rx} \subseteq ae_{rx} \\
&= [(ae_{rx})^\circ \setminus \ \overset{\bullet}{e}_x] \cup [\overset{\circ}{e}_x \setminus (ae_{rx})^\bullet] \cup \overset{\circ}{e}_{rx} \\
&= (ae_{rx})^\circ \cup \overset{\circ}{e}_x \cup \overset{\circ}{e}_{rx}
\end{aligned}$$

noting $(ae_{rx}) \cap e_x \subseteq (ae_{rx})^\bullet \cap \overset{\bullet}{e}_x$).

Since the $\overset{\circ}{e}_y \, (y \in Y)$ are pairwise disjoint, so are the$\overset{\circ}{f}_x \, (x \in X)$. Also,

$$\bigcup_{x \in X} \overset{\circ}{f}_x = \bigcup_{x \in X_*} (\overset{\circ}{f}_x \cup \overset{\circ}{f}_{rx}) = \bigcup_{y \in Y} (\overset{\circ}{e}_y \cup \{\, \{\, a \,\} \cup A \mid A \in \overset{\circ}{e}_y \,\}) = aK.$$

This completes the proof of Theorem (5.16). $\square$

We can now prove

5.20 Theorem. Let (W, R) be a Coxeter system, and $x, y \in W$ with $x \leq y$ (in the Bruhat order) and $\ell(y) \geq \ell(x)+2$. Then $\sum(x, y)$ is a $(\ell(y)-\ell(x)-2)$-sphere.

Proof First we prove by induction on $\ell(y)$ that there exists a $[1, y]$-celled complex $(K\colon (e_x)_{x \in [1,y]})$. If $y = 1$, one sets $K = \{\emptyset\} = e_1$ and the result holds. If $\ell(y) \geq 1$, choose $r \in R$ with $\ell(ry) < \ell(y)$. The interval $[1, y]$ satisfies the Jordan-Dedekind chain condition (e.g. from [De1]) and the action of $\{1, r\}$ on $[1, y]$ satisfies the Z-property in the sense of (5.15) (i) and (ii). By induction, there exists a $[1, ry]$-celled complex, and so (5.16) produces a $[1, y]$-celled complex $(L\colon (f_x)_{x \in [1,y]})$; in fact, there exists such a complex in which L is a $(\ell(y) - 1)$-simplex. Using (5.13) and (5.14), we see that $(K\colon (e_x)_{x \in [1,y]})$ is a $[1, y]$-celled complex, where $e_x = \sum[x, y)$, $(1 \leq x \leq y)$ and $K = \bigcup\limits_{x \in [1,y]} e_x$.

In particular, if $x \leq y$ and $\ell(y) \geq \ell(x) + 2$, then e_x is a $(\ell(y) - \ell(x) - 1)$-ball, so $\overset{\bullet}{e}_x$ is a $(\ell(y) - \ell(x) - 2)$-sphere. But $\overset{\bullet}{e}_x = \bigcup\limits_{z : x < z \leq y} \sum[z, y) = \sum(x, y)$, hence the result $\square$

5.21 Remark. Theorem (5.20) is a special case of a result of Björner and Wachs ([BW]). They prove that for any subset J of R, the simplicial complex of an open interval in the set W^J, with order induced by Bruhat order, is either a combinatorial ball or a combinatorial sphere. We wil need the following simpler result ([De1], [BW]):

(5.21.1) If $w, w' \in W^J$ and $w < w'$ then all maximal chains $w' = w_0 > w_1 > \ldots > w_n = w$ have the same length $n = \ell(w') - \ell(w)$. □

We conclude this chapter with a result which associates to any Bruhat interval in (W, R) an isomorphic Bruhat interval of some reflection subsystem (W', R') of (W, R).

5.22 Proposition. Let (W, R) be a Coxeter system, and $x, y \in W$ satisfy $x \leq y$. Let $x = x_0 < x_1 < \ldots < x_n = y$ be a maximal chain from x to y (thus, $n = \ell(y) - \ell(x)$). Let $W' = \langle x_i^{-1} x_{i-1} \mid 1 \leq i \leq n \rangle$. Then W' is a reflection subgroup of W. Let R' be the set of canonical generators of $W', \leq'$ denote the Bruhat order on (W', R'), and z denote the element of xW' with minimal length $\ell(z)$. Set $x' = z^{-1}x$, $y' = z^{-1}y$ and let I (resp. I') denote the interval $[x, y]$ in the Bruhat order of (WR) (resp. $[x', y']$ in the Bruhat order of (W', R')).

Then there is an isomorphism of posets $\theta \colon I' \longrightarrow I$ such that $\theta(w) = zw$ $(w \in I')$

<u>Proof</u> Since $x_{i-1} < x_i$ and $\ell(x_i) = \ell(x_{i-1}) + 1$, Definition (5.1) implies that $x_i^{-1} x_{i-1} \in T$ $(i = 1, \ldots, n)$. Hence $W' = \langle W' \cap T \rangle$, so W' is a reflection subgroup as claimed. Let ℓ' be the length function on (W', R'). Since $x_i^{-1} x_{i-1} \in W' \cap T$ and $\ell(x_{i-1}) < \ell(x_i)$, (1.13) (i) and (ii) imply that $\ell'(z^{-1} x_{i-1}) < \ell'(z^{-1} x_i)$; since $(z^{-1} x_i)^{-1} (z^{-1} x_{i-1}) \in W' \cap T$. Definition (5.1) now gives that $x' = z^{-1} x_0 <' z^{-1} x_1 <' \ldots <' z^{-1} x_n = y'$. In particular, $\ell'(y') - \ell'(x') \geq \ell(y) - \ell(x)$.

Now let $w \in [x', y']$. Let $x' = w_0 <' w_1 <' \ldots <' w_m = y'$ be a maximal chain from x' to y', containing w. By (5.1), $w_i^{-1} w_{i-1} \in W' \cap T$, and so by (1.13) again, it follows that $\ell(zw_0) < \ell(zw_1) < \ldots < \ell(zw_n)$. Since $(zw_i)^{-1}(zw_{i-1}) \in T$, (5.1) gives $x = zw_0 < zw_1 < \ldots < zw_m = y$. In particular, $m = \ell'(y') - \ell'(x') \leq \ell(y) - \ell(x)$.

The above shows that there is a strictly monotone map $\theta \colon I' \longrightarrow I$ such that $\theta(x) = zw$ $(w \in I')$, and that $\ell'(y') - \ell'(x') = \ell(y) - \ell(x) = n$. Now if $n \leq 1$, then θ is obviously an isomorphism of posets, so assume that $n \geq 2$. Let

$L' = \sum(x', y')$ (respectively $L = \sum(x, y)$) be the simplicial complex of the interval (x', y') in the order $\leq'$ (resp. of the interval (x, y) in the order $\leq$). By (5.20), both L' and L are $(n-2)$-spheres.

Now if $A = \{a_0, \ldots, a_k\}$ is a k-simplex of L', then $\theta(A) = \{\theta(a_0), \ldots, \theta(a_k)\}$ is a k-simplex of L. Hence $K = \{\, \theta(A) \mid A \in L' \,\}$ is a subcomplex of L, isomorphic to L' as a simplicial complex. In particular, $K \neq \emptyset$, K is $(n-2)$-homogeneous and $\overset{\bullet}{K} = \emptyset$. By (5.9) (ii), it follows that $K = L$. Therefore θ induces an isomorphism of the open intervals (x', y') and (x, y). But θ maps the minimal element x' of I' to the minimum element x of I, and similarly for y' and y, so θ is an isomorphism of posets. □

As an immediate consequence of (5.22), we have

5.23 Corollary. Let notation be as in (5.22). Then for any $z, z' \in I$ we have $z^{-1}z' \in W'$ □

5.24 Remark. In (5.22) and (5.23), it is possible that $W' = W$. This cannot happen if $\ell(y) - \ell(x) < \#(R)$. Later, we give conditions under which (W', R') is a dihedral reflection subsystem.

Chapter 6

THE POLYNOMIALS $R_{X,Y}$

This chapter begins with a construction that produces pairs of inverse elements of the incidence algebra (over the ring $\mathbb{Z}[u, u^{-1}]$, u an indeterminate) of a locally finite poset. Under an additional assumption, the inverse of one of these elements is obtained simply by applying the ring involution of the incidence algebra extending the involution $u \longmapsto u^{-1}$ of $\mathbb{Z}[u, u^{-1}]$, and one may define formal analogues of the Kazhdan-Lusztig polynomials in this context.

We then turn to the consideration of the polynomials $R_{x,y}$ defined for elements x, y of a Coxeter system by Kazhdan and Lusztig ([KL1]), and show how these polynomials $R_{x,y}$ arise from our incidence algebra construction; the data required for the construction is obtained from certain total orderings of the reflections.

We begin by recalling the definition of the incidence algebra of a poset ([Ai], Chapter IV).

6.1 Let P be a locally finite poset (i.e. all intervals in P are finite) and $\mathcal{A}$ be a commutative ring. The incidence algebra $\mathbf{A}_{\mathcal{A}}(P)$ is the set

$$\{\, f\colon P^2 \longrightarrow \mathcal{A} \mid f(x, y) = 0 \text{ if } x \not\leq y \,\}$$

regarded as an $\mathcal{A}$-module in the usual way, and equipped with the convolution product $*$ defined by

$$(f * g)(x, y) = \sum_{z : x \leq z \leq y} f(x, z)g(z, y) \ (f, g \in \mathbf{A}_{\mathcal{A}}(P)).$$

This is an associative $\mathcal{A}$-algebra with the Kronecker delta δ, defined by

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise,} \end{cases}$$

as identity. An element f of $\mathbf{A}_{\mathcal{A}}(P)$ is invertible iff $f(x,x)$ is a unit of $\mathcal{A}$, for all $x \in P$.

For our purposes, we assume given a fixed element α of $\mathcal{A}$, and a ring involution of $\mathcal{A}$, written $a \longmapsto \bar{a}$, such that $\bar{\alpha} = -\alpha$. Then there is an induced ring involution of $\mathbf{A}_{\mathcal{A}}(P)$, which we also denote by $f \longmapsto \bar{f}$, such that

$$\bar{f}(x,y) = \overline{f(x,y)} \ (f \in \mathbf{A}_{\mathcal{A}}(P),\ x,y \in P).$$

6.2 We now fix an arbitrary subset C_1 of $\{\,(x,y) \in P^2 \mid x < y\,\}$, and for $n \in \mathbb{N}$, define $C_n(x,y) = \{\,(x_0,\ldots,x_n) \in P^{n+1} \mid (x_{i-1},x_i) \in C_1$ for $i = 1,\ldots,n,\ x = x_0,\ y = x_n\,\}$. Also, set $C(x,y) = \bigcup_{n\in\mathbb{N}} C_n(x,y)$ (a finite set by the local finiteness of P) and $C_n = \bigcup_{(x,y)\in P^2} C_n(x,y)$ $(n \in \mathbb{N})$.

One could regard C_1 as the set of edges of a directed graph with vertex set P, and then $C_n(x,y)$ is the set of paths of length n from x to y. For $\tau \in C_n$, write $\ell(\tau) = n$.

For any subset I of C_2 and $\tau = (x_0,\ldots,x_n) \in C_n$, we now define

(6.2.1) $$d_I(\tau) = \{\, x_i \mid 1 \le i \le n-1,\ (x_{i-1},x_i,x_{i+1}) \in I\},$$

and set $a_I(\tau) = d_{C_2\setminus I}(\tau)$.

If one calls the chains of I decreasing chains and those of $C_2 \setminus I$ increasing, then $d_I(\tau)$ could be called the descent set of τ and $a_I(\tau)$ its ascent set. This terminology is motivated by the situation in which the elements of C_1 are labelled by elements of a poset, and I is the set of paths (x,y,z) in C_2 such that (x,y) has a greater label than (y,z) (e.g. see [BW]).

We now define an element r^I of the incidence algebra; on an interval $[x,y]$ of P, r^I is essentially the generating function for the numbers of increasing paths of lengths 0,1,2,... .

6.3 Definition. For any subset I of C_2, define $r^I \in \mathbf{A}_{\mathcal{A}}(P)$ by

(6.3.1) $$r^I(x,y) = \sum_{\substack{\tau\in C(x,y)\\ d_I(\tau)=\emptyset}} \alpha^{\ell(\tau)} \ (x,y \in P)$$

More generally, for $p \in \mathbb{N}$ let $r_p^I \in \mathbf{A}_{\mathcal{A}}(P)$ be defined by

$$(6.3.2) \qquad r_p^I(x,y) = \sum_{\substack{\tau \in C(x,y) \\ \#d_I(\tau)=p}} \alpha^{\ell(\tau)} \ (x,y \in P).$$

□

The following proposition describes the inverse of r^I.

6.4 Proposition. Let I be a fixed subset of C_2 and $r = r^I,\ s = r^{C_2 \setminus I}$. Then

$$r * \bar{s} = \bar{s} * r = \delta.$$

Proof It will suffice to show that $r * \bar{s} = \delta$ That is, we must show that for $x, y \in P$,

(6.4.1) $\sum_{z \in P} r(x,z)\bar{s}(z,y) = \delta(x,y).$

This is clear if $x = y$ or $x \not\leq y$, so suppose $x < y$. For any $z \in [x,y]$, we have an operation of concatenation of chains; if $\tau_1 = (x_0, \ldots, x_n) \in C_n(x,z)$ and $\tau_2 = (x_n, \ldots, x_{n+m}) \in C_m(z,y)$ we have $\tau_1 \cdot \tau_2 = (x_0, \ldots, x_n, \ldots, x_{n+m}) \in C_{n+m}(x,y)$. Note that $\ell(\tau_1 \cdot \tau_2) = \ell(\tau_1) + \ell(\tau_2)$.

For any $\tau = (x_0, \ldots, x_n) \in C(x,y)$, we abuse notation and write $z \in \tau$ if $z = x_i$ for some $i \in \{0, \ldots, n\}$ (in this case, i is uniquely determined). If $z = x_i$, we then define $\tau_1(z) \in C(x,z)$ and $\tau_2(z) \in C(z,y)$ by $\tau_1(z) = (x_0, \ldots, x_i)$ and $\tau_2(z) = (x_i, \ldots, x_n)$.

Supressing I from the notation, let

$$S_1 = \{(\sigma, \rho, z) \mid z \in P,\ \sigma \in C(x,z),\ \rho \in C(z,y),\ d(\sigma) = \emptyset,\ a(\sigma) = \emptyset\}$$

and

$$S_2 = \{\,(\tau, z) \mid \tau \in C(x,y),\ z \in \tau,\ d(\tau_1(z)) = \emptyset,\ a(\tau_2(z)) = \emptyset\,\}.$$

There are maps $S_1 \longrightarrow S_2$ and $S_2 \longrightarrow S_1$ given by $(\sigma, \rho, z) \longmapsto (\sigma \cdot \rho, z)$ and $(\tau, z) \longmapsto (\tau_1(z), \tau_2(z), z)$ respectively, and these are inverse bijections.

Therefore

$$\begin{aligned}
\sum_{z\in P} r(x,z)\bar{s}(z,y) &= \sum_{z\in P}\Big[\sum_{\substack{\sigma\in C(x,z)\\ d(\sigma)=\emptyset}} \alpha^{\ell(\sigma)}\Big]\Big[\sum_{\substack{\rho\in C(z,y)\\ a(\rho)=\emptyset}} \bar{\alpha}^{\ell(\rho)}\Big] \\
&= \sum_{(\sigma,\rho,z)\in S_1} (-1)^{\ell(\rho)}\alpha^{\ell(\sigma)+\ell(\rho)} \\
&= \sum_{(\tau,z)\in S_2} (-1)^{\ell(\tau_2(z))}\alpha^{\ell(\tau)} \\
&= \sum_{\tau\in C(x,y)}\Big[\sum_{\substack{z\in\tau\\ d(\tau_1(z))=\emptyset\\ a(\tau_2(z))=\emptyset}} (-1)^{\ell(\tau_2(z))}\Big]\alpha^{\ell(\tau)}
\end{aligned}$$

and it needs only be checked that if $x \neq y$ and $\tau \in C(x,y)$, then

$$\sum_{z\in F(\tau)} (-1)^{\ell(\tau_2(z))} = 0 \tag{6.4.1}$$

where $F(\tau) = \{\, z \in \tau \mid d(\tau_1(z)) = \emptyset,\ a(\tau_2(z)) = \emptyset \,\}$.

Write $\tau = (x_0, \ldots, x_n)$ (note $n \geq 1$). Then

$$\begin{aligned}
F(\tau) = \{\, x_i \mid (0 \leq i \leq n)(x_{j-1}, x_j, x_{j+1}) \notin I\ (1 \leq j \leq i-1), \\
x_{j-1}, x_j, x_{j+1}) \in I\ \ (i+1 \leq j \leq n-1)\ \ \}
\end{aligned} \; .$$

One sees that either $F(\tau) = \emptyset$, or $F(\tau) = \{x_{m-1},\ x_m\}$ for some m $(1 \leq m \leq n)$. In either case, (6.4.1) is satisfied. □

Here are two special cases of this result.

6.5 Examples.

(i) Let $C_1 = \{\, (x,y) \in P^2 \mid x < y \,\}$ and $I = \emptyset$. Then

$$r^{\emptyset}(x,y) = \begin{cases} 1 & x = y \\ \alpha & x < y \\ 0 & \text{otherwise} \end{cases}$$

and $r^{C_2}(x,y) = \sum_{\tau \in C(x,y)} \alpha^{\ell(\tau)}$. Taking our ring $\mathcal{A}$ to be the polynomial ring $\mathbb{Q}[\alpha]$ with the involution $\alpha \longmapsto -\alpha$, then specialising α to 1 in the identity

$$(r^{\emptyset})^{-1} = \overline{(r^{C_2})}$$

gives the well-known formula $\mu_P(x,y) = \zeta^{-1}(x,y) = \sum_{\tau \in C(x,y)} (-1)^{\ell(\tau)}$, where μ_P is the Möbius function on P and ζ denotes the zeta function of P (see [Ai]).

(ii) Take $\mathcal{A} = \mathbb{Q}[\alpha]$ as in (i), and suppose that P satisfies the Jordan-Dedekind chain conditon (all maximal chains from x to y have the same length, denoted $\ell(x,y)$). Let

$$\begin{aligned} C_1 &= \{ (x,y) \in P^2 \mid x < y \quad \text{and there is no } z \text{ with } x < z < y \} \\ &= \{ (x,y) \in P^2 \mid x < y \text{ and } \ell(x,y) = 1 \}. \end{aligned}$$

Suppose that $I \subseteq C_2$ is such that if $x, y \in P$ and $x \leq y$ then $r^I(x,y) = \alpha^{\ell(x,y)}$ (for example, P could be a lexicographically shellable poset in the sense of [BW], and I the set of chains of length 2 with label decreasing from top to bottom).

Specialising α to 1 in the identity

$$(r^I)^{-1} = \overline{(r^{C_2 \setminus I})}$$

gives $\mu_P(x,y) = (-1)^{\ell(x,y)} \#\{ \tau \in C(x,y) \mid a_I(\tau) = \emptyset \}$. More generally, one has a similar result to ([BW], Theorem 3.4) concerning the values of the Möbius function on a rank-selected subposet of P.

6.6 We assume as in (6.5) that $\mathcal{A} = \mathbb{Q}[\alpha]$ is a polynomial ring, and $\bar{\alpha} = -\alpha$. For any $m \in \mathbb{N}$ and $f \in \mathbf{A}_{\mathcal{A}}(P)$, define $\frac{d^m}{d\alpha^m} f \in \mathbf{A}_{\mathcal{A}}(P)$ in the obvious way:

$$\left[\frac{d^m}{d\alpha^m} f \right] (x,y) = \frac{d^m}{d\alpha^m} [f(x,y)] \; (x,y) \in P).$$

Our next result expresses the r_p^I ($p \in \mathbb{N}$) in terms of r^I (implicitly).

6.7 Proposition. For any subset I of C_2 and any $n \in \mathbb{N}$, $n \geq 1$,

$$(r^I)^n = \sum_{m=0}^{n-1} \frac{1}{m!} \frac{d^m}{d\alpha^m} (\alpha^m r^I_{n-1-m}) \tag{6.7.1}$$

<u>Proof</u> Evaluate both sides of (6.7.1) at $(x,y) \in P^2$. The right hand side of (6.7.1) gives

$$\sum_{m=0}^{n-1} \sum_{\substack{\tau \in C(x,y) \\ \#d(\tau)=n-1-m}} \binom{m+\ell(\tau)}{m} \alpha^{\ell(\tau)}$$

and the left hand side gives

$$\sum_{\substack{(z_0,\ldots,z_n)\in P^{n+1} \\ z_0=x, z_n=y}} \sum_{\substack{\tau_i \in C(z_{i-1},z_i) \\ d(\tau_i)=\emptyset \\ (i=1,\ldots,n)}} \prod_{i=1}^{n} \alpha^{\ell(\tau_i)} = \sum_{\tau \in C(x,y)} \sum_{\substack{\tau_i \in C(i=1,\ldots,n) \\ d(\tau_i)=\emptyset \\ \tau=\tau_1\cdot\ldots\cdot\tau_n}} \alpha^{\ell(\tau)}$$

where $C = \bigcup_{(x,y)\in P^2} C(x,y)$ and $\tau_1 \cdot \ldots \cdot \tau_n$ denotes the chain obtained by concatenating $\tau_1, \ldots, \tau_n$. Now if $\tau = \tau_1 \cdot \ldots \cdot \tau_n$ where $d(\tau_i) = \emptyset$, $(i = 1, \ldots, n)$, then $\#d(\tau) \leq n-1$.

Hence to prove (6.7.1), it suffices to show that if $\tau \in C(x,y)$ and $\#d(\tau) = n-1-m$, where $0 \leq m \leq n-1$, then
(6.7.2) $\#\{(\tau_1, \ldots, \tau_n) \in C^n \mid d(\tau_i) = \emptyset \ (i = 1, \ldots, n),\ \tau = \tau_1 \cdot \ldots \cdot \tau_n\} = \binom{m+\ell(\tau)}{m}$.

Suppose $\tau \in C_p(x,y)$, say $\tau = (x_0, \ldots, x_p)$, and that $d(\tau) = \{x_{i_1}, \ldots, x_{i_{n-m-1}}\}$ where $1 \leq i_1 < \ldots < i_{n-m-1} \leq p$ (thus $\ell(\tau) = p$).

Let $\{(\tau_1, \ldots, \tau_n) \in C^n \mid d(\tau_i) = \emptyset \ (i = 1, \ldots, n), \tau = \tau_1 \cdot \ldots \cdot \tau_n\} = K$. If $(\tau_1, \ldots, \tau_n) \in K$, then there exist integers $k_1, \ldots, k_{n-1}$ such that $0 \leq k_1 \leq \ldots \leq k_{n-1} \leq p$, and, with $k_0 = 0$ and $k_n = p$, $\tau_j \in C(x_{k_{j-1}}, x_{k_j})$ $(j = 1, \ldots, n)$. Moreover, since $\{x_{i_1}, \ldots, x_{i_{n-m-1}}\} = d(\tau_1 \cdot \ldots \cdot \tau_n)$, we must have $\{i_1, \ldots, i_{n-m-1}\} \subseteq \{k_1, \ldots, k_{n-1}\}$.

On the other hand, suppose given integers $k_1, \ldots, k_{n-1}$ with $0 \leq k_1 \leq \ldots \leq k_{n-1} \leq p$ and $\{i_1, \ldots, i_{n-m-1}\} \subseteq \{k_1, \ldots, k_{n-1}\}$. Set $k_0 = 0$, $k_n = p$, and

$\tau_j = (x_{k_{j-1}}, x_{k_{j-1}+1}, \ldots, x_{k_j})$ $(i = 1, \ldots, n)$. Then $\tau = \tau_1 \cdot \ldots \cdot \tau_n$ and $d(\tau_j) = \emptyset$ $(j = 1, \ldots, n)$.

The preceding two paragraphs prove that there is a bijection of K with the set of non-desceasing functions $f\colon \{1, \ldots, n-1\} \longrightarrow \{0, 1, \ldots, p\}$ such that the $(n-m-1)$-set $\{i_1, \ldots, i_{n-m-1}\}$ is contained in the image of f. Call this set of functions L.

Suppose given $f \in L$. For $k = 1, \ldots, n-m-1$, let $j_k = \min\{\, j \mid 1 \leq j \leq n-1,\ f(j) = i_k \,\}$. Let $h_f\colon \{1, \ldots, m\} \longrightarrow \{1, \ldots, n-1\}$ be the unique strictly increasing function such that j_k is not in the image of h_f, for all $k = 1, \ldots, n-m-1$. Then the map $f \longmapsto f \circ h_f$ is a bijection between L and the set M of non-decreasing functions $g\colon \{1, \ldots, m\} \longrightarrow \{0, 1, \ldots, p\}$. The map $g \longmapsto \{g(1),\ g(2)+1, \ldots, g(m)+m-1\}$ is a bijection between M and the set of m-subsets of $\{0, 1, \ldots, m+p-1\}$, of which there are $\binom{m+p}{m}$. Hence

$$\#(K) = \#(L) = \#(M) = \binom{m+p}{m} = \binom{m+\ell(\tau)}{m},$$

completing the proof of (6.7.2) and hence of the proposition. □

Now the coefficient of r^I_{n-1} on the right of (6.7.1) is 1, so one may solve in turn for $r^I_0, r^I_1, r^I_2, \ldots$, in terms of powers of r^I. This implies

6.8 Corollary. If I, J are subsets of C_2 and $r^I = r^J$, then $r^I_p = r^J_p$ $(p \in \mathbb{N})$. □

We will not make any essential use of (6.7) or (6.8), but give one application here.

6.9 Example. This is a continuation of Example (6.5) (ii), and we maintain the notation in force there. We need an identity involving binomial coefficients. For non-negative integers m, n $(m \geq 1)$, we have $\frac{d^{m-1}}{dx^{m-1}}[x^n(1+x)^m] = \sum\limits_{k=0}^{m} \binom{m}{k} \frac{(n+k)!}{(n+k+1-m)!} x^{k+n+1-m}$. Putting $x = -1$ and rearranging, one obtains

(6.9.1) $$\sum_{k=0}^{m} (-1)^k \binom{n+k}{k} \binom{n+1}{m-k} = \begin{cases} 0 & m > 0 \\ 1 & m = 0. \end{cases}$$

Fix $x, y \in P$ with $\ell(x,y) = n$, and for $i \in \mathbb{N}$, let γ_i and ζ_{i+1} denote the values of $r_i^I(x,y)$ and $(r^I)^{i+1}(x,y)$ at $\alpha = 1$ (thus, $\zeta_{i+1} = \zeta^{i+1}(x,y)$ is the value of a power of the zeta function).

The identity (6.7.1) gives

$$\zeta_{j+1} = \sum_{k=0}^{j} \binom{n+k}{k} \gamma_{j-k} \qquad (j \in \mathbb{N})$$

and it is clear that one may solve uniquely for γ_j in terms of $\zeta_{j+1}, \ldots, \zeta_1$. Indeed,

$$\begin{aligned} &\sum_{k=0}^{j} \binom{n+k}{k} \sum_{l=0}^{j-k} (-1)^l \binom{n+1}{l} \zeta_{j-k+1-l} \\ &= \sum_{m=0}^{j} \sum_{k=0}^{m} (-1)^{m-k} \binom{n+k}{k} \binom{n+1}{m-k} \zeta_{j+1-m} \\ &= \zeta_{j+1} \qquad \text{by (6.9.1) and so} \end{aligned}$$

$$\gamma_j = \sum_{k=0}^{j} (-1)^k \binom{n+1}{k} \zeta_{j+1-k}.$$

In particular,

$$\sum_{k=0}^{j} (-1)k \binom{\ell(x,y)+1}{k} \zeta^{j+1-k}(x,y) \geq 0 \ (j \in \mathbb{N})$$

(the left hand side being zero for $j \geq \ell(x,y)$). □

6.10 Henceforward we assume that $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ is the ring of Laurent polynomials in an indeterminate u, that the involution $a \longmapsto \bar{a}$ of $\mathcal{A}$ is the one determined by $\bar{u} = u^{-1}$, and that $\alpha = u^{-1} - u$. We also define another involution $a \longmapsto \hat{a}$ of $\mathcal{A}$ such that $\hat{u} = -u$.

We continue to let $\mathbf{A}_{\mathcal{A}}(P)$ denote the incidence algebra of a fixed locally finite poset P. The involutions $a \longmapsto \bar{a}$ and $a \longmapsto \hat{a}$ have extensions to $\mathbf{A}_{\mathcal{A}}(P)$ which we denote by $f \longmapsto \bar{f}$ and $f \longmapsto \hat{f}$ respectively (where $\bar{f}(x,y) = \overline{f(x,y)}$, $\hat{f}(x,y) = \widehat{f(x,y)}$ $(f \in \mathbf{A}_{\mathcal{A}}(P), x, y \in P)$).

Assume that there exists an element $r \in \mathbf{A}_{\mathcal{A}}(P)$ satisfying
(6.10.1) $r * \bar{r} = \bar{r} * r = \delta$
(6.10.2) $r(x,x) = 1$ $(x \in P)$

After a preliminary remark, the following sections describe some formal consequences of these two conditions.

6.11 Remark. Here are three situations in which such an element r exists.

(i) If (W,R) is any Coxeter system, take $P = W$ equipped with the Bruhat order. For $x, y \in P$, let $r(x,y) = u^{\ell(y)-\ell(x)}\overline{R_{x,y}(u^2)}$ where $R_{x,y}$ is the polynomial defined in [KL1]. Here, one also has
(6.11.1) $r(x,y)$ is a polynomial in $(u^{-1} - u) = \alpha$.

(ii) More generally, let $P = W^J$ be the set of shortest coset representatives of a parabolic subgroup W_J of W. In [De 4], Deodhar defines polynomials $R^J_{\tau,\sigma}$ and (6.10.1), (6.10.2) are satisfied with $r(x,y) = u^{\ell(y)-\ell(x)}\overline{R^J_{x,y}(u^2)}$ $(x, y \in W^J)$. However, $r(x,y)$ cannot in general be normalised here so as to satisfy (6.11.1).

(iii) Suppose given C_1 as in (6.2), and a subset I of C_2 such that, in the notation of (6.3), $r^I = r^{C_2 \setminus I}$. Set $r = r^I$. Then (6.10.2) and (6.11.1) hold, and so does (6.10.1) (by (6.4)). It will be seen later that the element r defined for a Coxeter system in (i) is produced by this construction.

Corresponding to the element $r \in \mathbf{A}_{\mathcal{A}}(P)$, we now define elements $p, q = \mathbf{A}_{\mathcal{A}}(P)$; in case r is as in (6.11) (i), the $p(x,y)$ $(x, y \in P)$ are, up to normalisation, the Kazhdan-Lusztig polynomials of the Coxeter system (W,R), and the $q(x,y)$ are the inverse Kazhdan-Lusztig polynomials (to within normalisation).

6.12 Proposition.

(i) There exists a unique element $p \in \mathbf{A}_{\mathcal{A}}(P)$ satisfying

(a) $p(x,x) = 1$ $(x \in P)$

(b) $p(x,y) \in u^{-1}\mathbb{Z}[u^{-1}]$ $(x, y \in P, x \neq y)$

(c) $p = r * \bar{p}$

(ii) There exists a unique element $q \in \mathbf{A}_{\mathcal{A}}(P)$ satisfying

(a) $q(x,x) = 1$ $(x \in P)$

(b) $q(x,y) \in u^{-1}\mathbb{Z}[u^{-1}]$ $(x, y \in P,\ x \neq y)$

(c) $q = \bar{q} * r$

(iii) If for all $x, y \in P$, $r(x,y)$ is a polynomial in $\alpha = u^{-1} - u$, then $p * \hat{q} = \hat{q} * p = \delta$

<u>Proof</u> (i) The proof here is by a standard argument ([L 1]). Set $p(x,x) = 1$ for all $x \in P$. Given $x, y \in P$ with $x < y$, one may suppose that $p(z,y)$ $(x < z \leq y)$ exists and is uniquely determined by the conditions $p(y,y) = 1$, $p(z,y) \in u^{-1}\mathbb{Z}[u^{-1}]$ $(z \neq y)$ and $p(z,y) = \sum\limits_{w: z \leq w \leq y} r(z,w)\overline{p(w,y)}$.

Let $\beta(x,y) = \sum\limits_{z: x < z \leq y} r(x,z)\overline{p(z,y)}$. One must show that there exists a unique element $p(x,y) \in u^{-1}Z[u^{-1}]$ satisfying $p(x,y) - \overline{p(x,y)} = \beta(x,y)$. This will be the case provided $\overline{\beta(x,y)} = -\beta(x,y)$. But

$$\begin{aligned}
\overline{\beta(x,y)} &= \sum_{z: x < z \leq y} \overline{r(x,z)} p(z,y) \\
&= \sum_{z: x < z \leq y} \quad \sum_{w: z \leq w \leq y} \overline{r(x,z)} r(z,w) \overline{p(w,y)} \\
&= \sum_{w: x < w \leq y} \quad \sum_{z: x < z \leq y} \overline{r(x,z)} r(z,w) \overline{p(w,y)} \\
&= -\sum_{w: x < w \leq y} r(x,w)\overline{p(w,y)} \text{ by (6.10.1) and (6.10.2)} \\
&= -\beta(x,y)
\end{aligned}$$

so (i) is proved, and (ii) is proved similarly.

(ii) Note $\hat{\alpha} = \bar{\alpha}$, so $\bar{r} = \hat{r}$, and that the involutions $-$ and $\hat{}$ of $\mathbf{A}_{\mathcal{A}}(P)$ commute. Then

$$\begin{aligned}
\hat{q} * p &= \hat{\bar{q}} * \hat{r} * p \\
&= \bar{\hat{q}} * \bar{r} * p \\
&= \bar{\hat{q}} * \bar{p} \\
&= \overline{(\hat{q} * p)}.
\end{aligned}$$

Now $(\hat{q} * p)(x,y) = 1$ if $x = y$, and $(\hat{q} * p)(x,y) \in u^{-1}\mathbb{Z}[u^{-1}]$ otherwise. Hence $(\hat{q} * p)(x,y) = 0$ if $x \neq y$, and so $\hat{q} * p = \delta$ ie $\hat{q}$ is a left inverse for p. This implies $p^{-1} = \hat{q}$ (p is invertible since $p(x,x) = 1$ is a unit in $\mathcal{A}$ for all $x \in P$). $\square$

6.13 Remark. For $\beta = \sum_{n \in \mathbb{Z}} b_n u^n$, define $\mathrm{supp}(\beta) = \{\, n \in \mathbb{Z} \mid b_n \neq 0 \,\}$. One could define a different element of $\mathbf{A}_{\mathcal{A}}(P)$ satisfying (6.12) (i) (a), (i) (c) and the following in place of (6.12) (i) (b): $p(x,y) \in \mathbb{N}[u, u^{-1}]$; $\mathrm{supp}(p(x,y)) \cap \mathrm{supp}(\overline{p(x,y)}) = \emptyset$ $(x, y \in P, x \neq y)$. $\square$

We now reinterpret the above facts as statements about a $\mathbb{Z}$-module involution on a certain $\mathcal{A}$-module defined in terms of the poset P, and make some simple remarks concerning the structure of the sets of invariant and anti-invariant elements.

6.14 Let $\hat{h}(P)$ be the set of formal $\mathcal{A}$-linear combinations $\sum_{x \in P} a_x t_x$ $(a_x \in \mathcal{A})$ of a family $\{t_x\}_{x \in P}$ of symbols, such that for each $x \in P$ there exist only finitely many $y \in P$ with $a_y \neq 0$ and $y \geq x$. Regard $\hat{h}(P)$ as an $\mathcal{A}$-module in the obvious way.

By (6.10.1), there is a $\mathbb{Z}$-module involution θ of $\hat{h}(P)$ such that $\theta(\sum_{x \in P} a_x t_x) = \sum_{x \in P} (\sum_{y \in P} r(x,y)\bar{a}_y) t_x$. Note that by (6.10.2)
(6.14.1) if $c = \sum_{x \in P} a_x t_x \in \hat{h}(P)$ $(a_x \in u^{-1}\mathbb{Z}[u^{-1}]$ for all $x)$ and $\theta(c) = c$, then $c = 0$.

In this context, (6.12) shows that for $x \in P$, there exists a unique element $c'_x \in \hat{h}(P)$ satisfying
(6.14.2) $\theta(c'_x) = c'_x\,;\ c'_x \in t_x + \sum_{y \in P} u^{-1}\mathbb{Z}[u^{-1}] t_y$,
and that in fact, $c'_x = \sum_{y \in P} p(y,x) t_y$, so
(6.14.3) $c'_x \in t_x + \sum_{y < x} u^{-1}\mathbb{Z}[u^{-1}] t_y$.

From (6.14.3), one sees that $\hat{h}(P)$ can also be regarded as the set of fomal $\mathcal{A}$-linear combinations $\sum_{x \in P} a_x c'_x$ such that for each $x \in P$ there exist only finitely many $y \in P$ with $a_y \neq 0$ and $y \geq x$.

Now one sees that for any $h \in \hat{h}(P)$ there exists a unique element $c'(h)$ of $\hat{h}(P)$ satisfying
(6.14.4) $\theta(c'(h)) = c'(h)\,;\ c'(h) \in h + \sum_{y \in P} u^{-1}\mathbb{Z}[u^{-1}]t_y$.
In fact, if $h = \sum_{x \in P} a_x c'_x \in \hat{h}(P)$, set $c'(h) = \sum_{x \in P} [(1-\lambda')a_x + \lambda'\bar{a}_x]c'_x$ where $\lambda'(\sum_{n \in \mathbb{Z}} a_n u^n) = \sum_{n<0} a_n u^n$; then (6.14.4) holds and (6.14.1) gives uniqueness.

Now define $\lambda\colon \hat{h}(P) \longrightarrow \hat{h}(P)$ by $\lambda(\sum_{x \in P} a_x t_x) = \sum_{x \in P} \lambda'(a_x)t_x$ and note that if $a_x \in \sum_{n<0} a_n u^n$, then $\lambda\theta(a_x t_x) \in \sum_{y<x} \mathcal{A}t_y$. It follows from this that for any $h \in \hat{h}(P)$ and $y \in P$, the element t_y occurs with non-zero coefficient in only finitely many of the elements $(\lambda\theta)^n(h)$ $(n \in \mathbb{N})$. This shows that for $h \in \hat{h}(P)$, the series $\sum_{n \in \mathbb{N}} (\lambda\theta)^n(h)$ converges (in an appropriate sense) to an element of $\hat{h}(P)$, and thus that $1-\lambda\theta$ is invertible in $\mathrm{End}_{\mathbb{Z}}(\hat{h}(P))$. Similarly, $(1-\theta\lambda)^{-1}$ exists in $\mathrm{End}_{\mathbb{Z}}(\hat{h}(P))$.

Now take $h \in \hat{h}(P)$; we compute

$$\begin{aligned}(1-\lambda\theta)c'(h) &= c'(h) - \lambda\theta c'(h)\\ &= c'(h) - \lambda c'(h)\\ &= (1-\lambda)c'(h) = (1-\lambda)h\end{aligned}$$

and conclude that $c'(h) = (1-\lambda\theta)^{-1}(1-\lambda)h$. Since $\theta(c'(h)) = c'(h)$ it follows that
(6.14.5) $(1-\theta)(1-\lambda\theta)^{-1}(1-\lambda) = 0$.

Now (6.14.5) is equivalent to each of (6.14.6)–(6.14.8) below
(6.14.6) $(1-\lambda)(1-\theta\lambda)^{-1}(1-\theta) = 0$
(6.14.7) $(1-\lambda\theta)^{-1}(1-\lambda) + (1-\theta\lambda)^{-1}(1-\theta) = 1$
(6.14.8) $(1-\theta)(1-\lambda\theta)^{-1} + (1-\lambda)(1-\theta\lambda)^{-1} = 1$
(the equivalence of (6.14.5)–(6.14.8) holds if θ, λ are elements of any initial ring and $(1-\theta\lambda), (1-\lambda\theta)$ are both invertible).

Now (6.14.8) gives a canonical decomposition of an element of $\hat{h}(P)$ as a sum of an θ-anti-invariant element, and an element of $\hat{h}(P)$ in $\sum_{x \in P} \mathbb{Z}[u]t_x$. Also, (6.14.7) gives a representation of an element of $\hat{h}(P)$ as a sum of a θ-invariant element and an element of $\sum_{x\in P} u^{-1}\mathbb{Z}[u^{-1}]t_x$, in $\hat{h}(P)$.

Similarly, one could show that there is a unique element $c(h)$ of $\hat{h}(P)$ such that $\theta(c(h)) = c(h)$ and $c(h) \in h + \sum_{y\in P} u\mathbb{Z}[u]t_y$, and prove analogous results for it.

Applying these constructions in the situation (6.11) (i), one obtains (essentially) the elements C_x, C'_x $(x \in W)$ of the corresponding Hecke algebra ([KL1]). Applying the construction to the reverse poset produces the elements D_x, D'_x ([L2]).

This completes our discussion of the formal consequences of the existence of an element $r \in \mathbf{A}_{\mathcal{A}}(P)$ satisfying (6.10.1) and (6.10.2), and we turn now to the justification of the remark in (6.11) (iii) that the polynomials $R_{x,y}$ for a Coxeter system are produced by our incidence algebra construction.

For the remainder of this chapter, (W, R) denotes a Coxeter system, and $T = \bigcup_{w\in W} wRw^{-1}$. The data required for the incidence algebra construction will be obtained from certain orderings of T.

6.15 Definition. A partial order $\preceq$ on T is called a natural order if for any dihedral reflection subgroup W' of W, either

$$r \prec rsr \prec \ldots \prec srs \prec s \qquad \text{or} \qquad s \prec srs \prec \ldots \prec rsr \prec r$$

where $\{r, s\} = S(W')$.

Here, for example, $r \prec rsr \prec \ldots \prec srs \prec s$ means that

$$\begin{cases} \overbrace{rs\ldots r}^{2m+1} \preceq \overbrace{rs\ldots r}^{2n+1} & (1 \le 2m+1 \le 2n+1 \le \operatorname{ord}(rs)) \\ \overbrace{sr\ldots s}^{2m+1} \preceq \overbrace{sr\ldots s}^{2n+1} & (1 \le 2n+1 \le 2m+1 \le \operatorname{ord}(rs)) \\ \overbrace{rs\ldots r}^{2m+1} \preceq \overbrace{sr\ldots s}^{2n+1} & (1 \le 2m+1,\ 2n+1 \le \operatorname{ord}(rs)). \end{cases}$$

$\square$

6.16 Remarks.

(i) A natural order is a total order on T.

(ii) The reverse of a natural order on T is a natural order on T.

(iii) Let $\preceq$ be a natural order on T, and (W', R') be a reflection subsystem of (W, R). Then the restriction of $\preceq$ to $W' \cap T$ is a natural order of the reflections of (W', R') (by (1.9) (i)).

(iv) Suppose that (W, R) is a finite Coxeter system, and write $T = \{t_1, \ldots, t_n\}$ where $n = \#(T)$. Then it may be shown that the partial order $\preceq$ on T such that $t_1 \prec t_2 \prec \ldots \prec t_n$ is a natural order iff there is a reduced expression $w_0 = r_1 \ldots r_n$ for the longest element w_0 of W, such that $t_i = r_1 \ldots r_i \ldots r_1$ $(i = 1, \ldots, n)$. $\square$

To show that natural orders exist we make use of the root system of (W, R).

6.17 Lemma. Suppose (W, R) is realised geometrically on a real vector space V with positive roots Ψ^+. Let W' be a dihedral reflection subgroup of W and write $S(W') = \{t, s\}$. Let $\preceq$ be the order on $W' \cap T$ such that $t \prec tst \prec \ldots \prec sts \prec s$, and set $\Psi'^+ = \{\, \alpha \varepsilon \Psi^+ \mid r_\alpha \in W' \,\}$.

If $\alpha, \beta, \gamma \in \Psi'^+$ and $r_\alpha \prec r_\beta \prec r_\gamma$ then $\beta = c\alpha + d\gamma$ for some $c > 0, d > 0$.

<u>Proof</u> Let δ,ε be the unique elements of Φ'^+ satisfying $r_\delta = t$, $r_\varepsilon = s$, and let p_n $(n \in \mathbb{N} \bigcup \{-1\})$ be the real numbers defined by the recurrence relation (2.1.3), with $\gamma = -(\delta \mid \varepsilon)$ (note that $\gamma \in \{\, \cos \frac{\pi}{m} \mid m \in \mathbb{N}, m \geq 2 \,\} \cup [1, \infty)$ by (3.9)).

Define an inner product on $U = \mathbb{R}\delta + \mathbb{R}\varepsilon$ so δ and ε form an orthonormal basis of U, and regard U as a two-dimensional Euclidean space. Every element of Φ'^+ is a non-negative linear combination of δ and ε, and so makes an angle of between 0 and $\frac{\pi}{2}$ (inclusive) with δ; write $\alpha \prec \beta$ $(\alpha, \beta \in \Phi'^+)$ if the angle made by the vectors α and δ is less than that formed by β and δ.

Now note that if $\operatorname{ord}(rs) = m$ is finite, then $W' \cap T = \{t, tst, \ldots, \overbrace{ts \ldots t}^{2m-1}\}$ and $t \prec tst \prec \ldots \prec \overbrace{ts \ldots t}^{2m-1}$. Whether $\operatorname{ord}(rs)$ is finite or not, (2.1.1) and (2.1.2) show that the positive root corresponding to $\overbrace{ts\dots t}^{2n+1}$ $(0 \leq n < \operatorname{ord}(ts))$ is $p_{n+1}\delta + p_n\varepsilon$, and that corresponding to $\overbrace{st\dots s}^{2n+1}$ $(0 \leq n < \operatorname{ord}(ts))$ is $p_n\delta + p_{n+1}\varepsilon$. Making use of the parts of (2.2) (iii), (iv) concerning the ratios $\frac{p_n}{p_{n+1}}$, it follows that for $\alpha, \beta \in \Phi'^{+}$, we have $\alpha \prec \beta$ iff $r_\alpha \prec r_\beta$. The assertion of the lemma is clear from this characterisation of $\prec$. □

6.18 Proposition. There exists a natural ordering of the reflections T of (W, R).

Proof Suppose (W, R) is realised geometrically as a group of isometries of a real vector space V, with simple roots Π and positive roots Φ^+. Let $U = \{ \sum_{\alpha\in\Pi} c_\alpha\alpha \in V \mid \sum_{\alpha\in\Pi} c_\alpha = 1, \}$ be the affine hyperplane spanned by Π, and $\Psi = \{ \beta \in U \mid \beta$ is non-isotropic and $r_\beta \in T \}$. The map $\rho\colon \Psi \longrightarrow T$ defined by $\beta \longmapsto r_\beta$ $(\beta \in \Psi)$ is a bijection (if $\beta = \sum_{\alpha\in\Pi} c_\alpha\alpha \in \Phi^+$, then $\rho^{-1}(r_\beta) = (\sum_{\alpha\in\Pi} c_\alpha)^{-1}\beta$).

Let A be a set of linear functions $V \longrightarrow \mathbb{R}$ separating the points of Ψ (if R is finite, A could be chosen to be a singleton set; in general, A can be taken as the set of all linear functions $V \longrightarrow \mathbb{R}$). We suppose that A is given some well-ordering, and define a relation $\preceq$ on T by the condition $t \preceq t'$ iff $t = t'$ or ($t \neq t'$ and $\varphi(\rho^{-1}(t)) \prec \varphi(\rho^{-1}(t'))$ where $\varphi = \min\{ f \in A \mid f(\rho^{-1}(t) \neq f(\rho^{-1}(t')) \}$) $(t, t' \in T)$.

The relation $\preceq$ is evidently reflexive and anti-symmetric. To show that $\preceq$ is transitive, it will suffice to show that if $t, t', t'' \in T$ are all distinct and $t \preceq t', t' \preceq t''$ then $t \preceq t''$. Let $\varphi = \min\{ f \in A \mid f(\rho^{-1}(t)) \neq f(\rho^{-1}(t')) \}$ and $\varphi' = \min\{ f \in A \mid f(\rho^{-1}(t') \neq f(\rho^{-1}(t'')) \}$. Setting $\varphi'' = \min\{\varphi, \varphi'\}$ we have $f(\rho^{-1}(t)) = f(\rho^{-1}(t')) = f(\rho^{-1}(t''))$ if $f \in A$ is less than φ'', and $\varphi''(\rho^{-1}(t)) < \varphi''(\rho^{-1}(t''))$, so $t \preceq t''$. Hence $\preceq$ is a partial order on T, in fact a total order.

Let W' be a dihedral reflection subgroup of W, and write $S(W') = \{s, t\}$ where $t \prec s$. Let $\preceq'$ be the total order on $W' \cap T$ such that $t \prec' tst \prec' \dots \prec' sts \prec' s$. To show that $\preceq$ is a natural order, it will be sufficient to prove that the restriction of $\preceq$ to $W' \cap T$ is $\preceq'$.

Set $\delta = \rho^{-1}(t)$, $\varepsilon = \rho^{-1}(s)$, $\Psi' = \rho^{-1}(W' \cap T)$ and $\varphi = \min\{ f \in A \mid f(\rho^{-1}(t) \neq f(\rho^{-1}(s)) \}$. It follows from (6.17) that if $\alpha, \beta, \gamma \in \Psi'$ and $r_\alpha \prec' r_\beta \prec' r_\gamma$ then for some $c \in \mathrm{I\!R}$ with $0 < c < 1$, we have $\beta = c\alpha + (1-c)\gamma$. This shows firstly $f(\beta) = f(\delta) = f(\varepsilon)$ for all $\beta \in \Psi'$ if $f \in A$ is less than φ, and that $\varphi(\delta) \prec \varphi(\beta) \prec \varphi(\varepsilon)$ for all $\beta \varepsilon \Psi' \setminus \{\delta, \varepsilon\}$. By a second application of the above consequence of (6.17), it follows that if $\alpha, \beta \in \psi'$ and $r_\alpha \prec' r_\beta$ then $\varphi(\alpha) \prec \varphi(\beta)$. Hence if $t', t'' \in W' \cap T$ and $t' \prec' t''$, then $t' \prec t''$. Since $\prec$ and $\prec'$ are total orders, $\prec'$ is the restriction of $\prec$ to $W' \cap T$. $\square$

In (6.20), we show how to construct new natural orders from a given natural order. First, we need the following simple

6.19 Lemma. Let $\preceq$ be a natural order on T, $r \in R$ and $t \in T$ $(t \neq r)$. If $t \prec r$ then $rtr \prec r$. If $r \prec t$ then $r \prec rtr$.

<u>Proof</u> Write $S(\langle t, r\rangle) = \{r, t'\}$. Suppose $t' \prec r$. Then $t' \prec t'rt' \prec \ldots \prec rt'r \prec r$. Hence $t \prec r$ and $rtr \prec r$. Similarly, if $r \prec t'$ then $r \prec t$ and $r \prec rtr$. $\square$

6.20 Proposition. Let $\preceq$ be a natural order on T, and $r \in R$. Then the relation $\preceq'$ on T defined by

$$t_1 \preceq' t_2 \text{ iff} \begin{cases} t_1 = r \\ \text{or } (t_1 \neq r, r \prec t_2 \text{ and } t_1 \preceq t_2) \\ \text{or } (t_1 \neq r, t_2 \prec r \text{ and } rt_1r \preceq rt_2r) \quad (t_1, t_2 \in T) \end{cases}$$

is a natural order on T.

<u>Proof</u> We first check that $\preceq'$ is a partial order. Now $\preceq'$ is clearly reflexive. To check that $\preceq'$ is anti-symmetric, suppose that $t_1, t_2 \in T$ and $t_1 \preceq' t_2, t_2 \preceq' t_1$. If $t_1, t_2 \prec r$ or $r \prec t_1, t_2$ or $r \in \{t_1, t_2\}$, the definition of $\preceq'$ implies that $t_1 = t_2$. Otherwise, we have, say, $t_1 \prec r \prec t_2$. By (6.19), $rt_1r \prec r \prec rt_2r$ contrary to $t_2 \preceq' t_1$. Hence this last case cannot occur, and $\preceq'$ is anti-symmetric. To prove that $\preceq'$ is transitive, it will suffice to show that if $t_1, t_2, t_3 \in T$ and $t_1 \prec' t_2, t_2 \prec' t_3$ then $t_1 \prec' t_3$. Note that $r \notin \{t_2, t_3\}$, so that $r \prec' t_3$. Hence we may assume that $t_1 \neq r$. If $t_3 \prec r$, then by (6.19) and the definition of $\preceq'$, we have $rt_2r \prec rt_3r \prec r$, hence $t_2 \prec r$. From $t_1 \prec' t_2$, we now get $rt_1r \prec rt_2r \prec rt_3r$ and so $t_1 \prec' t_3$. If $r \prec t_2$ and $r \prec t_3$, then $t_1 \prec t_2 \prec t_3$ so $t_1 \prec' t_3$. The remaining case is $t_2 \prec r \prec t_3$. Here $t_1 \prec' t_2$ and (6.19) give

$rt_1r \prec rt_2r \prec r$, hence $t_1 \prec r \prec t_3$ and $t_1 \prec' t_3$. Therefore $\preceq'$ is a partial order as claimed.

Now fix a dihedral reflection subgroup W' of W, and write $S(W') = \{t, s\}$ where $t \prec s$. We must check that either $t \prec' tst \prec' \ldots \prec' sts \prec' s$ or $s \prec' sts \prec' \ldots \prec' tst \prec' t$. Consider the following cases.

Case 1 $t \prec r \prec s$ Then $t \prec tst \prec \ldots \prec sts \prec s$. Now $S(rW'r) = \{rtr, rsr\}$ by (3.14) and $rtr \prec r \prec rsr$ by (6.19), so $rtr \prec rtstr \prec \ldots \prec rstsr \prec rsr$. Noting that if $t_1 \prec r \prec t_2$, then $t_1 \prec' t_2$, it follows that $t \prec' tst \prec' \ldots \prec' sts \prec' s$.

Case 2 $t \prec s \prec r$ Now $S(rW'r) = \{rtr, rsr\}$, so either $rtr \prec rtstr \prec \ldots \prec rstsr \prec rsr$ or $rsr \prec rstsr \prec \ldots \prec rtstr \prec rtr$. Since $t \prec tst \prec \ldots \prec sts \prec s \prec r$, we have either $t \prec' tst \prec' \ldots \prec' sts \prec' s$ or $s \prec' sts \prec' \ldots \prec' tst \prec' t$.

Case 3 $r \prec t \prec s$ Here $r \prec t \prec tst \prec \ldots \prec sts \prec s$ so $t \prec' tst \prec' \ldots \prec' sts \prec' s$.

Case 4 $t = r$ Here $r \prec rsr \prec \ldots \prec srs \prec s$ so $r \prec' rsr \prec' \ldots \prec' srs \prec' s$.

Case 5 $s = r$ Here $t \prec trt \prec \ldots \prec rtrtr \prec rtr \prec r$ so $r \prec' rtr \prec' \ldots \prec' trt \prec' t$.

Hence $\preceq'$ is a natural order as claimed. □

6.21 Remark. The natural order $\preceq'$ defined in (6.20) will be called the lower r-conjugate of $\preceq$. Define the upper r-conjugate $\preceq''$ of $\preceq$ to be the reverse of the lower r-conjugate of the reverse of $\preceq$. Thus

$$t_1 \preceq'' t_2 \text{ iff } \begin{cases} \qquad t_2 = r \\ \text{or } (t_2 \neq r, t_1 \prec r \text{ and } t_1 \preceq t_2) \\ \text{or } (t_2 \neq r, r \prec t_1 \text{ and } rt_1r \preceq rt_2r) \quad (t_1, t_2 \in T), \end{cases}$$

and $\preceq''$ is a natural order.

Note that the upper r-conjugate of the lower r-conjugate of $\preceq$ is equal to the upper r-conjugate of $\preceq$. □

6.22 Let P be the poset W equipped with Bruhat order, $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the ring of Laurent polynomials in an indeterminate $q^{\frac{1}{2}}$ and $\alpha = q^{-\frac{1}{2}} - q^{\frac{1}{2}}$. Let $C_1 = E_{(W,R)}$ be the edge set of the Bruhat graph $\Gamma_{(W,R)}$ and adopt the notations of (6.2).

Let $\preceq$ be a fixed natural order on T. We define a correponding element $r^{\prec}$ of the incidence algebra $\mathbf{A}_{\mathcal{A}}(P)$ as follows using (6.3); $r^{\prec} = r^I$ where $I = \{ (x,y,z) \in C_2 \mid y^{-1}z \prec x^{-1}y \}$. Thus, for any $x, y \in W$, $r^{\prec}(x,y)$ is a polynomial in α, the coefficient of α^m being the cardinality of the set

$$\{ (x_0, \dots, x_m) \in C_m(x,y) \mid x_0^{-1}x_1 \prec x_1^{-1}x_2 \prec \dots, x_{m-1}^{-1}x_m \}.$$

Recall the definition of the $\tilde{R}_{x,y}(x, y \in W)$ from Chapter 0. We may now state

6.23 Theorem. Let $\prec$ be a natural order on the reflections of the Coxeter system (W,R). Then for all $x, w \in W$,
(6.23.1) $r^{\prec}(x,w) = \tilde{R}_{x,w}$. □

Theorem (6.23) will be proved by induction on $\ell(w)$ in sections (6.24)–(6.26). If $w = 1$, the result is trivial. So assume $\ell(w) \geq 1$ and that (6.23.1) holds with w replaced by any element of W of length less than $\ell(w)$, for any natural order $\preceq$. In (6.24)–(6.26), w is fixed and s denotes a fixed element of R with $\ell(ws) < \ell(w)$

6.24 For any natural order $\preceq$, we define three elements $f^{\prec}, g^{\prec}, h^{\prec}$ of $\mathbf{A}_{\mathcal{A}}(P)$ as follows. Let $\preceq'$ be the upper s-conjugate of $\preceq$, and for $x, y \in W$ and $n \in \mathbb{N}$, define

$$X_n(x,y) = \{ (x_0, \dots, x_n) \in C_n(x,y) \mid s \preceq x_0^{-1}x_1 \prec \dots \prec x_{n-1}^{-1}x_n \}$$
$$Y_n(x,y) = \{ (y_0, \dots, y_n) \in C_n(x,y) \mid s \preceq y_0^{-1}y_1, y_0^{-1}y_1 \prec' \dots \prec' y_{n-1}^{-1}y_n \}$$
$$Z_n(x,y) = \{ (z_0, \dots, z_n) \in C_n(x,y) \mid z_0^{-1}z_1 \prec \dots \prec z_{n-1}^{-1}z_n \prec s \}.$$

We set

$$f^{\prec}(x,y) = \sum_{n \in \mathbb{N}} \# X_n(x,y) \alpha^n$$

$$g^{\prec}(x,y) = \sum_{n \in \mathbb{N}} \# Y_n(x,y) \alpha^n$$

$$h^{\prec}(x,y) = \sum_{n \in \mathbb{N}} \# Z_n(x,y) \alpha^n$$

Note that

$$\begin{aligned}\{\,(x_0,\dots,x_m)\in C_m(x,y)\mid x_0^{-1}x_1\prec\dots\prec x_{m-1}^{-1}x_m\,\}\\ =\dot{\bigcup_{n=0}^{m}}\{\,x_0,\dots,x_m)\mid(x_0,\dots,x_n)\in Z_n(x,x_n),\\ (x_n,\dots,x_m)\in X_{m-n}(x_n,y)\,\}.\end{aligned}$$

It follows that
(6.24.1) $r^{\prec}=h^{\prec}*f^{\prec}$.

For similar reasons
(6.24.2) $r^{\prec'}=h^{\prec}*g^{\prec}$. □

6.25 Lemma. For any $y,z\in W$ we have

(i) $f^{\prec}(y,z)=g^{\prec}(ys,zs)$ $(ys<y,zs<z)$

(ii) $g^{\prec}(y,z)-\alpha g^{\prec}(y,zs)=f^{\prec}(ys,zs)-\alpha f^{\prec}(y,zs)$ $(ys<y,zs<z)$

(iii) $g^{\prec}(ys,zs)=f^{\prec}(y,z)-\alpha f^{\prec}(ys,z)$ $(ys>y,zs<z)$

(iv) $f^{\prec}(y,z)=g^{\prec}(ys,zs)-\alpha g^{\prec}(ys,z)$ $(ys<y,zs>z)$

<u>Proof</u> For $y,z\in W$, and $n\in\mathbb{N}$, let

$$U_n(y,z)=\{\,(x_0,\dots,x_n)\in C_n(y,z)\mid s\prec x_0^{-1}x_1\prec\dots\prec x_{n-1}^{-1}x_n\,\}$$

and define $u^{\prec}(y,z)=\sum_{n\in\mathbb{N}}\#U_n(y,z)\alpha^n$. Now note that $X_n(y,z)=U_n(y,z)$ if $ys<y$, and that if $ys>y$, then

$$X_n(y,z)=U_n(y,z)\bigcup\{\,(y,y_0,\dots,y_{n-1})\mid(y_0,\dots,y_{n-1})\in X_{n-1}(ys,z)\,\}.$$

It follows that

(6.25.1) $$u^{\prec}(y,z)=\begin{cases}f^{\prec}(y,z) & (ys<y)\\ f^{\prec}(y,z)-\alpha f^{\prec}(ys,z) & (ys>y).\end{cases}$$

Similarly, define

$$
\begin{aligned}
&V_n(y,z) \\
&\quad = \{ (x_0,\dots,x_n) \in C_n(y,z) \mid s \prec x_0^{-1}x_1,\ x_0^{-1}x_1 \prec' \dots \prec' x_{n-1}^{-1}x_n \prec' s \}
\end{aligned}
$$

and $v^{\prec}(y,z) = \sum_{n\in\mathbb{N}} \#V_n(y,z)\alpha^n$. Here, we have

$$
(6.25.2) \qquad v^{\prec}(y,z) = \begin{cases} g^{\prec}(y,z) & (zs > z) \\ g^{\prec}(y,z) - \alpha g^{\prec}(y,zs) & (zs < z). \end{cases}
$$

Now note that by (1.20), (6.19) and the definition of $\prec'$, the map $W^{n+1} \longrightarrow W^{n+1}$ defined by $(x_0,\dots,x_n) \longmapsto (x_0 s,\dots,x_n s)$ restricts to a bijection

$$
U_n(y,z) \longrightarrow V_n(ys,zs).
$$

It follows that
(6.25.3) $u^{\prec}(y,z) = v^{\prec}(ys,zs)$ $(y,z\in W)$

The assertions (i)–(iv) all follow from (6.25.1)–(6.25.3). $\square$

We may now complete the proof of (6.23).

6.26 Let $\preceq$ be any natural order on T, $\preceq'$ be its upper s-conjugate and $\preceq''$ be the lower s-conjugate of $\preceq$. Define $f^{\prec}, g^{\prec}, h^{\prec}$ as in (6.24).

We first show that $f^{\prec}(x,ws) = g^{\prec}(x,ws)$ $(x\in W)$ by descending induction on $\ell(x)$. If $\ell(x) > \ell(ws)$, both sides are zero. Note that $r^{\prec}(x,ws) = \tilde{R}_{x,ws} = r^{\prec'}(x,ws)$. Hence by (6.24.1) and (6.24.2), for any $x\in W$ we have

$$
\begin{aligned}
f^{\prec}(x,ws) &= r^{\prec}(x,ws) - \sum_{y>x} h^{\prec}(x,y) f^{\prec}(y,ws) \\
&= r^{\prec'}(x,ws) - \sum_{y>x} h^{\prec}(x,y) g^{\prec}(y,ws) \text{ by induction} \\
&= g^{\prec}(x,ws).
\end{aligned}
$$

Now we make use of (6.25) to show $f^{\prec}(x,w) = g^{\prec}(x,w) (x \in W)$.

$$\begin{aligned}
\text{If } xs > x, \text{ then } g^{\prec}(x,w) &= \alpha g^{\prec}(x,ws) + f^{\prec}(xs,ws) \text{ by (6.25) (iv)} \\
&= \alpha g^{\prec}(x,ws) + g^{\prec}(xs,ws) \\
&= \alpha f^{\prec}(xs,w) + g^{\prec}(xs,ws) \text{ by (6.25) (i)} \\
&= f^{\prec}(x,w) \qquad\qquad\qquad \text{by (6.25) (iii).} \\
\text{If } xs < s, \text{ then } f^{\prec}(x,w) &= g^{\prec}(xs,ws) \qquad\qquad\qquad \text{by (6.25) (i)} \\
&= f^{\prec}(xs,ws) \\
&= g^{\prec}(x,w) + \alpha(f^{\prec}(x,ws) - g^{\prec}(x,ws)) \text{ by (6.25)(ii)} \\
&= g^{\prec}(x,w). \\
\text{For } x \in W, \ r^{\prec}(x,w) &= \sum_y h^{\prec}(x,y) f^{\prec}(y,w) \\
&= \sum_y h^{\prec}(x,y) g^{\prec}(y,w) \\
&= r^{\prec'}(x,y) \text{ by (6.24.1), (6.24.2) again.}
\end{aligned}$$

We have shown that
(6.26.1) $r^{\prec}(x,w) = r^{\prec'}(x,w)$ if $\preceq'$ is the upper s-conjugate of $\preceq$.

Since $\preceq'$ is also the upper s-conjugate of $\preceq''$, we have $r^{\prec''}(x,w) = r^{\prec'}(x,w)$. Hence $r^{\prec}(x,w) = r^{\prec''}(x,w)$, and to show $r^{\prec}(x,w) = \tilde{R}_{x,w}$, it suffices to show $r^{\prec''}(x,w) = \tilde{R}_{x,w}$. Since T has a minimum element s in the ordering $\preceq''$, it follows that there is no loss of generality in assuming that s is the minimum element of T in the natural ordering $\preceq$.

But then $f^{\prec} = r^{\prec}, \ g^{\prec} = r^{\prec'}$ so

$$\begin{aligned}
r^{\prec}(x,w) = f^{\prec}(x,w) &= \begin{cases} g^{\prec}(xs,ws) & (xs < s) \\ g^{\prec}(xs,ws) + \alpha g^{\prec}(x,ws) & (xs > x) \text{ by (6.25)} \end{cases} \\
&= \begin{cases} r^{\prec'}(xs,ws) & (xs < x) \\ r^{\prec'}(xs,ws) + \alpha r^{\prec'}(x,ws) & (xs > s) \end{cases} \\
&= \begin{cases} \tilde{R}_{xs,ws} & (xs < x) \\ \tilde{R}_{xs,ws} + \alpha \tilde{R}_{x,ws} & (xs > x) \end{cases} \\
&= \tilde{R}_{x,w} \text{ by (0.1).}
\end{aligned}$$

This complete the proof of (6.23). □

6.27 Remark. Let J be a subset of R. It may be shown that there is a natural order $\preceq$ of T such that $t \prec t'$ for all $t \in W_J \cap T$ and $t' \in T \setminus W_J$. Using (6.23), it may be shown that if the maximal chains $(x_0, \ldots, x_n)$ of the relative Bruhat interval $[x, y] \cap W^J$ ($x, y \in W^J$) are labelled by the successive ratios $(x_0^{-1}x_1, \ldots, x_{n-1}^{-1}x_n) \in T^n$, one obtains an L-labelling in the sense of ([BW], Definition 3.2) (except that here the labels lie in a totally ordered set rather than $\mathbb{N}$). $\square$

6.28 We conclude this chapter by mentioning one application of (6.23). Adopt the notation of (5.22). Let Γ_I (respectively $\Gamma_{I'}$) be the full subgraph of $\Gamma_{(W,R)}$ (respectively, $\Gamma_{(W',R')}$) with vertex set I (respectively). Attach to an edge (x, y) of Γ_I or $\Gamma_{I'}$ the label $x^{-1}y \in T$. The map $\theta\colon I' \longrightarrow I$ of (5.22) is an isomorphism of directed graphs $\Gamma_{I'} \longrightarrow \Gamma_I$ and $\theta(x)^{-1}\theta(y) = x^{-1}y$ for any edge (x, y) of $\Gamma_{I'}$. Using (6.16) (iii) and (6.23), it follows that for $v, w \in I'$, we have $\tilde{R}_{v,w} = \tilde{R}_{\theta(v),\theta(w)}$, hence $P_{v,w} = P_{\theta(v),\theta(w)}$ where the left-hand sides are computed in (W', R') and the right-hand sides in (W, R). Note that by (3.16) (i), $\#(R') \leq \ell(w) - \ell(v)$.

Now consider the case when (W, R) is a finite Weyl group. Every reflection subsystem of (W, R) corresponds to some Weyl group: now the only Kazhdan-Lusztig polynomials which occur for intervals of length 3 in finite Weyl groups of rank at most 3 are 1 and $1 + q$ ([Sh],pages 20 and 23). It follows that for any $v, w \in W$ with $v \leq w$ and $\ell(v) = \ell(w) - 3$, either $P_{v,w} = 1$ or $P_{v,w} = 1 + q$. $\square$

Chapter 7

POSITIVITY PROPERTIES OF HECKE ALGEBRAS

This chapter is devoted to a study of properties of structure constants of the Hecke algebra of a Coxeter group, taken with respect to various standard bases.

After giving a number of formal properties of these structure constants, we introduce four conjectural positivity properties. These hold for dihedral groups, and in Chapters 8 and 9, it will be shown that the structure constants of the Hecke algebra of a universal Coxeter system satisfy all four positivity porperties.

For general Coxeter systems, only isolated special cases of these positivity properties have been proved, and this chapter concludes with a number of such results. A criterion is given for a Bruhat interval to be isomorphic to a Bruhat interval in a dihedral group, and used to prove that the Kazhdan-Lusztig polynomials $P_{v,w}$ ($\ell(w) - \ell(v) = 3$ or 4) have non-negative coefficients.

7.1 Let (W, R) be a Coxeter system and $\mathcal{H}(W)$ the corresponding Hecke algebra over $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ where $q^{\frac{1}{2}}$ is an indeterminate; we adopt our standard notation concerning $(W, R,)$ and $\mathcal{H}(W)$, and, for the moment, find it particularly convenient to work with the $\mathcal{A}$-basis $\{\widetilde{T}_w\}_{w\in W}$ of $\mathcal{H}(W)$.

As in [L2], we let $\hat{\mathcal{H}}(W)$ denote the set of formal $\mathcal{A}$-linear combinations

$$\sum_{w\in W} a_w \widetilde{T}_w \ (a_w \in \mathcal{A},\ w \in W).$$

This is in a natural way a $\mathcal{H}(W)$-bimodule; for example

$$\begin{aligned}&\widetilde{T}_r \left(\sum_w a_w \widetilde{T}_w \right)\\ &\quad = \sum_{w:rw>w} a_{rw}\widetilde{T}_w + \sum_{w:rw<w} (a_{rw} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})a_w)\widetilde{T}_w \ (a_w \in \mathcal{A},\ r \in R),\end{aligned}$$

the action on the right being defined similarly.

Let $\tau \colon \hat{\mathcal{H}}(W) \longrightarrow \mathcal{A}$ denote the $\mathcal{A}$-linear map such that

$$\tau(\sum a_w \widetilde{T}_w) = a_1.$$

From [L2], one has the following properties of θ
(7.1.1) $\tau(\widetilde{T}_x \widetilde{T}_y) = \delta_{x,y^{-1}}$ $(x, y \in W)$
(7.1.2) $\tau(h_1 \hat{h}) = \tau(\hat{h} h_1)$ $(h_1 \in \mathcal{H}(W), \hat{h} \in \hat{\mathcal{H}}(W))$

We will require the following fact when W is finite.

7.2 Lemma. Let (W, R) be a finite Coxeter system with longest element w_0 If $h \in \mathcal{H}(W)$ and $h\widetilde{T}_w = \sum_{v \in W} a_{v,w} \widetilde{T}_v$ $(w \in W)$, then

$$\bar{h}\widetilde{T}_w = \sum_{v \in W} \overline{a_{vw_0, ww_0}}\, \widetilde{T}_v \ (w \in W)$$

<u>Proof</u> First we prove that for any $h \in \mathcal{H}(W)$,
(7.2.1) $\tau(\bar{h}) = \overline{\tau(\widetilde{T}_{w_0} h \widetilde{T}_{w_0})}$.

For this, it will be sufficient to prove

(7.2.2) $\overline{\tau(\bar{k}\widetilde{T}_{w_0})} = \tau(\widetilde{T}_{w_0} k)$ $(k \in \mathcal{H}(W))$.

Then (7.2.1) follows on setting $k = h\widetilde{T}_{w_0}$. Note that both sides of (7.2.2) are $\mathcal{A}$-linear as functions of k. Hence it suffices to check (7.2.2) when $k = \widetilde{T}_x$ $(x \in W)$. But then

$$\begin{aligned}
\tau(\overline{\widetilde{T}}_x \widetilde{T}_{w_0}) &= \tau(\widetilde{T}_{x^{-1}}^{-1} \widetilde{T}_{w_0}) \\
&= \tau(\widetilde{T}_{xw_0}) \\
&= \delta_{1,xw_0} \\
&= \delta_{w_0,x^{-1}} \\
&= \tau(\widetilde{T}_{w_0} \widetilde{T}_x).
\end{aligned}$$

Replacing h by $\widetilde{T}_v^{-1} h \widetilde{T}_{w^{-1}}^{-1}$ in (7.2.1) gives
(7.3.1) $\tau(\widetilde{T}_{v^{-1}} \bar{h} \widetilde{T}_w) = \overline{\tau(\widetilde{T}_{(vw_0)^{-1}} h \widetilde{T}_{ww_0})}$ $(h \in \hat{\mathcal{H}}(W),\ v, w \in W)$.

Now by (7.1.1),

$$k\widetilde{T}_w = \sum_{v\in W} \tau(\widetilde{T}_{v^{-1}}k\widetilde{T}_w)\widetilde{T}_v \ (w \in W);$$

taking $k = h$, then $k = \bar{h}$ in this and using (7.3.1) gives the result. $\square$

7.3 We return to the case of an arbitrary Coxeter system.

For $y, w \in W$, define $\tilde{P}_{y,w}, \tilde{Q}_{y,w} \in \mathcal{A}$ by the formulae
(7.3.1) $C_w = \sum_y \varepsilon_y\varepsilon_w \bar{\tilde{P}}_{y,w}\widetilde{T}_y$
(7.3.2) $\widetilde{T}_w = \sum_y \bar{\tilde{Q}}_{y,w}C_y$.

Thus, for $y, w \in W$
(7.3.3) $\tilde{P}_{y,w} = q_w^{-\frac{1}{2}}q_y^{\frac{1}{2}}P_{y,w}$
(7.3.4) $\tilde{Q}_{y,w} = q_w^{-\frac{1}{2}}q_y^{\frac{1}{2}}Q_{y,w}$
where $P_{y,w}$ and $Q_{y,w}$ denote the Kazhdan-Lusztig polynomials and inverse Kazhdan-Lusztig polynomials respectively.

In this notation, we have
(7.3.5) $C'_w = \sum_y \tilde{P}_{y,w}\widetilde{T}_y$
(7.3.6) $D_w = \sum_y \tilde{Q}_{w,y}\widetilde{T}_y$
(7.3.7) $D'_w = \sum_y \varepsilon_y\varepsilon_w\widetilde{Q}_{y,w}\widetilde{T}_y$
where $D_w, D'_w \in \hat{\mathcal{H}}(W)$ are defined in [L2].

We recall that
(7.3.8) $\tau(C_xD_y) = \tau(C'_xD'_y) = \delta_{x,y^{-1}} \ (x, y \in W)$

Also, from ([L3], 5.1) we have the following

7.4 Lemma. If W is finite with longest element w_0, then
(7.4.1) $D_w = C'_{ww_0}\widetilde{T}_{w_0} = \widetilde{T}_{w_0}C'_{w_0w}$ and
(7.4.2) $D'_w = C_{ww_0}\widetilde{T}_{w_0} = \widetilde{T}_{w_0}C_{w_0w}$. $\square$

7.5 Define elements $\tilde{R}^z_{y,w} \in \mathcal{A}$ by
(7.5.1) $\widetilde{T}^{-1}_{w^{-1}}\widetilde{T}_z = \sum\limits_{y \in W} \tilde{R}^z_{y,w}\widetilde{T}_y \ (w, z \in W)$;
note that $\tilde{R}^1_{y,w} = \tilde{R}_{y,w} \ (y, w \in W)$. We now define analogues of the polynomials $P_{y,w}$ and $Q_{y,w}$ as follows
(7.5.2) $C'_w\widetilde{T}_z = \sum\limits_{y \in W} \tilde{P}^z_{y,w}\widetilde{T}_y \ (w, z \in W)$
(7.5.3) $\widetilde{T}^{-1}_{w^{-1}}\widetilde{T}_z = \sum\limits_{y \in W} \tilde{Q}^z_{y,w}C_y \ (w, z \in W)$.

We remark right away that the $\tilde{P}^z_{y,w}$ are, up to some normalisation, the same as the $P^z_{y,w}$ defined in [Dy] but that the $\tilde{Q}^z_{y,w}$ are quite diferent from the $Q^z_{y,w}$ defined there.

Note that by (7.3.5) and (7.3.2), we have
(7.5.4) $\tilde{P}_{y,w} = \tilde{P}^1_{y,w} \ (y, w \in W)$
(7.5.5) $\tilde{Q}_{y,w} = \tilde{Q}^1_{y,w} \ (y, w \in W)$

We also have the following two facts
(7.5.6) $\tilde{P}^z_{x,w} = \sum\limits_{y} \tilde{R}^z_{x,y}\bar{\tilde{P}}_{y,w} \ (z, x, w \in W)$
(7.5.7) $\tilde{Q}^z_{y,w} = \sum\limits_{x} \bar{\tilde{Q}}_{y,x}\tilde{R}^z_{xw} \ (y, z, w \in W)$.

For example, (7.5.6) is proved by the computation

$$\begin{aligned} C'_w\widetilde{T}_z &= \sum_y \bar{\tilde{P}}_{y,w}\widetilde{T}^{-1}_{y^{-1}}\widetilde{T}_z \\ &= \sum_x \left(\sum_y \tilde{R}^z_{x,y}\bar{\tilde{P}}_{y,w}\right)\widetilde{T}_x \end{aligned}$$

and the proof of (7.5.7) is similar.

It is convenient to note at this point the following inductive formula for computing $\tilde{R}^z_{y,w}$:

7.6 Lemma.

(i) $\tilde{R}^z_{y,1} = \delta_{y,z} \ (y, z \in W)$

(ii) If $x, y, w \in W,\ r \in R$ and $wr < w$ then

$$\tilde{R}^y_{x,w} = \begin{cases} \tilde{R}^{ry}_{x,wr} & (ry < y) \\ \tilde{R}^{ry}_{x,wr} + \alpha \tilde{R}^y_{x,wr} & (ry > y), \end{cases}$$

where $\alpha = q^{-\frac{1}{2}} - q^{\frac{1}{2}}$.

Proof (i) follows trivially from the definition, and (ii) follows by a computation on writing $\widetilde{T}^{-1}_{w^{-1}} \widetilde{T}_y = \widetilde{T}^{-1}_{(wr)^{-1}} \widetilde{T}^{-1}_r \widetilde{T}_y$. $\square$

7.7 It follows easily from (7.6) that
(7.7.1) $\overline{\tilde{R}^z_{y,w}} = \varepsilon_y \varepsilon_z \varepsilon_w \tilde{R}^z_{y,w}$ and that
(7.7.2) $\tilde{R}^z_{y,w}(1) = \delta_{yz^{-1},w}$ where $\tilde{R}^z_{y,w}(1)$ denotes the value of the Laurent polynomial $\tilde{R}^z_{y,w}$ when $q^{\frac{1}{2}} = 1$.

From (7.5.6), (7.5.7) and (7.7.2) we have
(7.7.3) $\tilde{P}^z_{x,w}(1) = \tilde{P}_{xz^{-1},w}(1)$
(7.7.4) $\tilde{Q}^z_{x,w}(1) = Q_{x,wz}(1)$

The following proposition gives some simple symmetry properties of these structure constants.

7.8 Proposition. For any $y, z, w \in W$, we have
(7.8.1) $\tilde{R}^z_{y,w} = \tilde{R}^y_{z,w^{-1}}$
(7.8.2) $\tilde{P}^z_{y,w} = \tilde{P}^y_{z,w^{-1}}$
(7.8.3) $\tilde{Q}^z_{y,w} = \overline{\tilde{Q}^{w^{-1}}_{y^{-1},z^{-1}}}$

Proof Note that, from (7.1.1) and (7.3.8), we have
(7.8.4) $\tilde{R}^z_{y,w} = \tau(\widetilde{T}^{-1}_{w^{-1}} \widetilde{T}_z \widetilde{T}_{y^{-1}})$
(7.8.5) $\tilde{P}^z_{y,w} = \tau(C'_w \widetilde{T}_z \widetilde{T}_{y^{-1}})$
(7.8.6) $\tilde{Q}^z_{y,w} = \tau(\widetilde{T}^{-1}_{w^{-1}} \widetilde{T}_z D_{y^{-1}})$

Now there is an $\mathcal{A}$-algebra anti-involution ψ: $\sum\limits_{x \in W} a_x \widetilde{T}_x \longmapsto \sum\limits_{x \in W} a_x \widetilde{T}_{x^{-1}}$ of

$\mathcal{H}(W)$, which takes C'_w to $C'_{w^{-1}}$. Hence

$$\begin{aligned}\tilde{P}^z_{y,w} = \tau(C'_w\widetilde{T}_z\widetilde{T}_{y^{-1}}) &= \theta(\widetilde{T}_y\widetilde{T}_{z^{-1}}C'_{w^{-1}}) \\ &= \tau(C'_{w^{-1}}\widetilde{T}_y\widetilde{T}_{z^{-1}}) \qquad \text{by } (7.1.2) \\ &= \tilde{P}^y_{z,w^{-1}},\end{aligned}$$

proving (7.8.2), and (7.8.1) is proved similarly using (7.8.4).

To prove (7.8.3), note that the involution $h \longmapsto \bar{h}$ of $\mathcal{H}(W)$, applied to (7.5.3), gives
$\widetilde{T}_w\widetilde{T}^{-1}_{z^{-1}} = \sum_{y\in W} \overline{\tilde{Q}^z_{y,w}}\, C_y$. Applying ψ, we obtain
$\widetilde{T}^{-1}_z\widetilde{T}_{w^{-1}} = \sum_{y\in W} \overline{Q^z_{y,w}}\, C_{y^{-1}}$ and (7.8.3) follows on comparing with (7.5.3).

□

In case (W, R) is a finite Coxeter system, we have a number of additional symmetry properties

7.9 Proposition. Let (W, R) be a finite Coxeter system with longest element w_0. Then for $y, z, w \in W$, we have
(7.9.1) $\tilde{Q}^z_{y,w} = \tilde{P}^{z^{-1}}_{w_0w,w_0y}$
(7.9.2) $\overline{\tilde{P}^z_{y,w}} = \tilde{P}^{zw_0}_{yw_0,w}$
(7.9.3) $\tilde{Q}^z_{y,z} = \overline{\tilde{Q}^{w_0z}_{y,ww_0}}$

<u>Proof</u> From (7.8.6), we have

$$\begin{aligned}\tilde{Q}^z_{y,w} &= \tau(\widetilde{T}^{-1}_{w^{-1}}\widetilde{T}_zD_{y^{-1}}) \\ &= \tau(\widetilde{T}^{-1}_{w^{-1}}\widetilde{T}_zC'_{y^{-1}w_0}\widetilde{T}_{w_0}) \qquad \text{by } (7.4.1) \\ &= \tau(C'_{y^{-1}w_0}\widetilde{T}_{w_0w}\widetilde{T}_z) \qquad \text{by } (7.1.2) \\ &= \tilde{P}^{w_0w}_{z^{-1},y^{-1}w_0} \qquad \text{by } (7.8.5) \\ &= \tilde{P}^{z^{-1}}_{w_0w,w_0y} \qquad \text{by } (7.8.2),\end{aligned}$$

so (7.9.1) holds.

Using (7.2), we have
$C'_w\widetilde{T}_z = \sum_y \tilde{P}^z_{y,w}\widetilde{T}_y = \sum_y \overline{\tilde{P}^{zw_0}_{yw_0,w}}\widetilde{T}_y$ since $\bar{C}'_w = C'_w$, hence (7.9.2) is proved. Finally, (7.9.1) and (7.9.2) give (7.9.3) as follows:

$$\tilde{Q}^z_{y,w} = \tilde{P}^{z^{-1}}_{w_0w,w_0y} = \overline{\tilde{P}^{z^{-1}w_0}_{w_0ww_0,w_0y}} = \overline{\tilde{Q}^{w_0z}_{y,ww_0}}.$$

□

Our next result gives some information about the "supports" of these structure constants.

7.10 Proposition. If $y, z, w \in W$ then

(i) $\tilde{R}^z_{y,w} = 0$ unless $y \leq wz$ and $yz^{-1} \leq w$ and $z \leq w^{-1}y$.

(ii) $\tilde{P}^z_{y,w} = 0$ unless $yz^{-1} \leq w$

(iii) $\tilde{Q}^z_{y,w} = 0$ unless $y \leq wz$.

Proof (i) Suppose $\tilde{R}^z_{y,w} \neq 0$. We first show that $y \leq wz$ by induction on $\ell(w)$. If $w = 1$, then $y = z \leq wz$. If $w \neq 1$, choose $r \in R$ so $wr < w$. Then by (7.6), either $\tilde{R}^{rz}_{y,wr} \neq 0$ (whence $y \leq (wr)(rz) = wz$) or $rz > z$ and $\tilde{R}^z_{y,wr} \neq 0$ (whence $y \leq wrz$). But if $rz > z$, then $z^{-1}rz \in N(z) + z^{-1}N(w)z = N(wz)$, so $wrz = (wz)(zrz^{-1}) \leq wz$. Hence $y \leq wz$. The claim that $z \leq w^{-1}y$ follows from this and (7.8.1).

To prove that $yz^{-1} \leq w$, write $w = r_1 \dots r_n$ (reduced). Then $\widetilde{T}_y$ can only occur with non-zero coefficient in $\widetilde{T}^{-1}_{w^{-1}}\widetilde{T}_z$ if there exist $i_1, \dots, i_m$ $(i \leq i_1 < \dots \leq i_m \leq n)$ such that $y = r_{i_1} \dots r_{i_m} z$, and in that case, $yz^{-1} = r_{i_1} \dots r_{i_m} \leq w$ by (5.3) (ii). This proves (i).

Now $\tilde{P}_{y,w} = 0$ and $\tilde{Q}_{y,w} = 0$ unless $y \leq w$, so (ii) and (iii) follow from (i), (7.5.6) and (7.5.7). □

7.11 Remark. If (W, R) is any Coxeter system in which the Kazhdan-Lusztig polynomials $P_{y,w}$ have non-negative coefficients (e.g. if (W, R) is Crystallographic, or finite) then (7.10) (ii) and (7.7.3) prove that $\tilde{P}^z_{y,w} \neq 0$ iff $yz^{-1} \leq w$; a similar remark applies to $\tilde{Q}^z_{y,w}$.

Now we have some orthogonality relations for our structure constants.

7.12 Proposition. The following relations hold in $\mathcal{H}(W)$; for any $x, w \in W$

(i) $\sum\limits_y \varepsilon_x \varepsilon_y \varepsilon_z \tilde{R}^{z^{-1}}_{x,y} \tilde{R}^z_{y,w} = \delta_{x,w}$ $(z \in W)$.

(ii) $\sum\limits_z \varepsilon_x \varepsilon_y \varepsilon_z \tilde{P}^y_{x,z} \tilde{Q}^{y^{-1}}_{z,w} = \delta_{x,w}$ $(y \in W)$.

(iii) $\sum\limits_z \varepsilon_x \varepsilon_y \varepsilon_z \tilde{Q}^{y^{-1}}_{x,z} \tilde{P}^y_{z,w} = \delta_{x,w}$ $(y \in W)$.

<u>Proof</u> The involution $\sum a_w \widetilde{T}_w \longmapsto \sum \bar{a}_w \varepsilon_x \widetilde{T}_w$ takes C_y to $\varepsilon_y C'_y$; applying it to (7.5.3) gives

$$\begin{aligned} \widetilde{T}^{-1}_{w^{-1}} \widetilde{T}_z &= \sum_y \varepsilon_y \varepsilon_z \varepsilon_w \overline{\tilde{Q}^z_{y,w}} C'_y \\ &= \sum_y \varepsilon_y \varepsilon_z \varepsilon_w \tilde{Q}^{w^{-1}}_{y^{-1},z^{-1}} C'_y. \end{aligned}$$

Using the involution ψ defined in the proof of (7.8), we have

$$\widetilde{T}_{z^{-1}} \widetilde{T}^{-1}_w = \sum_y \varepsilon_y \varepsilon_z \varepsilon_w \tilde{Q}^{z^{-1}}_{y,z^{-1}} C'_y.$$

Hence, replacing z by x^{-1} and w by y^{-1},

$$\begin{aligned} \widetilde{T}_x &= \sum_z \varepsilon_x \varepsilon_y \varepsilon_z \tilde{Q}^{z^{-1}}_{y,x} C'_z \widetilde{T}_{y^{-1}} \\ &= \sum_w (\sum_z \varepsilon_x \varepsilon_y \varepsilon_z \tilde{Q}^y_{z,x} \tilde{P}^{y^{-1}}_{w,z}) \widetilde{T}_w \end{aligned}$$

which is equivalent to (ii).

Properties (i) and (iii) are proved by similar arguments. □

7.13 In some situations, it will prove more convenient to use the following notation
(7.13.1) $C'_w T_z = q_w^{-\frac{1}{2}} \sum\limits_y P^z_{y,w} T_y$ $(z, w \in W)$.
Comparing this with (7.5.2), we see that
(7.13.2) $\tilde{P}^z_{y,w} = q_z^{-\frac{1}{2}} q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} P^z_{y,w}$ $(y, z \in W)$.
The following formula permits calculations of the $P^z_{y,w}$.

7.14 Lemma.

(i) If $y, z \in W$ then $P^z_{y,1} = \delta_{y,z}$

(ii) If $y, z, w \in W,\ r \in R$ and $rw < w$ then

$$P^z_{y,w} = q^c P^z_{ry,rw} + q^{1-c} P^z_{y,rw} - \sum_{\substack{x \in W \\ rx < x}} \mu(x, rw) q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} P^z_{y,x}$$

where

$$c = \begin{cases} 1 & (ry > y) \\ 0 & (ry < y) \end{cases}$$

and $\mu(x, rw)$ is the coefficient of $q^{(\ell(w)-\ell(x)-2)/2}$ in $P_{x,rw} (\mu(x, rw)$ is zero unless $x < rw)$.

Proof (i) is trivial and (ii) follows simply from the formulae ([L3], 5.1)
(7.14.1) $C'_w = C'_r C'_{rw} - \sum\limits_{\substack{x \in W \\ rx < x}} \mu(x, rw) C'_x$ and $C'_r = q^{-\frac{1}{2}}(T_r + 1)$. □

This recurrence formula implies in particular that $P^z_{y,w}$ is a polynomial in q

7.15 Remark. It follows from (7.14) that for any $y, z, w \in W$, the polynomial $P^z_{y,w}$ has degree at most $\ell(w)$. Fix $z \in W$. One sees that there is, for any $w \in W$, a unique $y \in W$ such that the coefficient of q_w in $P^z_{y,w}$ is non-zero; denoting this y by $a(y)$, we see that $P^z_{a(y),w} = q_w$ and that $a(1) = z$,

$$a(w) = \begin{cases} a(rw) & (ra(rw) < a(rw)) \\ ra(rw) & (ra(rw) > a(rw)) \end{cases}$$

if $rw < w$ $(r \in R)$. It follows from (5.3.1) that $a(w)$ is the maximum element of the set $[1, w]z$ (in the Bruhat order). Similarly, for fixed z, there is for any

$w \in W$ a unique element $b(w)$ of W such that the coefficient of q^0 in $P^z_{b(w),w}$ is non-zero; we have $P^z_{b(w),w} = 1$ and $b(w)$ is the minimum element of $[1, w]z$. $\square$

7.16 It is known that if (W, R) is a crystallographic or finite Coxeter system then the Kazhdan-Lusztig polynomials $P_{x,w}$, $Q_{x,w}$ have non-negative coefficients, and that if (W, R) is a crystallographic Coxeter system, the structure constants of $\mathcal{H}(W)$ with respect to the $\mathcal{A}$-basis C'_w of $\mathcal{H}(W)$ are Laurent polynomials with non-negative coefficients (e.g. [KL2], [L2],[A1]). In view of (7.7.3) and (7.7.4),it is natural to ask whether the $\tilde{P}^z_{xw}$ and $\tilde{Q}^z_{xw}$ also have non-negative coefficients.

With these remarks as motivation, we now list four conjectural positivity properties [P1]–[P4] of the Hecke algebra of an arbitrary Coxeter system. We recall that

$$\mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] = \{ \sum_{n \in \mathbb{Z}} a_n q^{\frac{n}{2}} \mid a_n \in \mathbb{N} \ (n \in \mathbb{Z}) \}.$$

The four properties are

[P1] $C'_x \widetilde{T}_y \in \sum_{z \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \widetilde{T}_z \qquad (x, y \in W)$

[P2] $\widetilde{T}^{-1}_{x^{-1}} \widetilde{T}_y \in \sum_{z \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] C_z \qquad (x, y \in W)$

[P3] $C'_x C'_y \in \sum_{z \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] C'_z \qquad (x, y \in W)$

[P4] $C'_x C_y \in \sum_{z \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] C_z \qquad (x, y \in W)$

Conjectures [P1] and [P2] assert the positivity of the Laurent polynomials

$$\tilde{P}^y_{x,z} (x, y, z \in W)$$

and

$$\tilde{Q}^y_{x,z} \ (x, y, z \in W)$$

respectively. In terms of the function τ introduced in (7.1), [P1]–[P4] may be reformulated as follows, using (7.1.1) and (7.3.8):

[P1] $\tau(C'_x \widetilde{T}_y \widetilde{T}_z) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \ (x, y, z \in W)$

[P2] $\tau(\widetilde{T}^{-1}_{x^{-1}} \widetilde{T}_y D_z) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \ (x, y, z \in W)$

[P3] $\tau(C'_x C'_y D'_z) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \ (x, y, z \in W)$

[P4] $\tau(C'_x C_y D_z) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \ (x, y, z \in W)$.

Many other variants may be given; for instance, the last formulation of [P1] shows that [P1] is equivalent to the statement

$$\widetilde{T}_y \widetilde{T}_z \in \sum_{x \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] D'_x$$

where the right hand side is the set of infinite linear combination of the elements D'_x with non-negative coefficients (such linear combinations converge, in an appropriate sense, to elements of $\hat{\mathcal{H}}(W)$.)

The rest of this thesis is devoted to a study of these positivity properties and related facts. For finite Coxeter systems, only two of the properties [P1]–[P4] are independent of one another:

7.17 Proposition. If (W, R) is a finite Coxeter system, then [P1] is equivalent to [P2] and [P3] is equivalent to [P4].

Proof The equivalence of [P1] and [P2] follows from (7.9.1). Let w_0 be the longest element of W. By (7.4), for any $x, y, z \in W$,

$$\begin{aligned}\tau(C'_x C'_y D'_z) &= \tau(C'_x C'_y C_{zw_0} \widetilde{T}_{w_0}) \\ &= \tau(C'_y C_{zw_0} \widetilde{T}_{w_0} C'_x) \\ &= \tau(C'_y C_{zw_0} D_{w_0 x})\end{aligned}$$

which shows that [P3] and [P4] are equivalent. □

7.18 Remark Lascoux and Schützenberger have given a combinatorial proof that the Kazhdan-Lusztig polynomials have non-negative coefficients for certain pairs of elements in symmetric groups ([LS]); in the following chapters, [P1]–[P4] will be proved for universal Coxeter systems by elementary arguments. In other cases where positivity properties are known, the proofs depend on a study of the intersection cohomology of varieties constructed from algebraic groups; even using the "infinite-dimensional" groups associated with Kac-Moody Lie algebras, these methods only apply to crystallographic Coxeter systems.

For general Coxeter systems, only isolated results have been obtained. In the remainder of this chapter, we prove a number of such facts concerning the coefficient of q in polynomials $P_{v,w}$ and $Q_{v,w}$. □

7.19 For the remainder of this chapter, (W,R) denotes an arbitrary Coxeter system, and we adopt our usual notation concerning (W,R) and its Hecke algebra $\mathcal{H}(W)$. We begin with some properties of the polynomials $\tilde{R}_{y,w}$.

Recall the definition of the Bruhat graph $\Gamma_{(W,R)}$ of (W,R) (see (1.11)). An element $\tau = (x_0,\dots,x_n)$ of W^{n+1} will be called a path in $\Gamma_{(W,R)}$ if $(x_{i-1},x_i) \in E_{(W,R)}$ $(i = 1,\dots,n)$; τ is then said to be a path from x_0 to x_n, and we say that n is the length of τ.

The next result gives more evidence for close connections between the polynomials $\tilde{R}_{x,y}$ and the Bruhat graph.

7.20 Proposition. For $x,y \in W$, $m \in \mathbb{N}$ the coefficient of α^m in $\tilde{R}_{x,y}$ is non-zero iff there is a path of length m in $\Gamma_{(W,R)}$ from x to y.

<u>Proof</u> If the coefficient of α^m is non-zero, then (6.23) ensures the existence of a path of length m. The converse will be proved by induction on $\ell(y)$, being trivial for $\ell(y) = 0$.

Suppose $\ell(y) > 0$ and that there exists a path $(w_0, w_1, \dots, w_m)$ in $\Gamma_{(W,R)}$ with $w_0 = x$ and $w_m = y$. Choose $r \in R$ so that $yr < y$. If $w_j r \neq w_{j-1}$ for all $j = 1,\dots,m$, then by (1.20), $(w_0 r, w_1 r, \dots, w_m r)$ is a path in $\Gamma_{(W,R)}$ from xr to yr. By induction, the coefficient of α^m in $\tilde{R}_{xr,yr}$ is non-zero, and (0.1) shows that the coefficient of α^m in $\tilde{R}_{x,y}$ is non-zero.

The other possibility is that $w_j r = w_{j-1}$ for some $j \in \{1,\dots,m\}$; suppose without loss of generality that j is the largest element of $\{1,\dots,m\}$ with $w_j r = w_{j-1}$, and consider cases as follows:

Case 1. $xr < x$

In this case, $(xr, x = w_0, w_1, \dots, w_{j-1} = w_j r, w_{j+1} r, \dots, w_m r = yr)$ is a path in $\Gamma_{(W,R)}$ from xr to yr of length m, so the coefficient of α^m in $\tilde{R}_{xr,yr}$ is non-zero, hence so is that of α^m in $\tilde{R}_{x,y}$ (by (0.1)).

Case 2. $xr > x$

Here $(x = w_0, w_1, \dots, w_{j-1} = w_j r, w_{j+1} r, \dots, w_m r = yr)$ is a path of length $m-1$ from x to yr, so the coefficient of α^{m-1} in $\tilde{R}_{x,yr}$ is non-zero, and so is that of α^m in $\tilde{R}_{x,y}$ (by (0.1)). $\square$

7.21 Proposition. Let $x \in W$, $t \in T$ and suppose that $(x, xt) \in E_{(W,R)}$ (i.e. $xt > x$) and that $\ell(xt) - \ell(x) \geq 3$. Then there exist $t' \in T$ and $t_1, t_2, t_3 \in \langle t, t' \rangle \cap T$ such that $(x, xt_1, xt_1t_2, xt_1t_2t_3)$ is a path in $\Gamma_{(W,R)}$ from x to xt.

<u>Proof</u> Let $y = xt$ (note $\ell(y) \geq 1$). We prove the result by induction on $\ell(y)$. Choose $r \in R$ so that $yr < y$, and consider cases as follows:

Case 1. $xr > x$

Note that $t \neq r$; hence $(xr, yr)) \in E_{(W,R)}$ by (1.20). Take $t' = r$, $t_1 = t_3 = r$ and $t_2 = rtr$ to obtain the required path $(x, xt_1, xt_1t_2, xt_1t_2t_3) = (x, xr, yr, y)$.

Case 2. $xr < x$.

Here $yr = xr(rtr) > xr$ and $\ell(yr) - \ell(xr) > 3$ so there exist $t' \in T$ and $t'_1, t'_2, t'_3 \in \langle rtr, t' \rangle \cap T$ such that $(xr, xrt'_1, xrt'_1t'_2, xrt'_1t'_2t'_3)$ is a path in $\Gamma_{(W,R)}$ from xr to yr. Note that $t_3 \neq r$ (since $yrt_3 < yr$)

Case 2a. $r \notin \{t'_1, t'_2\}$.

By (1.20), $(x, xt_1, xt_1t_2, xt_1t_2t_3)$ (where $t_i = rt'_ir$, $i = 1, 2, 3$) is a path in $\Gamma_{(W,R)}$ from x to y, and we have $t_1, t_2, t_3 \in \langle t, rt'r \rangle$.

Case 2b. $r = t'_1$

Here, $(x, xt'_2, xt'_2t'_3, xt'_2t'_3r)$ is a path in $\Gamma_{(W,R)}$ from x to y, and $t'_2, t'_3, r \in \langle rtr, t' \rangle \cap T$, so $t'_2, t'_3, r \in \langle t, rt'r \rangle \cap T$.

Case 2c. $r = t'_2$

By (1.20),$(x, xt_1) \in E_{(W,R)}$ where $t_1 = rt_1r$. Note $t_1 = rt'_1t'_2$. Hence

$$(x, xt_1, xt_1t'_3, xt_1t'_3r)$$

is a path in $\Gamma_{(W,R)}$ from xr to yr. We have $t'_1, r, t'_3 \in \langle rtr, t' \rangle$, hence $t_1, t'_3, r \in \langle t, rt'r \rangle$. □

7.22 Corollary. If $x, y \in W$ and there is a path of length in $\Gamma_{(W,R)}$ from x to y $(m < \ell(y) - \ell(x))$ then there is a path of length $m+2$ from x to y in $\Gamma_{(W,R)}$. $\square$

In ([De 3], Prop 5.3), Deodhar proves the corresponding fact about the coefficients of $\tilde{R}_{x,y}$.

7.23 Let $x, y \in W$ with $x \leq y$. Then the coefficient of $\alpha^{\ell(y)-\ell(x)}$ in $\tilde{R}_{x,y}$ is 1, and (7.20) implies that $\tilde{R}_{x,y} = \alpha^{\ell(y)-\ell(x)}$ iff $(z, zt \in [x,y]$ and $t \in T$ imply $|\ell(zt) - \ell(z)| = 1)$. We now investigate the other extreme, when $\tilde{R}_{x,y}$ is as large as possible.

Define polynomials $\tilde{R}_n (n \in \mathbb{N})$ as follows:

$$\tilde{R}_0 = 1,\ \tilde{R}_1 = \alpha,\ \tilde{R}_2 = \alpha^2,\ \tilde{R}_{n+1} = \alpha\tilde{R}_n + \tilde{R}_{n-1}\ (n \geq 2).$$

From (0.1), it follows that for $x, y \in W$ with $x \leq y$, $\tilde{R}_{\ell(y)-\ell(x)} - \tilde{R}_{x,y}$ is a polynomial in α with non-negative coefficients.

7.24 Lemma. Let $x, y \in W$ with $x \leq y$ and $\ell(y) - \ell(x) = n \geq 2$. Then $\tilde{R}_{x,y} = \tilde{R}_n$ iff $W' = \langle v^{-1}w \mid v, w \in [x,y]\rangle$ is a dihedral reflection subgroup of (W, R).

<u>Proof</u> Recall that W' is always a reflection subgroup of (W,R); in fact, if $(x_0, \ldots, x_n)$ is a path in $\Gamma_{(W,R)}$ from x to y, then $W' = \langle x_i^{-1}x_{i-1} (i = 1, \ldots, n)\rangle$ (by (5.23)).

If $n = 2$, then $\tilde{R}_{x,y} = \alpha^2 = \tilde{R}_2$ and W' is dihedral (by the preceeding remark) so there is nothing to prove.

We now assume $\ell(y) - \ell(x) = n \geq 3$ and proceed by induction on $\ell(y)$. Choose $r \in R$ so that $yr < y$.

Case 1. $xr < x$

By (5.21.1), there is a path $(x'_0, x'_1, \ldots, x'_n)$ in $\Gamma_{(W,R)}$ from xr to yr, such that $\ell(x'_i r) > \ell(x'_i)$ $(i = 0, 1, \ldots, n)$. Then $(x'_0 r, \ldots, x'_n r)$ is a path from x to y, and $W' = \langle r x_i'^{-1} x'_{i-1} r\ (i = 1, \ldots, n)\rangle = rW''r$ where $W'' = \langle x_i'^{-1} x_{i-1}\ (i = 1, \ldots, n)\rangle = \langle v^{-1}w \mid v, w \in [xr, yr]\rangle$. Hence W' is dihedral iff W'' is dihedral.

Also, $\tilde{R}_{x,y} = \tilde{R}_{xr,yr}$ by (0.1), so $\tilde{R}_{x,y} = \tilde{R}_n$ iff $\tilde{R}_{xr,yr} = \tilde{R}_n$. The result follows by induction.

Case 2. $xr > x$

Suppose $\tilde{R}_{x,y} = \tilde{R}_n$. It follows by (0.1) and the definition of $\tilde{R}_n$ that $\tilde{R}_{x,yr} = \tilde{R}_{n-1}$ and $\tilde{R}_{xr,yr} = \tilde{R}_{n-2}$; in particular, $xr < yr$, since $\tilde{R}_{n-2} \neq 0$.

Choose a path $(w_0, \ldots, w_{n-2})$ in $\Gamma_{(W,R)}$ from xr to yr. Then

$$(x, xr = w_0, \ldots, w_{n-2})$$

is a path from x to yr, and so by induction, $W'' = \langle r, w_i^{-1} w_{i-1}\ (1 \leq i \leq n-2) \rangle$ is dihedral (since $\tilde{R}_{x,yr} = \tilde{R}_{n-1}$). But $(x, xr = w_0, \ldots, w_{n-2}, w_{n-2}r = y)$ is a path from x to y, so $W' = W''$ is also dihedral.

Conversely, suppose that W' is a dihedral reflection subgroup of (W, R). By (6.28), $\tilde{R}_{x,y} = \tilde{R}_{x',y'}$ for certain $x', y' \in W'$ with $\ell'(y') - \ell'(x') = n$ (where ℓ' is the length function on $(W', S(W'))$ and $\tilde{R}_{x',y'}$ is computed in $(W', S(W'))$). The result follows since $\tilde{R}_{x',y'} = \tilde{R}_n$ ($(W', S(W'))$ being dihedral). □

Recall that for any subset X of W, Γ_X denotes the full subgraph on the vertex set X of the Bruhat graph $\Gamma_{(W,R)}$. It will be convenient to say that a subgraph $\Gamma_{[v,w]}$ $(v, w \in W,\ v \leq w)$ is an interval of the Bruhat graph.

We now give some equivalent characterisations of intervals of "dihedral type" in the Bruhat order.

7.25 Proposition. Let $x, y \in W$ with $x \leq y$ and $\ell(y) - \ell(x) \geq 2$. Then conditions (i)–(vi) below are equiivalent:

(i) $[x, y]$ has 2 atoms

(ii) $[x, y]$ has 2 coatoms

(iii) $[x, y]$ is isomorphic to a Bruhat interval in a dihedral Coxeter system.

(iv) $\Gamma_{[x,y]}$ is isomorphic to an interval in the Bruhat graph of a dihedral Coxeter system.

(v) $\langle v^{-1}w \mid v, w \in [x, y]\rangle$ is a dihedral reflection subgroup.

(vi) $\tilde{R}_{x,y} = \tilde{R}_n$ where $n = \ell(y) - \ell(x)$.

Proof By (7.24), (v)$\Leftrightarrow$(vi). We show that (v)$\Longrightarrow$(iv) $\Longrightarrow$(iii) $\Longrightarrow$(i)$\Longrightarrow$(v). Assume $W' = \langle v^{-1}w \mid v, w \in [x, y]\rangle$ is dihedral, and let z be the minimum element of xW'. By (5.23), $[x, y] \subseteq zW'$, and (iv) follows by (1.13) (i) and (ii). Hence (v)$\Longrightarrow$(iv). Now (iv)$\Longrightarrow$(iii) by definition of Bruhat order, and the implication (iii)$\Longrightarrow$ (i) holds since an interval of length at least 2 in a dihedral group has two atoms.

Now suppose that $[x, y]$ has exactly two atoms x_1, x_2, and let W' be the maximal dihedral reflection subgroup containing $W'' = \langle x^{-1}x_1, x^{-1}x_2\rangle$ ((3.18)). Since any relfection subgroup of W containing W'' and contained in W' is itself dihedral, it will suffice to show that
(7.25.1) if (v, w) is an edge of $\Gamma_{[x,y]}$ and $\ell(w) = \ell(v) + 1$ then $v^{-1}w \in W'$.

This claim will be proved by induction on $n = \ell(v) - \ell(x)$. If $n = 0$, then $v = x$ and $w \in \{x_1, x_2\}$ by our assumption on $[x, y]$ so $v^{-1}w \in W'' \subseteq W'$.

Now assume that $n > 0$ and (7.25.1) holds for v, w with $\ell(v) - \ell(x) < n$. Take (v, w) to be an edge of $\Gamma_{[x,y]}$ with $\ell(w) = \ell(v) + 1$ and $\ell(v) - \ell(x) = n$. Choose $z \in [x, v]$ with $\ell(z) = \ell(v) - 1$. Then $[z, w]$ is an interval of length 2 in the Bruhat order and so has 2 atoms (by (5.20), for example) of which one is v; let v' be the other atom of $[z, w]$. Since $\ell(z) - \ell(x) < n$, the inductive assumption implies that the dihedral reflection subgroup $W''' = \langle z^{-1}v, z^{-1}v'\rangle$ is contained in W'. But by (5.23), the dihedral reflection subgroup $\langle v^{-1}w, z^{-1}v\rangle$ contains W'''; since W''' is contained in a unique maximal dihedral reflection subgroup, it follows that $\langle v^{-1}w, z^{-1}v\rangle \subseteq W'$. In particular, $v^{-1}w \in W'$ as required.

The proof of the implications (iii)$\Longrightarrow$(ii)$\Longrightarrow$(v) is entirely similar. □

The following result shows that the edges (v, w) of $\Gamma_{(W,R)}$ with $\ell(w) - \ell(v) = 3$ are determined by the Bruhat order alone.

7.26 Corollary. Let $v, w \in W$ with $\ell(w) = \ell(v) + 3$ and $v \leq w$. Then $(v, w) \in E_{(W,R)}$ iff the Bruhat interval $[v, w]$ has 2 coatoms.

Proof Suppose first that $(v, w) \in E_{(W,R)}$; then $w = vt$ for some $t \in T$. By (7.21), there is a dihedral reflection subgroup W' and reflections $t_1, t_2, t_3 \in W' \cap T$

such that $(v, vt_1, vt_1t_2, vt_1t_2t_3)$ is a path in $\Gamma_{(W,R)}$ from v to w. By (5.23), $\langle x^{-1}y \mid x, y \in [v,w]\rangle$ is a dihedral reflection subgroup, and so $[v,w]$ has 2 coatoms by (7.25).

Conversely, suppose $[v,w]$ has 2 coatoms. Then $\Gamma_{[v,w]}$ is isomorphic to an interval of length 3 in the Bruhat graph of a dihedral Coxeter system (W', R'); but $(x', y') \in E_{(W',R')}$ iff $\ell'(y') - \ell'(x')$ is an odd positive integer $(x', y' \in W')$ where ℓ' is the length function on (W', R'). Hence $(v, w) \in E_{(W,R)}$ as required. □

Proposition (7.25) also gives our first general positivity property.

7.27 Corollary. Let $x, y \in W$ with $x \leq y$ be such that the interval $[x, y]$ has 2 atoms (or 2 coatoms). Then $P_{x,y} = Q_{x,y} = 1$.

Proof Let $W' = \langle v^{-1}w \mid v, w \in [x,y]\rangle$ and $R' = S(W')$; then (W', R') is a dihedral Coxeter system. By (6.28), $P_{x,y} = P_{x',y'}$ for some $x', y' \in W'$ with $\ell'(x') \leq \ell'(y')$, where ℓ' is the length function on (W, R) and $P_{x',y'}$ is computed in (W', R'). Since the non-zero Kazhdan-Lusztig polynomials in a dihedral Coxeter system are all equal to 1, $P_{x,y} = P_{x',y'} = 1$. Similarly, $Q_{x,y} = 1$. □

We now need some information about coefficients of some powers of α in particular polynomials $\tilde{R}_{x,y}$.

For $x, y \in W$ let $a[x,y] = \#\{\, w \in [x,y] \mid \ell(w) = \ell(x) + 1 \,\}$ and $c[x,y] = \#\{\, w \in [x,y] \mid \ell(w) = \ell(y) - 1 \,\}$ be the numbers of atoms and coatoms of the interval $[x,y]$. Also, let $r_1[x,y]$ (respectively $p_1[x,y]$, $q_1[x,y]$) denote the coefficient of $\alpha^{\ell(y)-\ell(x)-2}$ in $\tilde{R}_{x,y}$ (respectively, of q in $P_{x,y}$, of q in $Q_{x,y}$).

7.28 Lemma. (i) If $x, y \in W$ and $x \leq y$, $\ell(y) - \ell x) \geq 2$ then
(7.28.1) $p_1[x,y] = c[x,y] + r_1[x,y] - \ell(y) + \ell(x)$
(7.28.2) $q_1[x,y] = a[x,y] + r_1[x,y] - \ell(y) + \ell(x)$

(ii) For any $y \in W$, $Q_{1,y} = 1$ and
(7.28.3) $p_1[1,y] = c[1,y] - a[1,y]$

Proof (i) Recall that if $x < z$ $\tilde{R}_{x,z}$ is a polynomial in $\alpha (= q^{-\frac{1}{2}} - q^{\frac{1}{2}})$ of degree $\ell(z) - \ell(x)$, and that for $z < y$, $\bar{\tilde{P}}_{z,y} \in q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$; by [KL1], the coefficient of

$q_y^{\frac{1}{2}} q_z^{-\frac{1}{2}}$ in $\tilde{\bar{P}}_{z,y}$) is $1(z \leq y)$. Now $p_1[x,y]$ is the coefficient of $q_y^{-\frac{1}{2}} q_x^{\frac{1}{2}} q$ in

$$\tilde{P}_{x,y} - \tilde{\bar{P}}_{x,y} = \sum_{z:x<z\leq y} \tilde{R}_{x,z} \tilde{\bar{P}}_{z,y}.$$

Let $n = \ell(y) - \ell(x)$. Then $p_1(x,y)$ is equal to the coefficient of

$$q^{\frac{-(n-2)}{2}} \text{ in } \tilde{R}_{x,y} + \sum_{\substack{z:x<z\leq y \\ \ell(z)=\ell(y)-1}} q^{\frac{1}{2}} \tilde{R}_{x,z}$$

and therefore to that of $q^{\frac{-(n-2)}{2}}$ in

$$[(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^n + r_1[x,y]\ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{n-2}] + \sum_{\substack{z:x<z\leq y \\ \ell(z)=\ell(y)-1}} q^{\frac{1}{2}} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{n-1}$$

i.e. $p_1[x,y] = r_1[x,y] + c[x,y] - n$, proving (7.28.1). The proof of (7.28.2) is similar.

(ii) By [KL1], for $w \in W$ and $r \in R$,

$$C_r C_w = \begin{cases} -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C_w & (rw < w) \\ C_{rw} + \sum\limits_{\substack{z<w \\ rz<z}} \mu(z,w) C_z & (rw > w). \end{cases}$$

In particular, $C_r C_w \in \sum\limits_{z\neq 1} \mathcal{A} C_z$. Since $\widetilde{T}_r = C_r + q^{\frac{1}{2}}$, it follows by induction on $\ell(y)$ that

$$\widetilde{T}_y \in q_y^{\frac{1}{2}} C_1 + \sum_{z\neq 1} \mathcal{A} C_z.$$

But $\widetilde{T}_y = \sum\limits_x \tilde{\bar{Q}}_{x,y} C_x$, hence $\tilde{Q}_{1,y} = q_y^{-\frac{1}{2}}$ and $Q_{1,y} = 1$.

Hence $p_1[1,y] = p_1[1,y] - q_1[1,y] = c[1,y] - a[1,y]$ by (i). □

7.29 Lemma. Let $x, y \in W$ with $x \leq y$. Then

(i) The coefficient of α in $\tilde{R}_{x,y}$ is

$$\begin{cases} 1 & (x^{-1}y \in T) \\ 0 & (x^{-1}y \notin T) \end{cases}$$

(ii) The coefficient of α^2 in $\tilde{R}_{x,y}$ is $\frac{1}{2}\#\{\, z \mid (x,z) \in E_{(W,R)}, (z,y) \in E_{(W,R)} \,\}$.

Proof Now (i) follows from (6.23).

To prove (ii), note that the coefficient of α^2 in $\tilde{R}_{x,y}$ can be non-zero only if $\ell(y) - \ell(x)$ is even and there can only exist a path (x, z, y) in $\Gamma_{(W,R)}$ if $\ell(y) - \ell(x)$ is even. Hence we may suppose that $\ell(y) - \ell(x)$ is even. We have, for $x \neq y$, $\sum\limits_{z} \tilde{R}_{x,z}\bar{\tilde{R}}_{z,y} = 0$; by (7.7.1), this gives

$$2\tilde{R}_{x,y} = -\sum_{z: x<z<y} \varepsilon_z \varepsilon_y \tilde{R}_{x,z}\tilde{R}_{z,y}$$

Since $\tilde{R}_{x,z}$,$\tilde{R}_{z,y} \in \alpha\mathbb{Z}[\alpha]$ for $z \neq x, y$, (ii) follows from (i). $\square$

It is known ([KL1]) that for $x, y \in W$ with $\ell(y) - \ell(x) \leq 2$ and $x \leq y$, $P_{x,y} = 1$; in fact, for $x \leq y$, the constant term of $P_{x,y}$ is always 1. The following settles the positivity of $P_{x,y}$ in the next simplest cases, when $\ell(y) - \ell(x) = 3$ or 4.

7.30 Proposition. (i) For $x \leq y$ with $\ell(y) - \ell(x) = 3$,

$$P_{x,y} = 1 + \max(0, c[x,y] - 3)q = Q_{x,y}$$

(ii) For $x \leq y$ with $\ell(y) - \ell(x) = 4$, $P_{x,y}$ and $Q_{x,y}$ have non-negative coefficients.

Proof We will only prove (i) and (ii) for $P_{x,y}$; the proofs for $Q_{x,y}$ are essentially the same arguments, applied to the reverse poset.

(i) We $P_{x,y} = 1 + p_1(x,y)q$ where

$$p_1[x,y] = c[x,y] - 3 + r_1[x,y],$$

and

$$r_1[x,y] = \begin{cases} 1 & (x^{-1}y \in T) \\ 0 & (x^{-1}y \notin T) \end{cases}$$

by (7.29) (i). Now $c[x,y] \geq 2$ and $x^{-1}y \in T$ iff $c(x,y) = 2$, by (7.26). Hence the result.

(ii) Here, $P_{x,y} = 1 + p_1(x,y)q$ where by (7.29.2), (7.28.1) $p_1[x,y] = c[x,y] - 4 + r_1[x,y]$ and $r_1[x,y] = \frac{1}{2}\#\{\, z \mid x < z < y,\ x^{-1}z \in T,\ z^{-1}y \in T\,\}$. Now $c[x,y] \geq 2$, and if $c[x,y] = 2$, then $P_{x,y} = 1$ by (7.27). If $c[x,y] \geq 4$ then $p_1[x,y] \geq 0$, so we need only consider the case $c[x,y] = 3$.

Let a_1, a_2, a_3 be the 3 coatoms of the interval $[x,y]$, and let $x_1, \ldots, x_n$ be the atoms of the interval $[x,y]$ ($n \geq 2$). Note that $n \geq 3$ (for if $n = 2$, then $c[x,y] = 2$ by (7.25)). Also, write $\{z \in [x,y] \mid \ell(z) = \ell(x) + 2\,\} = \{z_1, \ldots, z_m,\}$ (where $z_1, \ldots, z_m$ are distinct).

Now for any $v, w \in [x,y]$, we have $\mu_W(v,w) = (-1)^{\ell(w)-\ell(v)}$ where μ_W denotes the Möbius function on W (with Bruhat order) (e.g. see [De1]). It follows that
(7.30.1) $\sum_{j=\ell(x)}^{\ell(y)} (-1)^j h_j(x,y) = 0$
where $h_j(x,y) = \#\{\, z \in [x,y] \mid \ell(z) = j\,\}$ (e.g. [Ai]. Prop. 4.10; another way of seeing this is to compute the Euler characteristic of the sphere of dimension $\ell(y) - \ell(x) - 2$ by using the decomposition of the simplicial complex $\sum(x,y)$ into cells $\sum(x,z]$ ($z \in (x,y)$)).

In any case, (7.30.1) gives $1 - n + m - 3 + 1 = 0$, hence $m = n + 1$. Similarly, in each interval $[x, a_i]$, the number of atoms is equal to the number of coatoms ($i = 1,2,3$).

We show that we cannot have $x_i < a_j$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1,2,3\}$. For if $a_j > x_i$ ($i = 1, \ldots, n$), then the interval $[x, a_j]$ has n atoms, hence n coatoms, and so there is exactly one z_{k_j} ($1 \leq k_j \leq m$) with $z_{k_j} \notin [x, a_j]$. If $a_j > x_i$ for all $j \in \{1,2,3\}$ and $i \in \{1, \ldots, n\}$, choose $k \in \{1, \ldots, m\}$ such that $k \neq k_1, k_2, k_3$ (this being possible since $m = n + 1 \geq 4$); then $z_k \in [x, a_j]$ ($j = 1,2,3$), so $a_j \in [z_k, y]$ ($y = 1,2,3$). This states that the interval $[z_k, y]$, of length 2, has three coatoms a_1, a_2, a_3, which is absurd.

Hence there exist i, j with $x_i \not< a_j$ ($i \in \{1, \ldots, n\},\ j \in \{1,2,3\}$). This means that the interval $[x_i, y]$ has two coatoms.

Since $\ell(y) - \ell(x_i) = 3$, it follows by (7.26) that $x_i^{-1}y \in T$. But $x^{-1}x_i \in T$ since $x < x_i$ and $\ell(x_i) = \ell(x) + 1$. Hence $x_i \in \{\, z \mid x < z < y, x^{-1}z \in T,\ z^{-1}y \in T\,\}$

and since $r_1[x,y]$ is an integer, $r_1[x,y] \geq 1$ (in fact, $r_1[x,y] = 1$ by (7.25) but we don't need this). Hence $p_1[x,y] = c[x,y] - 4 + r_1[x,y] = r_1[x,y] - 1 \geq 0$ as required. □

7.31 Remark. It is interesting to note that, if $x \leq y$ and $\ell(y) - \ell(x) \leq 4$, then $\tilde{R}_{x,y}, P_{x,y}, Q_{x,y}$ depend only on the (isomorphism type of the) poset $[x,y]$, □

We finish this chapter with one more positivity property; this one follows immediately from (7.28.3) and (5.5).

7.32 Proposition. For any $w \in W$, the coefficient of q in $P_{1,w}$ is non-negative. □

Chapter 8

UNIVERSAL COXETER SYSTEMS: [P3], [P4] AND [P1]

In the following chapters, it will be shown that the Hecke algebra of a universal Coxeter system satisfies all four positivity properties [P1]–[P4] defined in (7.16). Recall that these conjectures state that certain Laurent polynomials in $q^{\frac{1}{2}}$, arising as the structure constants of the Hecke algebra with respect to various combinations of bases, have non-negative coefficients. Our general technique for proving non-negativity of the coefficients is to explicitly construct sets whose cardinalities are the coefficients in question. The proofs thus involve only elementary combinatorics, and provide explicit information about the coefficients.

In this chapter, we give proofs of [P3], [P4] and [P1]. The proof of [P2] will be given in Chapter 9. We begin with some facts needed in the proofs of all properties [P1]–[P4].

8.1 In all this chapter, (W, R) denotes a fixed universal Coxeter system. Thus, for any $r, s \in R$ with $r \neq s$, rs has infinite order, and W is therefore isomorphic (as a group) to a free product of cyclic groups of order two.

If $w \in W$, then w has a unique reduced expression $w = r_1 \ldots r_n$ $(n = \ell(w))$. We set
(8.1.1) $\mathcal{L}(w) = \{\, r \in R \mid \ell(rw) < \ell(w) \,\},\ \mathcal{R}(w) = \mathcal{L}(w^{-1})$ $(w \in W)$. Note that either
(8.1.2) $(w = 1,\ \mathcal{L}(w) = \mathcal{R}(w) = \emptyset)$ or $(w \neq 1,\ \#\mathcal{L}(w) = \#\mathcal{R}(w) = 1)$.

We adopt the notation concerning the Hecke algebra $\mathcal{H}(W)$ of (W,R) from Chapter 0. For $x, y \in W$, we let $\mu(x, y)$ denote the coefficient of $q^{(\ell(y)-\ell(x)-1)/2}$ in the Kazhdan-Lusztig polynomial $P_{x,y}$; thus $\mu(x, y)$ is an integer which is zero unless $x \leq y$ and $\ell(y) - \ell(x)$ is odd.

It is convenient to let $\ll$ denote the relation on W defined by the condition (8.1.3) $y \ll w$ iff $y \leq w$ and $\ell(y) = \ell(w) - 1$.

The next simple lemma ([Dy]) is fundamental for our proofs of [P1]–[P4] for universal Coxeter systems.

8.2 Lemma.

(i) Suppose $x, y \in W$, $\mu(x, y) \neq 0$ and $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$. Then $x = sy$ where $\mathcal{L}(y) = \{s\}$. In particular, $x \ll y$.

(ii) Suppose $y \in W \setminus \{1\}$; let $\mathcal{L}(y) = \{s\}$ $(s \in R)$. Then $\mu(sy, y) = 1$.

Proof

(i) Since $\mu(x, y) \neq 0$, we have $x < y$ and so $y \neq 1$. Let $\mathcal{L}(y) = \{s\}$ $(s \in R)$. Now $\mathcal{L}(x) \neq \emptyset$, so $\mathcal{L}(x) = \{r\}$ for some $r \in R$. Since $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$, we must have $r \neq s$. Hence $sy < y$, $sx > x$ and $\mu(x, y) \neq 0$. By ([KL1], (2.3.e)), $x = sy$. The claim (ii) also follows from ([KL1], (2.3.e)). $\square$

In (8.4), we will give an explicit formula for the products $C'_v C'_w$ $(v, w \in W)$ in the Hecke algebra which will make it obvious that the positivity property [P3] $C'_v C'_w \in \sum_{y \in W} \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] C'_y$ $(v, w \in W)$ holds. For the statement of (8.4), the following notation will be required.

8.3 Definition. If $w \in W$ has reduced expression $w = r_1 \ldots r_n$ and $2 \leq j \leq n-1$, let

$$w_{(j)} = r_1 r_2 \ldots r_{j-1} r_{j+2} \ldots r_n$$

and note that this is a reduced expression for $w_{(j)}$ if $r_{j-1} = r_{j+1}$. Define elements $C'(w, i)$ $(w \in W,\ 0 \leq i \leq \ell(w))$ recursively as follows:

$$C'(w,i) = \begin{cases} C'_{w(i)} + C'(w_{(i)}, i-1) & (2 \leq i \leq n-1 \text{ and } r_{i-1} = r_{i+1}) \\ 0 & (\text{otherwise}) \end{cases}$$

$\square$

8.4 Theorem. For any $v, w \in W$,

(8.4.1)
$$C'_v C'_w = \begin{cases} C'_{vw} + C'(vw, n) + C'(vw, n+1) & (\mathcal{L}(w) \cap \mathcal{R}(v) = \emptyset) \\ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})[C'_{vrw} + C'(vrw, n)] & (\mathcal{L}(w) \cap \mathcal{R}(v) = \{r\},\ r \in R) \end{cases}$$

where $n = \ell(v)$. □

This is a restatement of ([Dy], Theorem (3.12)). The proof will occupy (8.5)–(8.7). The formula (8.4.1) holds if $\ell(v) = 0$ or $\ell(w) = 0$; the next lemma deals with the case $\ell(w) = 1$ or $\ell(v) = 1$.

8.5 Lemma. Let $r \in R$, and $w \in W$ have reduced expression $w = r_1 \dots r_n$. Then

$$C'_r C'_w = \begin{cases} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C'_w & (n > 0 \text{ and } r = r_1) \\ C'_{rr_1 \dots r_n} + C'_{r_2} \dots r_n & (n \geq 2 \text{ and } r_2 = r) \\ C'_{rr_1 \dots r_n} & (\text{otherwise}) \end{cases}$$

There is a similar formula for $C'_w C'_r$.

Proof The result follows from the formula ([L3], (5.1.15))

$$C'_r C'_w = \begin{cases} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C'_w & (rw < w) \\ C'_{rw} + \sum\limits_{y: ry < y} \mu(y, w) C'_y & (rw > w) \end{cases}$$

and (8.2). □

8.6 In this section, we prove (8.4.1) when $\mathcal{L}(w) \cap \mathcal{R}(v) = \emptyset$. The proof is by induction on $\ell(v)$. By (8.5), we may assume $\ell(w) \geq 2,\ \ell(v) \geq 2$ and write $v = v'rs$ ($v' \in W,\ r, s \in R,\ \ell(v) = \ell(v') + 2$), $w = r's'w'$ ($w' \in W,\ r', s' \in R$ and $\ell(w) = \ell(w') + 2$); inductively, assume (8.4.1) holds with v replaced by $v'r$ or v'. There are four cases to consider.

Case 1. $s \notin \mathcal{R}(v'),\ s \neq s'$.

Now $C'(vw, n-1) = 0$ since $s \notin \mathcal{R}(v')$ and $C'(vw, n+1) = 0$ since $s \neq s'$. Hence

$$\begin{aligned} C'_v C'_w &= C'_{v'r} C'_s C'_w \qquad \text{(by (8.5), since } s \notin \mathcal{R}(v')) \\ &= C'_{v'r} C'_{sw} \qquad \text{(by (8.5), since } s \neq s') \\ &= C'_{vw} + C'(vw, n-1) + C'(vw, n) \\ &= C'_{vw} + C'(vw, n) + C'(vw, n+1) \end{aligned}$$

Case 2. $s \notin \mathcal{R}(v'),\ s = s'$.

Here, $C'(vw, n-1) = 0$ since $s \notin \mathcal{R}(v'),\ C'(vw', n-1) = 0$ since $s \notin \mathcal{R}(v')$, and $C'(vw, n+1) = C'_{vw'} + C'(vw', n)$ since $s = s'$.

Therefore

$$\begin{aligned} C'_v C'_w &= C'_{v'r} C'_s C'_w \qquad \text{(by (8.5))} \\ &= C'_{v'r}(C'_{sw} + C'_{sw'}) \qquad \text{(by (8.5))} \\ &= C'_{vw} + C'(vw, n-1) + C'(vw, n) + C'_{vw'} \\ &\quad + C'(vw', n-1) + C'(vw', n) \\ &= C'_{vw} + C'(vw, n) + C'(vw, n+1). \end{aligned}$$

Case 3. $s \in \mathcal{R}(v'),\ s \neq s'$.

In this case, $C'(vw, n-1) = C'_{v'w} + C'(v'w, n-2)$ since $s \in \mathcal{R}(v'),\ C'(v'w, n-1) = 0$ since $s' \notin \mathcal{R}(v') = \{s\}$ and $C'(vw, n+1) = 0$ since $s = s'$. So

$$\begin{aligned} C'_v C'_w &= (C'_{v'r} C'_s - C'_{v'}) C'_w \qquad \text{(by (8.5))} \\ &= C'_{v'r} C'_{sw} - C'_{v'} C'_w \qquad \text{(by (8.5))} \\ &= C'_{vw} + C'(vw, n-1) + C'(vw, n) \\ &\quad - [C'_{v'w} + C'(v'w, n-2) + C'(v'w, n-1)] \\ &= C'_{vw} + C'_{v'w} + C'(v'w, n-2) + C'(vw, n) - [C'_{v'w} + C'(v'w, n-2)] \\ &= C'_{vw} + C'(vw, n) + C'(vw, n+1) \end{aligned}$$

Case 4. $s \in \mathcal{R}(v')\ s = s'$

Now $C'(vw, n+1) = C'_{vw'} + C'(vw', n) \qquad$ since $s = s'$,

$$C'(v'w,n-1) = C_{v'w'} + C'(v'w',n-2) \qquad \text{since } s' \in \mathcal{R}(v')$$
$$C'(vw',n-1) = C'_{v'w'} + C'(v'w',n-2) \qquad \text{since } s \in \mathcal{R}(v') \text{ and}$$
$$C'(vw,n-1) = C'_{v'w} + C'(v'w,n-2) \qquad \text{since } s \in \mathcal{R}(v').$$

$$\begin{aligned}
\text{So } C'_vC'_w &= [C'_{v'r}C'_s - C'_{v'}]C'_w \qquad \text{(by (8.5))}\\
&= C'_{v'r}[C'_{sw} + C'_{sw'}] - C'_{v'}C'_w \qquad \text{(by (8.5))}\\
&= C'_{vw} + C'(vw,n-1) + C'(vw,n) + C'_{vw'} + C'(vw',n-1)\\
&\quad + C'(vw',n) - [C'_{v'w} + C'(v'w,n-2) + C'(v'w,n-1)]\\
&= C'_{vw} + C'_{v'w} + C'(v'w,n-2) + C'(vw,n) + C'_{vw'}\\
&\quad + C'_{v'w'} + C'(v'w',n-2) + C'(vw',n) - [C'_{v'w} + C'(v'w,n-2)\\
&\quad + C'_{v'w'} + C'(v'w',n-2)]\\
&= C'_{vw} + C'(vw,n) + C'_{vw'} + C'(vw',n)\\
&= C'_{vw} + C'(vw,n) + C'(vw,n+1)
\end{aligned}$$

Hence (8.4.1) holds if $\mathcal{L}(w) \cap \mathcal{R}(v) = \emptyset$ □

8.7 This section proves (8.4.1) when $\mathcal{L}(w) \cap \mathcal{R}(v) = \{r\}$ $(r \in R)$. If $\ell(v) = 1$, (8.5) gives the result. Hence assume $\ell(v) \geq 2$ and write $v = v'sr$ where $v' \in W$, $s \in R$ and $\ell(v) = \ell(v') + 2$. Write $w = rw'$ $(w' \in W)$. Assume inductively that the result holds with v' replacing v. This time there are two cases.

Case 1. $r \notin \mathcal{R}(v')$.

Here $C'(v'sw,n-1) = 0$ since $r \in \mathcal{L}(w)$ and $r \notin \mathcal{L}(v')$, so

$$\begin{aligned}
C'_vC'_w &= C'_{v's}C'_rC'_w \qquad \text{(by (8.5))}\\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C'_{v's}C'_w \qquad \text{(by (8.5))}\\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})[C'_{v'sw} + C'(v'sw,n-1) + C'(v'sw,n)] \quad \text{(by (8.6))}\\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})[C'_{vrw} + C'(vrw,n)]
\end{aligned}$$

Case 2 $r \in \mathcal{R}(v')$

In this case, $C'(v'sw, n-1) = C'_{v'w'} + C'(v'w', n-2)$. Therefore,

$$\begin{aligned}
C'_v C'_w &= [C'_{v's} C'_r - C'_{v'}] C'_w \qquad \text{(by (8.5))} \\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_{v's} C'_w - C'_{v'} C'_w \\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) [C'_{v'sw} + C'(v'sw, n-1) + C'(v'sw, n)] \\
&\quad - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) [C'_{v'rw} + C'(v'rw, n-2)] \quad \text{(by (8.6) and induction)} \\
&= (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) [C'_{vrw} + C'(vrw, n)].
\end{aligned}$$

This completes the proof of Theorem (8.4) $\square$

We turn now to the investigation of the coefficients arising when the products $C'_v C_w$ are exposed as $\mathcal{A}$-linear combinations of elements $\{C_x\}_{x \in W}$. In (8.9), we give a formula for these coefficients involving certain sequences of elements of $W \times \mathbb{N}$, defined in the next section.

8.8 Definition.

(i) For $e = ((w_0, i_0), \ldots, (w_n, i_n)) \in (W \times \mathbb{N})^{n+1}$ $(n \in \mathbb{N})$ define $\nu(e) = i_n \in \mathbb{N}$ and $\rho(e) = w_n \in W$.

(ii) Let $y, w \in W$ and $w = r_n \ldots r_1$ $(n = \ell(w))$ be the reduced expression for w. Let $\mathcal{B}(w, y) \subseteq (W \times \mathbb{N})^{n+1}$ be the set of those $e = ((w_0, i_0), \ldots, (w_n, i_n)) \in (W \times \mathbb{N})^{n+1}$ satisfying (8.8.1), $(8.8.2)_k$ $(k = 1, \ldots, n)$, $(8.8.3)_k$ $(k = 1, \ldots, n)$ and $(8.8.4)_k$ $(k = 3, \ldots, n)$ below:

(8.8.1) $w_0 = y,\ i_0 = 0$

$(8.8.2)_k$
$$w_k w_{k-1}^{-1} \in \begin{cases} \{1\} & (r_k \notin \mathcal{L}(w_k)) \\ R & (r_k \in \mathcal{L}(w_k)) \end{cases}$$

$(8.8.3)_k$
$$|i_k - i_{k-1}| = \begin{cases} 1 & (r_k \notin \mathcal{L}(w_k)) \\ 0 & (r_k \in \mathcal{L}(w_k)) \end{cases}$$

$(8.8.4)_k$ If $r_k = r_{k-2}$ and $(x_{k-1}, i_{k-1}) = (x_{k-3}, i_{k-3})$, then either

(a) $\ell(x_{k-2}) < \ell(x_{k-1})$ or

(b) $x_{k-2} = x_{k-1}$ and $i_{k-2} < i_{k-1}$. $\square$

We may now state

8.9 Theorem. For any $y, w \in W$,
(8.9.1)

$$C'_w C_y = \sum_{e \in \mathcal{B}(w,y)} \tau_{\nu(e)} C_{\rho(e)}$$

where for $i \in \mathbb{N}$, $\tau_i = q^{\frac{i}{2}} + q^{\frac{i-2}{2}} + \ldots + q^{-\frac{i-2}{2}} + q^{-\frac{i}{2}}$ and the other notation is as in (8.8). □

Sections (8.10)–(8.14) contain a proof of (8.9).

8.10 Lemma. For $s \in R$ and $x \in W$,

$$C_r C_x = \begin{cases} -(q^{\frac{1}{2}} + q^{-1/2}) C_x & (r \in \mathcal{L}(x)) \\ \sum\limits_{\substack{x \in R \\ r \in \mathcal{L}(sx)}} C_{sx} & (r \notin \mathcal{L}(x)) \end{cases}$$

<u>Proof</u> This follows from (8.2) and the formula ([L3],(5.1.12))

$$C_r C_x = \begin{cases} -(q^{\frac{1}{2}} + q^{-1/2}) C_x & (r \in \mathcal{L}(x)) \\ C_{rx} + \sum\limits_{y : sy < y} \mu(y, x) C_x & (r \notin \mathcal{L}(x)) \end{cases}$$

□

8.11 In order to facilitate comparison between the various sets $\mathcal{B}(w, y)$ we now define sets $\mathcal{B}'(w, y) \subseteq (W \times \mathbb{N})^{n+1}$ $(n = \ell(w))$ as follows:

For y, w as in (8.8) (ii), let $\mathcal{B}'(w, y) \subseteq (W \times \mathbb{N})^{n+1}$ be the set of those $e = ((w_0, i_0), \ldots, (w_n, i_n)) \in (W \times \mathbb{N})^{n+1}$ satisfying (8.8.1), $(8.8.2)_k$ $(k = 1, \ldots, n)$, $(8.8.3)_k$ $(k = 1, \ldots, n)$ and $(8.8.4)_k$ $(k = 3, \ldots, n-1)$.

Note that the definition of $\mathcal{B}'(w, y)$ differs from that of $\mathcal{B}(w, y)$ only in the range of values which k may take in the condition $(8.8.4)_k$.

The proof of (8.9) will be by induction on $\ell(w)$; note that (8.9) holds trivially for $w = 1$. Suppose that $w \in W$, $\ell(w) \geq 1$ and assume inductively that

$$C'_v C_y = \sum_{e \in \mathcal{B}(v,y)} \tau_{\nu(e)} C_{\rho(e)}$$

for all $v \in W$ with $\ell(v) < \ell(w)$.

Let $\mathcal{L}(w) = \{r\}$ $(r \in R)$ and set $x = rw$. Write $w = r_n \dots r_1$ $(n = \ell(w))$.

8.12 Lemma. With the above notation,

$$C'_r C'_x C'_y = \sum_{e \in \mathcal{B}'(w,y)} \tau_{\nu(e)} C_{\rho(e)}.$$

<u>Proof</u> Note that for any $m \in \mathbb{N}$

$$\tag{8.12.1} \tau_1 \tau_m = \begin{cases} \tau_{m-1} + \tau_{m+1} & (m \geq 1) \\ \tau_1 & (m = 0), \end{cases}$$

that $C'_r = C_r + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ and so $C'_r C_z = 0$ if $r \in \mathcal{L}(z)$ (by 8.10). By induction, we have

$$\begin{aligned}
C'_r C_x C_y = & \sum_{\substack{e' \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}(\rho(e'))}} \tau_{\nu(e')} C_r C_{\rho(e')} + \sum_{\substack{e \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}(\rho(e'))}} \tau_1 \tau_{\nu(e')} C_{\rho(e')} \\
= & \sum_{\substack{e' \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}(\rho(e'))}} \tau_{\nu(e')} \sum_{\substack{s \in R \\ r \in \mathcal{L}(s\rho(e'))}} C_{s\rho(e')} \\
& + \sum_{\substack{e' \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}'(\rho(e')) \\ \nu(e') > 0}} [\tau_{\nu(e')+1} + \tau_{\nu(e')-1}] C_{\rho(e')} + \sum_{\substack{e' \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}(\rho(e')) \\ \nu(e') = 0}} \tau_1 C_{\rho(e')} \\
= & \sum_{e' \in \mathcal{B}(x,y)} \sum_{\substack{z \in W \\ z\rho(e')^{-1} \in R \\ r \in \mathcal{L}(z)}} \tau_{\nu(((z,\nu(e'),e'))} C_{\rho(((z,\nu(e')),e'))} \\
& + \sum_{\substack{e' \in \mathcal{B}(x,y) \\ r \notin \mathcal{L}(\rho(e'))}} \sum_{\substack{i \in \mathbb{N} \\ |i - \nu(e')| = 1}} \tau_{\nu(((\rho(e')i),e'))} C_{\rho((\rho(e'),i),e')} \\
= & \sum_{e \in \mathcal{B}'(w,y)} \tau_{\nu(e)} C_{\rho(e)}.
\end{aligned}$$

since every element of $\mathcal{B}'(w,y)$ is uniquely expressible in one of the forms

$$((z,\nu(e')),e') \qquad (e'\in\mathcal{B}(x,y), z\in W, z\rho(e')^{-1}\in R,\ r\in\mathcal{L}(z))$$
$$((\rho(e'),i),e') \qquad (e'\in\mathcal{B}(x,y),\ r\notin\mathcal{L}(\rho(e')), i\in\mathbb{N},\ |i-\nu(e')|=1)$$

and each such element of $(W\times\mathbb{N})^{\ell(w)+1}$ is in $\mathcal{B}'(w,y)$. (When convenient, we identify $(W\times\mathbb{N})^{n+1}$ and $(W\times\mathbb{N})\times(W\times\mathbb{N})^n$). □

Now if $n\le 2$ or ($n\ge 3$ and $r_n\ne r_{n-2}$ ($n=\ell(w)$)), then $C'_rC'_x=C'_{rx}$ and $\mathcal{B}'(w,y)=\mathcal{B}(w,y)$ so (8.9.1) holds. Hence we may assume for the remainder of the proof that $n\ge 3$ and $r_n=r_{n-2}$. Let $z=r_{n-2}\ldots r_1$.

8.13 Lemma. With the above notation

$$\sum_{e\in\mathcal{B}'(w,y)\setminus\mathcal{B}(w,y)}\tau_{\nu(e)}C_{\rho(e)}=\sum_{e'\in\mathcal{B}(z,y)}\tau_{\nu(e')}C_{\rho(e')}$$

Proof It will suffice to show that the map $\theta'\colon(W\times\mathbb{N})^{\ell(w)+1}\longrightarrow(W\times\mathbb{N})^{\ell(w)-1}$ defined by $((w_0,i_o),\ldots,(w_n,i_n))\longmapsto((w_o,i_o),\ldots,(w_{n-3}i_{n-3}),\ (w_n,i_n))$ restricts to a bijection $\theta\colon\mathcal{B}'(w,y)\setminus\mathcal{B}(w,y)\longrightarrow\mathcal{B}(z,y)$ such that $\nu(\theta(e))=\nu(e)$ and $\rho(\theta(e))=\rho(e)$ ($e\in\mathcal{B}'(w,y)\setminus\mathcal{B}(w,y)$).

But for $e\in(W\times\mathbb{N})^{\ell(w)+1}$, $\nu(\theta'(e))=\nu(e)$ and $\rho(\theta'(e))=\rho(e)$ so we need only check that θ' restricts to a bijection as claimed.

Before checking this claim, it is convenient to note the following:
(8.13.1) if $((w_0,i_0),\ldots,(w_{n-1},i_{n-1}),(w_n,i_n))\in\mathcal{B}(w,y)$ and $(w'_n,i'_n)\in W\times\mathbb{N}$ then $((w_0,i_o),\ldots,(w_{n-1},i_{n-1}),(w'_n,i'_n))\in\mathcal{B}(w,y)$ iff

$$w'_nw_{n-1}^{-1}\in\begin{cases}\{1\} & (r_n\notin\mathcal{L}(w'_n))\\ R & (r_n\in\mathcal{L}(w'_n))\end{cases}$$

and

$$|i'_n-i_{n-1}|=\begin{cases}1 & (r_n\notin\mathcal{L}(w'_n))\\ 0 & (r_n\in\mathcal{L}(w'_n)).\end{cases}$$

Let $e=((w_0,i_0),\ldots,(w_n,i_n))\in\mathcal{B}'(w,y)\setminus\mathcal{B}(w,y)$. Then $((w_0,i_0),\ldots,(w_{n-2},i_{n-2}))\in\mathcal{B}(z,y)$ and $(w_{n-1},i_{n-1})=(w_{n-3},i_{n-3})$, so

$$w_nw_{n-3}^{-1}=w_nw_{n-1}^{-1}\in\begin{cases}\{1\} & (r_{n-2}=r_n\notin\mathcal{L}(w_n))\\ R & (r_{n-2}=r_n\in\mathcal{L}(w_n))\end{cases}$$

and

$$|i_n - i_{n-3}| = |i_n - i_{n-1}| = \begin{cases} 1 & (r_{n-2} = r_n \notin \mathcal{L}(w_n)) \\ 0 & (r_{n-2} = r_n \in \mathcal{L}(w_n)). \end{cases}$$

By the claim (8.13.1), it follows that $\theta'(e) \in \mathcal{B}(z,y)$; hence θ' restricts to a map $\theta: \mathcal{B}'(w,y) \setminus \mathcal{B}(w,y) \longrightarrow \mathcal{B}(z,y)$.

To show θ is a bijection, define a map $\psi': \mathcal{B}(z,y) \longrightarrow (W \times \mathbb{N})^{\ell(w)+1}$ as follows. For $e' = ((w_0,i_0),\ldots,(w_{n-2},i_{n-2})) \in \mathcal{B}(z,y)$, set $\psi'(e') =$

$$((w_0,i_0),\ldots,(w_{n-3},i_{n-3}),(w_{n-3},i_{n-3}+1),(w_{n-3},i_{n-3}),(w_{n-2},i_{n-2}))$$

if $r_{n-1} \notin \mathcal{L}(w_{n-3})$ and $\psi'(e') =$

$$((w_0,i_0),\ldots,(w_{n-3},\ i_{n-3}),(r_{n-2}w_{n-3},i_{n-3}),(w_{n-3},i_{n-3}),(w_{n-2},i_{n-2}))$$

if $r_{n-1} \in \mathcal{L}(w_{n-3})$. Fix e' and write $\psi'(e') = (w'_0,i'_0),\ldots,(w'_n,i'_n))$.

Now we prove that $\psi'(e') \in \mathcal{B}'(w,y) \setminus \mathcal{B}(w,y)$. Since $e' \in \mathcal{B}(z,y)$, the conditions $(8.8.2)_k$ and $(8.8.3)_k$ hold for $1 \le k \le n-3$, (8.8.1) holds and $(8.8.4)_k$ holds for $3 \le k \le n-2$. To check the other conditions, we distinguish the cases $r_{n-1} \notin \mathcal{L}(w_{n-3}), r_{n-1} \in \mathcal{L}(w_{n-3})$.

Suppose first that $r_{n-1} \notin \mathcal{L}(w_{n-3})$. Now $r_{n-2} \notin \mathcal{L}(w_{n-3}) = \mathcal{L}(w'_{n-2})$ by $(8.8.2)_{n-2}$ for e', $r_{n-1} \notin \mathcal{L}(w_{n-3})$ by assumption and so $(8.8.2)_{n-2}$, $(8.8.2)_{n-1}$, $(8.8.3)_{n-2}$ and $(8.8.3)_{n-1}$ all hold for $\psi'(e')$. Also

(8.13.2) $$w'_n w'^{-1}_{n-1} = w_{n-2}w^{-1}_{n-3} \in \begin{cases} \{1\} & (r_n = r_{n-2} \notin \mathcal{L}(w_{n-2}) = \mathcal{L}(w'_n)) \\ R & (r_n \in \mathcal{L}(w'_n)) \end{cases}$$

and

(8.13.3) $$|i'_n - i'_{n-1}| = |i_{n-2} - i_{n-3}| = \begin{cases} 1 & (r_n \notin \mathcal{L}(w'_n)) \\ 0 & (r_n \in \mathcal{L}(w'_n)) \end{cases}$$

so $(8.8.2)_n$ and $(8.8.3)_n$ are satisfied by $\psi'(e')$. Now $(8.8.4)_{n-1}$ holds since $i'_{n-2} = i_{n-3}+1 > i_{n-3} = i'_{n-3}$. Hence $\psi'(e') \in \mathcal{B}'(w,y)$. But $\psi'(e') \notin \mathcal{B}(w,y)$ since $r_n = r_{n-2}$, while $i'_{n-2} = i_{n-3}+1 > i_{n-3} = i'_{n-1}$.

Now suppose that $r_{n-1} \in \mathcal{L}(w_{n-3})$; then $r_{n-2} \in \mathcal{L}(r_{n-2}w_{n-2}) = \mathcal{L}(w'_{n-2})$ and $r_{n-1} \in \mathcal{L}(w'_{n-1})$ so $(8.8.2)_{n-2}$, $(8.8.2)_{n-1}$, $(8.8.3)_{n-2}$ and $(8..8.3)_{n-1}$ are all satisfied by $\psi'(e')$; moreover, (8.13.2) and (8.13.3) continue to hold, and show that $(8.8.2)_n$, $(8.8.3)_n$ are valid in this case. Also,

$(8.8.4)_{n-1}$ holds since $\ell(w'_{n-2}) = 1 + \ell(w_{n-3}) > \ell(w_{n-3}) = \ell(w'_{n-3})$; hence $\psi'(e') \in \mathcal{B}'(w,y)$. But $\psi'(e') \notin \mathcal{B}(w,y)$ since $(w'_{n-3}, i'_{n-3}) = (w'_{n-1}, i'_{n-1})$ and $\ell(w'_{n-2}) = \ell(w_{n-3}) + 1 > \ell(w_{n-3}) = \ell(w'_{n-1})$.

Thus, ψ' induces a function $\psi: \mathcal{B}(z,y) \longrightarrow \mathcal{B}'(w,y) \setminus \mathcal{B}(w,y)$, and $\theta\psi$ is the identity on $\mathcal{B}(z,y)$. To complete the proof of bijectivity of θ, and hence of the lemma, it will be shown that $\psi\theta(e) = e$ for any $e = ((w_0,i_0),\ldots,(w_n,i_n)) \in \mathcal{B}'(w,y) \setminus \mathcal{B}(w,y)$.

Case 1. $r_{n-1} \in \mathcal{L}(w_{n-1})$

Now $(8.8.4)_n$ is false and $w_{n-2} \neq w_{n-1}$, so it follows that $(w_{n-1}, i_{n-1}) = (w_{n-3}, i_{n-3})$ and $\ell(w_{n-2}) > \ell(w_{n-1})$. This gives $\ell(w_{n-2}) > \ell(w_{n-3})$; by $(8.8.2)_{n-2}$, $w_{n-2}w_{n-3}^{-1} \in R$ and $r_{n-2} \in \mathcal{L}(w_{n-2})$. Hence

$$w_{n-2} = r_{n-2}w_{n-3}, i_{n-2} = i_{n-3}.$$

Therefore,

$$\begin{aligned}\psi\theta(e) &= \psi((w_0,i_0),\ldots,(w_{n-3},i_{n-3}),(w_n,i_n)) \\ &= ((w_0,i_0),\ldots,(w_{n-3},i_{n-3}),(r_{n-2}w_{n-3},i_{n-3}),(w_{n-3},i_{n-3}),(w_n,i_n)) \\ &= e \qquad \text{as required.}\end{aligned}$$

Case 2. $r_{n-1} \notin \mathcal{L}(w_{n-1})$

Here $(8.8.4)_n$ is false and $w_{n-2} = w_{n-1}$, so it follows that $(w_{n-1}, i_{n-1}) = (w_{n-3}, i_{n-3})$ and $i_{n-2} > i_{n-1}$. Since $w_{n-2} = w_{n-3}$, it follows that $r_{n-2} \notin \mathcal{L}(w_{n-2})$ and $|i_{n-2} - i_{n-3}| = 1$. Hence

$$\begin{aligned}\psi\theta(e) &= \psi((w_0,i_0),\ldots,(w_{n-3},i_{n-3}),(w_n,i_n)) \\ &= ((w_0,i_0),\ldots,(w_{n-3},i_{n-3}),(w_{n-3},i_{n-3}+1),(w_{n-3},i_{n-3}),(w_n,i_n)) \\ &= e.\end{aligned}$$

□

8.14 This section completes the proof of (8.9).

By (8.5), we have $C'_w = C'_r C'_x - C'_z$ and so

$$
\begin{aligned}
C'_w C_y &= C'_r C'_x C_y - C'_z C_y \\
&= \sum_{e \in \mathcal{B}'(w,y)} \tau_{\nu(e)} C_{\rho(e)} - \sum_{e \in \mathcal{B}(z,y)} \tau_{\nu(e)} C_{\rho(e)} \text{ by (8.12) and induction} \\
&= \sum_{e \in \mathcal{B}'(w,y)} \tau_{\nu(e)} C_{\rho(e)} - \sum_{e \in \mathcal{B}'(w,y) \setminus \mathcal{B}(w,y)} \tau_{\nu(e)} C_{\rho(e)} \text{ by (8.13)} \\
& \sum_{e \in \mathcal{B}(w,y)} \tau_{\nu(e)} C_{\rho(e)}.
\end{aligned}
$$

□

8.15 Remark. If $\mathcal{R}(w) \cap \mathcal{L}(y) \neq \emptyset$ then $\mathcal{B}(w,y) = \emptyset$ and so $C'_w C_y = 0$. That this holds in the Hecke algebra of an arbitrary Coxeter system can be seen as follows: if $s \in R$, $ws < w$ and $sy < y$, then $(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_w C_y = C'_w C'_s C_y = C'_w \cdot 0 = 0$ and so $C'_w C_y = 0$ (because the Hecke algebra is a free $\mathcal{A}$-module and $\mathcal{A}$ is an integral domain). □

We now give an explicit formula, similar to (8.9.1), for the polynomials $P^y_{x,w}$. Before stating this formula in (8.17), we need to define a function $\rho_w \colon W^{n+1} \longrightarrow \mathbb{N}$ and a set $\mathcal{P}_w(y,x) \subseteq W^{n+1}$, where $x, y, w \in W$ and $n = \ell(w)$.

8.16 Definition. Fix $w \in W$, and let $w = r_n \ldots r_1$ be the reduced expression for w.

(i) For any $e = (x_0, \ldots, x_n) \in W^{n+1}$, define

$$\rho_w(e) = \#\{\, j \mid 1 \leq j \leq n,\ r_j \in \mathcal{L}(x_{j-1}) \,\}$$

(ii) For any $x, y \in W$, let $\mathcal{P}_w(y,x)$ be the set of those $(x_0, \ldots, x_n) \in W^{n+1}$ satisfying (8.16.1), $(8.16.2)_i$ $(i = 1, \ldots, n)$ and $(8.16.3)_i$ $(i = 2, \ldots, n-1)$ below:

(8.16.1) $x_0 = y$ and $x_n = x$
$(8.16.2)_i$ $x_i x_{i-1}^{-1} \in \{1, r_i\}$
$(8.16.3)_i$ If $r_{i-1} = r_{i+1}$, then one or both of conditions (a), (b) below hold:

(a) $x_{i-1} \neq x_i$

(b) $r_{i-1} \notin \mathcal{L}(x_{i-1})$. □

The theorem below ([Dy], Theorem 3.8) will be proved in (8.18)–(8.23).

8.17 Theorem. With the notation of (8.16), for any $x, y, w \in W$,
(8.17.1) $P^y_{x,w} = \sum\limits_{e \in \mathcal{P}_w(y,x)} q^{\rho_w(e)}$. □

8.18 In the case of a universal Coxeter system, the recurrence formula (7.14) for the $P^y_{x,w}$ may be rewritten, using (8.2), as follows: for $x, y, w \in W$ and $r \in R$ with $rw < w$,
(8.18.1) $P^y_{x,w} = q^c P^y_{rx,rw} + q^{1-c} P^y_{x,rw} - q \sum\limits_{\substack{z \ll rw \\ rz < z}} P^y_{x,z}$, where

$$c = \begin{cases} 1 & (rx > x) \\ 0 & (rx < x). \end{cases}$$

Now (8.17.1) holds if $w = 1$, and will be proved by induction on $\ell(w)$. Fix $w \in W \setminus \{1\}$ and assume inductively that (8.17.1) holds when w is replaced by any $v \in W$ with $\ell(v) < \ell(w)$. To relate the sets $\mathcal{P}_w(y,x)$ for varying $x, y \in W$, it is convenient to define sets $\mathcal{P}'_w(y,x) \subseteq W^{n+1}$ as follows:

For x, y, w as in (8.16) (ii), let $\mathcal{P}'_w(y,x) \subseteq W^{n+1}$ be the set of those $(x_0, \ldots, x_n) \in W^{n+1}$ satisfying (8.16.1), $(8.16.2)_i$ $(i = 1, \ldots, n)$ and $(8.16.3)_i$ $(i = 2, \ldots, n-2)$. Note that the only difference between the definitions of $\mathcal{P}_w(y,x)$ and $\mathcal{P}'_w(y,x)$ is the range of values i may take in the condition $(8.16.3)_i$.

Let $w = r_n \ldots r_1$ be the reduced expression for w $(n \geq 1)$ and let $r_n = r$. The relationship between $\mathcal{P}_w(y,x)$ and $\mathcal{P}'_w(y,x)$ is contained in the following

8.19 Lemma. Fix $e = (x_0, \ldots, x_n) \in W^{n+1}$ and $x, y \in W$. Then

(i) $e \in \mathcal{P}'_w(y,x)$ iff $(x_0, \ldots, x_{n-1}) \in \mathcal{P}_{rw}(y, x_{n-1})$, $x_n x_{n-1}^{-1} \in \{1, r_n\}$ and $x_n = x$

(ii) $e \in \mathcal{P}_w(y,x)$ iff $e \in \mathcal{P}'_w(y,x)$ and, if $n \geq 2$ and $r_n = r_{n-2}$, then either (a) $x_{n-2} \neq x_{n-1}$ or (b) $r_{n-2} \notin \mathcal{L}(x_{n-2})$, or both.

(iii) $e \in \mathcal{P}_w(y,x)$ iff $(x_0,\ldots,x_{n-1},rx_n) \in \mathcal{P}_w(y,rx)$.

<u>Proof</u> Parts (i) and (ii) are immediate consequences of the definitions and (iii) follows from (i) and (ii). □

To prove (8.17.1), we need only show that the right hand side of (8.17.1) is equal to the right hand side of (8.18.1). Hence it will be sufficient to prove (8.20) and (8.21) below:

8.20 Lemma. For any $x, y \in W$,

$$\sum_{e\in\mathcal{P}'_w(y,x)} q^{\rho_w(e)} = q^c P^y_{rx,rw} + q^{1-c} P^y_{x,rw}$$

where

$$c = \begin{cases} 1 & (rx > x) \\ 0 & (rx < x) \end{cases}$$

□

8.21 Lemma. For any $x, y \in W$,

(8.21.1) $$\sum_{e\in\mathcal{P}'_w(y,x)} q^{\rho_w(e)} = \sum_{e\in\mathcal{P}_w(y,x)} q^{\rho_w(e)} + q \sum_{\substack{z\ll rw \\ rz<z}} P^y_{x,z}$$

□

The proofs of these lemmas are given in (8.22) and (8.23) below.

8.22 Proof of Lemma (8.20)

The function $\theta' \colon W^{n+1} \longrightarrow W^n$ which maps $(x_0,\ldots,x_n)$ to $(x_0,\ldots,x_{n-1})$ restricts (by (8.19)) to a bijection

$$\theta \colon \mathcal{P}'_w(y,x) \longrightarrow \mathcal{P}_{rw}(y,rx) \cup \mathcal{P}_{rw}(y,x).$$

Furthermore, if $e' = (x_0, \ldots, x_{n-1}) \in \mathcal{P}_{rw}(y, rx) \cup \mathcal{P}_{rw}(y, x)$, then

$$\begin{aligned}
\rho_w(\theta^{-1}(e')) &= \begin{cases} \rho_{rw}(e') + 1 & (r_n \in \mathcal{L}(x_{n-1})) \\ \rho_{rw}(e') & (r_n \notin \mathcal{L}(x_{n-1})) \end{cases} \qquad \text{(by (8.16) (i))} \\
&= \begin{cases} \rho_{rw}(e') + c & (e' \in \mathcal{P}_{rw}(y, rx) \\ \rho_{rw}(e') + 1 - c & (e' \in \mathcal{P}_{rw}(y, x)). \end{cases}
\end{aligned}$$

By induction,

$$\begin{aligned}
q^c P^y_{rx,rw} + q^{1-c} P^y_{x,rw} &= \sum_{e' \in \mathcal{P}_{rw}(y,rx)} q^{\rho_{rw}(e')+c} + \sum_{e' \in \mathcal{P}_{rw}(y,x)} q^{\rho_{rw}(e')+1-c} \\
&= \sum_{e' \in \mathcal{P}_{rw}(y,rx) \cup \mathcal{P}_{rw}(y,x)} q^{\rho_w(\theta'(e'))} \\
&= \sum_{e' \in \mathcal{P}'_w(y,x)} q^{\rho_w(e)} \text{ as claimed}
\end{aligned}$$

□

8.23 Proof of Lemma (8.21)

If $n \leq 2$, or if $n \geq 3$ and $r_n \neq r_{n-2}$, then $\mathcal{P}'_w(y, x) = \mathcal{P}_w(y, x)$ (by (8.19) (ii)) and $\{ z \in W \mid z \ll rw, rz < z \} = \emptyset$ so (8.21.1) holds.

Hence it may be assumed that $n \geq 3$ and $r_n = r_{n-2}$. In this case there exists a unique $z \in W$ satisfying $z \ll rw$ and $rz < z$, namely $z = r_{n-2} \ldots r_1$. To prove (8.21.1), it will be sufficient to show that

$$\sum_{e \in \mathcal{P}'_w(y,x) \setminus \mathcal{P}_w(y,x)} q^{\rho_w(e)} = \sum_{e' \in \mathcal{P}_z(y,x)} q^{\rho_z(e')+1}.$$

In order to do this, it will be shown that the function $\theta' \colon W^{n+1} \longrightarrow W^{n-1}$ defined by $(x_0, \ldots, x_n) \longmapsto (x_0, \ldots, x_{n-3}, x)$ restricts to a bijection

$$\theta \colon \mathcal{P}'_w(y, x) \setminus \mathcal{P}_w(y, x) \longrightarrow \mathcal{P}_z(y, x)$$

such that if $e \in \mathcal{P}'_w(y, x) \setminus \mathcal{P}_w(y, x)$, then

$$\rho_w(e) = \rho_z(\theta(e)) + 1.$$

For the proof of this claim, it will be convenient to set

$$x' = \begin{cases} x & (r \in \mathcal{L}(x)) \\ rx & (r \notin \mathcal{L}(x)). \end{cases}$$

Suppose that $e = (x_0, \ldots, x_n) \in \mathcal{P}'_w(y, x) \setminus \mathcal{P}_w(y, x)$. Then (8.19) (i), (ii) yield $r_n = r_{n-2}, x_{n-2} = x_{n-1},\ r_{n-2} \in \mathcal{L}(x_{n-2}),\ (x_0, \ldots, x_{n-1}) \in \mathcal{P}_{rw}(y, x_{n-1})$ and $x_{n-1} \in \{x, rx\}$. These facts imply $x \in \{x_{n-2}, rx_{n-2}\}$ so (8.19) gives $\theta'(e) = (x_0, \ldots, x_{n-3}, x) \in \mathcal{P}_z(y, x)$. Hence θ' restricts to a function $\theta \colon \mathcal{P}'_w(y, x) \setminus \mathcal{P}_w(y, x) \longrightarrow \mathcal{P}_z(y, x)$. Note also that since $x_{n-2} = x_{n-1} \in \{x, rx\}$ and $r \in \mathcal{L}(x_{n-2})$, we must have $x_{n-2} = x_{n-1} = x'$; this observation proves that θ is injective.

We check that $\rho_w(e) = \rho_z(\theta(e)) + 1$. Set $e'' = (x_0, \ldots, x_{n-3}) \in \mathcal{P}_{rz}(y, x_{n-3})$ and $\theta(e) = e' = (x_0, \ldots, x_{n-3}, x) \in \mathcal{P}_z(y, x)$. Then since $r_{n-1} \notin \mathcal{L}(x_{n-2}) = \{r\}$ but $r_n \in \mathcal{L}(x_{n-1}) = \{r\}$, we have

$$\begin{aligned}\rho_w(e) &= \rho_{rz}(e'') + \#\{\, j \mid n-2 \le j \le n,\ r_j \in \mathcal{L}(x_{j-1})\,\} \\ &= \begin{cases} \rho_{rz}(e'') + 1 & (r_{n-2} \notin \mathcal{L}(x_{n-3})) \\ \rho_{rz}(e'') + 2 & (r_{n-2} \in \mathcal{L}(x_{n-3})) \end{cases}\end{aligned}$$

while

$$\rho_z(e') = \begin{cases} \rho_{rz}(e'') & (r_{n-2} \notin \mathcal{L}(x_{n-3})) \\ \rho_{rz}(e'') + 1 & (r_{n-2} \in \mathcal{L}(x_{n-3})) \end{cases}$$

so

$$\rho_w(e) = \rho_z(e') + 1 = \rho_z(\theta(e)) + 1.$$

It remains to check that θ is surjective. Take $e' = (x_0, \ldots, x_{n-2}) \in \mathcal{P}_z(y, x)$. Since $r_{n-2} = r$, (8.19) (iii) shows that $(x_0, \ldots, x_{n-3}, x') \in \mathcal{P}_z(y, x')$. Now using (8.19) (i), (ii), it follows that $(x_0, \ldots, x_{n-3}, x', x') \in \mathcal{P}_{rw}(y, x')$ (since if $x_{n-3} = x'$, then $\mathcal{L}(x_{n-3}) = \mathcal{L}(x') = \{r\} = \{r_{n-2}\}$ so $r_{n-3} \notin \mathcal{L}(x_{n-3})$). Since $xx'^{-1} \in \{1, r\}$, (8.19) (i) now shows that $e = (x_0, \ldots, x_{n-3}, x', x', x) \in \mathcal{P}'_w(y, x)$. However, $e \notin \mathcal{P}_w(y, x)$ since $r_{n-2} = r_n,\ x' = x'$ and $r_{n-2} \in \mathcal{L}(x')$. So $e \in \mathcal{P}'_w(y, x) \setminus \mathcal{P}_w(y, x)$ and $\theta(e) = (x_0, \ldots,_{n-3}, x) = e'$; hence θ is surjective as claimed. ∎

We conclude this chapter by exhibiting the polynomials $1 + nq$ ($n \in \mathbb{N}$) as Kazhdan-Lusztig polynomials

8.24 Example.

(i) Consider a universal Coxeter system (W,R) in which $R=\{r,s,t\}$. Fix $n\in\mathbb{N}$ and let

$$w=\overbrace{(\ldots trstrst)}^{n+3},\qquad y=\overbrace{(\ldots trst)}^{n}.$$

For fixed j $(0\le j\le n)$ set

$$y_i=\begin{cases}\overbrace{(\ldots trst)}^{i} & (0\le i\le j)\\ \overbrace{(\ldots trst)}^{j} & (j+1\le i\le j+3)\\ \overbrace{(\ldots trst)}^{i-3} & (j+4\le i\le n+4).\end{cases}$$

Define $e_j=(y_0,\ldots,y_n)\in W^{n+4}$. Then

$$\rho_w(e_j)=\begin{cases}0 & (j=0)\\ 1 & (j\in\{1,\ldots,n\})\end{cases}$$

and $\mathcal{P}_w(1,x)=\{e_0,e_1,\ldots,e_n\}$.

Hence $P_{y,w}=P^1_{y,w}=q^0+nq^1=1+nq$.

(ii) Take $w=trst$ in (i). Then $\mathcal{P}_w(s,s)=\{f_1,f_2\}$ where $f_1=(s,s,s,s,s)$ and $f_2=(s,ts,ts,ts,s)$. We have $\rho_w(f_1)=\rho_w(f_2)=1$, and hence $P^s_{s,w}=2q$. $\square$

Chapter 9

UNIVERSAL COXETER SYSTEMS: [P2]

This chapter is devoted to a proof that the Hecke algebra of a universal Coxeter system satisfies the positivity property [P2] of (7.21). Recall that [P2] asserts that

$$\widetilde{T}_{x^{-1}}^{-1}\widetilde{T}_y \in \sum_z \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]C_z.$$

In contrast to [P1], [P3] and [P4], this involves three separate bases of the Hecke algebra (the others involve at most two); the proof of [P2] is correspondingly more intricate. The coefficients in the Laurent polynomials arising as structure constants will again turn out to be the cardinalities of combinatorially defined sets, but this time we will not explicitly describe these sets.

9.1 Throughout this chapter, (W, R) denotes a fixed universal Coxeter system, and we use the notation of (8.1). For the proof of [P2], we will need an ancillary result that expresses the products $\widetilde{T}_{x^{-1}}^{-1}C_y$ $(x, y \in W)$ in terms of the basis $\{C_z\}_{z\in W}$. This result is a restatement of ([Dy], Theorem (3.9)) but we will give a different proof. Following is some notation required for the statement of this auxiliary result in (9.3) below.

9.2 Definition.

(i) For any $(y_0, \ldots, y_n) \in W^{n+1}$, let

$$\gamma(e) = \frac{1}{2}\#\{\, j \mid 1 \le j \le n,\ y_j = y_{j-1}\,\}$$

(ii) Let $x, y \in W$, and $x = r_n \ldots r_1$ be the reduced expression for x. Let $\mathcal{Q}_x(y, z) \subseteq W^{n+1}$ be the set of those $(y_0, \ldots, y_n) \in W^{n+1}$ satisfying (9.2.1), $(9.2.2)_j$ $(j = 1, \ldots, n)$ and $(9.2.3)_j$ $(j = 1, \ldots, n-1)$ below:

(9.2.1) $y_0 = y,\ y_n = z$

$$(9.2.2)_j \qquad y_j y_{j-1}^{-1} \in \begin{cases} \{1\} & (r_j \notin \mathcal{L}(y_j)) \\ R & (r_j \in \mathcal{L}(y_j)) \end{cases}$$

$(9.2.3)_j$ $\ell(y_j) \leq \ell(y_{j-1})$ or $\ell(y_j) \leq \ell(y_{j+1})$ □

9.3 Proposition. For any $x, y \in W$ with $\mathcal{R}(x) \cap \mathcal{L}(y) = \emptyset$,

$$\widetilde{T}_{x^{-1}}^{-1} C_y = \sum_{z \in W} \left(\sum_{e \in \mathcal{Q}_x(y,z)} q^{-\gamma(e)} \right) C_z$$

<u>Proof</u> For any $r \in R$, we have $\widetilde{T}_r^{-1} = C_r + q^{-\frac{1}{2}}$, so (8.10) gives

$$\widetilde{T}_r^{-1} C_w = \begin{cases} -q^{\frac{1}{2}} C_w & (rw < w) \\ q^{-\frac{1}{2}} C_w + \displaystyle\sum_{\substack{s \in R \\ r \in \mathcal{L}(sw)}} C_{sw} & (rw > w) \end{cases}$$

i.e.

$$(9.3.1) \qquad \widetilde{T}_r^{-1} C_w = \begin{cases} -q^{\frac{1}{2}} C_w + \displaystyle\sum_{\substack{s \in R \\ r \in \mathcal{L}(sw)}} C_{sw} & (rw < w) \\ q^{-\frac{1}{2}} C_w + \displaystyle\sum_{\substack{s \in R \\ r \in \mathcal{L}(sw)}} C_{sw} & (rw > w) \end{cases}$$

since $\{ s \in R \mid r \in \mathcal{L}(sw) \} = \emptyset$ if $rw < w$.

Now let $x = r_n \dots r_1$ be the reduced expression for x. For any $e = (y_0, \dots, y_n) \in W^{n+1}$, let

$$\delta_j(e) = \begin{cases} 1 & (y_j \neq y_{j-1}) \\ -q^{\frac{1}{2}} & (y_j = y_{j-1},\ r_j \in \mathcal{L}(y_j)) \\ q^{-\frac{1}{2}} & (y_j = y_{j-1},\ r_j \notin \mathcal{L}(y_j)) \qquad (j = 1, \dots, n) \end{cases}$$

and $\delta(e) = \delta_1(e) \dots \delta_n(e)$.

Note that
(9.3.2) $\delta(e) = q^{-\gamma(e)}$ if $e \in \mathcal{Q}_x(y, z)$

Define $A_x(y,z)$ to be the subset of W^{n+1} consisting of those $(y_0,\ldots,y_n)\in W^{n+1}$ satisfying (9.3.3), (9.3.4) and (9.3.5) below:
(9.3.3) $y_0=y,\ y_n=z$
(9.3.4) $y_jy_{j-1}^{-1}\in\{1\}\cup R\ (j=1,\ldots,n)$
(9.3.5) If $1\le j\le n$ and $y_jy_{j-1}^{-1}\in R$ then $r_j\in\mathcal{L}(y_j)$.

Using (9.3.1), it follows by induction on $\ell(x)$ that

$$\widetilde{T}_{x^{-1}}^{-1}C_y=\sum_{z\in W}\left(\sum_{e\in A_x(y,z)}\delta(e)\right)C_z \tag{9.3.6}$$

We now fix x,y,z and show that

$$\sum_{e\in A_x(y,z)}\delta(e)=\sum_{e\in\mathcal{Q}_x(y,z)}q^{-\gamma(e)}. \tag{9.3.7}$$

For any $e=(y_0,\ldots,y_n)\in A_x(y,z)$, define

$$\begin{aligned}J(e)=\{\,j\mid 1\le j\le n-1,\ y_{j-1}=y_j=y_{j+1}\text{ and }r_{j+1}\in\mathcal{L}(y_{j+1})\,\}\\ \cup\{\,j\mid 1\le j\le n-1,\ y_{j-1}=y_{j+1},\ell(y_j)>\ell(y_{j-1})\,\}.\end{aligned}$$

Now introduce an equivalence relation $\sim$ on $A_x(y,z)$ as follows: for e as above and $e'=(y_0,\ldots,y_n')\in A_x(y,z)$, write $e\sim e'$ iff $J(e)=J(e')$ and $y_i=y_i'$ $(i\in\{0,\ldots,n\}\setminus J(e))$.

Fix $e=(y_0,\ldots,y_n)\in A_x(y,z)$. For any $e'=(y_0',\ldots,y_n')\in W^{n+1}$, we show that
(9.3.8) ($e'\in A_x(y,z)$ and $e'\sim e$) iff

$$\begin{cases} y_j'y_j^{-1}=1 & (j\in\{0,\ldots,n\}\setminus J(e))\\ y_j'y_j^{-1}\in\{1,r_j\} & (j\in J(e)).\end{cases}$$

Note firstly that if $j,k\in J(e)$ and $j\ne k$ then $|j-k|\ge 2$. Suppose that $e'\in A_x(y,z)$ and $e'\sim e$. Then for any $j\in J(e)$, we have $y_{j-1}=y_{j+1}=y_{j+1}'=y_{j-1}'$. Also, either $y_j=y_{j-1}$ or $y_jy_{j-1}^{-1}\in R,\ r_j\in\mathcal{L}(y_j)$; since in the latter case we have $\ell(y_j)>\ell(y_{j-1})$ by definition of $J(e)$, it follows that either $y_j=y_{j-1}$ or

$y_j = r_j y_{j-1}$. Similarly, either $y'_j = y'_{j-1}$ or $y'_j = r_j y'_{j-1}$. Thus, the left hand condition in (9.3.8) implies that on the right.

Conversely, suppose that $e' \in W^{n+1}$ satisfies the conditions on the right of (9.3.8). Let $j \in J(e)$. We have either $y_{j-1} = y_j = y_{j+1}$ or $y_{j-1} = r_j y_j = y_{j+1}$ and $r_j \in \mathcal{L}(y_j)$; it follows that $r_{j+1} \in \mathcal{L}(y_{j+1})$ (from the definition of $J(e)$ in the first case, and (9.3.5) in the other). Now $y'_j y'^{-1}_{j-1} \in \{1, r_j\}$ and $y'_{j+1} y'^{-1}_j \in \{1, r_j\}$; if $y'_j = r_j y'_{j-1}$, then $r_j \in \mathcal{L}(y'_j)$ (since $r_{j+1} \in \mathcal{L}(y_{j+1})$ implies $r_j \notin \mathcal{L}(y'_{j-1})$). These facts imply that $e' \in A_x(y,z)$, and that $J(e) \subseteq J(e')$. Interchanging the roles of e and e', $J(e') \subseteq J(e)$. Hence $e \sim e'$ as claimed.

Now for $e = (y_0, \ldots, y_n) \in A_x(y,z)$ and $j \in J(e)$, we have

$$\delta_{j+1}(e)\delta_j(e) = \begin{cases} -1 & (y_j = y_{j+1}) \\ +1 & (y_j \neq y_{j+1}). \end{cases}$$

It follows that if also $e' = (y'_0, \ldots, y'_n) \in A_x\ (y,z)$ and $e' \sim e$, then $\delta(e') = (-1)^k \delta(e)$ where $k = \#\{\, j \in J(e) \mid y'_j \neq y_j \,\}$. By (9.3.8), we have now that if $J(e) \neq \emptyset$, then

$$\sum_{\substack{e' \in A_x(y,z) \\ e' \sim e}} \delta(e) = 0, \text{ hence } \sum_{e \in A_x(y,z)} \delta(e) = \sum_{\substack{e \in A_x(y,z) \\ J(e) = \emptyset}} \delta(e).$$

To prove (9.3.7), it need only be checked that

$$\{\, e \in A_x(y,z) \mid J(e) = \emptyset \,\} = \mathcal{Q}_x(y,z)$$

(by (9.3.2)).

Fix $e = (y_0, \ldots, y_n) \in W^{n+1}$. Suppose first that $e \in A_x(y,z)$ and $J(e) = \emptyset$. If $1 \leq j \leq n-1$ and $(9.2.3)_j$ failed, (9.3.4) and (9.3.5) would give $y_{j-1} = r_j r_j = y_{j+1}$, $y_j > y_{j-1}$ and so $j \in J(e)$, a contradiction. To check $(9.2.2)_j$ for $1 \leq j \leq n$, it is enough by (9.3.4) and (9.3.5) to show that if $y_j = y_{j-1}$ then $r_j \notin \mathcal{L}(y_j)$. But if $y_1 = y_0$ and $r_1 \in \mathcal{L}(y_1)$, then $r_1 \in \mathcal{L}(y) \cap \mathcal{R}(x)$ contrary to the hypothesis of (9.3). If $2 \leq j \leq n$ $y_j = y_{j-1}$ and $r_j \in \mathcal{L}(y_j)$, then $r_{j-1} \notin \mathcal{L}(y_{j-1})$. Thus, (9.3.4) and (9.3.5) imply that $y_{j-2} = y_{j-1}$; hence $j-1 \in J(e)$, again contrary to assumption. Since e satisfies (9.3.3), we have $e \in \mathcal{Q}_x(y,z)$.

Conversely, suppose that $e \in \mathcal{Q}_x(y,z)$. Then a fortiori, $e \in A_x(y,z)$. If $j \in J(e)$, then $(9.2.3)_j$ implies that $y_{j-1} = y_j = y_{j+1}$ and $r_{j+1} \in \mathcal{L}(y_{j+1})$, contrary to

$(9.2.2)_{j+1}$. Hence $J(e) = \emptyset$, completing the proof of (9.3.7) and hence of the proposition. $\square$

9.4 Corollary. For any $x \in W$,

$$\widetilde{T}_x = \sum_{z\in W} \left(\sum_{e\in \mathcal{Q}'_x(1,z)} q^{\gamma(e)} \right) C_z$$

where $\mathcal{Q}'_x(1,z) = \{ (y_0,\ldots,y_n) \in W^{\ell(x)+1} \mid (y_0^{-1},\ldots,y_n^{-1}) \in \mathcal{Q}_{x^{-1}}(1,z^{-1}) \}$.

Proof Applying the anti-involution $\sum a_z \widetilde{T}_z \longmapsto \sum \bar{a}_z \widetilde{T}_z^{-1}$ of the Hecke algebra to (9.3) gives

$$\widetilde{T}_{x^{-1}} = \sum_{z\in W} \left(\sum_{e\in \mathcal{Q}_x(1,z)} q^{\gamma(e)} \right) C_{z^{-1}}$$

and the corollary follows on noting that $\gamma((y_0,\ldots,y_n)) = \gamma((y_0^{-1},\ldots,y_n^{-1}))$ $\square$

Following is the main result of this chapter.

9.5 Theorem. Fix $y, w \in W$. Then for any $x \in W$, there is a subset $\mathcal{B}_x$ of $\bigcup_{z\in W} (\mathcal{Q}_w(z,x) \times \mathcal{Q'}_y(1,z))$ such that

$$\widetilde{T}_{w^{-1}}^{-1}\widetilde{T}_y = \sum_{x\in W} \left(\sum_{(e,e')\in \mathcal{B}_x} q^{\gamma(e')-\gamma(e)} \right) C_x.$$

$\square$

The proof of (9.5) will be given in (9.6)–(9.13).

9.6 If $w = 1$ then (9.5) is trivial. Henceforward, we assume $w \neq 1$. Let $\mathcal{R}(w) = \{s\}$ $(s \in R)$ and write $v = ws$. Note that $\widetilde{T}_s^{-1} C_x = -q^{\frac{1}{2}} C_x$ if $sx < x$.

Hence

$$
\begin{aligned}
\widetilde{T}_{w^{-1}}^{-1}\widetilde{T}_y &= \widetilde{T}_{w^{-1}}^{-1} \sum_{\substack{z\in w\\ e'\in\mathcal{Q}'_y(1,z)}} q^{\gamma(e')}C_z \\
&= \sum_{\substack{z\in W\\ e'\in\mathcal{Q}'_y(1,z)\\ sz>z}} q^{\gamma(e')}\widetilde{T}_{w^{-1}}^{-1}C_z \\
&\quad - \sum_{\substack{z\in W\\ e'\in\mathcal{Q}'_y(1,z)\\ sz<z}} q^{\gamma(e')+\frac{1}{2}}\widetilde{T}_{v^{-1}}^{-1}C_z \\
&= \sum_{\substack{z\in W\\ e'\in\mathcal{Q}'_y(1,z)\\ sz>z}} \sum_{\substack{x\in W\\ e\in\mathcal{Q}_w(z,x)}} q^{\gamma(e')-\gamma(e)}C_x \\
&\quad - \sum_{\substack{z\in W\\ e'\in\mathcal{Q}'_y(1,z)\\ sz<z}} \sum_{\substack{x\in W\\ e'\in\mathcal{Q}_v(z,x)}} q^{\gamma(e')+\frac{1}{2}-\gamma(e)}C_x \\
&= \sum_{x\in W} \Bigg[\sum_{\substack{z\in W\\ (e,e')\in\mathcal{Q}_w(z,x)\times\mathcal{Q}'_y(1,z)}} q^{\gamma(e')-\gamma(e)} \\
&\quad - \sum_{\substack{z\in W\\ sz<z\\ (e,e')\in\mathcal{Q}_v(z,x)\times\mathcal{Q}'_y(1,z)}} q^{\gamma(e')+\frac{1}{2}-\gamma(e)}\Bigg] C_x
\end{aligned}
$$

noting that $\mathcal{Q}_w(z,x)=\emptyset$ unless $sz>z$.

Hence to prove (9.5), it will suffice to prove the following

9.7 Lemma. For any $x\in W$, there exists an injection

$$
\theta_x\colon \bigcup_{\substack{z\in W\\ sz<z}} (\mathcal{Q}_v(z,x)\times\mathcal{Q}'_y(1,z)) \longrightarrow \bigcup_{z\in W} (\mathcal{Q}_w(z,x)\times\mathcal{Q}'_y(1,z))
$$

such that if $\theta_x((f,e)=(f',e')$, then $\gamma(e')-\gamma(f')=\gamma(e)-\gamma(f)+\frac{1}{2}$. $\square$

(For (9.5) is then satisfied by taking

$$\mathcal{B}_x = \bigcup_{z\in W} (\mathcal{Q}_w(z,x) \times \mathcal{Q}'_y(1,z)) \setminus Im\theta_x)$$

9.8 Let $y = r_1 \dots r_m$ be the reduced expression for y. It is convenient to write out the definition of $\mathcal{Q}'_y(1,z)$ explicitly:

$\mathcal{Q}'_y(1,z)$ is the set of those sequences $(y_0,\dots,y_m) \in W^{n+1}$ such that (9.8.1), $(9.8.2)_j$ $(j = 1,\dots,m)$ and $(9.8.3)_j$ $(j = 1,\dots,m-1)$ below hold:
(9.8.1) $y_0 = 1,\ y_m = z$

$(9.8.2)_j$
$$y_{j-1}^{-1}y_j \in \begin{cases} \{1\} & (r_j \notin \mathcal{R}(y_j)) \\ R & (r_j \in \mathcal{R}(y_j)) \end{cases}$$

$(9.8.3)_j$ $\ell(y_j) \le \ell(y_{j-1})$ or $\ell(y_j) \le \ell(y_{j+1})$.

Note that if $(y_0,\dots,y_m) \in \mathcal{Q}'_y(1,z)$ and $s \in R,\ sz < z$ then there exists $j (1 \le j \le m)$ such that $y_i = 1\ (0 \le i < j),\ y_j = s$ and $s \in \mathcal{L}(y_i)$ for $j \le i \le m$.

9.9 Now let $w = s_n \dots s_1$ be the reduced expression for w. Suppose $z \in W,\ sz < z$; let $e = (y_0,\dots,y_m) \in \mathcal{Q}'_y(1,z)$ and $f = (z_1,\dots,z_n) \in \mathcal{Q}_v(z,x)$.

Define $j_0,\ k_0,\ l_0$ as follows:

$$\begin{aligned} j_0 = \ & \max\{\, j \mid 1 \le j \le n,\ \ell(z_j) < \dots < \ell(z_1)\,\} \\ k_0 = \ & \min\{\, i \mid 1 \le i \le m,\ \ell(y_0) \le \dots \le \ell(y_{i-1}) < \ell(y_i) \text{ and} \\ & \quad \ell(y_j) \ge \ell(y_i) \text{ for all } j\ (i \le j \le m)\,\} \\ l_0 = \ & \ell(y_{k_0}). \end{aligned}$$

To define, $\theta_x((f,e))$, the three cases below must be considered separately:
Case 1 $l_0 > j_0$
Case 2 $l_0 = j_0$ and there does not exist $j\ (0 \le j \le m)$ with $\ell(y_j) = l_0 + 1$
Case 3 $l_0 < j_0$ or ($l_0 = j_0$ and there exists $j\ (0 \le j \le m)$ with $\ell(y_j) = l_0 + 1$).

We begin with Case 1.

9.10 Case 1. $l_0 > j_0$

For $j = 0, \ldots, j_0 + 1$ let $i_j = \min\{\, i \mid 0 \leq i \leq m,\ \ell(y_i) = j \,\}$. Then $y_{i_j} = r_{i_1} \ldots r_{i_j}$ $(0 \leq j \leq j_0 + 1)$ and for $i \leq i_{j_0+1}$,
(9.10.1) $y_i = y_{i_j}$ where $j = \max\{\, k \mid 0 \leq k \leq j_0 + 1,\ i_k \leq i \,\}$.

Since $e \in \mathcal{Q}'_y(1, z)$, it follows that
(9.10.2) $i_j = \max\{\, i \mid 1 \leq i \leq i_{j+1},\ r_i = r_{i_j} \,\}$ $(1 \leq j \leq j_0)$.

Since $f \in \mathcal{Q}_v(z, x)$ and $s_1 \in \mathcal{L}(z_1) = \mathcal{L}(z)$,
(9.10.3) $s_i \in \mathcal{L}(z_i)$ $(1 \leq i \leq j_0)$
and since $z_j z_{j-1}^{-1} \in R$ $(2 \leq j \leq j_0)$,
(9.10.4) $z_j = s_{j-1} z_{j-1}$ $(2 \leq j \leq j_0)$

Now $y_{i_{j_0}+1} = r_{i_1} \ldots r_{i_{j_0}+1}$; for $i \geq i_{j_0+1}, y_i^{-1} y_{i+1} \in \{1\} \cup R$ and $\ell(y_i) \geq j_0 + 1$. This implies
(9.10.5) $\ell(y_i) = \ell(y_{i_{j_0+1}}) + \ell(y_{i_{j_0+1}}^{-1} y_i)$ $(i_{j_0+1} \leq i \leq m)$.

In particular, we may write, for some $x' \in W$,
(9.10.6) $z_1 = y_m = r_{i_1} \ldots r_{i_{j_0+1}} x'$ where $\ell(z_1) = j_0 + 1 + \ell(x')$

From (9.10.3), (9.10.4) and (9.10.6), it now follows that
(9.10.7) $r_{i_j} = s_j$ $(1 \leq j \leq j_0)$.

For $m \geq i \geq i_{j_0+1}$, set $y'_i = y_{i_{j_0}}^{-1} y_i$. From (9.10.5), we have
(9.10.8) $r_{i_{j_0+1}} \in \mathcal{L}(y'_i)$ and $\mathcal{R}(y'_i) = \mathcal{R}(y_i)$ $(i_{j_0+1} \leq i \leq m)$.

Note that $r_1 \ldots r_{i_{j_0+1}} \not\leq s_1 \ldots s_{j_0-1}$ (where $\leq$ denotes Bruhat order) since $\ell(r_1 \ldots r_{i_{j_o+1}}) = i_{j_0+1} \geq j_0 + 1 > j_0 - 1 = \ell(s_1 \ldots s_{j_0-1})$.

Let $k = \max\{\, i \mid 1 \leq i \leq i_{j_0+1}, r_i \ldots r_{i_{j_0}+1} \not\leq s_1 \ldots s_{j_0-1} \,\}$ and define

$$
\begin{aligned}
y''_i &= \begin{cases} 1 & (0 \leq i < k) \\ r_k \ldots r_i & (k \leq i < i_{j_0+1}) \\ r_k \ldots r_{i_{j_0+1}-1} y'_i & (i_{j_0+1} \leq i \leq m), \end{cases} \\
e' &= (y''_0, \ldots, y''_m) \in W^{m+1}.
\end{aligned}
$$

Since $y'_{i_{j_0+1}} = r_{i_{j_0+1}}$, it follows from (9.10.8) that
(9.10.9) $e' \in \mathcal{Q}'_y\ (1, y''_m)$

Now define $p_0 = j_0$ and define $p_1, \ldots, p_{i_{j_0+1}-k}$ in turn by $p_j = \max\{\, l \mid 1 \le l < p_{j-1},\ s_l = r_{i_{j_0+1}-j+1} \,\}$; these are all defined since by choice of $k, r_{k+1} \ldots r_{i_{j_0+1}} \le s_1 \ldots s_{j_0-1}$. Note also, since $r_k \ldots r_{i_{j_0+1}} \not\le s_1 \ldots s_{j_0-1}$, it follows that
(9.10.10) $s_l \ne r_k$ $(1 \le l < p_{i_{j_0+1}-k})$.

Now let $p_{i_{j_0+1}-k+1} = 0$ and define

$$z_i' = \begin{cases} z_i & (n \ge i \ge j_0) \\ s_{j_0} z_{j_0} & (j_0 > i \ge p_1) \\ r_{i_{j_o+1}-l} \ldots r_{i_{j_0+1}-1} s_{j_0} z_{j_0} & (p_l > i \ge p_{l+1};\ 1 \le l \le i_{j_0+1} - k) \end{cases}$$

We now check that $(z_0', \ldots, z_n')$ satisfies (9.2.2), (9.2.3).

We have $s_{j_0} z_{j_0} = s_{j_0} s_{j_0-1} \ldots s_1 y_m$ by (9.10.4), so (9.10.6) and (9.10.7) give $\mathcal{L}(s_{j_0} z_{j_0}) = \{r_{i_{j_0+1}}\}$. It follows that
(9.10.11) $\mathcal{L}(z_i') = \{r_{i_{j_0}-l}\}$ $(p_l > i \ge p_{l+1}, 0 \le l \le i_{j_0} - k)$.

Now for $p_l > i > p_{l+1},\ s_i \ne r_{i_{j_0+1}-l} = \mathcal{L}(z_i')$ and $z_i' = z_{i-1}'$. If $i_{j_0+1} - k \ge l > 0$ then $\mathcal{L}(z_{p_l}') = \{r_{i_{j_0+1}-l} + 1\} = \{s_{p_l}\}$ and $z_{p_l}' = r_{i_{j_0+1}-l} z_{p_l-1}'$. This shows that $(9.2.2)_i$ holds for $1 \le i < j_0$, and, moreover, that $\ell(z_0') \ge \ldots \ge \ell(z_{j_0-1}')$, so $(9.2.3)_i$ holds for $1 \le i < j_0$.

Also, $s_{j_0} \in \mathcal{L}(z_{j_0}')$ and $z_{j_0-1}' = s_{j_0} z_{j_0}' < z_{j_0}'$, so $(9.2.2)_{j_0}$ holds; if $j_0 < n$, then the definition of j_0 implies that $(9.2.3)_{j_0}$ holds. Since $(9.2.2)_i$ and $(9.2.3)_i$ hold for $i > j_0$ (because $f \in \mathcal{Q}_v(z, x)$), we may conclude that
(9.10.12) $f' = (z_0', \ldots, z_n') \in \mathcal{Q}_w(z_0', z_n')$.

Note that

$$\begin{aligned} z_0' &= r_k \ldots r_{i_{j_0+1}-1} s_{j_0} z_{j_0} \\ &= r_k \ldots r_{i_{j_0+1}-1} (s_{j_0} \ldots s_1 y_m) \\ &= r_k \ldots r_{i_{j_o+1}-1} y_m' \\ &= y_m'' \end{aligned}$$

Hence $(f', e') \in \bigcup_{z \in W} (\mathcal{Q}_w(z, x) \times \mathcal{Q}_y'(1, z))$, and we set $\theta_x((f, e)) = (f', e')$.

Now

$$\gamma(e') - \gamma(f') = \frac{1}{2} \#\{\, i \mid y_i'' = y_{i-1}'' \,\} - \frac{1}{2} \{\, j \mid z_j' = z_{j-1}' \,\}$$

$$= \frac{1}{2}[\#\{\, i > i_{j_0+1} \mid y_i = y_{i-1}\,\} + k - 1] - \frac{1}{2}[\#\{\, j > j_0 \mid z_j = z_{j-1}\,\}$$
$$+ ((j_0 - 1) - (\ell(z'_0) - \ell(z'_{j_0-1})))]$$

and $\ell(z'_0) - \ell(z'_{j_0-1}) = i_{j_0+1} - k$. Also

$$\begin{aligned}\gamma(e) - \gamma(f) &= \frac{1}{2}\#\{i \mid y_i = y_{i-1}\,\} - \frac{1}{2}\#\{\, j \mid z_j = z_{j-1}\,\} \\ &= \frac{1}{2}[\#\{\, i > i_{j_0+1} \mid y_i = y_{i-1}\,\} + (i_{j_0+1} - 1 - j_0)] \\ &\quad - \frac{1}{2}\#\{\, j > j_0 \mid z_j = z_{j-1}\,\}\end{aligned}$$

and so we have that $\gamma(e) - \gamma(f) = \gamma(e') - \gamma(f') - \frac{1}{2}$ as wanted.

The following observations will show that (f, e) is determined by (f', e').

Firstly, j_0 is determined by
(9.10.13) $j_0 - 1 = \max\,\{\, j \mid 0 \le j \le n,\ \ell(z'_j) \le \ldots \le \ell(z'_0)\,\}$.

Then by (9.10.4), $f = (z_1, \ldots, z_m)$ is determined by
(9.10.4) $z_j = z'_j$ $(n \ge j \ge j_0)$; $z_j = s_j \ldots s_{j_0-1} z'_{j_0}$ $(1 \le j \le j_0 - 1)$.

Now k is determined by the condition
(9.10.15) $k = \min\{\, i \mid 0 \le i \le m,\ \ell(y''_i) = 1\,\}$
and then i_{j_0+1} is determined by
(9.10.16) $i_{j_0+1} - k = \ell(z'_0) - \ell(z'_{j_0-1})$.

By (9.10.7) and (9.10.2), $i_{j_0}, \ldots, 1$, are given by
(9.10.17) $i_j = \max\{\, i \mid 1 \le i \le i_{j+1}, r_i = s_j\ \ (j = j_0, \ldots, 1)$.

Now (9.10.1) gives
(9.10.18) for $i \le i_{j_0+1}, y_i = r_{i_1} \ldots r_{i_j}$ where $j = \max\{\, k \mid i_k \le i\,\}$.

The remaining y_i, and hence e, are finally given by
(9.10.19) $y_i = r_{i_1} \ldots r_{i_{j_0+1}}\ (r_k \ldots r_{i_{j_0+1}})^{-1} y''_i\ (m \ge i \ge i_{j_0+1})$.

When we have finished defining θ_x the observations (9.10.13–19) will show that the restriction of θ_x to the set of pairs (f, e) in Case 1 is injective. In order to compare the image of pairs (f_1, e_1), (f_2, e_2) belonging to different cases, we shall also need the following fact (9.10.21). Let

(9.10.20) $a = \max\{\, j \mid \ell(z'_j) \le \ldots \le \ell(z'_0)\,\}$
$b = \ell(z'_0) - \ell(z'_a)$
$c = \max\,\{\, i \mid \ell(y''_i) = 0\,\}$
$d = \max\{\, i \mid \ell(y''_c) < \ldots < \ell(y''_i), \ell(y''_j) \ge \ell(y''_i) \text{ for } j \ge i\,\}.$

Then
(9.10.21) $\ell(y''_m) \ge 1$ and $1 + b \le d - c$.

Here, this follows because $a = j_0 - 1,\ b = i_{j_0+1} - k,\ c = k - 1$ and $d \ge i_{j_0+1}$.

□

9.11 Case 2. $l_0 = j_0$ and for all j with $0 \le j \le m$, we have $\ell(y_j) \ne l_0 + 1$.

For $j = 0, \ldots, j_0$ let $i_j = \min\{\, i \mid 0 \le i \le m,\ \ell(y_i) = j\,\}$. Also, define $i_{j_0+1} = m + 1$. Then for f$0 \le i \le m$,
(9.11.1) $y_i = y_{r_1} \ldots y_{r_j}$ where $j = \max\,\{\, k \mid 0 \le k \le j_0 + 1,\ i_k \le i\,\}$.

Since $e \in \mathcal{Q}'_y(1, z)$, it follows that
(9.11.2) $i_j = \max\,\{\, i \mid 1 \le i < i_{j+1}, r_i = r_{i_j}\,\}$ $(1 \le j \le j_0)$.

Since $f \in \mathcal{Q}_v(z, x)$ and $s_1 \in \mathcal{L}(z_1)$, we have
(9.11.3) $s_i \in \mathcal{L}(z_i)$ $(1 \le i \le j_0)$

and because $z_j z_{j-1}^{-1} \in R$ $(2 \le j \le j_0)$, it follows that
(9.11.4) $z_j = s_{j-1} z_{j-1}$ $(2 \le j \le j_0)$.

Now
(9.11.5) $z_1 = y_m = r_{i_1} \ldots r_{i_{j_0}}$ and so from (9.11.3), (9.11.4) we get
(9.11.6) $r_{i_j} = s_j$ $(1 \le j \le j_0)$

Note that $z_{j_0} = s_{j_0-1} \ldots s_1 z_1 = s_{j_0}$. Set $y''_i = 1$ $(0 \le i \le m)$ and $e' = (y''_0, \ldots, y''_m)$. Also, define

$$z'_j = \begin{cases} 1 & (0 \le j \le j_0 - 1) \\ z_j & (j_0 \le j \le n) \end{cases}$$

and $f' = (z'_0, \ldots, z'_n)$.

Then $e' \in \mathcal{Q}'_y(1,1)$ and, noting that f' satisfies $(9.2.3)_{j_0}$ (if $j_0 < n$) by definition of j_0, we also have $f' \in \mathcal{Q}_w(1, z'_n) = \mathcal{Q}_w(1, x)$.

We set $\theta_x((f,e)) = (f', e')$, and note that

$$\begin{aligned}\gamma(e') - \gamma(f') &= \frac{m}{2} - \frac{1}{2}[(j_0 - 1) + \#\{\, j > j_0 \mid z_j = z_{j-1}\,\}] \\ &= \frac{1}{2}(m - j_0) - \frac{1}{2}\#\{\, j > j_0 \mid z_j = z_{j-1}\,\} + \frac{1}{2} \\ &= \gamma(e) - \gamma(f) + \frac{1}{2}\end{aligned}$$

as wanted.

The following points show how (f,e) may be reconstructed from (f', e'). Firstly,
(9.11.7) $j_0 = \min\{\, j \mid \ell(z'_j) > \ell(z'_{j-1})\,\}$

and then f is given by
(9.11.8) $z_j = z'_j$ $(n \geq j \geq j_0)$; $z_j = s_j \ldots s_{j_0-1} z'_{j_0}$ $(1 \leq j \leq j_0)$ (from (9.11.4)).

From (9.11.2) and (9.11.6),
(9.11.9) $i_j = \max\{\, i \mid 1 \leq i < i_{j+1}\ r_i = s_j\,\}$

$(1 \leq j \leq j_0)$ and e is now determined using (9.11.1) as follows:
(9.11.10) $y_i = r_{i_1} \ldots r_{i_j}$ where $j = \max\{k \mid i_k \leq i\}$ $(0 \leq i \leq m)$.

For comparison with other cases, this time we need merely note that
(9.11.11) $y''_m = 1$. □

9.12 Case 3. $l_0 < j_0$ or ($l_0 = j_0$ and $\ell(y_j) = l_0 + 1$ for some j with $0 \leq j \leq m$).

We begin be reformulating the conditions defining this case.

Suppose first that $l_0 < j_0$. Since $s_{j_0} \in \mathcal{L}(z_{j_0})$, we have $\ell(y_m) = \ell(z_1) = \ell(z_{j_0}) + j_0 - 1 \geq j_0 > l_0$. This implies that there exists j $(0 \leq j \leq m)$ with $\ell(y_j) = l_0 + 1$; this also holds (by assumption) if $l_0 = j_0$.

Hence we may define $p = \min\{\, j \mid 0 \leq j \leq m,\ \ell(y_j) = l_0 + 1\,\}$. Now $p > k_0$ and $\ell(y_0) \leq \ldots \leq \ell(y_{p-1}) < \ell(y_p)$, so by definition of k_0, there exists $p' > p$ with $\ell(y_{p'}) = l_0$. Take p' minimal with respect to these two conditions: then $\ell(y_{p'-1}) = l_0 + 1$ and $\ell(y_{p'}) = l_0$.

This shows that the conditions defining this case are equivalent to
(9.12.1) $l_0 \le j_0$ and for some j $(0 \le j \le m)$, $\ell(y_j) = l_0 + 1$ and $\ell(y_{j+1}) = l_0$.

For $j = 0, \ldots, l_0 + 1$, let $i_j = \min\{\, i \mid 0 \le i \le m,\ \ell(y_i) = j \,\}$. Then
(9.12.2) $y_i = r_{i_1} \ldots r_{i_j}$ where $j = \max\{\, k \mid i_k \le i \,\}$ $(0 \le i \le i_{l_0+1})$.

Since $e \in \mathcal{Q}'_y(1, z)$, it follows that
(9.12.3) $i_j = \max\{\, i \mid 1 \le i \le i_{j+1},\ r_i = r_{i_j} \,\}$ $(1 \le j \le l_0)$.
Now $f \in \mathcal{Q}_v(z, x)$ and $s_1 \in \mathcal{L}(z_1)$, so
(9.12.4) $s_i \in \mathcal{L}(z_i)$ $(1 \le i \le l_0)$.

This implies, since $z_j z_{j-1}^{-1} \in R$ $(2 \le j \le l_0)$, that
(9.12.5) $z_j = s_{j-1} z_{j-1}$ $(2 \le j \le l_0)$.

Let $p = \min\{\, j \mid m \ge j > i_{l_0+1}\ \ell(y_j) = l_0 \,\}$. Because $e \in \mathcal{Q}'_y(1, z)$, we have
(9.12.6) $p > i_{l_0+1} + 1$, $\ell(y_{p-1}) = l_0 + 1$.

Now $y_{i_{l_0+1}} = r_{i_1} \ldots r_{i_{l_0+1}}$. For $i_{l_0+1} < j \le p - 1$, $y_{j-1}^{-1} y_j \in \{1\} \cup R$ and $\ell(y_j) \ge l_0 + 1$, so by induction, we may write, for $i_{l_o+1} \le j \le p - 1$
(9.12.7) $y_j = r_{i_1} \ldots r_{i_{l_0+1}} x_j = y_{i_{l_0}} r_{i_{l_0+1}} x_j$ where $\ell(y_j) = l_0 + 1 + \ell(x_j)$.

Since $\ell(y_{p-1}) = l_0 + 1$, it follows that $y_{p-1} = r_{i_1} \ldots r_{i_{l_0+1}}$; since $\ell(y_p) < \ell(y_{p-1})$ and $y_{p-1}^{-1} y_p \in R$, it follows that
(9.12.8) $y_p = r_{i_1} \ldots r_{i_{l_0}}$.

But $r_p \in \mathcal{R}(y_p)$, and therefore
(9.12.9) $r_p = r_{i_{l_0}}$.

Now $y_{i_{l_0}} = r_{i_1} \ldots r_{i_{l_o}}$, and for $j > i_{l_0}$, $\ell(y_j) \ge l_0$ and $y_{j-1}^{-1} y_j \in \{1\} \cup R$. It follows that we may write
(9.12.10) $y_j = y_{i_{l_0}} x'_j$ where $\ell(y_j) = l_0 + \ell(x'_j)$ $(j \ge i_{l_0})$.

Taking $j = m$ in (9.12.10), using (9.12.4) and (9.12.5), we have
(9.12.11) $r_{i_j} = s_j$ $(1 \le j \le l_0)$.

For $i \ge i_{l_0+1}$, let

$$y'_i = \begin{cases} y_{i_{l_0}}^{-1} y_i & (i < p) \\ r_{i_{l_0+1}} r_{i_{l_0}} y_{i_{l_0}}^{-1} y_i & (i \ge p) \end{cases}$$

i.e.

$$y'_i = \begin{cases} r_{i_{l_0+1}} x_i & (i < p) \\ r_{i_{l_0+1}} r_{i_{l_0}} x'_i & (i \geq p) \end{cases}$$

Now $x_{p-1} = 1 = x'_p$, so $y'_{p-1} = r_{i_{l_0+1}}$ and $y'_p = r_{i_{l_0+1}} r_{i_{l_0}}$; also, $r_p = r_{i_{l_0}} \in \mathcal{R}(y'^{-1}_{p-1} y'_p)$. For $i \geq i_{l_0+1}$, we have

$$\mathcal{R}(y'_i) = \begin{cases} \{r_{i_{l_0+1}}\} & (i < p, \quad x_i = 1) \\ \mathcal{R}(x_i) & (i < p, \quad x_i \neq 1) \\ \{r_{i_{l_0}}\} & (i \geq p, \quad x'_i = 1) \\ \mathcal{R}(x'_i) & (i \geq p, \quad x'_i \neq 1) \end{cases}$$

noting that, by (9.12.7) and (9.12.10), $r_{i_{l_0+1}} \notin \mathcal{L}(x_i)$ $(i < p)$ and $r_{i_{l_0}} \notin \mathcal{L}(x'_i)$ $(i \geq p)$.

From this, we conclude that
(9.12.12) $\mathcal{R}(y'_j) = \mathcal{R}(y_j), \mathcal{L}(y'_j) = \{r_{i_{l_0+1}}\}$ $(i_{l_0+1} \leq j \leq m)$.

The remainder of the argument here is similar to that in Case 1.

Note that $r_1 \ldots r_{i_{l_0+1}} \not\leq s_1 \ldots s_{l_0-1}$, since $\ell(r_1 \ldots r_{i_{l_0+1}}) = i_{l_0+1} \geq l_0 + 1 > l_0 - 1 = \ell(s_1 \ldots s_{l_0-1})$. Let $k = \max\{\, i \mid 1 \leq i \leq i_{l_0+1},\ r_i \ldots r_{i_{l_0+1}} \not\leq s_1 \ldots s_{l_0-1} \,\}$ and set

$$y''_i = \begin{cases} 1 & (0 \leq i < k) \\ r_k \ldots r_i & (k \leq i \leq i_{l_0+1}) \\ r_k \ldots r_{i_{l_0+1}-1} y'_i & (m \geq i > i_{l_0+1}) \end{cases}$$

We define $e' = (y''_0, \ldots, y''_m) \in W^{m+1}$ and claim that $e' \in \mathcal{Q}'_y(1, y''_m)$. Firstly, note $y''^{-1}_{j-1} y''_j = y^{-1}_{j-1} y_j$ and $\mathcal{R}(y''_j) = \mathcal{R}(y_j)$ $(j > i_{l_0+1},\ j \neq p)$; it follows that e' satisfies $(9.8.2)_j$ $(j > i_{l_0+1}, j \neq p)$. But $y''^{-1}_{p-1} y''_p = r_{i_{l_0}} = r_p \in \mathcal{R}(y''_p)$, so $(9.8.2)_p$ holds, and $(9.8.2)_j$ clearly holds for $1 \leq j \leq i_{l_0+1}$.

Now $(9.8.3)_j$ holds for $1 \leq j < i_{l_0+1}$. If $i_{l_0+1} \leq j < p-1$ or $p \leq j \leq n-1$, then $\ell(y''_{j+1}) - \ell(y''_j) = \ell(y_{j+1}) - \ell(y_j)$; this shows that $(9.8.3)_j$ holds for such j (noting that, in case $j = i_{l_0+1}$, then $\ell(y_{j+1}) \geq \ell(y_j)$ by (9.12.6) and that, if $j = p$, then

$\ell(y_{j+1}) \geq \ell(y_j) = l_0$). Finally, $(9.8.3)_{p-1}$ holds since $\ell(y''_p) > \ell(y''_{p-1})$. Hence $e' \in \mathcal{Q}'_y(1, y''_m)$ as claimed.

Now define $p_0 = l_0$ and define $p_1, \ldots, p_{i_{l_0+1}-k}$ in turn by $p_i = \max\{\, l \mid 1 \leq l \leq p_{i-1},\ s_l = r_{i_{l_0+1}-i+1} \,\}$ (these are all defined since $r_{k+1} \ldots r_{i_{l_0+1}} \leq s_1 \ldots s_{l_0-1}$). Note that, since $r_k \ldots r_{i_{l_0+1}} \not\leq s_1 \ldots s_{l_0-1}$, $s_\ell \neq r_k$ for all $l < p_{i_{l_0+1}-k}$. Set $p_{i_{l_0+1}-k+1} = 0$ and define

$$z'_j = \begin{cases} z_j & (j \geq l_0) \\ r_{i_{l_0+1}-l} \dots r_{i_{l_0+1}} z_{l_0} & (p_l > j \geq p_{l+1},\ 0 \leq l \leq i_{l_0+1} - k) \end{cases}$$

Note that

$$\mathcal{L}(z_{l_0}) = \{s_{l_0}\} = \{r_{i_{l_0}}\}$$

and hence that

$$\mathcal{L}(z'_j) = \{r_{i_{l_0+1}-l}\}\ (p_l > j \geq p_{l+1},\ 0 \leq l \leq i_{l_0+1} - k).$$

An argument like the corresponding one from Case 1 now shows that $f'' = (z'_0, \dots, z'_n) \in \mathcal{Q}_w(z'_0, z'_n)$. Here, $z'_n = z_n = x$ and $z'_0 = r_k \dots r_{i_{l_0+1}} z_{l_0} = r_k \dots r_{i_{l_0+1}} s_{l_0-1} \dots s_1 z_1$; hence $y''_m = r_k \dots r_{i_{l_0+1}-1} r_{i_{l_0+1}} r_{i_{l_0}} y^{-1}_{i_{l_0}} y_m = z'_0$, since $y_{i_{l_0}} = s_1 \dots s_{l_0},\ s_{l_0} = r_{i_{l_0}}$.

Thus, we may set $\theta_x((f, e)) = (f', e')$. Now

$$\begin{aligned} \gamma(e) - \gamma(f) = \frac{1}{2}[\#\{\, j > i_{l_0+1} \mid y_j = y_{j-1} \,\} + (i_{l_0+1} - (l_0 + 1))] \\ - \frac{1}{2}\#\{\, j > l_0 \mid z_j = z_{j-1} \,\} \end{aligned}$$

while

$$\begin{aligned} \gamma(e') - \gamma(f') = \frac{1}{2}\#\{\, j > i_{l_0+1} \mid y''_j = y''_{j-1} \,\} + k - 1] \\ - \frac{1}{2}[\ \#\{\, j > l_0 \mid z'_j = z'_{j-1} \,\} + l_0 - (l(z'_0) - l(z'_{l_0}))]. \end{aligned}$$

But for $j > i_{l_0+1},\ y''_j = y''_{j-1}$ iff $y_j = y_{j-1}$, and also $l(z'_0) - l(z'_{l_0}) = i_{l_0+1} - k + 1$. Hence $\gamma(e') - \gamma(f') = \gamma(e) - \gamma(f) + \frac{1}{2}$ as desired.

We now show that (f', e') determines (f, e). Firstly,
(9.12.13) $k = \min\{\, j \mid \ell(y''_j) = 1 \,\}$.

Now for $i_{l_0+1} \leq i < p$,

$$\begin{aligned} \ell(y''_i) &= i_{l_0+1} - k + \ell(y'_i) \\ &= i_{l_0+1} - k + 1 + \ell(x_i) \\ &= i_{l_0+1} - k + \ell(y_j) - l_0 \end{aligned}$$

and for $p \le i$,

$$\begin{aligned}\ell(y_i'') &= i_{l_0+1} - k + \ell(y_i') \\ &= i_{l_0+1} - k + 2 + \ell(x_i') \\ &= i_{l_0+1} - k + 2 + \ell(y_j) - l_0.\end{aligned}$$

For $i_{l_0+1} \le i < p$, $\ell(y_i) \ge l_0+1$ so $\ell(y_i'') \ge i_{l_0+1}-k+1$, and for $i \ge p$, $\ell(y_j) \ge l_0$ so $\ell(y_i'') \ge i_{l_0+1} - k + 2$. Now $\ell(y''_{i_{l_0+1}}) = i_{l_0+1} - k + 1$. Since $p - 1 > i_{l_0+1}$ (by (9.12.6)) and $y''_{p-1} = r_k \ldots r_{i_{l_0+1}} = y''_{i_{l_0+1}}$, it follows that
(9.12.14) $i_{l_0+1} = \max\{\, i \mid k \le i \le m,\ \ell(y_k'') < \ldots < \ell(y_i''), \ell(y_j'') \ge \ell(y_i'') \text{ for all } j \ge i \,\}$ and that
(9.12.15) $p - 1 = \max\{\, i \mid \ell(y_i'') = \ell(y''_{i_{l_0+1}}) \,\}$

Then l_0 is determined by the condition
(9.12.16) $l_0 = \min\{\, j \mid \ell(z_j') = \ell(z_0') - i_{l_0+1} + k - 1 \,\}$.

Now from (9.12.5), we have
(9.12.17) $z_j = z_j'$ $(j \ge l_0)$; $z_j = s_j \ldots s_{l_0-1} z_{l_0}$ $(j < l_0)$

and so $f = (z_1, \ldots, z_n)$ is determined.

Now $i_{l_0} \ldots, i_1$, are determined (from (9.12.3), (9.12.11)) by
(9.12.18) $i_j = \max\{\, i \mid 1 \le i \le i_{j+1},\ r_i = s_j \,\}$ $(j = l_0, \ldots, 1)$.

For $i_{l_0+1} \le i \le m$, y_i' is given by
(9.12.19) $y_i' = r_{i_{l_0+1}-1} \ldots r_k y_i''$

and finally, $e = (y_0, \ldots, y_m)$ is given by

$$\text{(9.12.20)} \qquad y_i = \begin{cases} r_{i_1} \ldots r_{i_j},\ j = \max\{\, k \mid i_k \le i \,\} & (0 \le i \le i_{l_0+1}) \\ y_{i_{l_0}} y_i' & (i_{l_0+1} < i < p) \\ y_{i_{l_0}} r_{i_{l_0}} r_{i_{l_0+1}} y_i' & (i \ge p). \end{cases}$$

Hence (f', e') determines (f, e) as claimed.

In order to compare with other cases, define a, b, c, d as in (9.10.20). Here, we have $a \ge l_0$, hence $b = \ell(z_0) - \ell(z_a') \ge \ell(z_0') - \ell(z_{l_0}') = i_{l_0+1} - k + 1$. Further, $c = k - 1$ and, by (9.12.14), $d = i_{l_0+1}$. Hence
(9.12.21) $\ell(y_m'') \ge 1$ and $b \ge d - c$. □

9.13 In sections (9.10)–(9.12), we have constructed a function

$$\theta_x\colon \bigcup_{\substack{z\in W\\ sz<z}} (\mathcal{Q}_v(z,x)\times \mathcal{Q}'_y(1,z)) \longrightarrow \bigcup_{z\in W} (\mathcal{Q}_w(z,x)\times \mathcal{Q}'_y(1,z))$$

such that if $\theta_x((f,e))=(f',e')$, then $\gamma(e')-\gamma(f')=\gamma(e)-\gamma(f)+\frac{1}{2}$. Moreover, it has been shown that the restriction of θ_x to the set of $(e,f)\in\cup(\mathcal{Q}_v(z,x)\times \mathcal{Q}'_y(1,z))$ lying in each particular case (i.e. Case 1, Case 2 or Case 3) is injective. To complete the proof of the injectivity of θ_x, and hence the proof of (9.7), it will therefore suffice to show that if $(f,e)\in\cup(\mathcal{Q}_v(z,x)\times \mathcal{Q}'_y(1,z))$ and $\theta_x((f,e))=(f',e')$, then the case in which (f,e) lies (i.e. Case 1, 2 or 3) is determined by (f',e'). Let $f'=(z'_0,\ldots,z'_n)$ and $e'=(y''_0,\ldots,y''_m)$; define a,b,c,d as in (9.10.20). Then by (9.10.21), (9.11.11) and (9.12.21), the following are the only possibilities:

$$\begin{cases} \ell(y''_m)\geq 1 \text{ and } 1+b\leq d-c; & \text{here } (f,e) \text{ is in Case 1}\\ \ell(y''_m)=0; & \text{here, } (f,e) \text{ is in Case 2}\\ \ell(y''_m)\geq 1 \text{ and } b\geq d-c; & \text{here } (f,e) \text{ is in Case 3.}\end{cases}$$

The proof of Theorem (9.5) is now complete. □

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