MODULES FOR THE DUAL NIL HECKE RING

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Introduction

Fix a finitely generated Coxeter system (W, S) in a suitable reflection representation on a finite-dimensional real vector space V, and let $\mathcal{S}$ denote the symmetric algebra of V with natural W-action. Kostant and Kumar [26] have associated to this data a graded $\mathcal{S}$-algebra Λ (the dual nil Hecke ring) on which W acts as a group of graded $\mathcal{S}$-algebra automorphisms; let Λ^L (resp., $\mathcal{S}^L$) denote the ring of W_L-invariants of Λ (resp., $\mathcal{S}$), for any standard parabolic subgroup W_L of W. The definition of Λ was motivated by geometric questions; if W is the Weyl group of a suitable Kac-Moody group G, $\Lambda^L \otimes_{\mathcal{S}} \mathbb{R}$ is isomorphic to the cohomology ring of the generalized flag variety G/P_L for a standard parabolic subgroup P_L corresponding to W_L (see also [1]). This paper studies properties of Λ^L, and particularly some Λ^L-modules, which play an important role in the study of certain representation theories associated to the W-action on V.

These representation theories are defined for each suitable (finite here, for simplicity) interval Γ^L of minimal W_L-coset representatives in orders on W analogous to Chevalley (Bruhat) order. For finite Weyl groups, some of them are known via [32, 4] to be essentially blocks of Harish-Chandra bimodules for a semisimple complex Lie group or blocks of $\mathcal{O}$ for the corresponding semisimple complex Lie algebra, and they are conjecturally closely analogous in general. A construction of one category associated to Γ^L has been sketched in [21]. First, for $R^L = \mathcal{S}^L \otimes_{\mathbb{R}} \mathcal{S}$ (if W_L is finite) or $R^L = \Lambda^L$ (in general), one constructs an exact (in the sense of Quillen) category $\mathcal{C}^L$ of graded R^L-modules, with filtrations of a prescribed type having in particular certain R^L-modules N_x^L for $x \in \Gamma^L$ as successive subquotients. The category $\mathcal{P}^L$ of "projective" objects of $\mathcal{C}^L$ is equivalent to the category of finitely generated graded projective modules for a graded R^L-algebra $\mathcal{A}^L$. The algebra $\mathcal{A}^L$ and its module category are the objects of basic interest. Some of their properties can be established using "translation functors" between categories $\mathcal{C}^L$ and $\mathcal{C}^\emptyset$, induced by the usual restriction and extension of scalars between R^L-mod and $R^\emptyset$-mod.

The first goal of this paper is to provide proofs for a number of results stated without proof in [21], and used in an essential way in the theory surveyed there. First, for construction of the categories $\mathcal{C}^L$, one needs to compute $\mathrm{Ext}^1_{R^L}(N_x^L, N_y^L)$ for $x \neq y$ in Γ^L. Actually, we shall obtain more general results on Ext-groups which imply that $\mathcal{C}^L$ and $\mathcal{P}^L$ can be given a combinatorial description using the posets

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(with edges labelled by roots) studied in [18]. To define translation functors, one needs to know that for finite W_L, $R := R^{\emptyset}$ is free over R^L and that $\mathrm{Hom}_{R^L}(R, R^L) \cong R$ as R-module up to degree shift (these last facts are well-known for $R^L = \mathcal{S}^L \otimes \mathcal{S}$). The proof of compatibility of the two definitions of $\mathcal{C}^L$ and the translation functors for finite W_L (using $\mathcal{S}^L \otimes \mathcal{S}$ or Λ^L in the definition) requires the fact $\mathcal{S} \otimes_{\mathcal{S}^L} \Lambda^L \cong \Lambda$, together with a main result of Kostant-Kumar's on the structure of the nil Hecke ring, for which we provide a simpler proof suggested by the Ext-computations.

The most important open question about the representation categories discussed above is the Kazhdan-Lusztig conjecture for them. According to this, graded characters of "Verma-modules" in $\mathcal{A}^L$-mod are characterized by invariance under a combinatorially defined involution on the module of formal characters, together with some degree and support conditions. For finite W_L, these properties would follow from existence of suitable dualities (contravariant self-equivalences) of the categories $\mathcal{P}^L$, compatible with translation functors, together with some degree conditions on the projective indecomposable objects of $\mathcal{P}^L$ (with suitably normalized gradations). In this paper, we construct for each Γ^L an R^L-module $\mathbb{M}^L$ (actually in $\mathcal{C}^L$) which conjecturally functions as a "dualizing object" for $\mathcal{P}^L$, in the sense that $\mathrm{Hom}_{R^L}(?, \mathbb{M}_L)$ should be the desired contravariant equivalence on $\mathcal{P}^L$. Compatibility of these dualities with translation functors would then follow using the result $\Lambda \otimes_{\Lambda^L} \mathbb{M}^L \cong \mathbb{M}^{\emptyset}$ proved in this paper. The appropriate degree normalization of the projective indecomposables would simply be the one requiring them to be fixed under the duality. A much stronger conjecture concerning $\mathbb{M}^L$ (asserting roughly that all objects in $\mathcal{P}^L$ may be obtained by iterating extensions which "come from" $\mathbb{M}^L$) receives some support from results here which imply that it is true after localizing at any height one prime of $\mathcal{S}$.

In this paper, we study mainly modules for $R^L = \Lambda^L$ and for $R^L = \mathcal{S}^L \otimes_{\mathbb{R}} \mathcal{S}$; the categories $\mathcal{P}^L$, $\mathcal{C}^L$ and ring $\mathcal{A}^L$ are not even defined here and will be studied elsewhere (though we do describe here in a general setting a few simple properties of objects of $\mathcal{C}^L$ regarded just as R^L-modules). For crystallographic groups W, our main results all hold as well for an integral form $\Lambda_{\mathbb{Z}}$ of Λ (essentially, the graded ring of the filtered equivariant K-theory ring Ω of the flag variety of an associated Kac-Moody group). This gives rise to canonically associated highest weight representation theories (and others more akin to blocks of Harish-Chandra bimodules) over arbitrary fields, for instance, reducing to the ones from Λ_L for the field $\mathbb{R}$. A sequel to this paper will give the analogous results for Ω, needed for the definition and study of (probably closely related) integral representation theories which may be constructed from Ω itself (see [21]) For crystallographic groups W, several representation theories obtained from those mentioned above are known or conjectured to be closely related to more familiar representation theories arising in Lie theory; also, it seems likely that for such W, many of the objects constructed in this paper and its sequel have geometric significance in relation to the generalized Bruhat decompositions and Schubert-like varieties defined in [6].

We now describe the contents of this paper in some detail. Section 1 gives, mainly without proofs, a listing of some of the (mostly well known) properties of Coxeter groups and root systems used in this paper. We also recall from [16, 17] the definitions of the orders $\leq_A$ on W used in subsequent constructions; they generalize and are closely analogous to Chevalley order and its reverse. The class of reflection representations of Coxeter groups used is a natural one which includes those of Weyl groups of Kac-Moody Lie algebras and the standard reflection representations in [7] (more generally, also those from “root bases” in [28]). For a given non-crystallographic reflection representation of W, it may not be possible to choose a corresponding “reduced” root system. This phenomenon arises for reasons similar to those for existence of non-symmetrizeable generalized Cartan matices; it substantially complicates the statements and proofs of results, so in subsequent sections we work with reduced root systems and at one point briefly indicate how the main results can be extended to non-reduced ones.

The main results of the paper are given in Sections 2–6, where we study a fixed order $\leq_A$ on W, and a suitable subset Γ of W in the order $\leq_A$, with $W_L\Gamma \subseteq \Gamma$. Let Q be the quotient field of $\mathcal{S}$. In Section 2, we consider for $J \subseteq L$ (essentially) the ring H_J of functions $Q \to Q$ generated by left multiplications by elements of $\mathcal{S}$ and the BGG-Demazure operators $q \mapsto \alpha^{-1}(s_\alpha(q) - q)$ for simple roots α with the reflection $s_\alpha \in J$ (H_J is (anti)isomorphic to the “nil Hecke ring” of W_J as defined in [26]). We construct a H_J-module M_Γ which, in the case $\Gamma = W$ in Chevalley order and $W_J = W$, reduces to the left regular module for $H := H_S$. There is a similar result for suitable reflection subgroups W'_K of W (associated to the order $\leq_A$) with $\Gamma W'_K \subseteq \Gamma$. Actually, in Section 2, many of the results have analogues replacing the BGG-Demazure operators by operators $q \mapsto X\alpha^{-1}(s_\alpha(q) - q) + Y s_\alpha(q)$ (for suitable scalars X, Y in $\mathbb{R}$) arising in Lusztig's study [30, 31] of graded Hecke algebras (I wish to thank George Lusztig for suggesting this possibility). The arguments in Section 2 are given in a form that applies simultaneously to both situations.

In Section 3, we recall Kostant-Kumar's “dualization” of H to obtain the dual nil Hecke ring Λ with its operators from H, and then give a similar dualization of M_Γ to obtain a graded Λ-module Λ_Γ with operators from H_L. In particular, one has an action of W_L on Λ and Λ_Γ satisfying $w(\psi\psi') = w(\psi)w(\psi')$ for $w \in W_L$, $\psi \in \Lambda$ and $\psi' \in \Lambda_\Gamma$. If $\Gamma = W$ in Chevalley order, Λ_Γ reduces to the left regular module for Λ; many of the formulae we give for Λ_Γ are formally obtained simply by replacing Chevalley order by $\leq_A$ in the corresponding formulae for Λ as obtained in [26].

In Section 4, we describe a formalism suggested by the well-known Schubert calculus, and use it to obtain a number of previously mentioned results concerning the ring Λ^L and the Λ^L-module $(\Lambda_\Gamma)^L$ of W_L invariants (for finite posets, $(\Lambda_\Gamma)^L$ is the candidate “dualizing object” $\mathbb{M}^L$).

Section 5 gives a “local” characterization of the nil Hecke ring, which is used first to give a simple proof of another description of H from [26]. The local description of H is then extended to some closely related situations, and applied to give the computation 5.7 of the previously mentioned Ext-groups $\mathrm{Ext}^1(N^L_x, N^L_y)$ for the rings $R^L = \Lambda^L$ and, for finite W_L, $R^L = \mathcal{S}^L \otimes_{\mathbb{R}} \mathcal{S}$. We also give a basic technical fact 5.8 concerning the candidate dualizing object $\mathbb{M}^L$.

In Section 6, we study a homomorphism from the Iwahori-Hecke algebra of W to the split Grothendieck group of a category of (Λ, Λ)-bimodules; the results here do not extend in this form to $\Lambda_{\mathbb{Z}}$ or to the equivariant K-theory ring Ω discussed in the sequel to this paper. Much of Sections 5 and 6 depends only on properties of the nil Hecke ring and its dual proved in [26] and the first part of Section 4.

A brief Section 7, which also is largely independent of earlier sections, discusses the analogues for modules associated to polyhedral cones of some of the main results from Sections 2–5 (again, these are needed in the study of representation categories [21] which are not discussed here).

Some of the arguments used in Sections 2–7 require some easy general properties of modules with filtrations of a certain type; modules in the categories $\mathcal{C}^L$ previously mentioned (but not defined) are examples. These arguments have been collected together and given at a general commutative algebra level in Section 8; they do not depend on the previous sections at all. Some of the problems considered in this paper for Coxeter groups have also been formulated in a more general context in Section 8.

Sections 9–10 are appendixes to the paper. Section 9 discusses an extension of Matsumoto's well-known monoid lemma for Coxeter groups; it is useful in constructing versions associated to orders $\leq_A$ of standard objects parametrized by W. Using this result, we indicate a possible more conceptual approach to the construction of the modules from Section 2; unfortunately, this approach contains a gap which can at present only be filled in special cases.

In Section 10, we quote some general facts from commutative algebra and describe some results in the invariant theory of finite pseudoreflection groups arising as special cases. Though these results are not new (especially for Coxeter groups, in which case some of them may also be easily obtained using the Schubert calculus formalism as in Section 4), it seems difficult to find an explicit reference for them in the literature.

1. Root systems and orders on Coxeter groups

In this section, we summarize, with only occasional indications of proof, some basic properties of Coxeter groups and root systems, and facts we shall require on certain partial orders on Coxeter groups which generalize the well-known Chevalley (Bruhat) order on W and its reverse. As general references on Coxeter groups, one has [7] and [22]; for the standard Chevalley order, see [10].

1.1. Let k denote $\mathbb{R}$ or $\mathbb{Z}$. We say that a subset X of a free k-module V is pointed if there exists $\phi \in V^* := \operatorname{Hom}_k(V, k)$ with $\phi(\alpha) > 0$ for all $\alpha \in X$. We say that X is reduced if $m\alpha = n\beta$ for $\alpha, \beta \in X$ and $m, n \in k_{\geq 0}$ implies $m = n$.

Now fix two free k-modules V and V' with a given k-bilinear map $\langle -, - \rangle \colon V \times V' \to k$, and pointed, reduced subsets $\Pi \subseteq V$, $\Pi^\vee \subseteq V'$ with a given bijection $\alpha \mapsto \alpha^\vee \colon \Pi \to \Pi^\vee$ satisfying

$$\text{(i)} \qquad \langle \alpha, \alpha^\vee \rangle = 2 \text{ for all } \alpha \in \Pi.$$

For $\alpha \in \Pi$, define $s_\alpha \colon v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$ in $\operatorname{GL}(V)$, $S = \{s_\alpha\}_{\alpha \in \Pi}$, and $W = \langle S \rangle$ (the subgroup of $\operatorname{GL}(V)$ generated by S). Define $\Phi = \{\, w(\alpha) \mid \alpha \in \Pi,\ w \in W \,\}$ and $\Phi^+ = \{\, \gamma \in \Phi \mid \gamma \in \sum_{\alpha \in \Pi} k_{\geq 0} \alpha \,\}$. Dually, define $s_{\alpha^\vee} \in S' \subseteq W'$ in $\operatorname{GL}(V')$ and $\Phi^{\vee +} \subseteq \Phi^\vee \subseteq V'$. For $\alpha \neq \beta \in \Pi$, let $c_{\alpha,\beta} := \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in k$.

Lemma. *In the above situation, one has $\Phi = \Phi^+ \cup (-\Phi^+)$ iff* (ii)–(iii) *below hold;*
(ii) *for $\alpha \neq \beta \in \Pi$, $\langle \alpha, \beta^\vee \rangle \leq 0$. Further, $\langle \beta, \alpha^\vee \rangle = 0$ if $\langle \alpha, \beta^\vee \rangle = 0$.*
(iii) *for $\alpha \neq \beta \in \Pi$, either $c_{\alpha,\beta} \geq 4$ (in which case we set $m_{\alpha,\beta} = \infty$) or $c_{\alpha,\beta} = 4\cos^2 \frac{\pi}{m_{\alpha,\beta}}$ for some $m_{\alpha,\beta} \in \mathbb{N}_{\geq 2}$.*

Proof. The proof is very similar to that of the special case [13, 4.11], by reduction to rank two as in [12].

1.2. If the conditions (i)-(iii) of the previous section hold (and Π, $\Pi^\vee$ are pointed and reduced), as we now assume, we call (V, Π, Φ) and $(V', \Pi^\vee, \Phi^\vee)$ (dual) based root systems over k. One calls elements of Φ, Φ^+ and Π roots, positive roots and simple roots respectively. Elements of S are called simple reflections.

For integral dual based root systems (i.e. ones with $k = \mathbb{Z}$), we call V the weight lattice. From such integral root systems, one obtains dual based root system over $\mathbb{R}$ (and a natural identification of their corresponding groups W, W') by extension of scalars to $\mathbb{R}$ (i.e. regarding $V \subseteq V \otimes_\mathbb{Z} \mathbb{R}$ and similarly for V').

Though we mainly consider dual based root systems (V, Π, Φ) and $(V', \Pi^\vee, \Phi^\vee)$ over $k = \mathbb{R}$ in this paper, we sometimes assume they arise by extension of scalars from integral systems as above, and refer to their weight lattice (which is regarded as a lattice in V).

1.3. Consider fixed dual based root systems as above. Let $l' \colon W \to \mathbb{N}$ be the standard length function of (W, S), defined by $l'(w) = n$ if $w = r_1 \ldots r_n$ for some $r_i \in S$ with n minimal; then $r_1 \ldots r_n$ is called a reduced expression for w. For $r, s \in S$, define $n_{r,s} := \mathrm{ord}(rs) \in \mathbb{N} \cup \{\infty\}$. For any monoid M, a family of elements $\{x_s\}_{s \in S}$ of M is said to satisfy the braid relations for W if for each $r \neq s \in S$ such that $n_{r,s} \neq \infty$, one has $x_{r_1} \ldots x_{r_n} = x_{r_0} \ldots x_{r_{n-1}}$ where $n = n_{r,s}$ and $r_i = r$ for even i and $r_i = s$ for odd i. Let T be the set of W-conjugates of elements of S (the set of reflections of (W, S)) and regard the power set $\mathcal{P}(T)$ as an abelian group under symmetric difference $A + B = (A \cup B) \setminus (A \cap B)$. For $w \in W$, define $N(w) = \{\, t \in T \mid l'(tw) < l'(w) \,\} \in \mathcal{P}(T)$.

Lemma. (a) *For $\alpha, \beta \in \Pi$, n_{s_α, s_β} is equal to* 1 *if $\alpha = \beta$ and to $m_{\alpha,\beta}$ otherwise.*
(b) *(W, S) is a Coxeter system i.e.*

$$W \cong \langle\, S \mid (rs)^{n_{r,s}} = 1 \text{ for all } r, s \in S \text{ with } n_{r,s} \neq \infty \,\rangle$$

naturally.
(c) *The map $S \to S'$ given by $s_\alpha \mapsto s_{\alpha^\vee}$ extends to a (unique) isomorphism $W \to W'$ which we use to identify W' with W. Then $\langle w(v), v' \rangle = \langle v, w^{-1}(v') \rangle$ for $w \in W$, $v \in V$ and $v' \in V'$.*
(d) *The bijection $\Pi \to \Pi^\vee$ extends to a W-equivariant bijection $\Phi \to \Phi^\vee$. Define $s_\alpha \colon v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$ in $\mathrm{GL}(V)$ for $\alpha \in \Phi$. Then for any $w \in W$ and $\alpha \in \Phi$, $w s_\alpha w^{-1} = s_{w(\alpha)}$.*
(e) *The "reflection cocycle" $N \colon W \to \mathcal{P}(T)$ is characterized by $N(s) = \{s\}$ for $s \in S$ and $N(xy) = N(x) + xN(y)x^{-1}$ for $x, y \in W$. For $w \in W$, $\sharp(N(w)) = l'(w)$ and $N(w) = \{\, s_\alpha \mid \alpha \in \Phi^+ \cap w(-\Phi^+) \,\}$*
(f) (Monoid Lemma) *Suppose elements x_r of a monoid M, for $r \in S$, satisfy the braid relations of (W, S). Then there is a unique family of elements $\{x_w\}_{w \in W}$ of M such that $x_w = x_{r_1} \ldots x_{r_n}$ for any reduced expression $w = r_1 \ldots r_n$.*
(g) *For any $J \subseteq S$, let W_J be the standard parabolic subgroup of W generated by J, let $\Pi_J = \{\, \alpha \in \Pi \mid s_\alpha \in J \,\}$ and let $\Phi_J = \{\, w(\alpha) \mid \alpha \in \Pi_J,\ w \in W_J \,\}$. Then (V, Π_J, Φ_J), and the obvious dual (V', Π'_J, Φ'_J), are dual based root system with associated Coxeter system (W_J, J).*
(h) Φ *(or equivalently $\Phi^\vee$) is reduced (in the sense of* 1.1*) iff for all $\alpha \neq \beta \in \Pi$ with $m_{\alpha,\beta}$ finite and odd, $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle$. In that case, the map $\alpha \mapsto s_\alpha$ is a bijection $\Phi^+ \to T$, and if $w = s_{\alpha_1} \ldots s_{\alpha_n}$, $\alpha_i \in \Pi$, is a reduced expression for $w \in W$, then $\Phi^+ \cap w(-\Phi^+) = \{\beta_1, \ldots, \beta_n\}$ where $\beta_i := s_{\alpha_1} \ldots s_{\alpha_{i-1}}(\alpha_i)$.*

Proof. The proofs of these facts from the decomposition $\Phi = \Phi^+ \cup (-\Phi^+)$ of the root system into positive and negative roots are well known [35, 12] if the root system is reduced. In general, one can prove the following two claims together by induction on n;

(i) for $\alpha, \alpha_i, \beta \in \Pi$ with $s_{\alpha_1} \dots s_{\alpha_n}(\alpha) = c\beta$ for some $c \in \mathbb{R}_{>0}$ and with $s_{\alpha_1} \dots s_{\alpha_n}$ reduced, one has $s_{\alpha_1^\vee} \dots s_{\alpha_n^\vee}(\alpha^\vee) = c^{-1}(\beta^\vee)$
(ii) any reduced expression of an element w of W can be converted to any other reduced expression for w by successive application of the braid relations, if $l'(w) \leq n$.

For each n, one first proves (i) by a reduction to the dihedral case similar to that in the proof of Lemma 1.1, and then proves (ii) essentially by the usual proof of the monoid lemma. Once (i) and (ii) are proved, the rest of the proof is standard.

Remarks. The W-conjugates of the subgroups W_J in (g) are called parabolic subgroups of W. Note that Φ is necessarily reduced (in the sense of 1.1) if $k = \mathbb{Z}$, by (h).

1.4. Define the "fundamental chambers" $C = \{ v \in V \mid \langle v, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi \}$ and $C' = \{ v' \in V' \mid \langle \alpha, v' \rangle \geq 0 \text{ for all } \alpha \in \Pi \}$ of W on V and V'. A standard argument [23, 3.12] shows that

(1) each W-orbit on the "Tit's cone" $\cup_{w \in W} w(C)$ contains a unique point of C, and the stabilizer in W of $v \in C$ is generated by the reflections s_α which fix v.

By (1), no non-trivial element of W fixes C elementwise iff

(i) for each $\alpha \in \Pi$, there exists $\phi \in C$ with $\langle \phi, \alpha^\vee \rangle > 0$.

If (i) holds, we say C is sufficiently large. If V is of finite dimension over $k = \mathbb{R}$ and Π is finite, C is sufficiently large iff the interior of C (in the Euclidean topology) is non-empty.

An element w of W is called a pseudoreflection if $(1 - w)(V)$ is a free k-module of rank one. The following sufficient conditions (b) for all pseudoreflections in W to be reflections (i.e. of the form s_α for $\alpha \in \Phi^+$) are implicit in [26, 4.8], from which the following proof is adapted.

Lemma. (a) *Any pseudoreflection $w \in W$ of order 2 is a reflection i.e. equal to s_α for some $\alpha \in \Phi^+$.*
(b) *If the fundamental chamber of W on V is sufficiently large, then every pseudoreflection of W on V is a reflection.*

Proof. The case $k = \mathbb{Z}$ follows from the case $k = \mathbb{R}$, so we assume $k = \mathbb{R}$. In general, it is known ([33], [11]) that any involution w in a Coxeter group can be expressed as a product of commuting reflections, say $w = s_{\beta_1} \dots s_{\beta_n}$ for $\beta_i \in \Phi^+$. One must have $\langle \beta_i, \beta_j^\vee \rangle = 0$ for $i \neq j$, hence $(1 - w)(V) = \oplus_i \mathbb{R}\beta_i$. This makes (a) obvious. For (b), note that there is no loss of generality in assuming that the fundamental chamber on V' is sufficiently large (see the remark below). It is enough by (a) to show that if $w \in W$ is pseudoreflection, then $w^2 = e$. Write $w = s_{\alpha_1} \dots s_{\alpha_n}$ with $\alpha_i \in \Pi$. The standard parabolic subgroup of W generated by the s_{α_i} acts faithfully on the $\mathbb{R}$-subspace U of V spanned by the α_i (using 1.2(e), for instance). On U, w has determinant $(-1)^n$ and fixes a hyperplane pointwise, so, considering the Jordan form of w (on U), either $w^2 = e$ or $(w - e)^2 = 0$. Suppose for a contradiction that $e - w = -(e - w^{-1})$. By assumption, there exist $\phi \in C$ and $\phi' \in C'$ with

$\langle \alpha_i, \phi' \rangle > 0$ and $\langle \phi, \alpha_i^\vee \rangle > 0$ for $i = 1, \dots, n$. A standard argument [23, 3.12] shows that $(e-w)(\phi)$ and $(e-w^{-1})(\phi)$ are expressible as non-zero non-negative $\mathbb{R}$-linear combinations of the α_i so $0 < \langle (e-w)\phi, \phi' \rangle = -\langle (e-w^{-1})\phi, \phi' \rangle < 0$.

Remark. Note that if U, U' are free k-submodules of V containing Π, $\Pi^\vee$ respectively, one gets dual based root systems (U, Π, Φ) and $(U', \Pi^\vee, \Phi^\vee)$, with associated Coxeter system naturally identified with that of the original root systems (by restriction of W-action to U and U'). Call the new dual based root systems restrictions of the original ones. The following observation is useful.

(2) any dual based root systems (U, Π, Φ) and $(U', \Pi^\vee, \Phi^\vee)$ may be regarded as restrictions of dual based root systems (V, Π, Φ) and $(U', \Pi^\vee, \Phi^\vee)$ such that the fundamental chamber on V is sufficiently large. In fact, one may take $V := U \oplus k\phi$ for any $\phi \in \mathrm{Hom}_k(U', k)$ with $\langle \phi, \Pi^\vee \rangle \subseteq k_{>0}$.

1.5. We now recall from [16,17] the main facts we shall need concerning certain partial orders $\leq_A$ on W. Fix an "initial section of a reflection order" $A \subseteq T$, where T is the set of reflections; we don't repeat the definition here, but mention that the basic example is $A = \{ s_\alpha \mid \alpha \in \Phi^+ \cap P \}$ where P is the cone of positive elements of some vector space total ordering of V (if $k = \mathbb{R}$). We let $l_A : W \to \mathbb{Z}$ denote the length function defined by $l_A(w) = l'(w) - 2\sharp(N(w^{-1}) \cap A)$. Let $\leq_A$ denote the associated order on W as in [16]; thus, $\leq_A$ is the partial order on W generated by the relations

$$x \leq_A s_\alpha x \quad \text{if } x \in W,\ \alpha \in \Phi^+ \text{ and } l_A(x) < l_A(s_\alpha x). \tag{1}$$

For $A = \emptyset$ (resp., $A = T$), $\leq_A$ is Chevalley order on W (resp., its reverse).

Let $\mathcal{P}_A$ be the set of non-empty, finite, closed intervals $[u, v] := \{ x \in W \mid u \leq_A x \leq_A v \}$ in W in the order $\leq_A$ such that the open interval $[u, v] \setminus \{u, v\}$ has a combinatorial sphere as its order complex (if it is non-empty). Any interval in $\mathcal{P}_A$ is finite and all of its non-empty closed subintervals are also in $\mathcal{P}_A$. In this paper, we use mainly the recursive characterization of $\mathcal{P}_A$ given by (d) of the following proposition.

Proposition. (a) *For $x \leq_A w$, there is a chain $x = x_0 \leq_A \dots \leq_A x_n = w$ with $l_A(x_n) = l_A(x) + n$.*
(b) *For $x \in W$, $s \in S$, one has $l_A(sx) = l_A(x) \pm 1$*
(c) (the "Z-property") *for $x, y \in X$ and $s \in S$ with $sx <_A x$, $sy <_A y$ one has $sx \leq_A sy$ iff $x \leq_A y$ iff $sx \leq_A y$.*
(d) *$\mathcal{P}_A$ is the smallest set of non-empty closed intervals in W in the order $\leq_A$ which contains $[e, e]$ and satisfies $[sx, sy] \in \mathcal{P}_A \iff [x, y] \in \mathcal{P}_A \implies [sx, y] \in \mathcal{P}_A$ for all $s \in S$, $x, y \in W$ with $sx <_A x$, $sy <_A y$ (see* [16, 2.5]*).*

Remark. It is known that for some A (e.g. $A = W_K \cap T$ for some $K \subseteq S$) every closed interval in W in the order $\leq_A$ is in $\mathcal{P}_A$. In particular, this holds for Chevalley order and its reverse.

1.6. We define a spherical poset in the order $\leq_A$ to be a subset Γ of W in the order $\leq_A$ such that (i) below holds:

(i) for any $u \leq_A v$ in Γ, one has $[u, v] \subseteq \Gamma$ and $[u, v] \in \mathcal{P}_A$.

For any $\Gamma \subseteq W$ and any standard parabolic subgroup W_L of W, define $\Gamma^L = \{ w \in \Gamma \mid l_A(sw) \geq l_A(w) \text{ for all } s \in J \}$.

Lemma. *Let $\Gamma \subseteq W$ be spherical in the order $\leq_A$, and fix $L \subseteq S$.*
(a) *For any any $w \in \Gamma^L$, the map $x \mapsto xw$ is an order-isomorphism between W_L (in Chevalley order) and $W_L w$ (in the order induced by $\leq_A$) satisfying $l_A(xw) = l'(x) + l_A(w)$ for all $x \in W_L$. Moreover, $W_L\Gamma^L$ is spherical.*
(b) *If $W_L\Gamma^L \subseteq \Gamma^L$ and either W_L is finite or $A = \emptyset$, then $W_L\Gamma^L = \Gamma$*
(c) *For $K \subseteq L$, any $x \in W_L\Gamma^L$ may be uniquely written $x = x_K x^K$ where $x_K \in W_K$ and $x^K \in \Gamma^K$.*

Remark. These results apply in particular to Chevalley order, where they are well known. We write W^J for $\{ w \in W \mid l'(xw) = l'(x) + l'(w) \text{ for all } x \in W_J \}$.

2. Modules for the nil Hecke ring

In this section, we construct some modules for Kostant-Kumar's nil Hecke ring (case 2.2(i) below); these play an important role in the rest of the paper. We also obtain by the same procedure modules for (the analogue for general Coxeter systems of) Lusztig's graded affine Hecke algebra (case 2.2(ii) below), but these are not used subsequently in this paper.

2.1. Throughout this paper, we fix dual based root systems (V, Π, Φ), $(V', \Pi^\vee, \Phi^\vee)$ over $\mathbb{R}$ with associated Coxeter system (W, S). Unless otherwise stated, we assume that V and V' are finite-dimensional, that S is finite and that Φ (and hence $\Phi^\vee$) is reduced. Let $\mathcal{S} = \sum_{n\in\mathbb{N}} \mathcal{S}_n$ denote the symmetric algebra of V over $\mathbb{R}$, graded so $\mathcal{S}_0 = \mathbb{R}$ and $\mathcal{S}_2 = V$; thus, $\mathcal{S}$ is non-canonically isomorphic to a graded polynomial ring over $\mathbb{R}$, in $\dim V$ indeterminates of degree 2. If our based root system arises by extension of scalars from an integral based root system, we define $\mathcal{S}_\mathbb{Z}$ to be the the symmetric algebra over $\mathbb{Z}$ of the weight lattice, naturally regarded as a graded subring of $\mathcal{S}$.

Fix a partial order $\leq_A$ on W associated to an initial section of a reflection order. We sometimes denote $l_A(w) - l_A(v)$ by $l_A(v, w)$. Write $\gamma_A(u, w)$ (resp., $u \xrightarrow{\gamma}_A w$ to indicate that $\gamma \in \Phi^+$, $u <_A w \in W$ and $u = s_\gamma w$ (resp., that $\gamma_A(u, w)$ and $v \lessdot_A w$, where $v \lessdot_A w$ means $v \leq w$ and $l_A(v) = l_A(w) - 1$). If we need to make dependence of some object or relation on the "parameter" A explicit, we will attatch a subscript or superscript A. If there is no danger of confusion, we may omit the subscript or superscript when referring to $\leq_A$. In particular, we write $\leq$ for $\leq_A$ and l for l_A; standard Chevalley order will be denoted $\leq_\emptyset$ or $\leq'$, and the standard length function is $l' = l_\emptyset$.

2.2. Fix elements $\{X_\alpha\}_{\alpha\in\Phi}$ and X in the subalgebra of $\mathcal{S}$ generated by W-invariant elements of V, such that for $\alpha \in \Phi$, one has $X_\alpha = X_{w\alpha}$ for all $w \in W$, and such that either (i) or (ii) below holds.

(i) $X = 0$ and all $X_\alpha = 1$. In this case, let $\theta\colon V \to V$ be the identity map.
(ii) $X \neq 0$ and there is a direct sum decomposition $V = U_1 \oplus U_2 \oplus \mathbb{R}X$ such that $\Pi \subseteq U_1$, the distinct X_α form a basis of U_2, and $\langle U_2 \oplus \mathbb{R}X, \Phi^\vee\rangle = 0$. In this case, let $\theta\colon V \to V$ be the $\mathbb{R}$-linear map fixing $U_1 \oplus U_2$ pointwise and with $\theta(X) = -X$.

In situation (ii), one regards U_1 as given and the X_α, X as indeterminates over the symmetric algebra of U_1. Given the reflection representation of W on U_1, one could for instance extend it to one on a suitable V as above so that $X_\alpha = X_\beta$ iff α, β are in the same W-orbit on Φ.

2.3. Let Q denote the quotient field of $\mathcal{S}$. There is a natural action of W as a group of $\mathbb{R}$-algebra automorphisms of $\mathcal{S}$ and of Q extending the natural action on V. This action fixes X and all X_α for $\alpha \in \Phi$. We also extend θ to a $\mathbb{R}$-algebra automorphism θ of $\mathcal{S}$ and of Q. Note that in either case 2.2(i)–(ii), θ commutes with each element of W on Q, fixes each simple root, fixes the X_α with $\alpha \in \Phi$, and satisfies $\theta(X) = -X$. We let $\mathbb{R}[X_\alpha, X]$ denote the $\mathbb{R}$-subalgebra of $\mathcal{S}$ generated by X and all the X_α with $\alpha \in \Phi$.

2.4. Define elements $S_x = S_x^A$ of Q for $x \in W$ by

$$(1) \qquad S_x^A = \prod_{\alpha \in \Phi^+ \cap x(-\Phi^+)} \left(\frac{\epsilon_A(\alpha, x)\alpha}{X_\alpha + \epsilon_A(\alpha, x)\alpha X} \right)^{\epsilon_A(\alpha, x)}$$

where $\epsilon_A(\alpha, x)$ denotes 1 if $x <_A s_\alpha x$ and -1 otherwise. Then using 1.3(h),

$$(2) \qquad S_{s_\alpha x}^A = \frac{X_\alpha - \alpha X}{-\alpha} s_\alpha(S_x^A) \qquad \text{if } \alpha \in \Pi \text{ and } \ s_\alpha x >_A x,$$

$$(3) \qquad (S_x^A)^{-1} = (-1)^{l'(x)} \theta(S_x^{T \setminus A}).$$

2.5. We now define elements $S_{x,w} = S_{x,w}^A$ of Q for $[x, w] \in \mathcal{P}_A$.

Lemma. *There is a unique family of elements $S_{x,w}^A \in Q$, defined for $[x, w] \in \mathcal{P}_A$, such that $S_{x,w}^A = S_x^A$ if $x = w$, and such that for any $\chi \in V$,*

$$(1) \qquad (\chi - xw^{-1}(\chi)) S_{x,w}^A = - \sum_{\gamma_A(u,w)} \langle \chi \mid \gamma^\vee \rangle X^{l_A(u,w)-1} X_\gamma S_{x,u}^A$$

if $x <_A w$. Here and later, any term $S_{x,u}^A$ with $x \not\leq_A u$ is interpreted as zero, and we interpret X^0 as 1 if $X = 0$ (i.e. in case 2.2(i)).

Proof. Fix a point $\chi_e \in V$ with no W-isotropy, and set $\chi_w = w(\chi_e)$ for all $w \in W$. Then $v(\chi_w) \neq \chi_w$ for all $v \neq e$ and w in W. There is obviously a unique family of elements $S_{x,w}$ defined for $[x, w] \in \mathcal{P}$ such that $S_{x,x} = S_x$ and

$$S_{x,w} = - \sum_{\gamma(u,w)} \langle \chi_w \mid \gamma^\vee \rangle (\chi_w - xw^{-1}(\chi_w))^{-1} X_\gamma X^{l(u,w)-1} S_{x,u}$$

if $x < w$. For example,

$$(2) \qquad S_{x,w} = -X_\gamma S_x / \gamma \qquad \text{if } x \xrightarrow{\gamma} w.$$

Fix $x \in W$ and $r \in S$ with $rx \geq x$, say $r = s_\alpha$ with $\alpha \in \Pi$. We will show by induction on $l(u)$ that for $u \geq x$ in W with $ru > u$ and $[x, u] \in \mathcal{P}$, one has

$$(3) \qquad X_\alpha S_{x',u} + (X_\alpha - \alpha X) r(S_{rx',u}) = -\alpha S_{x',ru} \quad \text{if } x' \in \{x, rx\}.$$

Now if $l(u) = l(x)$, then $u = x$ and (3) follows from (2) and 2.4(2). Suppose that $w > x$, $rw > w$, $[x, w] \in \mathcal{P}$ and (3) holds for all u with $x \leq u < w$ and $ru > u$. It follows that

$$(4) \qquad X_\alpha S_{x',ru} + (X_\alpha - \alpha X) r(S_{rx',ru}) = -\alpha X^2 S_{x',u} \qquad \text{if } x' \in \{x, rx\}$$

for such u. Note that for $r \in S$, $w \in W$ with $rw >_A w$, the map

$$\{\,(u,\gamma) \mid ru >_A u,\ \gamma(u,w)\,\} \to \{\,(v,\beta) \mid v \neq w,\ \beta(v,rw)\,\}$$

given by $(u, \gamma) \to (ru, r(\gamma))$ is a bijection. Set $\chi = \chi_w$. Then

$$\begin{aligned}
&\Big(r(\chi) - x'w^{-1}(\chi)\Big)\Big(X_\alpha S_{x',w} + (X_\alpha - \alpha X) r(S_{rx',w})\Big)\\
&= (r\chi - \chi) X_\alpha S_{x',w} + \Big(\chi - x'w^{-1}(\chi)\Big) X_\alpha S_{x',w}\\
&\qquad + (X_\alpha - \alpha X) r\Big[\Big(\chi - rx'w^{-1}(\chi)\Big) S_{rx',w}\Big]\\
&= -\langle\, \chi \mid \alpha^\vee \,\rangle \alpha X_\alpha S_{x',w}\\
&\qquad - \sum_{\gamma(u,w)} \langle \chi \mid \gamma^\vee \rangle X_\gamma X^{l(u,w)-1}\Big(X_\alpha S_{x',u} + (X_\alpha - \alpha X) r(S_{rx',u})\Big)\\
&= \alpha\Bigg(\langle\, r(\chi) \mid \alpha^\vee \,\rangle X_\alpha S_{x',w}\\
&\qquad + \sum_{\gamma(u,w)} \langle\, r(\chi) \mid r(\gamma^\vee)\,\rangle X_\gamma X^{l(ru,w)} S_{x',ru}\Bigg) \qquad \text{by (3), (4)}\\
&= \alpha \sum_{\beta(v,rw)} \langle\, r(\chi) \mid \beta^\vee \,\rangle X_\beta X^{l(v,rw)-1} S_{x',v} \qquad \text{from above}\\
&= -\alpha\Big(r(\chi) - x'(rw)^{-1} r(\chi)\Big) S_{x',rw} \qquad \text{since } \chi_{rw} = r(\chi)
\end{aligned}$$

which proves (3) for $u = w$ since $r(\chi) - x'w^{-1}(\chi) \neq 0$.

Note that by (3), (4) we have now established the recurrence formula

$$(5) \qquad X_\alpha S^A_{x',w} + (X_\alpha - \alpha X) r(S^A_{rx',w}) = \begin{cases} -\alpha S^A_{x',rw} & \text{if } rw >_A w \\ -\alpha X^2 S^A_{x',rw} & \text{if } rw <_A w \end{cases}$$

which holds for $\alpha \in \Pi$, $r = s_\alpha$, $[x, w] \in \mathcal{P}_A$ with $rx >_A x$ and $x' \in \{x, rx\}$. It follows using 1.5(d) that for $[x, w] \in \mathcal{P}$ the value of $S_{x,w} \in Q$ is uniquely determined by this recurrence equation and the initial condition $S_{e,e} = 1$. In particular $S_{x,w}$ is independent of the initial choice of χ_e. As one varies χ_e over the points in V with no W-isotropy, $\chi_w = w(\chi_e)$ ranges over a dense (in the Euclidean toplology) subset of V. The equation (1) holds if $\chi = \chi_w$, and both sides are linear in χ, so (1) holds for all $\chi \in V$ as required.

2.6. In the situation 2.2(i), for Chevalley order and its reverse, the following result was proved in [19].

Proposition. *Fix* $[x,w] \in \mathcal{P}_A$. *Then*
(a) $S^A_{x,w} = (-1)^{l_A(x,w)} S^A_x h^A_{x,w}/g^A_{x,w}$ *for some* $h^A_{x,w}, g^A_{x,w} \in \mathcal{S}$ *expressible as non-zero linear combinations (with positive real coefficients) of products of elements from* $\Pi \cup \{X_\alpha, X\}$*; in particular,* $S^A_{x,w} \neq 0$.
(b) *There exist elements* $f^A_{x,w} \in \mathcal{S}$ *such that* $S^A_{x,w} = S^A_x f^A_{x,w} \big(\prod_{\substack{\alpha\in\Phi^+\\ x<_A s_\alpha x \leq_A w}} \alpha\big)^{-1}$
(c) *In* (a), $h^A_{x,w}$, $g^A_{x,w}$ *may be chosen so as to have expressions as homogeneous real polynomials in elements of* $\Pi \cup \{X_\alpha\} \cup \{X\}$ *(with respect to the grading giving elements of* Π *degree* 2*, the* X_α *degree* 0 *and, in case 2.2(ii),* X *degree* -2*), with degrees satisfying* $\deg(g^A_{x,w}) - \deg(h^A_{x,w}) = 2l_A(x,w)$.

Proof. For the proof, one may assume without loss of generality that the fundamental chamber for W on V is sufficiently large, by 1.4(2). Parts (a) and (c) are proved by induction on $l_A(x,w)$ by considering 2.5(1) for points $\chi \in V$ taken in the interior of the fundamental chamber on V. A proof of a more general version of (b) is given in 8.14 (cf. also 7.5). We omit the details, which are esentially the same as in [19].

2.7. Comparing degrees in the two expressions for $S^A_{x,w}$ in this proposition in case 2.2(i) immediately gives the following. For $A = \emptyset$, it reduces to a conjecture of Deodhar proved in [19].

Corollary. *For any* $[v,w] \in \mathcal{P}_A$, $\sharp\{\alpha \in \Phi^+ \mid v <_A s_\alpha v \leq_A w\} \geq l_A(v,w)$.

2.8. Define the ring Q_W as in [26, (4.1)]. Then Q_W is a free right Q-module with basis $\{\delta_w\}_{w\in W}$ and the multiplication is determined by

$$(\delta_v q_v)(\delta_w q_w) = \delta_{vw}(w^{-1}q_v)q_w \quad \text{for } v,w \in W \text{ and } q_v, q_w \in Q.$$

The δ_w are also a basis of Q_W in the left Q-module structure $q(\delta_w q_w) = \delta_w(w^{-1}q)q_w$ of Q_W. We identify Q with the subring $Q\delta_e = \delta_e Q$ of Q_W. Note that Q is not central in Q_W but X and the X_α are central. For $J \subseteq S$, define the subring $Q_{W_J} := \sum_{w\in W_J} \delta_w Q$ of Q_W. Also define $\tilde{Q}_W$ to be the set of unrestricted formal linear combinations $\sum_{w\in W} \delta_w q_w$ with all $q_w \in Q$. This may be naturally regarded as a (Q_W, Q_W)-bimodule.

2.9. For a poset X, X^{op} denotes the opposite poset of X. An upper ray in X is a set $\{x \in X \mid x \geq y\}$ for some fixed $y \in X$. A (finitely generated) coideal of X is a (finite) union of upper rays. Lower rays and (finitely generated) ideals of X are defined dually, replacing $\geq$ by $\leq$. Frequently, we consider formal sums $\sum_{x\in X} a_x$ of elements of a module where the support of a_x (i.e. the set of $x \in X$ with $a_x \neq 0$) is required to be a subset of some finitely generated coideal (resp., ideal) of X (we say the a_x, or the sum, is supported on a finitely generated coideal (resp., ideal); similarly, we define functions from Γ to the module which are supported on a finitely generated coideal (resp., ideal)). A sum as above will be written $\sum^{\uparrow}_{x\in X} a_x$ (resp., $\sum^{\downarrow}_{x\in X} a_x$) to indicate the assumed condition on its support. To obviate the need for case-by-case discussions of convergence, we introduce in the next subsection a topology on the most frequently occurring (in this paper) spaces of such formal sums.

2.10. For the remainder of this paper, unless otherwise stated, we fix a (non-empty) spherical subset Γ of W and some $L \subseteq S$ with $W_L\Gamma \subseteq \Gamma$. In referring to this fixed Γ, we write $\gamma_A(u,w)$ or $\gamma(u,w)$ to indicate that $u, w \in \Gamma$, $\gamma \in \Phi^+$ and $u <_A w = s_\gamma u$ (previously we allowed $u, w \in W$), and similarly for $u \xrightarrow{\gamma}_A w$.

For any commutative ring B and any symbol D, introduce formal symbols D_w for $w \in \Gamma$ and define the right B-module $K_B(\Gamma, D)$ of formal B-linear combinations $\sum_{w\in\Gamma} D_w b_w$ with all $b_w \in B$. This has a B-submodule $K = K_B^{\downarrow}(\Gamma, D)$ consisting of those sums $\sum_{w\in\Gamma}^{\downarrow} d_w b_w$ in $K_B(\Gamma, D)$ supported in a finitely generated ideal of Γ. For a finitely generated coideal X of Γ, let K_X denote the B-submodule of elements $\sum_{w\in\Gamma}^{\downarrow} D_w b_w$ in K such that $b_w = 0$ if $w \in X$. Then K/K_X is a free B-module (with basis consisting of the cosets $D_x + K_X$ for $x \in X$). Give $K = K_B^{\downarrow}(\Gamma, D)$ the linear topology in which the sets K_X, for finitely generated coideals X, form a basis of neighbourhoods of 0. For instance, if Γ is finitely generated as ideal (in particular, in the case $A = \emptyset$ of Chevalley order on $\Gamma = W$), then K has the discrete topology and the elements D_w for $w \in W$ form a right B-module basis of K. Note that if $\phi\colon K_B^{\downarrow}(\Gamma, D) \to K_B^{\downarrow}(\Gamma, D)$ is a continuous B-linear map, then

$$\phi\Big(\sum_w^{\downarrow} D_w b_w\Big) = \sum_w^{\downarrow} \phi(D_w) b_w \tag{1}$$

in the sense that the (countable) sum on the right hand side converges to the left hand side (in any order). For example,

(2) for $u \in W_L$, there is a continuous B-linear map $K \to K$ given by $\sum_w^{\downarrow} D_w b_w \mapsto \sum_w^{\downarrow} D_w b_{uw}$.

Indeed, it is enough to check this when $u \in L$, when it follows easily using the Z-property of the order $\leq_A$.

For use later, we also define the topological right B-module $K_B^{\uparrow}(\Gamma, E)$ by setting it equal to $K_B^{\downarrow}(\Gamma^{\mathrm{op}}, E)$ for any formal symbol E. If B is given the discrete topology, there is a bilinear map $K_B^{\uparrow}(\Gamma, E) \times K_B^{\downarrow}(\Gamma, D) \to B$ given by

$$\Big(\sum_w^{\uparrow} E_w b_w, \sum_w^{\downarrow} D_w b'_w\Big) \mapsto \sum_w b_w b'_w \tag{3}$$

which is readily checked to induce an isomorphism of B-modules

$$K_B^{\uparrow}(\Gamma, E) \to \mathrm{Hom}_{\mathrm{cont.}\ B}(K_B^{\downarrow}(\Gamma, D), B), \tag{4}$$

where on the right we have the continuous B-module homorphisms. We define $K_B^{\downarrow}(\Gamma^J, D)$ and $K_B^{\uparrow}(\Gamma^J, E)$ similarly for any standard parabolic subgroup W_J with $W_J\Gamma \subseteq \Gamma$.

2.11. For $\alpha \in \Pi$, define

$$t_\alpha = -X_\alpha \alpha^{-1}(\delta_{s_\alpha} + \delta_e) + X\delta_{s_\alpha} \in Q_W, \tag{1}$$

where e is the identity of W, and note that $t_\alpha^2 = X^2\delta_e$. For any $J \subseteq S$, let H_J be the subring of Q_{W_J} generated by $\mathcal{S}$ and the t_α such that $s_\alpha \in J$. Regard all the H_J, in particular $H := H_S$, as topological rings with the discrete topology. Also, define the topological right $\mathcal{S}$-module $M = M_\Gamma := K_{\mathcal{S}}^{\downarrow}(\Gamma, m^A)$ of formal $\mathcal{S}$-linear combinations $\sum_{w\in\Gamma}^{\downarrow} m_w^A q_w$ of symbols m_w^A for $w \in \Gamma$.

Proposition. *The map* $\sum^{\downarrow}_{w\in\Gamma} m^A_w q_w \mapsto \sum_{x\in\Gamma}\delta_x\big(\sum_{\substack{w\in\Gamma\\ w\geq x}} q_w x^{-1}(S^A_{x,w})\big)$ *gives an injective homomorphism* $M \to \tilde{Q}_W$ *of right* $\mathcal{S}$*-modules, the image of which is a* $(H_L,\mathcal{S})$*-subbimodule of* $\tilde{Q}_W$*. Identifying* M *with its image, the resulting* $(H_L,\mathcal{S})$*-bimodule structure on* M *makes* M *a topological* H_L*-module (i.e. the elements of* H_L *act continuously on* M*) satisfying*

$$\chi m^A_w = m^A_w(w^{-1}\chi) - \sum_{\gamma_A(u,w)} m^A_u \langle \chi \mid \gamma^\vee\rangle X_\gamma X^{l_A(u,w)-1}, \tag{2}$$

$$t_r m^A_w = \begin{cases} m^A_{rw} & \text{if } rw >_A w \\ m^A_{rw}X^2 & \text{if } rw <_A w. \end{cases} \tag{3}$$

for all $r \in L$*,* $w \in \Gamma$ *and* $\chi \in V$*.*

Proof. First, there is an obvious continuous inclusion $K^{\downarrow}_{\mathcal{S}}(\Gamma, m^A) \to K^{\downarrow}_Q(\Gamma, m^A)$ of right $\mathcal{S}$-modules. Next, let δ be a formal symbol. Then the map

$$\sum^{\downarrow}_{w\in\Gamma} m^A_w q_w \mapsto \sum^{\downarrow}_{x\in\Gamma}\delta_x\Bigg(\sum_{\substack{w\in\Gamma\\ w\geq x}} q_w x^{-1}(S^A_{x,w})\Bigg)$$

gives an isomorphism $K^{\downarrow}_Q(\Gamma, m^A) \to K^{\downarrow}_Q(\Gamma,\delta)$ of topological right Q-modules, since $S^A_{x,y} = 0$ unless $x \leq_A y$ and $S^A_x \neq 0$, for $x, y \in \Gamma$. Finally, we may naturally regard $K^{\downarrow}_Q(\Gamma,\delta)$ as a right Q-submodule of $\tilde{Q}_W$. The composite of these maps gives the injective right $\mathcal{S}$-module map $M \to \tilde{Q}_W$ as in the statement of the proposition.

Now $K^{\downarrow}_Q(\Gamma,\delta) \subseteq \tilde{Q}_W$ is stable under left multiplication by elements of Q_{W_L}, in particular by all elements of H_L, and the left multiplications by elements of Q_{W_L} on $K^{\downarrow}_Q(\Gamma,\delta)$ are continuous; it is enough to check continuity of left multiplication by elements $q \in Q$, which is clear, and by elements δ_w for $w \in W_L$, which is just 2.10(2). Now regarding $M \subseteq K^{\downarrow}_Q(\Gamma,\delta) \subseteq \tilde{Q}_W$, the recurrence formulae 2.5(1) and 2.5(5) are respectively seen to be equivalent to (2) (cf. 8.10) and (3) above. The continuity of left multiplications by the elements of H_L now implies that M is a left H_L-submodule of $K^{\downarrow}_Q(\Gamma,\delta) \subseteq \tilde{Q}_W$, and the proposition is completely proved.

Remark. It is possible to reverse the order of development of this section by first proving the above theorem and then using it to define the $S^A_{x,w}$ and establish the recurrence formulae for them. The main point is then to give an independent construction of the M_Γ as $(\mathcal{S},\mathcal{S})$-bimodules. In the situation 2.2(i), this was done in [20]. Using properties [14, 3.4] and [16, 2.6] of spherical intervals, essentially the same reduction to dihedral groups can also be used to get the $(\mathcal{S},\mathcal{S})$-bimodule structure on M_Γ in case 2.2(ii).

2.12. In general, we always regard M_Γ as a $(H_L,\mathcal{S})$-subbimodule of $\tilde{Q}_W$, via the embedding given by Proposition 2.11. Suppose in this subsection only that $A = \emptyset$, $\Gamma = W$ and $L = S$; thus, Γ is W in Chevalley order. Recall the notation $H = H_S$.

Proposition. *Suppose* $A = \emptyset$*,* $\Gamma = W$ *and* $L = S$*. As left* H*-module,* $M_\emptyset$ *is just the left regular module for* H*. Write* $t_w := m^\emptyset_w$*. Then the elements* $\{t_w\}_{w\in W}$ *form* *a basis of H as left $\mathcal{S}$-module (and also as right $\mathcal{S}$-module), and $t_w = t_{\alpha_1} \dots t_{\alpha_n}$ for any reduced expression $w = s_{\alpha_1} \dots s_{\alpha_n}$ (and $t_e = \delta_e$).*

Moreover, for $J \subseteq S$, the elements $\{t_w\}_{w \in W}$ form a left (and right) $\mathcal{S}$-module basis of H_J.

Proof. As observed in 2.10, the elements t_w defined in the statement of the proposition are a right $\mathcal{S}$ basis of $M_\emptyset$ since e is the minimum element of W in Chevalley order. From 2.11(2), it follows that they form a left $\mathcal{S}$-module basis of $M_\emptyset$ as well. Note $t_e = \delta_e$ is the identity element of H. The relations 2.11(3) now implies $t_w = t_{\alpha_1} \dots t_{\alpha_n}$ for any reduced expression $w = s_{\alpha_1} \dots s_{\alpha_n}$, and hence that $M_\emptyset$ is generated as H-module by t_e, hence $M_\emptyset = H$. The final assertion in the proposition follows easily.

Remark. In case 2.2(i), the ring H is (anti-isomorphic to) the nil Hecke ring defined in [26,(4.12)]; the elements t_w here are the $\overline{x}_w$ there. In the situation 2.2(ii), with W a finite Weyl group, the ring H can be specialized to Lusztig's graded affine Hecke algebra, as defined in [30, 0.1] (see also [31]).

2.13. The following gives a presentation by generators and relations for H_J as $\mathbb{R}$-algebra, for $J \subseteq S$.

Corollary. *The ring H_J is generated as $\mathbb{R}$-algebra with identity t_e by generators t_r with $r \in J$ and χ with $\chi \in V$ subject to the following relations:*

(a) *the linear relations on V and $\chi_1\chi_2 = \chi_2\chi_1$ for $\chi_1, \chi_2 \in V$.*
(b) $t_r^2 = X^2 t_e$
(c) $\chi t_r = t_r r(\chi) - \langle \chi, \alpha^\vee \rangle X_\alpha t_e$
(d) *the braid relations of (W, S) on the t_r.*

Proof. Consider the ring H'_J generated by elements t'_r ($r \in J$), χ ($\chi \in V$) and identity t'_e subject to relations as above. Let $\mathcal{S}'$ denote the (commutative) subring of H'_J generated by t'_e and V. There is a (surjective) ring homomorphism $\phi: H'_J \to H_J$ mapping $\chi \mapsto \chi$, $t'_r \mapsto t_r$, $t'_e \mapsto t_e$. By the monoid lemma 1.3(f), the braid relations imply that there are well-defined elements t'_w of H'_J, for $w \in W_J$, with $t'_w = t'_{r_1} \dots t'_{r_n}$ whenever $r_1 \dots r_n$ is a reduced expression for w; one must have $\phi(t'_w) = t_w$ for $w \in W_J$. For $r \in J$, $t'_r t'_w = t'_{rw}$ if $l'(rw) > l'(w)$ and $t'_r t'_w = t'_{rw} X^2$ otherwise, using (b). Now using (c), one sees that the t_w span H'_J as right $\mathcal{S}'$-module. Since $\phi(\mathcal{S}') \subseteq \mathcal{S}$ and the t_w are a right $\mathcal{S}$-basis of H_J, it is now easily seen that ϕ is an isomorphism.

Remark. It follows from the corollary that there is a ring anti-involution of H fixing $\mathcal{S}$ elementwise and mapping t_w to $t_{w^{-1}}$ for $w \in W$; it is the restriction of the ring anti-involution of Q_W which fixes Q elementwise and maps δ_r to $\delta_r \frac{\alpha X + X_\alpha}{\alpha X - X_\alpha}$ for $\alpha \in \Pi$, $r = s_\alpha$.

2.14. We describe the inverse of the (in general infinite) upper triangular matrix $\left(x^{-1}(S^A_{x,w})\right)_{x,w \in \Gamma}$ associated to $\leq_A$ in terms of the matrix $\left(x^{-1}(S^{T\setminus A}_{x,w})\right)_{x,w \in \Gamma}$ associated to the reverse order $\leq_{T \setminus A}$.

Proposition. *For any $x, z \in \Gamma$, $\sum_{y \in [z,x]} (-1)^{l'(y)} y^{-1}\left(S^A_{y,x} \theta(S^{T\setminus A}_{y,z})\right) = \delta_{x,z}$*

Proof. Since $S^A_{x,y} = 0$ unless $x \leq_A y$ and $S^A_x \neq 0$, it is enough to show that

$$(1) \qquad \delta_x = \sum_{v\in\Gamma} m^A_v (-1)^{l'(x)} x^{-1}\big(\theta(S^{T\setminus A}_{x,v})\big).$$

One may even assume without loss of generality that Γ is finite. The result now follows for general reasons (see 8.13 and 8.10) from 2.4(3).

Remark. The involution θ of Q fixes all elements $S^A_{x,w}/S^A_x$ with $x, w \in \Gamma$.

2.15. Note that if $A = \emptyset$ and $\Gamma = W$, then $M_\Gamma = H$ has, as well as the left H-module structure, also a right H-module structure; we now establish the analogous result in general. It will not be used anywhere else in this paper.

For $B \in \mathcal{P}(T)$ and $w \in W$, write $w \cdot B = N(w) + wBw^{-1} \in \mathcal{P}(T)$. Let $S' := \{\, r \in T \mid r \cdot A = A + \{r\}\,\}$ and let W' be the reflection subgroup of W generated by S'.

Lemma. (a) *(W', S') is a Coxeter system with reflection cocycle $N'\colon W' \to W' \cap T$ given by $N'(w) = w \cdot A + A$ for $w \in W'$. For $K \subseteq S'$, let W'_K be the parabolic subgroup of W' generated by K, and let $l''\colon W' \to \mathbb{N}$ denote the standard length function of (W', S').*
(b) *For any $r \in S'$ and $w \in W$, one has $l_A(wr) = l_A(w) \pm 1$.*
(c) *if $v, w \in W$ and $t \in S'$ with $wt >_A w$ and $vt >_A v$, then $v \leq_A w$ iff $v \leq_A wt$ iff $vt \leq_A wt$ (the "right Z-property")*
(d) *if $K \subseteq S'$ and $w \in W$ with $l_A(wr) \leq l_A(w)$ for all $r \in K$, then the map $x \mapsto wx\colon W'_K \to wW'_K$ gives an isomorphism between W'_K (in the order acquired as a subset of (W', S') in its Chevalley order) and the spherical subset xW'_K of W in the order $\leq_A$, with $l_A(wx) = l_A(w) + l''(x)$.*

Proof. Parts (a)–(c) are all in [16, 1.8]. For $\sharp(K') = 2$, which is the only case required in this paper, it is easy to prove (d) by an ad hoc argument using (b) and (c).

Remark. Much more general facts will be proved in another paper which requires a systematic study of shortest (W_J, W'_K) double coset representatives in the orders $\leq_A$. In general, S' may be empty, and it is likely there are also examples with S finite and S' infinite.

2.16. For each $r \in S'$, define the element

$$(1) \qquad \delta'_r := \delta_r \prod_{\substack{\alpha\in\Phi^+\cap r(-\Phi^+)\\ s_\alpha\neq r}} \left(\frac{-\epsilon_A(\alpha,e)\alpha}{X_\alpha - \epsilon_A(\alpha,e)\alpha X}\right)^{\epsilon_A(\alpha,e)}$$

of Q_W, where $\epsilon_A(\alpha, x)$ is defined in 2.4. Also, for $\gamma \in \Phi^+$ with $r = s_\gamma \in S'$, set

$$(2) \qquad t'_\gamma = \delta'_r \frac{X_\gamma + \gamma X}{\gamma} - \epsilon_A(\gamma, e)\,\delta_e \frac{X_\gamma}{\gamma}.$$

For $K \subseteq S'$, let H'_K denote the subalgebra of Q_W generated by the elements $\delta_e\chi$ for $\chi \in \mathcal{S}$ and t'_γ for $\gamma \in \Phi^+$ with $s_\gamma \in K$. Regard H'_K as a topological ring with discrete topology. Also, define $H' := H'_{S'}$.

Proposition. (a) *If* $K \subseteq S'$ *with* $\Gamma K \subseteq \Gamma$*, then* $M_\Gamma \subseteq \tilde{Q}_W$ *is a topological* (H_L, H'_K)*-bimodule. For* $\gamma \in \Phi^+$ *and* $w \in \Gamma$ *with* $r = s_\gamma \in K$*,*

$$m^A_w t'_\gamma = \begin{cases} m^A_{wr} & \text{if } wr >_A w \\ m^A_{wr} X^2 & \text{if } wr <_A w. \end{cases}$$

(b) *The* $\mathbb{R}$*-subalgebra* H' *of* Q_W *has a basis* $\{\, t'_w \,\}_{w \in W'}$ *as left (or right)* $\mathcal{S}$*-module such that* $t'_e = \delta_e$ *and for* $w \in W$ *and* $r = s_\gamma \in S'$ *with* $\gamma \in \Phi^+$*,*

$$t'_\gamma t'_w = \begin{cases} t'_{rw} & \text{if } l''(rw) > l''(w) \\ t'_{rw} X^2 & \text{if } l''(rw) < l''(w). \end{cases}$$

Proof. Fix r and γ as in (a). One easily sees that

$$\chi t'_\gamma = t'_\gamma r(\chi) - \epsilon_A(\gamma, e)\langle \chi, \gamma^\vee \rangle X_\gamma \delta_e \tag{3}$$

$$t'^2_\gamma = X^2 \delta_e. \tag{4}$$

A somewhat more tedious calculation using 1.3(e) shows that

$$\delta_y y^{-1}(S_y) = \delta_x x^{-1}(S_x)\, \delta'_r \frac{X_\gamma + \gamma X}{\gamma} \qquad \text{for all } x <_A xr = y \in W. \tag{5}$$

Now we prove (a) (the proof is very similar to that of [26, 4.2]). Recall $M_\Gamma \subseteq K^\downarrow_Q(\Gamma, \delta) \subseteq \tilde{Q}_W$ is certainly stable under right multiplication by elements of $\mathcal{S}$. As before, right multiplication by an element $\sum_{w \in W'_K} \delta_w q_w$ in Q_W is continuous on $K^\downarrow_Q(\Gamma, \delta)$, using the right Z-property. To prove (a), it remains only to establish the formula there in the case $wr >_A w$ (the other case then follows immediately by (4)).

Fix $w \in \Gamma$ with $wr >_A w$. Write $m^A_w t'_\gamma = \sum_v \delta_v v^{-1}(S'_{v,wr})$. Note $S'_{v,wr} = 0$ unless $v \leq wr$, by the right Z-property. To prove (a) in this case, one has to show that $S_{v,wr} = S'_{v,wr}$ for $v \leq wr$, and for this, it is sufficient to consider the case that Γ is a finite spherical interval $\Gamma = [y, wr]$ where $yr >_A y$ (since v is contained in such an interval). Now by induction we may assume that (a) holds with w replaced by any $w' \in \Gamma$ except perhaps $w' = w$ or $w' = wr$. We calculate using (3) and 2.11(2) that for $\chi \in V$,

$$\begin{aligned} \chi m^A_w t'_\gamma &= m^A_w w^{-1}(\chi) t'_\gamma - \sum_{\alpha(u,w)} \langle \chi, \alpha^\vee \rangle X_\alpha X^{l_A(u,w)-1} m^A_u t'_\gamma \\ &= m^A_w t'_\gamma (wr)^{-1}(\chi) - \epsilon_A(\gamma, e)\langle w^{-1}(\chi), \gamma^\vee \rangle X_\gamma m^A_w \\ &\qquad - \sum_{\alpha(u,w)} \langle \chi, \alpha^\vee \rangle X_\alpha X^{l_A(u,w)-1} X^{1+l_A(ru,u)} m^A_{ur} \\ &= m^A_w t'_\gamma (wr)^{-1}(\chi) - \sum_{\beta(x,wr)} \langle \chi, \beta^\vee \rangle X_\beta X^{l_A(x,wr)-1} m^A_x. \end{aligned}$$

We have also used here the fact that $\alpha(u, wt)$ iff either $\alpha(ut, w)$, or $u = w$ and α is the positive root in $\pm w\gamma$. Now this calculation gives that for $z \in [y, wr]$,

$$(\chi - z(wr)^{-1}\chi)S'_{z,wr} = -\sum_{\beta(x,wr)} \langle \chi, \beta^\vee \rangle X_\beta X^{l_A(x,wr)-1} S^A_{z,x}.$$

The same formula holds if $S'_{z,wr}$ is replaced by $S^A_{z,wr}$; if $z \neq wr$, choosing χ so $z(wr)^{-1}(\chi) \neq \chi$ shows $S'_{z,wr} = S^A_{z,wr}$. On the other hand, for $z = wr$ this last equality follows from (5). This completes the proof of (a).

Next, we show that the t'_γ satisfy the braid relations of (W', S'). Choose positive roots $\beta \neq \gamma$ with $s_\gamma = r \in S'$, $s_\beta = s \in S'$ such that rs has finite order n, say. Let γ_i denote β for even i and γ for odd i. Let $W'_{r,s}$ be the parabolic subgroup of W' generated by r and s, with longest element $\omega = s_{\gamma_1} \dots s_{\gamma_n} = s_{\gamma_2} \dots s_{\gamma_{n+1}}$. As a finite subset of W in the order $\leq_A$, $W'_{r,s}$ has an element u of minimal length $l_A(u)$ with respect to l_A. By 2.15(d), the map $w' \mapsto uw'$ is an isomorphism $W'_{r,s} \to \Gamma' := [u, u\omega]$ of posets satisfying $l_A(uw') = l_A(u) + l''(w')$, where W'_{rs} has the usual Chevalley order as a finite dihedral group with generators r, s and Γ' is the indicated interval of W in the order $\leq_A$. In particular, Γ' is spherical and $\Gamma' W'_{r,s} = \Gamma'$. Assume temporarily that $\Gamma = \Gamma'$. Now part (a) applied to $\Gamma = \Gamma'$ implies that

$$m^A_u t'_{\gamma_1} \dots t'_{\gamma_n} = m^A_{u\omega} = m^A_u t'_{\gamma_2} \dots t'_{\gamma_{n+1}}$$

and the braid relation for r and s follows since (for $\Gamma = \Gamma'$) $m^A_u = S^A_u \delta_u$ is a unit in Q_W. The monoid lemma now gives elements t'_w of Q_W for $w \in W'$, satisfying $t'_{s_\gamma} t'_w = t'_{s_\gamma w}$ for $w \in W''$, $\gamma \in \Phi^+$ with $s_\gamma \in S'$ and $l''(s_\gamma w) > l''(w)$. By (3), $t'_{s_\gamma} t'_w = t'_{s_\gamma w} X^2$ if $l''(s_\gamma w) < l''(w)$, so the right S-module H'' spanned by the t'_w for $w \in W'$ is closed under left multiplication by the t'_γ. One easily sees

$$t'_w \in \delta_w Q^\bullet + \sum_{\substack{v \in W' \\ l''(v) < l''(w)}} \delta_v Q,$$

(where in general $R^\bullet$ denotes the unit group of a ring R), so the t_w for $w \in W'$ are right (or left) Q-linearly independent. From (3), H'' is also closed under left multiplication by elements of V, so $H'' = H'$. This completes the proof of (b), and hence of the proposition.

2.17. The proof of the following result, showing the very close similarity between H' and H, is essentially the same as that of 2.13, and is therefore omitted.

Corollary. *For any parabolic subgroup (W'_K, K) of (W', S'), $\sum_{w \in W'_K} t'_w \mathcal{S}$ is a subring of H'. It may be identified with the $\mathbb{R}$-algebra generated by (identity element t'_e and) the elements t'_r for $r \in K$, $\chi t'_e$ for $\chi \in V$ subject to the following relations:*

- (a) *the linear relations on V, and $\chi\chi' = \chi'\chi$ for $\chi, \chi' \in V$*
- (b) $t'^2_r = X^2 t'_e$
- (c) $\chi t'_r = t'_r r(\chi) - \epsilon_A(\gamma, e)\langle \chi, \gamma^\vee \rangle X_\gamma t'_e$ *for $r = s_\gamma \in K$, $\gamma \in \Phi^+$*
- (d) *the braid relations for (W'_K, K) on the t'_r.*

2.18. In this subsection, we briefly indicate the changes necessary in case 2.2(i) if one works with non-reduced root systems. First, in that case one may still define t_α as before. Note that a rank two root system (in our sense) can always be replaced by a reduced rank two root system, without changing the action of W on V or V', simply by multiplying all roots in one W-orbit by a suitable $c \in \mathbb{R}_{>0}$ and multiplying the corresponding coroots by c^{-1}. It follows that (in the non-reduced case) the t_α satisfy the braid relations up to multiplication of one side by a positive scalar. Hence one obtains a (left or right) $\mathcal{S}$-basis t_w of H with $t_w \in \mathbb{R}_{>0} t_{r_1} \dots t_{r_n}$ for any reduced expression $w = r_1 \dots r_n$. To get 2.11 for non-reduced root systems, one can first show that there is at most one (up to isomorphism) $(\mathcal{S}, \mathcal{S})$-bimodule structure on the right $\mathcal{S}$-module $M_\Gamma := K_{\mathcal{S}}^{\downarrow}(\Gamma, m^A)$ (with continuous left $\mathcal{S}$-action) such that

$$\chi m_w^A = m_w^A(w^{-1}\chi) - \sum_{u \xrightarrow{\gamma}_A w} m_u^A c_{u,w} \langle \chi \mid \gamma^\vee \rangle \tag{1}$$

for some non-zero scalars $c_{u,w}$ (the proof of this can be reduced to its special case in which Γ is a length two spherical interval and $\sharp(S) = 2$, where it is easily checked). Moreover, using 8.8, one can show that if such a module M_Γ exists, it has an embedding as a $(\mathcal{S}, \mathcal{S})$-subbimodule of $\tilde{Q}_W$. Using the uniqueness, one can prove existence of such a module M_Γ, with all $c_{u,w} > 0$, satisfying in addition

$$t_r m_w^A = \begin{cases} d_{r,w} m_{rw}^A & \text{if } rw >_A w \\ 0 & \text{if } rw <_A w \end{cases} \tag{2}$$

for $r \in S$ with $r\Gamma \subseteq \Gamma$, for some scalars $d_{r,w} > 0$ in $\mathbb{R}$. First, one builds a suitable $(\mathcal{S}, \mathcal{S})$-bimodule $M = M_{[x,y]}$ of $\tilde{Q}_W$ recursively by 1.5(d), for finite spherical intervals $[x, y]$, as follows. Suppose $r \in S$, $rx > x$, $ry < y$ and $[x, y] \in \mathcal{P}_A$. If $M_{[x,ry]} \subseteq \tilde{Q}_W$ is already defined, one sets $m_w^A = t_r m_{rw}^A$ for $w \in [x, y] \setminus [x, ry]$ and $M_{[x,y]} = \sum_{w \in [x,y]} m_w^A \mathcal{S}$ for $\Gamma = [x, y]$, and defines $M_{[rx,y]}$ as the quotient of $M_{[x,y]}$ by its subbimodule $\sum_{z \in [x,y] \setminus [rx,y]} m_z^A \mathcal{S}$. If instead $M_{[rx,y]}$ is defined, one can first dualize (see 8.13), apply the preceeding argument for the reverse order $\leq_{T \setminus A}$ and then dualize again to get $M_{[x,y]}$ and $M_{[x,ry]}$. The construction of M_Γ for general Γ reduces easily to the case of finite intervals Γ just sketched.

With similar changes (insertion of positive unit factors from $\mathbb{R}$ in appropriate places in the statements, and generally more complicated proofs), all results proved in this paper for reduced root systems (except those in the situation 2.2(ii)) can be extended to non-reduced root systems.

3. Modules for the dual nil Hecke ring

In this section, we dualize the modules for the nil Hecke ring defined in the previous section to obtain a family of modules Λ_Γ for the dual nil Hecke ring Λ.

3.1. We use the following conventions for graded modules $M = \oplus_{n \in \mathbb{Z}} M_n$, N over a positively graded, commutative ring $R = \oplus_{n \in \mathbb{N}} R_n$. We denote M with degrees shifted up by $p \in \mathbb{Z}$ as $M\langle p \rangle$, so $(M\langle p \rangle)_n = M_{n-p}$. We give $M \otimes_R N$ the natural structure of graded R-module so that its homogeneous component of degree i is spanned by elements $m\otimes n$ with $m\in M_p$, $n\in N_q$ and $p+q=i$. If M is in addition finitely generated over R, $\mathrm{Hom}_R(M,N)$ has a natural structure of graded R-module with $\mathrm{Hom}_R(M,N)_p=\{\, f\in \mathrm{Hom}_R(M,N)\mid f(M_n)\subseteq N_{n+p} \text{ for all } n\in\mathbb{Z}\,\}$.

3.2. For the remainder of the paper, unless otherwise specified, all notations used from Section 1 will refer to the situation 2.2(i), for a fixed spherical poset Γ in the order $\leq_A$ and fixed $L\subseteq S$ with $W_L\Gamma\subseteq\Gamma$. Any finite standard parabolic subgroup W_K of W, has a unique "longest element" which will always be denoted ω_K; it is characterized by the condition $l'(\omega_K)\geq l'(w)$ for all $w\in W_K$. Define $\Phi_K^+=\{\,\alpha\in\Phi^+\mid s_\alpha\in W_K\,\}$, and write $\mathcal{S}^K=\{\, f\in S\mid w(f)=f \text{ for all } w\in W^K\,\}$ for the ring of W_K-invariant elements of $\mathcal{S}$.

We give some additional properties of the elements $S^A_{x,w}$ which are special to the situation 2.2(i). First, note that the field automorphism θ of Q is now the identity, and the recurrence formula 2.5(5) simplifies to

$$(1)\qquad S^A_{x',w}+r(S^A_{rx',w})=\begin{cases}-\alpha S^A_{x',rw} & \text{if } rw>_A w\\ 0 & \text{if } rw<_A w.\end{cases}$$

This shows in particular that if $[y,z]\in\mathcal{P}_A$ and $sz<z$ for all $s\in J\subseteq S$, then

$$(2)\qquad (-1)^{l'(xy)}(xy)^{-1}\Big(S^A_{xy,z}\Big)=(-1)^{l'(y)}y^{-1}\Big(S^A_{y,z}\Big)\qquad \text{for all } x\in W_J.$$

From the definition 2.4(1),

$$(3)\qquad S^T_x=\prod_{\alpha\in\Phi^+\cap x(-\Phi^+)}\alpha,\qquad S^\emptyset_x=(-1)^{l'(x)}\prod_{\alpha\in\Phi^+\cap x(-\Phi^+)}\alpha^{-1}$$

and combining this with (2),

$$(4)\qquad S^\emptyset_{x,\omega_K}=(-1)^{l'(\omega_K)}\prod_{\alpha\in\Phi_K^+}\alpha^{-1}\qquad \text{if } x\in W_K$$

for any finite standard parabolic subgroup W_K of W. Recall the ring H, defined in 2.11 as a subring of Q_W generated by $\mathcal{S}$ and all elements $t_\alpha=-\alpha^{-1}(\delta_{s\alpha}+\delta_e)$ for $\alpha\in\Pi$. In the situation of (4), one has

$$(5)\qquad t_{\omega_K}=\sum_{w\in W_K}\delta_w(-1)^{l'(\omega_K)-l'(w)}\prod_{\alpha\in\Phi_K^+}\alpha^{-1}.$$

It follows that for $J\subseteq L$ and $[x,w]\in\mathcal{P}_A$ that

(6)
$$\sum_{v\in W_J}(vx)^{-1}(S^A_{vx,w})$$
$$=\begin{cases}0 & \text{if } w\notin\Gamma^J \text{ and } W_Jx\cap\Gamma^J\neq\emptyset\\ (-1)^{l'(\omega_J)}x^{-1}(S^A_{x,\omega_J w}\prod_{\alpha\in\Phi_J^+}\alpha) & \text{if } w\in\Gamma^J \text{ and } \sharp(W_J)<\infty.\end{cases}$$

Indeed, for $w \notin \Gamma^J$, (6) follows from (1) and in the other case, (6) follows from (5) since $t_{\omega_J} m_w^A = m_{\omega_J w}^A$. In general, the standard basis elements δ_w of Q_W, with $w \in W$, are in H since $\delta_{s_\alpha} = t_\alpha \alpha + \delta_e \in H$, and so by 2.11 and 2.14(1),

(7) $$S_{x,y}^T \in \mathcal{S} \qquad \text{for all } x, y \in W.$$

From (1), one computes $S_{x,y}^T$ for $l'(y) \leq 1$ explicitly as (8)–(9) below:

(8) $$S_{x,e}^T = (-1)^{l'(x)} \qquad \text{for } x \in W$$

(9) For a simple reflection $r \in S$, define $\sigma^{(r)}: W \to V$ by $\sigma^{(r)}(w) = (-1)^{l'(w)} S_{w,r}^T$. Then $\sigma^{(r)} \in Z^1(W, V)$ (i.e. $\sigma^{(r)}$ is a 1-cocycle of W on V) with $\sigma^{(r)}(s) = 0$ for simple $s \neq r$, and $\sigma^{(r)}(r) = -\alpha$, where $\alpha \in \Pi$ has $s_\alpha = r$. If there exists an element $\chi_\alpha \in V$ with $\langle \chi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for all $\beta \in \Pi$, then $S_{w,r}^T = (-1)^{l'(w)}(w(\chi_\alpha) - \chi_\alpha)$.

3.3. Let $x \mapsto \overline{x}$ denote the involutory ring anti-isomorphism of Q_W defined by $\delta_w q \mapsto \delta_{w^{-1}} w(q)$, and let $\overline{H}$ (resp., $\overline{H}_J$ for $J \subseteq S$) denote the subring of Q_W into which H (resp., H_J) is mapped by this involution. Give Q the left Q_W-module structure with $(\delta_w q) \cdot q' = w(qq')$ for $w \in W$, $q, q' \in Q$. Then

(1) $$x \cdot \mathcal{S} \subseteq \mathcal{S} \text{ if } x \in \overline{H}.$$

In fact, $\overline{H}$ is generated as ring by $\mathcal{S}$ and the elements $\overline{t}_\alpha = -(\delta_{s_\alpha} + \delta_e)\frac{1}{\alpha}$ for $\alpha \in \Pi$. Now one need only check that $\frac{1}{\alpha}(s(f) - f) \in \mathcal{S}$ for $f \in \mathcal{S}$, which is immediate since $\mathcal{S} = \mathcal{S}^{s_\alpha} + \alpha \mathcal{S}^{s_\alpha}$.

3.4. For any positive root $\beta \in \Phi^+$, let $\mathcal{S}^{(\beta)} = \mathcal{S}[\gamma^{-1} \mid \gamma \in \Phi^+ \setminus \{\beta\}]$ denote the localization of $\mathcal{S}$ at the multiplicative subset generated by the positive roots other than β.

Lemma. *Let $v \in W$, $\beta \in \Phi^+$ be such that $v <_A w := vs_\beta$ and $[v, w] \in \mathcal{P}_A$. Then there exists a unit u of $S^{(\beta)}$ such that $v^{-1}(S_{v,w}^A / S_v^A) \equiv u\beta^{-1} \pmod{S^{(\beta)}}$.*

Proof. For any $[x, y] \in \mathcal{P}_A$, define $S'_{x,y} = x^{-1}(S_{x,y}^A / S_x^A)$. From the above proposition

(1) $$S'_{x,y} \in \Big(\prod_{\substack{\alpha \in \Phi^+ \\ x < xs_\alpha \leq y}} \alpha\Big)^{-1} \mathcal{S}.$$

Suppose $\alpha \in \Pi$ and $r = s_\alpha \in S$ satisfy $rx >_A x$ and $ry <_A y$. From 2.5(5),

(2) $$S'_{x,y} = -\frac{1}{x^{-1}(\alpha)} S'_{rx,y},$$

(3) $$S'_{x,ry} = -\frac{1}{x^{-1}(\alpha)} S'_{rx,ry} + S'_{rx,y}.$$

Now consider $\beta \in \Phi^+$. If $ry = xs_\beta$, then $y = rxs_\beta$, the first term on the right of (3) is in $S^{(\beta)}$ by (1) and so the lemma holds for the pair $[v, w] = [x, ry]$ iff it holds for $[v, w] = [rx, y]$. Similarly, if $y = xs_\beta$, then $ry = rxs_\beta$ and $[rx, ry] \in \mathcal{P}_A$ unless $rx \not\leq$

ry i.e. unless $y = rx$. If $y \neq rx$, (1) and $S'_{x,y} = -(x^{-1}\alpha)^{-1}S'_{x,ry} - \bigl(x^{-1}(\alpha)\bigr)^{-2}S'_{rx,ry}$ show that the lemma holds for $[v,w]=[x,y]$ iff it holds for $[v,w]=[rx,ry]$. Using 1.5(d), the lemma is therefore true if it holds for intervals $[v,w]=[x,y]$ with $y = xs_\beta$ such that $y = s_\alpha x$ for some $\alpha \in \Pi$. But then $S'_{x,y} = -\frac{1}{x^{-1}(\alpha)} = \pm\frac{1}{\beta}$ by (2). This completes the proof of the lemma.

Remark. Suppose that the root system arises by extension of scalars from an integral root system in which the roots are indivisible elements of the weight lattice. Recall that $\mathcal{S}_\mathbb{Z}$ denotes the symmetric algebra over $\mathbb{Z}$ of the weight lattice, regarded as a subring of $\mathcal{S}$. The recurrence formulae above shows that $S^A_{x,y}$ and $S'_{x,y}$ are elements of the localization of $\mathcal{S}_\mathbb{Z}$ at its multiplicative subset generated by Φ^+. It follows that (1), and hence also the lemma, remains true if the symmetric algebra $\mathcal{S}$ is replaced everywhere by $\mathcal{S}_\mathbb{Z}$.

3.5. Let $M = M_\Gamma$ be the left H_L-module associated to Γ in the previous section.

Let $Q_W \otimes_Q Q_W$ denote the tensor product with both sides considered as right Q-modules. Define the diagonal map $\Delta\colon Q_W \to Q_W \otimes_Q Q_W$ by

$$\Delta(\delta_w q) = \delta_w q \otimes \delta_w = \delta_w \otimes \delta_w q \qquad \text{for } w \in W \text{ and } q \in Q.$$

Then Δ is associative and commutative with a counit $\epsilon\colon Q_W \to Q$ defined by $\epsilon(\delta_w q) = q$. There is an associative product structure $\odot$ on $Q_W \otimes_Q Q_W$, defined by

$$(\delta_x \otimes \delta_y q_y) \odot (\delta_z \otimes \delta_w q_w) = \delta_{zw^{-1}xw} \otimes \delta_{yw}(w^{-1}q_y)q_w,$$

such that Δ is a ring homomorphism. If Γ is finitely generated as a coideal (in particular, if Γ is finite or $\Gamma = W$ in Chevalley order), one has

$$\begin{aligned}\Delta\Bigl(m^A_w\Bigr) &= \Delta\Biggl(\sum_{y\in\Gamma}\delta_y y^{-1}(S^A_{y,w})\Biggr) = \sum_{y\in\Gamma}\delta_y \otimes \delta_y y^{-1}(S^A_{y,w})\\ &= \sum_{z\in\Gamma}\Omega^A_{z,w}\otimes m^A_z = \sum_{\substack{z\in\Gamma\\ v\in W}} t_v \otimes m^A_z p^{w,A}_{v,z}\end{aligned}$$

where by 2.14

$$\Omega^A_{z,w} = \sum_{y\in\Gamma}(-1)^{l'(y)}\delta_y y^{-1}\Bigl(S^{T\setminus A}_{y,z}S^A_{y,w}\Bigr), \tag{1}$$

$$p^{w,A}_{v,z} = \sum_{y\in\Gamma} y^{-1}\Bigl(S^T_{y,v}S^{T\setminus A}_{y,z}S^A_{y,w}\Bigr) \tag{2}$$

(with the assumed condition on Γ, all the sums above have only finitely many non-zero terms).

3.6. In general, for any $v, w, z \in W$ with $[z,w] \in \mathcal{P}_A$ define elements $p^{w,A}_{v,z} \in Q$ and $\Omega^A_{z,w} \in Q_W$ by 3.5(1) and (2) (note that these sums involve only finitely many non-zero terms, even if Γ is not a finitely generated coideal). Also, set $P^w_{v,z} = p^{w,\emptyset}_{v,z}$. We record the following.

Lemma. *Assume $[z,w]\in\mathcal{P}_A$ and $v\in W$ Then*
(a) $\Omega^A_{z,w}=0$ *unless* $z\leq_A w$, *and* $p^{w,A}_{v,z}=0$ *unless* $v\leq_\emptyset y$ *for some* y *with* $z\leq_A y\leq_A w$
(b) $\Omega^A_{w,w}=\delta_w$ *and* $p^{w,A}_{v,w}=(-1)^{l'(w)}w^{-1}(S^T_{w,v})$, *independent of* A
(c) $\Omega^A_{z,w}=\Omega^{T\setminus A}_{w,z}$ *and* $p^{w,A}_{v,z}=p^{z,T\setminus A}_{v,w}$.
(d) $p^{w,A}_{e,v}=\delta_{v,w}$, *independent of* A
(e) *if* $z\xrightarrow{\gamma}_A w$, $r=s_\alpha$ *where* $\alpha\in\Pi$, *and an element* $\chi_\alpha\in V$ *exists as in* 3.2(9), *then* $p^{w,A}_{r,z}=\langle\chi_\alpha,\gamma^\vee\rangle$
(f) $(w^{-1}\chi-z^{-1}\chi)p^{w,A}_{v,z}=\sum_{x\xrightarrow{\gamma}_A w}\langle\chi\mid\gamma^\vee\rangle p^{x,A}_{v,z}-\sum_{z\xrightarrow{\gamma}_A u}\langle\chi\mid\gamma^\vee\rangle p^{w,A}_{v,u}$ *for* $\chi\in V$.
(g) $p^{x,A}_{y,x}=P^z_{y,z}$ *if* $y\in W^J$, $x\in W$ *and* $z\in W_Jx$

Proof. Parts (a)-(c) are immediate consequences of the definitions, and (d) follows from 3.2(8) and 2.14. Part (e) follows from 3.2(9) (the more general result when χ_α doesn't exist is left to the reader). Part (f) (and an equivalent formula involving $\Omega^A_{z,w}$) follows easily using 2.5(1) for A and $T\setminus A$, while (g) follows from (b) and 3.2(2).

3.7. Part (a) of the following lemma provides the basis for the subsequent "dualization" of H and M.

Lemma. *Fix $v,z,w\in W$ with $[z,w]\in\mathcal{P}_A$. Then*
(a) $p^{w,A}_{v,z}\in\mathcal{S}_{2(l'(v)-l_A(z,w))}$, *and in particular*, $p^{w,A}_{v,z}=0$ *unless* $l_A(z,w)\leq l'(v)$.
(b) $\Omega^A_{z,w}\in H$ *and* $\overline{\Omega^A_{z,w}}\cdot\mathcal{S}\subseteq\mathcal{S}$
(c) *if* $u,v,w\in W^J$, $\beta\in\Phi^+$ *with* $u\in W_Jws_\beta$, *then* $\beta\mid(P^w_{v,w}-P^u_{v,u})$.

Proof. First, we show that if $[z,w]\in\mathcal{P}_A$ and $r=s_\alpha$ for some $\alpha\in\Pi$ satisfy $s_\alpha z>z$ and $s_\alpha w<w$, then

(1)
$$\begin{aligned}\Omega^A_{z,w}&=t_r\Omega^A_{rz,w}\\ \Omega^A_{z,rw}&=\delta_r\Omega^A_{rz,w}+t_r\Omega^A_{rz,rw}.\end{aligned}$$

To prove the formula for $\Omega^A_{z,w}$, for example, one calculates (using 3.2(1) repeatedly) that

$$\begin{aligned}\Omega^A_{z,w}&=\sum_{y\in\Gamma}(-1)^{l'(y)}\delta_y y^{-1}\Big(S^{T\setminus A}_{y,z}S^A_{y,w}\Big)\\
&=\sum_{y\in\Gamma}(-1)^{l'(y)}\delta_y y^{-1}(S^{T\setminus A}_{y,z})\frac{y^{-1}(S^A_{y,rw})+(ry)^{-1}(S^A_{ry,rw})}{-y^{-1}(\alpha)}\\
&=-\frac{1}{\alpha}\sum_{y\in\Gamma}(-1)^{l'(y)}\delta_y\Big(y^{-1}(S^{T\setminus A}_{y,z}S^A_{y,rw})-(ry)^{-1}(S^{T\setminus A}_{ry,z}S^A_{ry,rw})\Big)\\
&=-\frac{1}{\alpha}(\Omega_{z,rw}+\delta_r\Omega_{z,rw})\\
&=t_r\Omega_{z,rw}.\end{aligned}$$

The second part of (1) can be proved in a similar way. Note $\delta_r\in H$ and $t_r\in H$. Since $\Omega^A_{e,e}=\delta_e\in H$, one sees from (1) and 1.5(d) that $\Omega^A_{z,w}\in H$ for $[z,w]\in\mathcal{P}_A$. The second part of (b) then follows by 3.3(1). Since $\Omega^A_{z,w}=\sum_{v\in\Gamma}t_v p^{w,A}_{v,z}$, it follows that $p^{w,A}_{v,z} \in \mathcal{S}$. Using 2.6 and 3.5(2), $p^{w,A}_{v,z}$ is expressible as a quotient f/g of homogeneous elements f, g of $\mathcal{S}$ satisfying $\deg(f) - \deg(g) = 2\big(l'(v) - l_A(z,w)\big)$, giving the remaining part of (a).

Finally, for (c), write $\beta = x(\alpha)$ with $x \in W$ and $\alpha \in \Pi$. Then for $w \in W$,

$$(\delta_w - \delta_{ws_\beta})\frac{1}{\beta} = \delta_{wx}(\delta_e - \delta_{s_\alpha})\frac{1}{\alpha}\delta_{x^{-1}} \in H$$

since all $\delta_y \in H$. But

$$(\delta_w - \delta_{ws_\beta})\frac{1}{\beta} = \sum_{v \in W} t_v(P^w_{v,w} - P^{ws_\beta}_{v,ws_\beta})\frac{1}{\beta}$$

by 2.14(1) and 3.6(b), so $\beta \mid (P^w_{v,w} - P^{ws_\beta}_{v,ws_\beta})$ by 2.12. The result now follows by 3.6(g).

Remark. The fact that $\overline{\Omega^A_{z,w}} \cdot \mathcal{S} \subseteq \mathcal{S}$ is true for general reasons, see 8.10–8.11.

3.8. Following [26], we dualize H to obtain the "dual nil Hecke ring" Λ. Let Q^W denote the set of all functions $W \to Q$, endowed with the structure of commutative Q-algebra with pointwise addition, multiplication and scalar multiplication. Identify Q^W and $\mathrm{Hom}_Q(Q_W, Q)$, regarding Q_W as a right Q-module, by setting $f(\sum_{w \in W} \delta_w q_w) = \sum_w q_w f(w)$ for any $f \in Q^W$. Then the algebra structure on Q^W is the one obtained by dualizing the Q-linear comultiplication in Q_W (see 3.5).

Define a left Q_W-module structure on H by

(1) $$(x \cdot \psi)(y) = \psi(\overline{x}y) \text{ for } x, y \in Q_W \text{ and } \psi \in Q^W.$$

Now define the dual nil Hecke ring Λ to be the set of $\psi \in Q^W$ such that $\psi(H) \subseteq \mathcal{S}$ and $\psi(t_w) = 0$ for almost all $w \in W$. For $w \in W$, let ξ^w be the element of Q^W defined by

(2) $$\xi^w(t_v) = \delta_{v,w}, \quad \text{i.e.} \quad \xi^w(y) = P^y_{w,y} = (-1)^{l'(y)} y^{-1}(S^T_{y,w}) \quad \text{for } y, v, w \in W.$$

The elements $\{\xi^w\}_{w \in W}$ then form a right $\mathcal{S}$-module basis of Λ, and we give Λ the unique graded $\mathcal{S}$-module structure so that $\xi^w \in \Lambda_{2l'(w)}$.

Proposition. (a) *Λ is a $\mathcal{S}$-subalgebra of Q^W, with identity element ξ^e, and is even a graded $\mathcal{S}$-algebra in the grading defined above.*
(b) *Λ is stable under the left action of $\overline{H} \subseteq Q^W$.*

Proof. First, the calculation in 3.5 (for $\Gamma = W$, $A = \emptyset$) implies that

(3) $$\xi^x \xi^y = \sum_{w \in W} P^w_{x,y} \xi^w$$

where $P^w_{x,y} := p^{w,\emptyset}_{x,y}$ was defined in 3.5(2). Together with 3.7(a) and 3.6(d), this proves (a). Part (b) follows immediately from the definitions.

3.9. If Γ is finite, one can dualize the comodule M_Γ for the coalgebra H (see 3.5 and 3.7(a)). In general, we now give a similar "dualization" of M_Γ, to obtain a graded Λ-module Λ_Γ.

Recall the ring H_L with its discrete topology; we also give Q^W and its subring Λ the discrete topology. Now let Q^Γ denote the set of all functions $\Gamma \to Q$ which are supported in a finitely generated coideal of Γ. For any formal symbol D, the map $f \mapsto \sum_{w\in\Gamma} D_w f(w)$ is a bijection $Q^\Gamma \to K_Q^{\uparrow}(\Gamma, D)$ which we use to give Q^Γ the structure of topological (right) Q-module. From 2.10, we hence obtain isomorphisms

$$Q^\Gamma \cong K_Q^{\uparrow}(\Gamma, D) \cong \mathrm{Hom}_{\mathrm{cont.}\ Q}\big(K_Q^{\downarrow}(\Gamma, \delta), Q\big) \tag{1}$$

$$K_B^{\uparrow}(\Gamma, \eta^A) \cong \mathrm{Hom}_{\mathrm{cont.}\ B}\big(K_B^{\downarrow}(\Gamma, m^A), B\big) \qquad \text{for } B = Q \text{ or } B = \mathcal{S} \tag{2}$$

of Q-modules (resp., B-modules), where η^A is a new formal symbol. Using the isomorphism $K_Q^{\downarrow}(\Gamma, m^A) \cong K_Q^{\downarrow}(\Gamma, \delta)$ from the proof of 2.11, we have isomorphisms of right Q-modules

$$K_Q^{\uparrow}(\Gamma, \eta^A) \cong \mathrm{Hom}_{\mathrm{cont.}\ Q}(K_Q^{\downarrow}(\Gamma, \delta), Q) \cong Q^\Gamma. \tag{3}$$

Now Q^Γ has a natural structure of topological Q^W-module, with

$$(\Psi\Psi')(w) = \Psi(w)\Psi'(w) \qquad \text{for all } \Psi \in Q^W,\ \Psi' \in Q^\Gamma \text{ and } w \in \Gamma. \tag{4}$$

Moreover, since $K_Q^{\downarrow}(\Gamma, \delta)$ has a natural structure of topological left Q_{W_L}-module, we have a natural Q_{W_L}-module structure on $\mathrm{Hom}_{\mathrm{cont.}\ Q}\big(K_Q^{\downarrow}(\Gamma, \delta), Q\big)$, given by

$$(h \cdot \psi')(m) = \psi'(\overline{h}m) \tag{5}$$

for $h \in Q_{W_L}$, $\psi \in \mathrm{Hom}_{\mathrm{cont.}\ Q}\big(K_Q^{\downarrow}(\Gamma, \delta), Q\big)$ and $m \in K_Q^{\downarrow}(\Gamma, \delta)$. Using (3), we transfer the Q_{W_L}-module structure and Q^W-module structure to $K_Q^{\uparrow}(\Gamma, \eta^A)$.

Proposition. *For $n \in \mathbb{Z}$, define Λ'_n as the set of formal $\mathcal{S}$-linear combinations $\sum^{\uparrow}_{w\in\Gamma} \eta^A_w a_w$ in $K_{\mathcal{S}}^{\uparrow}(\Gamma, \eta^A)$ with $a_w \in \mathcal{S}_{n-2l_A(w)}$ for all $w \in \Gamma$, and define the right $\mathcal{S}$-submodule $\Lambda_\Gamma = \Lambda' := \oplus_{n\in\mathbb{Z}}\Lambda'_n$ of $K_{\mathcal{S}}^{\uparrow}(\Gamma, \eta^A)$. Then the structures of Q_{W_L}-module and Q^W-module defined above on $K_Q^{\uparrow}(\Gamma, \eta^A)$ induce by restriction natural structures of topological $\overline{H}_L$-module and graded topological Λ-module on Λ_Γ.*

Proof. Now from the definitions, for $x, v \in \Gamma$,

$$\delta_{x,v} = \eta^A_v(m^A_x) = \sum^{\downarrow}_{y\in\Gamma} \eta^A_v(\delta_y) y^{-1}(S^A_{y,x}),$$

so inverting using 2.14,

$$\eta^A_v(y) = \eta^A_v(\delta_y) = (-1)^{l'(y)} y^{-1}(S^{T\setminus A}_{y,v}) \quad \text{for } y \in \Gamma \tag{6}$$

or equivalently,

$$\eta^A_v = \sum^{\uparrow}_{y\in\Gamma} D_y (-1)^{l(y)} y^{-1}(S^{T\setminus A}_{y,v}). \tag{7}$$

Now in (3), the first and last modules are actually topological Q-modules, and (7) shows the isomorphism between them given by (3) is a homeomorphism; hence the Q^W-module structure on $K_Q^{\uparrow}(\Gamma, \eta^A)$ is continuous by its definition. Inverting (7) again gives

$$D_v = \sum_{y\in\Gamma}^{\uparrow} \eta_y^A v^{-1}(S_{v,y}^A). \tag{8}$$

Hence for any $u \in W$ and $v \in \Gamma$,

$$\xi^u \eta_v^A = \sum_{y\in\Gamma} D_y \xi^u(y)\eta_v^A(y) = \sum_{w\in\Gamma} \eta_w^A y^{-1}\big(S_{y,w}^A S_{y,u}^T S_{y,v}^{T\setminus A}\big).$$

That is,

$$\xi^u \eta_v^A = \sum_{w\in\Gamma} \eta_w^A\, p_{u,v}^{w,A}. \tag{9}$$

Since $K_Q^{\uparrow}(\Gamma, \eta^A)$ is a topological Λ-module (by restiction of its topological Q^W-module structure) and the ξ^u form a graded right S-basis of Λ, it follows from 3.7(a) that $\Lambda_n\Lambda'_m \subseteq \Lambda'_{n+m}$ and that $\Lambda_\Gamma := \oplus_{n\in\mathbb{Z}}\Lambda'_n$ is a graded topological Λ-module.

Now for the $\overline{H}_L$-module structure. First, the definitions give that

$$(\delta_w \cdot \psi')(v) = \psi'(w^{-1}v) \qquad \text{for } w \in W_L, \psi' \in Q^\Gamma \text{ and } v \in \Gamma \tag{10}$$

$$(q \cdot \psi')(v) = v^{-1}(q)\psi'(v) \qquad \text{for } q \in Q,\ \psi' \in Q^\Gamma \text{ and } v \in \Gamma, \tag{11}$$

from which (using the Z-property for (10)) the Q_{W_L}-module structure on Q^Γ (and hence the one on $K_Q^{\uparrow}(\Gamma, \eta^A)$) is continuous. By dualizing 2.11(2)–(3), one obtains

$$\chi \cdot \eta_w^A = \eta_w^A w^{-1}(\chi) - \sum_{w\xrightarrow{\gamma}u} \eta_u^A \langle\, \chi \mid \gamma^\vee \,\rangle \qquad \text{for } \chi \in V, \tag{12}$$

$$\overline{t_r} \cdot \eta_w^A = \begin{cases} \eta_{rw}^A & \text{if } rw <_A w \\ 0 & \text{if } rw >_A w. \end{cases} \tag{13}$$

This and continuity shows that

(14) $\bar{t}_r \cdot \Lambda'_n \subseteq \Lambda'_{n-2}$ and $\chi \cdot \Lambda'_n \subseteq \Lambda'_{n+2}$ for all $\chi \in V$ and $r \in L$.

Since V and elements $\bar{t}_r$, for $r \in L$ generate $\overline{H}_L$, this completes the proof of the proposition.

Remark. If $\Gamma = W$ in Chevalley order, so $A = \emptyset$, then Λ_Γ is the left regular Λ-module ${}_\Lambda\Lambda$ and $\eta_w^A = \xi^w$. Also, taking $L = S$, the $\overline{H}_L$-module structure on Λ_Γ coincides with the $\overline{H}$-module structure on Λ. Therefore, formulae involving Λ_Γ established so far or subsequently apply in particular to Λ.

3.10. Introduce the following notation. Write $w(\psi) = \delta_w \cdot \psi$ and $\Delta_w(\psi) = \overline{t_{w^{-1}}} \cdot \psi$ for $w \in W$ and $\psi \in \Lambda$, or for $\psi \in \Lambda_\Gamma$ and $w \in W_L$. There should be no confusion between the notation Δ_w and that for comultiplication. For $\chi \in \mathcal{S}$ and $m \in \Lambda_\Gamma$ or $m \in \Lambda$, we sometimes write χm in place of $m\chi$. Note that in general, $\chi \cdot m \neq \chi m$ for $\chi \in \mathcal{S}$ (see 3.9(12)). The following lemma lists a number of additional facts concerning Λ and the graded Λ-module $\Lambda' = \Lambda_\Gamma$.

Lemma. *Let $\psi \in \Lambda$, $\psi' \in \Lambda_\Gamma$, $w \in W_L$, $u \in \Gamma$ and $r = s_\alpha$ with $\alpha \in \Pi$. Then*
(a) *the W_L action on Λ_Γ is as a group of graded $\mathcal{S}$-module automorphisms satisfying $w(\psi\psi') = w(\psi)w(\psi')$. In particular, the W-action on Λ is as a group of graded $\mathcal{S}$-algebra automorphisms.*
(b) $\Delta_r(\psi\psi') = \psi\Delta_r(\Psi') + \Delta_r(\psi)r(\psi') = r(\psi)\Delta_r(\psi') + \Delta_r(\psi)\psi'$ *if $r \in L$*
(c) $\Delta_r\big(\Delta_r(\psi)\psi'\big) = \Delta_r\big(\psi\Delta_r(\psi')\big)$ *if $r \in L$*
(d) $\Delta_w(\eta_u^A)$ *equals η_{wu}^A if $l_A(u) = l'(w^{-1}) + l_A(wu)$ and is zero otherwise.*
(e) $w(\eta_u^A) = \sum_{z\in\Gamma} \eta_z^A g_{w,u}^{z,A}$ *where*

$$g_{w,u}^{z,A} = \sum_{y\in\Gamma} (-1)^{l'(y)} y^{-1}(S_{y,u}^{T\setminus A})(wy)^{-1}(S_{wy,z}^A) \in \mathcal{S}_{2l_A(z,u)}$$

(f) *if $r \in L$, $r(\eta_u^A)$ is equal to $\eta_u^A + \eta_{ru}^A u^{-1}(\alpha) - \sum_{ru \xrightarrow{\gamma}_A y} \langle\, \alpha \mid \gamma^\vee \,\rangle \eta_y^A$ if $ru \leq_A u$ and to η_u^A if $ru >_A u$.*
(g) *assuming for simplicity that there is an element $\chi_\alpha \in V$ as in 3.2(9), one has*

$$\xi^r \eta_u^A u = \eta_u^A\big(\chi_\alpha - u^{-1}(\chi_\alpha)\big) + \sum_{u \xrightarrow{\gamma}_A z} \eta_z^A \langle\, \chi_\alpha \mid \gamma^\vee \,\rangle \qquad \text{for } r \in L.$$

(h) *for $\chi \in \mathcal{S}$, write $\chi \cdot m_u^A = \sum_{y\in\Gamma} m_y^A \Omega_{y,u}(\chi)$ for some (unique) $\Omega_{y,u}(\chi) \in \mathcal{S}$. Then $\chi \cdot \eta_y^A = \sum_{u\in\Gamma} \eta_u^A \Omega_{y,u}(\chi)$.*
(j) $\chi \cdot \psi' = (\chi \cdot \xi^e)\psi'$.

Proof. The formula in (f) follows from 3.9(12)–(13) on writing $\delta_r \cdot \psi' = (\alpha\bar{t}_r + \delta_e)\cdot\psi'$, and the claim in (a) that the W_L-action preserves the grading follows from (f), for instance. By 3.9(10), one has $\big(w(\psi')\big)(u) = \psi'(w^{-1}u)$ and the analogous formula for ψ, so the rest of (a) follows easily from the definition $(\psi\psi')(u) = \psi(u)\psi'(u)$ of the Λ-module structure on Λ_Γ. By 3.9(10)–(11),

$$(1) \qquad \big(\Delta_r(\psi')\big)(u) = \frac{\psi'(ru) - \psi'(u)}{u^{-1}(\alpha)}.$$

Then (b) follows by easy computations from (1) and the analogous formula for ψ, and (c) follows from (b) noting $\Delta_r^2 = 0$ since $\bar{t}_r^2 = 0$. Part (d) follows immediately from 3.9(13) by induction on $l'(w)$. Part (e) may be proved writing $\delta_w \cdot \eta_u^A = \sum_{y\in\Gamma} D_{wy}(\delta_w \cdot \eta_u^A)(wy)$, using 3.9(10), (6) and (8) and noting $g_{w,u}^{z,A} \in \mathcal{S}_{2l_A(z,u)}$ since w is a graded $\mathcal{S}$-module automorphism of Λ_Γ. For (g), note by 3.7(a) that $p_{r,u}^{y,A} = 0$ unless $y \geq u$ and $l(u,y) \leq 1$, in which cases the value is given by 3.6(e), or 3.6(b) and 3.2(9). Finally, (h) and (j) are immediate consequences of the definitions.

Remarks. (a) In the terminology of [24], Λ equipped with operators Δ_r for $r \in L$ is a ring with twisted derivations, and Λ_Γ equipped with the Δ_r for $r \in L$ is a module with twisted derivations over Λ.
(b) There may be infinitely many non-zero terms in the sums in 3.6(f)–(g).

3.11. Let $\Lambda' = \Lambda_\Gamma$ and $\Lambda'' = \Lambda_{\Gamma^{\text{op}}}$ be the graded Λ-modules with $\overline{H}_L$-module structure associated to the poset Γ in the orders $\leq_A$ and $\leq_{T\setminus A}$ respectively.

Lemma. *There is a bilinear pairing $\langle\cdot,\cdot\rangle\colon \Lambda'\times\Lambda''\to\mathcal{S}$, separately continuous in each variable, given by*

$$\langle \sum_w^{\uparrow} \eta_w^A a_w, \sum_w^{\downarrow} \eta_w^{T\setminus A} b_w \rangle = \sum_w a_w b_w.$$

For $\psi\in\Lambda$, $\psi'\in\Lambda'$, $\psi''\in\Lambda''$ and $h\in\overline{H}_L$
(a) $\langle\psi\psi',\psi''\rangle = \langle\psi',\psi\psi''\rangle$
(b) $\langle h\cdot\psi',\psi''\rangle = \langle\psi',\iota(h)\psi''\rangle$ *where $\iota\colon\overline{H}\to\overline{H}$ is the ring anti-involution determined by $\iota(\overline{t_w}) = \overline{t_{w^{-1}}}$ for $w\in W$ and $\chi\mapsto\chi$ for $\chi\in V$ (see 2.13).*

Proof. The claimed continuity is easily checked, and (a) follows from 3.6(c). To check (b), it is enough to check the cases $h=\overline{t}_r$ for $r\in L$ and $h\in V$, which are clear from 3.9(12)–(13).

3.12. Suppose that our fixed dual based root systems over $\mathbb{R}$ arise by extension of scalars from root systems over $\mathbb{R}$. Recall that we have then defined $\mathcal{S}_{\mathbb{Z}}$ as the symmetric algebra of the weight lattice in V. Let $\Lambda_{\mathbb{Z}}$ be the graded $\mathcal{S}_{\mathbb{Z}}$ submodule of Λ spanned by the elements ξ^w for $w\in W$. Similarly, let $(\Lambda_\Gamma)_{\mathbb{Z}}$ be the graded $\mathcal{S}_{\mathbb{Z}}$-submodule of Λ_Γ with n-th homogeneous component consisting of elements $\sum^{\uparrow}_{w\in\Gamma}\eta_w^A a_w$ with all $a_w\in\mathcal{S}_{\mathbb{Z}}$ homogeneous of degree $n-2l_A(w)$.

Proposition. (a) *The $\mathcal{S}_{\mathbb{Z}}$ submodule $(\Lambda_\Gamma)_{\mathbb{Z}}$ of Λ_Γ is stable under action by elements of $\Lambda_{\mathbb{Z}}$. Hence, $\Lambda_{\mathbb{Z}}$ is a graded $\mathcal{S}_{\mathbb{Z}}$-algebra, and $(\Lambda_\Gamma)_{\mathbb{Z}}$ is a graded $\Lambda_{\mathbb{Z}}$-module. Moreover, $\Lambda_{\mathbb{Z}}$ (resp., $(\Lambda_\Gamma)_{\mathbb{Z}}$) is stable under the Coxeter group operators from W on Λ (resp., from W_L on Λ_Γ), the operators Δ_w for $w\in W$ (resp., $w\in W_L$) and $\psi\mapsto\chi\cdot\psi$ for $\chi\in\mathcal{S}_{\mathbb{Z}}$ and $\psi\in\Lambda_{\mathbb{Z}}$ (resp., $\psi\in(\Lambda_\Gamma)_{\mathbb{Z}}$).*
(b) *If $u,v,w\in W^J$, $\beta\in\Phi^+$ and $u\in W_J w s_\beta$, then $\beta\mid(P^w_{v,w}-P^u_{v,u})$ in $\mathcal{S}_{\mathbb{Z}}$.*

Proof. Observe first that $H_{\mathbb{Z}} := \sum_{w\in W} t_w\mathcal{S}_{\mathbb{Z}}$ is a subring of H, by 2.11(2)–(3) for $A=\emptyset$. It is easy to see from 3.7(1) that all $\Omega_{y,w}\in H_{\mathbb{Z}}$ and hence all $p^{w,A}_{x,y}\in\mathcal{S}_{\mathbb{Z}}$. This implies the first claim in (a), and the second follows immediately. Stability of $\Lambda_{\mathbb{Z}}$ (resp., $(\Lambda_\Gamma)_{\mathbb{Z}}$) under the indicated operators in $\overline{H}$ (resp., $\overline{H}_L$) is clear from previously given formulae for them. For (b), one need only note that the the elements of H considered in the proof of 3.7(c) actually lie in $H_{\mathbb{Z}}$.

Remark. The sequel to this paper will study the analogues of the situation here for a ring Ω which is essentially the equivariant K-theory ring Ω of the flag variety of a Kac-Moody group (see [27]). The ring $\Lambda_{\mathbb{Z}}$ arises as the graded ring of a filtered form of Ω. It is known for some classes of orders $\leq_A$ (and is probably true in general) that the $(\Lambda_\Gamma)_{\mathbb{Z}}$ arise as the graded modules for certain filtered Ω-modules Ω_Γ.

4. Schubert calculus

This section develops some properties of the rings of W_J-invariants of the ring Λ, and for $J\subseteq L$, of the W_J-invariants of the Λ-module Λ_Γ. We obtain these results for Λ by well known arguments from the Schubert calculus, using the following formalism which is extended afterward to Λ_Γ.

4.1. Suppose given a (finitely generated) Coxeter system (W, S), a commutative, graded ring B and a commutative, graded B-algebra $\mathbb{W}$. We make the following assumptions (i)–(v) on this situation.

(i) There is a given family $\{\xi^w\}_{w\in W}$ of elements of $\mathbb{W}$, with $\xi^w \in \mathbb{W}_{2l'(w)}$, which span $\mathbb{W}$ as B-module, and such that ξ^e is the identity of $\mathbb{W}$ (where e is the identity element of W).
(ii) There is a given action of W as a group of graded B-algebra automorphisms of $\mathbb{W}$
(iii) For each $s \in S$, there is a given B-linear map $\Delta_s\colon \mathbb{W} \to \mathbb{W}$ such that $\Delta_s(\psi_1\psi_2) = \Delta_s(\psi_1)s(\psi_2) + \psi_1\Delta_s(\psi_2)$ for all $\psi_1, \psi_2 \in \mathbb{W}$
(iv) for a simple reflection s and any $w \in W$,

$$\Delta_s(\xi^w) = \begin{cases} \xi^{sw} & \text{if } l'(sw) < l'(w) \\ 0 & \text{otherwise.} \end{cases}$$

(v) $s(\psi) = \psi$ iff $\Delta_s(\psi) = 0$, for any simple reflection s and any $\psi \in \mathbb{W}$.

There is then a B-linear operator $\Delta_x\colon \mathbb{W} \to \mathbb{W}$ for each $x \in W$, defined by setting $\Delta_x = \Delta_{s_1} \ldots \Delta_{s_n}$ for any reduced expression $x = s_1 \ldots s_n$. For $x, y \in W$, let $I^A_{x,y} \in \mathbb{W}$ be defined to be ξ^e if $l_A(y) = l'(x^{-1}) + l_A(xy)$ and 0 otherwise, and set $I'_{x,y} = I^{\emptyset}_{x,y}$. Then

$$(1) \qquad \Delta_x(\xi^y) = I'_{x,y}\xi^{xy}.$$

4.2. Before describing the examples to which the preceeding formalism is intended to apply, we record its main consequences.

Proposition. *Let $W_J \subseteq W_K$ be standard parabolic subgroups of W, with rings of invariants $\mathbb{W}^J \supseteq \mathbb{W}^K$ on $\mathbb{W}$. Then*
(a) *the elements $\{\xi^w\}_{w\in W^J}$ form a graded basis of $\mathbb{W}^J$ over B.*
(b) *the elements $\{\xi^w\}_{w\in W_K\cap W^J}$ form a graded $\mathbb{W}^K$-basis of $\mathbb{W}^J$.*

Proof. If there were a non-trivial linear relation $\sum \xi^w a_w = 0$ with all $a_w \in B$, one could choose $u \in W$ so $l'(u)$ is maximal with $a_u \neq 0$ and apply $\Delta_{u^{-1}}$ to the relation to get the contradiction $a_u = a_u\xi^e = 0$. Hence $\{\xi^w\}_{w\in W}$ is B-linearly independent; in fact, it is a graded B-basis of $\mathbb{W}$. Part (a) follows immediately from 4.1(iv)–(v) on noting that $\mathbb{W}^K = \cap_{s\in K}\mathbb{W}^s$. Now one sees by induction on $l'(x)$ that

$$(1) \qquad \Delta_x(\psi_1\psi_2) = \psi_1\Delta_x(\psi_2) \qquad \text{for } \psi_1 \in \mathbb{W}^K, \psi_2 \in \mathbb{W} \text{ and } x \in W_K.$$

For any $w \in W$, we may write (uniquely) $w = w_k w^K$ where $w_K \in W_K$ and $w^K \in W^K$. Note $W^J = (W_K \cap W^J)W^K$ with uniqueness of expression of elements from the left as a product of elements as on the right. Consider $w \in W^J$. Then w_K, w^K are both in W^J. Write $\xi^{w_K}\xi^{w^K} = \sum_{v\in W^J} \xi^v a_v$ for some $a_v \in B_{2(l'(w)-l'(v))}$. We claim that $a_v = 0$ unless $v_K \leq' w_K$, and that $a_v = \delta_{v,w}$ if $v_K = w_K$ (recall $\leq'$ denotes Chevalley order). To prove this claim, take $x \in W_K$. Then $\Delta_x(\xi^{w_K}\xi^{w^K}) = \Delta_x(\sum_{v\in W^J} \xi^v a_v)$ i.e.

$$\xi^{xw_K}\xi^{w^K} I'_{x,w_K} = \sum_{v\in W^J} I'_{x,v_K}\xi^{xv}a_v$$

by (1) and 4.1(1). For $x^{-1} \not\leq' w_K$ (resp., $x^{-1} = w_K$) the left hand side of this equation is 0 (resp. ξ^{w^K}). Examining the coefficient of ξ^{xv} on the right for $v \in W^J$ with $v_K = x^{-1}$ gives the claim. The claim just proved can be equivalently restated as

$$(2) \qquad \xi^x \xi^y \in \xi^{xy} + \sum_{\substack{v \in W^J \\ v_K <' x}} \xi^v B_{2(l'(x)+l'(y)-l'(v))} \quad \text{for } x \in W_K \cap W^J \text{ and } y \in W^K.$$

It follows that the elements $\{\xi^x \xi^y\}$ with $x \in W_K \cap W^J$ and $y \in W^K$ form another graded B-basis of $\mathbb{W}^J$ (related to the basis $\{\xi^w\}_{w \in W^J}$ by an infinite, upper unitriangular change of basis matrix, with respect to suitable row and column orderings). Since the elements $\{\xi^y\}_{y \in W^K}$ form a graded B-basis of $\mathbb{W}^K$, (b) is proved.

4.3. Let W_J, W_K be as in the preceeding proposition, and assume also that they are finite. Denote their longest elements by ω_J and ω_K respectively. It is well-known that the map $x \mapsto \hat{x}: W_K \cap W^J \to W_K \cap W^J$ given by $\hat{x} = w_J x w_K$ is an order-reversing (in Chevalley order) bijection of $W_K \cap W^J$ with itself, satisfying $l'(x) + l'(\hat{x}) = l'(\omega_J \omega_K)$. In particular, $\hat{1} = \omega_J \omega_K$ is the maximum element of $W_K \cap W^J$ in the order induced by Chevalley order.

By 4.2(b) with $J = \emptyset$, there are for $x \in W^K$ unique $\mathbb{W}^K$-linear maps (of degree zero) $c_x: \mathbb{W} \to \mathbb{W}^K \langle 2l'(x) \rangle$ such that $\psi = \sum_{x \in W^K} c_x(\psi) \xi^x$ for all $\psi \in \mathbb{W}$.

Proposition. *If $W_J \subseteq W_K$ are finite, there is is an isomorphism of graded $\mathbb{W}^J$-modules $\theta: \mathbb{W}^J \to \mathrm{Hom}_{\mathbb{W}^K}(\mathbb{W}^J, \mathbb{W}^K)\langle 2l'(\omega_J \omega_K) \rangle$ given by $\theta(a)(b) = c_{w_K}(a \xi^{w_J} b)$ for $a, b \in \mathbb{W}^J$.*

Proof. Note first that for $\psi \in \mathbb{W}$, $\Delta_{\omega_K}(\psi) = \Delta_{\omega_K}(\sum_{x \in W^K} c_x(\psi) \xi^x) = c_{\omega_K}(\psi)$ by 4.1(1) and 4.2(1). Also, for a simple reflection s and any $\psi_1, \psi_2 \in \mathbb{W}$, we have

$$\Delta_s(\Delta_s(\psi_1)\psi_2) = \Delta_s(\Delta_s(\psi_1)\Delta_s(\psi_2)) = \Delta_s(\psi_1 \Delta_s(\psi_2))$$

since $\Delta_s^2 = 0$. We show now that

$$(1) \qquad \Delta_{\omega_K}(\xi^x \xi^y) = \delta_{x, y\omega_K} \qquad \text{if } x, y \in W_K \text{ with } l'(x) + l'(y) \leq l'(\omega_K).$$

In the proof, we write $I(x <' y)$ to denote ξ^e if $x <' y$ and 0 otherwise. Now if $l'(x) + l'(y) < l'(\omega_K)$, then (1) is trivial, so we assume $l'(x) + l'(y) = l'(\omega_K)$ and proceed by downward induction on $l'(y)$. If $l'(y) = l'(\omega_K)$, then (1) is again trivial. Otherwise, choose $s \in K$ with $l'(sy) > l'(y)$ and observe that

$$\begin{aligned} \Delta_{\omega_K}(\xi^x \xi^y) &= \Delta_{\omega_K s} \Delta_s(\xi^x \Delta_s(\xi^{sy})) = \Delta_{\omega_K s} \Delta_s(\Delta_s(\xi^x) \xi^{sy}) \\ &= I(sx <' x) \Delta_{\omega_K}(\xi^{sx} \xi^{sy}) = I(sx <' x) \delta_{sx, sy\omega_K} = \delta_{sx, sy\omega_K} \end{aligned}$$

as required to finish the proof of (1). For $x, y \in W_K \cap W^J$ with $l'(x) + l'(y) \leq l'(\omega_J \omega_K)$, we now have

$$(2) \qquad \begin{aligned} \Delta_{\omega_K}(\xi^x \xi^{\omega_J} \xi^y) &= \Delta_{\omega_K \omega_J} \Delta_{\omega_J}(\xi^x \xi^y \xi^{\omega_J}) = \Delta_{\omega_K \omega_J}(\xi^x \xi^y) \\ &= \Delta_{\omega_K \omega_J} \Delta_{\omega_J}(\xi^x \xi^{\omega_J y}) = \Delta_{\omega_K}(\xi^x \xi^{\omega_J y}) = \delta_{x, \hat{y}} \end{aligned}$$

by (1) and 4.2(1). This implies that the matrix $\big(\theta(\xi^x)(\xi^{\hat{y}})\big)_{x,y\in W_K\cap W^J}$ of elements of W is upper unitriangular for a suitable ordering of its rows and colums, in particular it has determinant $\pm\xi^e \in \mathbb{W}^K$. Since the elements $\{\xi^x\}_{x\in W_K\cap W^J}$ form a $\mathbb{W}^K$-basis of $\mathbb{W}^J$, it is clear that θ is an isomorphism of ungraded $\mathbb{W}_J$-modules. Since θ is homogeneous of degree zero, it is a graded isomorphism also.

4.4. From data $\mathbb{W}$, B, W etc satisfying the conditions 4.1(i)–(v), one may obtain similar sets of data by base change, under suitable conditions. For example, let C be any commutative, graded B-algebra. Then

(1) if 2 is not a zero-divisor in C, the C-algebra $\mathbb{W}\otimes_B C$ with natural C-basis $\{\xi^w\otimes 1_C\}_{w\in W}$, W-automorphisms $w\otimes_B \mathrm{Id}_C$ and operators $\Delta_s\otimes_B \mathrm{Id}_C$ for simple reflections s also satisfy the conditions 4.1(i)–(v).

Indeed, all conditions except (v) are clear without the hypothesis on $2\in C$. For (v), let $f\in\mathbb{W}\otimes_B C$ and s be a simple reflection. If $s(f)=f$, then $2f=f+s(f)\in \mathbb{W}^s\otimes_B C$ and so $\Delta_s(2f)=0$. Since 2 is not a zero-divisor in C and $W^s\otimes_B C$ is a free C-module, $\Delta_s(f)=0$ as wanted. Conversely, suppose $\Delta_s(f)=0$. Writing $f=\sum_{w\in W}\xi^w\otimes c_w$ with $c_w\in C$, one has $c_w=0$ unless $sw>' w$. Since $s(\xi^w)=\xi^w$ when $sw>w$, this gives $s(f)=f$.

Remark. If C is an ungraded B-algebra in which 2 is not a zero divisor, then the data in (1) satisfies the ungraded analogues of 4.1(1)–(v) and the ungraded analogues of conclusions of the Propositions 4.2 and 4.3.

4.5. This subsection describes some situations (a)–(c') in which the formalism from 4.1 can be applied. First, we have (a)–(c) immediately below.

(a) Let $\mathbb{W}=\Lambda$, $B=\mathcal{S}$, and define the elements ξ^w as in 3.8. The operators w and Δ_w for $w\in W$ are as defined in 3.10. Then 4.1(i)–(v) hold. The only point not previously checked is 4.1(v), which follows readily from the formula 3.10(f), for instance.

(b) Take $\mathbb{W}=\Lambda_{\mathbb{Z}}$, $B=S_{\mathbb{Z}}$, and ξ^w as in 3.12. The Weyl group automorphisms and operators Δ_s are also as defined there. Again, 4.1(i)–(v) are easily seen to hold.

(c) Here we assume that W is finite with longest element ω. Take $\mathbb{W}=\mathcal{S}$, with the natural W-action induced by that on $V=\mathcal{S}_2$, and let B denote the subalgebra of W-invariant elements of $\mathcal{S}$. Recall from 3.3 the elements $\overline{t_w}$ of the nil Hecke ring and their action on $\mathcal{S}$; for instance, for a parabolic subgroup W_K of W,

$$\overline{t_{\omega_K}}(f)=\frac{1}{\prod_{\alpha\in\Phi_K^+}\alpha}\sum_{w\in W_K}(-1)^{l'(\omega_K)-l'(w)}w(f). \tag{1}$$

Set $\Delta_r=\overline{t_r}$ for $r\in S$, and, for any $y\in W$, set $\xi^{y\omega}=\overline{t_{y^{-1}}}(\frac{(-1)^{l'(\omega)}}{\sharp(W)}D)$ where $D=\prod_{\alpha\in\Phi^+}\alpha$. Then conditions 4.1(i)–(v) are known to hold; we indicate a proof below.

First, for $w\in W$, $w(D)=(-1)^{l'(w)}D$, so (1) shows that

$$\overline{t_{\omega_K}}(\xi^\omega)=(-1)^{l'(\omega)-l'(\omega_K)}\frac{\sharp(W_K)}{\sharp(W)}\prod_{\alpha\in\Phi^+\setminus\Phi_K^+}\alpha, \tag{2}$$

and in particular $\xi^e = 1$. Next, since $\overline{t_v}\,\overline{t_w}$ is equal to $\overline{t_{wv}}$ if $l'(wv) = l'(v) + l'(w)$ and to 0 otherwise, one has $\overline{t_{y^{-1}}}\xi^w = I'_{y,w}\xi^{yw}$ which proves 4.1(iv). For 4.1(v), note that if $\alpha \in \Pi$, then $s_\alpha(f) = \alpha\bar{t}_{s_\alpha}(f) + f$ for $f \in \mathcal{S}$, and $\mathcal{S}$ is an integral domain. The part 4.1(ii) is trivial, 4.1(iii) is an easy computation from the definitions, and clearly $\xi^y \in \mathcal{S}_{2l'(y)}$, so it remains to prove the claim in 4.1(i) that $\{\xi^w\}_{w\in W}$ spans $\mathcal{S}$ over B. Let $F = \mathcal{S}\mathcal{S}^W_+$ be the ideal of S generated by homogeneous W-invariant elements of $\mathcal{S}$ of positive degree. By well-known facts ([34, 4.2.6 and 4.2.8] and 10.1(2)), a family $\{d^w\}_{w\in W}$ of homogeneous elements of $\mathcal{S}$ are a B-module basis of $\mathcal{S}$ iff the classes $d^w + F$ of the d^w in the quotient algebra S/F are $\mathbb{R}$-linearly independent. Now $\bar{t}_r(F) \subseteq F$ for $r \in S$, so for $w \in W$, $\overline{t_w}$ acts naturally on the quotient $\mathcal{S}/F$. Linear independence of the classes $\xi^w + F$ in $\mathcal{S}/F$ over $\mathbb{R}$ follows as in the start of the proof of 4.2, so the ξ^w are a B-basis of $\mathcal{S}$ as required to complete the verification of the conditions 4.1(i)–(v) in the situation (c) above.

For later use, we record the following well-known fact (which can also be easily seen from the above discussion):

(3) the map $f \mapsto \overline{t_{\omega_K}}(f)$ is a $\mathcal{S}^K$-linear surjection of $\mathcal{S}$ onto $\mathcal{S}^K$.

Of course, (3) requires only the assumption of finiteness of W_K and not of W.

Additional examples of data satisfying 4.1(i)–(v) arise from base change as in 4.4. In particular, we explicitly list the main examples (a')–(c') arising this way.

(a') the example $\mathbb{W} = \Lambda \otimes_{\mathcal{S}} \mathbb{R}$, $B = \mathbb{R}$ obtained by base change $? \otimes_{\mathcal{S}} \mathbb{R}$ from (a)

(b') the example $\mathbb{W} = \Lambda_{\mathbb{Z}} \otimes_{\mathcal{S}_{\mathbb{Z}}} \mathbb{Z}$, $B = \mathbb{Z}$ obtained by base change $? \otimes_{\mathcal{S}_{\mathbb{Z}}} \mathbb{Z}$ from (b)

(c') for finite W, the example $\mathbb{W} = \mathcal{S} \otimes_{\mathcal{S}'} \mathbb{R}$, $B = \mathbb{R}$ arising from base change $? \otimes_{\mathcal{S}'} \mathbb{R}$ from situation (c), where $\mathcal{S}'$ is the subalgebra of W-invariants of $\mathcal{S}$.

The results obtained from the formalism in 4.1 may be summarized as follows.

Corollary. *The conclusions of* 4.1 *and* 4.3 *hold in all situations* 4.5(*a*)–(*c'*) *listed above.*

Moreover, in situations (a) *or* (c)*, the canonical maps* $\mathbb{W}^J \otimes_B C \to (\mathbb{W} \otimes_B C)^J$ *and (if* $\sharp W_K < \infty$*)* $\mathrm{Hom}_{\mathbb{W}^K}(\mathbb{W}^J, \mathbb{W}^K) \otimes_B C \to \mathrm{Hom}_{\mathbb{W}^K \otimes_B C}(\mathbb{W}^J \otimes_B C, \mathbb{W}^K \otimes_B C)$ *are graded* B*-algebra isomorphisms, for any commutative, graded* B*-algebra* C*.*

Proof. The first assertion is known, and the second claim follows from the fact that in situations (a) or (c) one can actually apply base-change $? \otimes_B C$ as in 4.4(1) for any graded B-algebra C.

Remark. In the situation (c′), there is a natural identification of $\mathbb{W}$ with the coinvariant algebra $\mathcal{S}/F$. The results in situations 4.5(c)-(c′) comprise part of the classical Schubert calculus (see e.g. [5], from which several of the proofs in this section are adapted). Some of them hold in somewhat greater generality (see Section 10). The result 4.3 in situation (a) could also be deduced from 4.3 in situation (c), using the proposition in the next subsection. In the situation (a), the result 4.2(a) is proved in [26].

4.6. Regard Λ as a graded $\mathcal{S} \otimes_{\mathbb{R}} \mathcal{S}$ algebra with $(\chi \otimes_{\mathbb{R}} \chi')\xi^e = \chi \cdot (\xi^e\chi')$. We call the $\mathcal{S}$-algebra structure on Λ given by action of the subring $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{R}$ (resp., $\mathbb{R} \otimes_{\mathbb{R}} \mathcal{S}$) of $\mathcal{S} \otimes \mathcal{S}$ the left (resp., right) S-algebra structure.

Corollary. *Fix finite standard parabolic subgroups* $W_J \subseteq W_K$ *of* W*.*

(a) *The map* $\chi \otimes_{\mathbb{R}} \xi \mapsto \chi \cdot \xi$ *for* $\chi \in \mathcal{S}$*,* $\xi \in \Lambda$ *restricts to an isomorphism of graded* $\mathcal{S}^J \otimes_{\mathbb{R}} \mathcal{S}$*-algebras* $\mathcal{S}^J \otimes_{\mathcal{S}^K} \Lambda^K \to \Lambda^J$*.*

(b) *If W itself is finite and $K = S$, the map $\chi \otimes \chi' \mapsto \chi \cdot \xi^e \chi'$ for $\chi, \chi' \in \mathcal{S}$ restricts to an isomorphism of graded $\mathcal{S}_J \otimes_{\mathbb{R}} \mathcal{S}$-algebras $\mathcal{S}^J \otimes_{\mathcal{S}^K} \mathcal{S} \to \Lambda^J$.*

Proof. Note that if $W = W_K$ is finite, then $\Lambda^K = \xi^e \mathcal{S}$, so (b) is just a special case of (a). Hence it will suffice to prove (a). First, we prove (a) in case $J = \emptyset$. For a simple root α with $s = s_\alpha \in K$, one has operators $s \otimes \mathrm{Id}$ and $\Delta_s = \frac{1}{\alpha}(s - e) \otimes \mathrm{Id}$ on $\mathcal{S} \otimes_{\mathbb{R}} \Lambda^K$, and the operators s and $\bar{t}_s$ on Λ from the action of the nil Hecke ring. From the definitions, one easily sees that these operators intertwine the map $m' : \mathcal{S} \otimes_{\mathbb{R}} \Lambda^K \to \Lambda$ given by $\chi \otimes_{\mathbb{R}} \xi \mapsto \chi \cdot \xi$ for $\chi \in \mathcal{S}$, $\xi \in \Lambda^K$:

(1) $m' \circ (s \otimes \mathrm{Id}) = s \circ m'$ and $m' \circ (\Delta_s) = \bar{t}_s \circ m'$. In particular, the image of m' is closed under action by the operators w and $\bar{t}_w$ for any $w \in W_K$.

This implies that the left graded $\mathcal{S}$-algebra structure on Λ restricts to a left graded $\mathcal{S}^K$-algebra structure on Λ^K, and so m' factors through $\mathcal{S} \otimes_{\mathcal{S}^K} \Lambda^K$ to give a $\mathcal{S} \otimes_{\mathcal{S}^K} \mathcal{S}$-algebra homomorphism $m : \mathcal{S} \otimes_{\mathcal{S}^K} \Lambda^K \to \Lambda$.

Let $D = \frac{(-1)^{l'(\omega_K)}}{\sharp(W_K)} \prod_{\alpha \in \Phi_K^+} \alpha$. Write $m(D \otimes \xi^e) = \sum_w \xi^w a_w$ for some $a_w \in \mathcal{S}$; note $a_w = 0$ if $2l'(w) > \deg D = 2l'(\omega_K)$. Using 3.10(h) and 4.5(2), one sees that $a_{\omega_K} = 1$. From 4.2(b) and 4.2(2), it follows that $d := m(D \otimes \xi^e) \in \xi^{\omega_K} + \sum_{\substack{w \in W_K \\ l'(w) < l'(\omega_K)}} \xi^w \Lambda^K$. Hence for any $u \in W^K$, the image of m contains an element $\Delta_{u\omega_K}(d) \in \xi^u + \sum_{\substack{w \in W_K \\ l'(w) < l'(u)}} \xi^w \Lambda^K$. By induction on $l'(u)$, it follows that ξ^u is in the image of m for all $u \in W_K$. By 4.2(b), it follows that m is surjective. As Λ_K-modules, both Λ and $\mathcal{S} \otimes_{\mathcal{S}^K} \Lambda^K$ are free of the same finite rank $\sharp(W_K)$ (by 4.2(b)). In general, a surjective homomorphism of isomorphic finitely generated (free) modules over a commutative ring is an isomorphism. Hence m is an isomorphism of ungraded Λ^K-modules, and so m is also an isomorphism of graded $\mathcal{S} \otimes_{\mathcal{S}^K} \mathcal{S}$-algebras. This proves (a) if $J = \emptyset$. Now for general J, note first that

(2) Λ is graded free as left $\mathcal{S}^K$-module.

Indeed, for any graded basis $\{d_i\}_{i \in I}$ of $\mathcal{S}$ over $\mathcal{S}^K$, the elements $\xi^w w^{-1}(d_i)$ for $i \in I$ and $w \in W^K$ are easily seen (using 4.2(a)) to form a graded basis of Λ^K as left $\mathcal{S}^K$-module. Now taking W_J invariants in the W_J-equivariant isomorphism m above gives isomorphisms $\mathcal{S}^J \otimes_{\mathcal{S}^K} \Lambda^K \cong (\mathcal{S} \otimes_{\mathcal{S}^K} \Lambda^K)^J \cong \Lambda^J$ of the rings of invariants (the first map is seen to be an isomorphism using (2), for instance). The composite isomorphism is clearly the restriction of the map defined in (a).

4.7. Suppose given data $\mathbb{W}$, W, B etc as in 4.1. We assume here also that $B = \oplus_{n \in \mathbb{N}} B_n$ is positively graded. We give $\mathbb{W}$ and its subrings the discrete topology. We assume for simplicity in the following results that $\Gamma = W_L \Gamma^L$; then $\Gamma = W_J \Gamma^J$ for any $J \subseteq L$. We now consider any topological, graded $\mathbb{W}$-module $\mathbb{M} = \oplus_{n \in \mathbb{Z}} M_n$ satisfying the following conditions (i)–(v) analogous to those imposed on $\mathbb{W}$ (the intended examples will be described later).

(i) with the B-module structure induced by that on $\mathbb{W}$, $\mathbb{M}$ is a B-submodule of $K_B^{\uparrow}(\Gamma, \eta)$ (with the the subspace topology) and $\mathbb{M}_n$ consists of all formal sums $\sum_{w \in \Gamma}^{\uparrow} \eta_w b_w \in K_B^{\uparrow}(\Gamma, \eta)$ with all $b_w \in B_{n - 2l_A(w)}$.
(ii) W_L acts as a group of continuous graded B-module automorphisms of M; the action satisfies $w(\psi\psi') = w(\psi)w(\psi')$ for any $w \in W_L$, $\psi \in \mathbb{W}$ and $\psi' \in \mathbb{M}$
(iii) For each $s \in L$, there is a given continuous, B-linear map $\Delta_s : \mathbb{M} \to \mathbb{M}$ such that $\Delta_s(\psi\psi') = \Delta_s(\psi)s(\psi') + \psi\Delta_s(\psi') = s(\psi)\Delta_s(\psi') + \Delta_s(\psi)\psi'$ for all $\psi \in \mathbb{W}$,

$\psi' \in \mathbb{M}$
(iv) for $s \in L$ and any $w \in W$,

$$\Delta_s(\eta_w) = \begin{cases} \eta_{sw} & \text{if } sw <_A w \\ 0 & \text{otherwise.} \end{cases}$$

(v) $s(\psi) = \psi$ iff $\Delta_s(\psi) = 0$, for any $s \in L$ and any $\psi \in \mathbb{W}$.

It follows that here is a continuous B-linear operator $\Delta_x : \mathbb{M} \to \mathbb{M}$ for each $x \in W_L$, defined by setting $\Delta_x = \Delta_{s_1} \dots \Delta_{s_n}$ for any reduced expression $x = s_1 \dots s_n$. One has

$$(1) \qquad \Delta_x(\eta_y) = I^A_{x,y} \eta_{xy}$$

where $I^A_{x,y}$ is defined in 4.1.

4.8. For any $J \subseteq L$, denote the set of W_J-invariant elements of $\mathbb{M}$ by $\mathbb{M}^J$. Then $\mathbb{M}^J$ is a graded B-submodule of $\mathbb{M}$. Now we have the following result analogous to 4.2.

Proposition. *Let $W_J \subseteq W_K \subseteq W_L$ be standard parabolic subgroups of W_L. Then*
(a) *$\mathbb{M}^J$ is a graded $\mathbb{W}^J$-submodule of $\mathbb{M}$, with $\mathbb{M}^J_n = \sum^{\uparrow}_{w \in \Gamma^J} \eta_w B_{n-2l_A(w)}$.*
(b) *the map $\psi \otimes \psi' \mapsto \psi\psi'$, for $\psi \in \mathbb{W}^J$ and $\psi' \in \mathbb{M}^K$, induces an isomorphism $\mathbb{W}^J \otimes_{\mathbb{W}^K} \mathbb{M}^K \cong \mathbb{M}^J$ of graded $\mathbb{W}_J$-modules.*

Proof. The fact that $\mathbb{M}^J$ is a graded $\mathbb{W}^J$-module follows from 4.7(ii). The explicit description of $\mathbb{M}^J$ in (a) follows immediately from the conditions in 4.7(iv)–(v) on noting that $\mathbb{M}^K = \cap_{s \in K} \mathbb{M}^s$. Next, we prove that
(1) each $m \in M^J_n$ is uniqely expressible as a finite sum $m = \sum_{x \in W_K \cap W^J} \xi^x m_x$ with $m_x \in \mathbb{M}^K_{n-2l'(x)}$.
By 4.2(b), which is actually a special case of (1), this will prove (b). In the proof, for any $v \in \Gamma$, write $v = v_K v^K$ with $v_K \in W_K$ and $v^K \in W\Gamma^K$, so $l_A(v) = l'(v_K) + l_A(v^K)$. First, note the formula

$$(2) \qquad \Delta_x(\psi m) = \Delta_x(\psi) m \qquad \text{for } \psi \in \mathbb{W},\ m \in \mathbb{M}^K \text{ and } x \in W_K$$

which is proved by induction on $l'(x)$. Consider $x \in W_K \cap W^J$ and $y \in \Gamma^K$. As in the proof of 4.2(b), one sees using (2) that

$$(3) \qquad \xi^x \eta_y \in \eta_{xy} + \sum_{\substack{v \in \Gamma^J \\ v_K <' x \\ v^K \geq y}} \eta_v B_{2l_A(v,xy)} \qquad \text{for } x \in W_K \cap W^J \text{ and } y \in \Gamma^K.$$

The uniqueness claim in (1) follows from this. For let $m = \sum_{x \in W_K \cap W^J} \xi^x m_x = 0$ with $m_x \in \mathbb{M}^K_{n-2l'(x)}$. If some $m_x \neq 0$, choose x' of maximal length $l'(x')$ with $m_{x'} \neq 0$, and write $m_{x'} = \sum^{\uparrow}_{y \in \Gamma^K} \eta_y c_y$, Choose y' minimal in the order $\leq_A$ with $c_{y'} \neq 0$. Then the coefficient of $\eta_{x'y'}$ in the expression of $m = 0$ as a formal B-linear combination of elements η_z is $c_{y'} \neq 0$ by (3), a contradiction. From (3), it follows that for $z \in \Gamma^J$,

$$\eta_z \in \xi^{z_K}\eta_{z^K} + \sum^{\uparrow}_{\substack{x\in W_K\cap W^J,\, x<z_K\\ y\in\Gamma^K,\, y\geq z^K}} \xi^x \eta_y B_{2l_A(xy,z)}. \tag{4}$$

Consider an element $m = \sum^{\uparrow}_{z\in\Gamma^J} \eta_z c_z$ of $\mathbb{M}^J_n$. Then $c_z \in B_{n-2l_A(z)}$. There are elements $z_1, \dots, z_p$ of W^J such that $c_z = 0$ unless $z \geq z_i$ for some i. If $c_z \neq 0$, then $z^K \geq z_i^K$ for some i, and $l'(z_K) \leq l_A(z) - l_A(z_i^K) \leq n/2 - l_A(z_i^K)$; in particular, for some $N \in \mathbb{N}$, $l'(z_K) \leq N$ for $z \in \Gamma$ with $c_z \neq 0$. Note that there are only finitely many elements w of W_K of length $l'(w) \leq N$. It follows that substituting (4) into $m = \sum^{\uparrow}_{z\in\Gamma^J} \eta_z c_z$ gives an expression for m as required in (1), completing the proof.

4.9. We now describe the main situations to which the formalism from 4.8 can be applied.
(a) Let $\mathbb{W} = \Lambda$, $B = \mathcal{S}$ etc be as in 4.5(a), and $\mathbb{M} = \Lambda_\Gamma$ with $\eta_w = \eta^A_w$ as in 3.9. The W_L-action and operators Δ_s on $\mathbb{M}$ are as defined in 3.10. Then all conditions 4.7(i)–(v) hold.
(b) Take $\mathbb{W} = \Lambda_\mathbb{Z}$ $B = \mathcal{S}_\mathbb{Z}$, etc as in 4.5(b). Let $\mathbb{M} = (\Lambda_\Gamma)_\mathbb{Z}$ with W_L-automorphisms and operators Δ_s as in 3.12. Again, all conditions 4.7(i)–(v) hold.

Additional examples arise from (a), (b) by base change (to a commutative, positively graded B-algebra C in which 2 is not a zero divisor) as follows. Let $\mathbb{M}(C)$ be the right C-submodule of $K^{\uparrow}_C(\Gamma, \eta')$ with $\mathbb{M}(C)_n = \sum^{\uparrow} \eta'_w C_{n-2l_A(w)}$. Define the continuous left $\Lambda \otimes_B C$-module structure (and left action of Δ_s, for $s \in L$, and $w \in W_L$) on $\mathbb{M}(C)$ as follows; define the structure constants of $\mathbb{M}(C)$ with respect to standard elements $\xi^w \otimes 1_C$, η'_w by applying the structural homomorphism $B \to C$ to the corresponding structure constants of $\mathbb{M}$ with respect to standard elements ξ^w, η^A_w. The conditions in 4.7 are easily verified. We explicitly record the two main examples arising by base change.
(a′) the module $\mathbb{M}(\mathbb{R})$ arising by base change from $B = \mathcal{S}$ to $C = \mathbb{R}$ in (a)
(b′) the module $\mathbb{M}(\mathbb{Z})$ arising by base change from $B = \mathcal{S}_\mathbb{Z}$ to $C = \mathbb{Z}$ in (b).
We summarize the results obtained with this formalism in the following proposition.

Proposition. *The conclusions of propositions* 4.8 *hold in all situations* (a)–(b′) *listed above. In situation* (a) *with $K \subseteq J \subseteq L$ and W_J finite (so $\mathbb{M} = \Lambda_\Gamma$), $\mathbb{M}^J$ has a natural graded $(\mathcal{S}^J, \mathcal{S})$-bimodule structure induced by its Λ^J-module structure, and, moreover, $\mathcal{S}^K \otimes_{\mathcal{S}^J} \mathbb{M}^J \cong \mathbb{M}^K$ as graded $(\mathcal{S}^K, \mathcal{S})$-bimodule.*

Proof. The first statement is known, and the second follows from 4.6.

Remark. Recall the Coxeter system (W', S') from 2.15. If W_J, W'_K are finite parabolic subgroups of (W, S), (W', S') respectively with $W_J\Gamma W'_K \subseteq \Gamma$ (and Γ finite, for simplicity) there is a natural $(\mathcal{S}^J, \mathcal{S}^K)$-subbimodule $\mathbb{M}^{(J,K)}$ of $\mathbb{M}$ with $\mathbb{M} \cong \mathcal{S} \otimes_{\mathcal{S}^J} \mathbb{M}^{(J,K)} \otimes_{\mathcal{S}^K} \mathcal{S}$ as $(\mathcal{S}, \mathcal{S})$-bimodule, and there is an interesting representation theory associated to shortest (W_J, W'_K) double coset representatives in Γ for which one might hope $\mathbb{M}^{(J,K)}$ might function as a dualizing object. The problem of extending these definitions to infinite W'_K motivated 2.16–2.17. For crystallographic W, this problem (for $J = \emptyset$ and $W'_K = W'$) is expected to be relevant (see

[16, 21]) to the study of the category $\mathcal{O}$ of modules for Kac-Moody Lie algebras (W arises as an "integral Weyl group" of a weight λ and W' arises as the reflection subgroup of W generated by reflections fixing λ, in the standard dot action of W on the weights).

5. Extensions of some modules for the dual nil Hecke ring

This section gives some alternative descriptions of the nil Hecke ring and its dual, and some calculations of certain Ext^1 groups (e.g. for some Λ-modules). Throughout this section, we always assume (if the root system arises by extension of scalars from an integral root system) that the roots are all indivisible elements of the weight lattice.

5.1. Recall that Q is a Q_W-module with $(\sum_w q_w\delta_w)\cdot q = \sum_{w\in W} q_w w(q)$. For $\beta\in\Phi^+$, define the localization $\mathcal{S}^{(\beta)} := \mathcal{S}[\alpha^{-1}\mid \alpha\in\Phi^+\setminus\{\beta\}]$ of $\mathcal{S}$; we regard it as a subring of Q. Then

$$\mathcal{S} = \cap_{\beta\in\Phi^+}\mathcal{S}^{(\beta)}. \tag{1}$$

Lemma. *Assume that the fundamental chamber of W on V is sufficiently large (see 1.4). Then an element $h=\sum_w q_w\delta_w\in Q_W$ satisfies $h\cdot\mathcal{S}\subseteq\mathcal{S}$ iff for all $w\in W$ and $\alpha\in\Phi^+$, one has $q_w\in\alpha^{-1}S^{(\alpha)}$ and $q_w+q_{s_\alpha w}\in S^{(\alpha)}$.*

Proof. Let p be any prime element of $\mathcal{S}$, and $x,y\in W$. Then $p\mid(\delta_x-\delta_y)(\chi)$ for all $\chi\in\mathcal{S}$ iff $p\mid(\delta_x-\delta_y)(\chi)$ for all $\chi\in V$, which in turn holds iff p is associate to some (unique) $\alpha\in\Phi^+$ and $y=s_\alpha x$, by 1.4. By 8.2, $h\cdot\mathcal{S}\subseteq\mathcal{S}$ iff (i)–(ii) below hold:
(i) for each prime element $p\in\mathcal{S}$ not associate to any root, $q_w\delta_w\cdot(\mathcal{S})\subseteq\mathcal{S}_{p\mathcal{S}}$
(ii) for each $\alpha\in\Phi^+$ and $w\in W$, $(q_w\delta_w+q_{s_\alpha w}\delta_{s_\alpha w})\cdot\chi\in S_{w(\alpha)\mathcal{S}}$ for all $\chi\in\mathcal{S}$.

Here, $\mathcal{S}_{p\mathcal{S}}$ is the localization of $\mathcal{S}$ at the prime ideal generated by p. For fixed w and α, condition (ii) is equivalent to

$$q_w\chi+q_{s_\alpha w}s_\alpha(\chi)\in\mathcal{S}_{\alpha\mathcal{S}} \tag{iii}$$

for all $\chi\in\mathcal{S}$. Note $\mathcal{S}=\mathcal{S}^{s_\alpha}+\alpha\mathcal{S}^{s_\alpha}$ where $\mathcal{S}^{s_\alpha}$ is the subring of s_α-invariant elements of $\mathcal{S}$. The left hand term in (iii) is $\mathcal{S}^{s_\alpha}$-linear in χ, so (iii) holds iff it holds for $\chi=1$ and for $\chi=\alpha$. From this, it readily follows that (iii) holds iff $q_w\in\alpha^{-1}\mathcal{S}_{\alpha\mathcal{S}}$ and $q_w+q_{s_\alpha w}\in\mathcal{S}_{\alpha\mathcal{S}}$. Now note that condition (i) is just $q_w\in\mathcal{S}_{p\mathcal{S}}$ for any prime element p not associate to a root i.e. only products of roots occur in the denominators of each q_w (when written in lowest terms). The lemma now follows immediately.

5.2. The following description of the nil Hecke ring $\overline{H}$ was obtained in [26] by a more complicated argument.

Corollary. *Assume that the fundamental chamber of W on V is sufficiently large. Define $H' := \{\, h\in Q_w \mid h\cdot\mathcal{S}\subseteq\mathcal{S}\,\}$. Then*
(a) $H'=\overline{H}$
(b) $H\cap\overline{H}=\sum_{w\in W}\delta_w\mathcal{S}$.

Proof. Let $\hat{\Lambda}$ denote the set of functions $\psi: W \to \mathcal{S}$ satisfying the following condition:

$$\psi(w) \equiv \psi(ws_\beta) \pmod{\beta\mathcal{S}} \qquad \text{if } w \in W \text{ and } \beta \in \Phi^+ \tag{1}$$

Regard $\hat{\Lambda}$ as a subset of $\mathrm{Hom}_Q(Q_W, Q)$ by setting $f(\sum_w \delta_w q_w) = \sum_w f(w) q_w$; by 3.7(c) and 3.8(2), one has $\Lambda \subseteq \hat{\Lambda}$.

Fix $\psi \in \hat{\Lambda}$, and $h = \sum_w q_w \delta_w \in H'$. Then $\overline{h} = \sum_w \delta_{w^{-1}} q_w = \sum_w \delta_w a_w$ where $a_w := q_{w^{-1}}$ satisfies $a_w \in \alpha^{-1}\mathcal{S}^{(\alpha)}$ and $a_w + a_{ws_\alpha} \in \mathcal{S}^{(\alpha)}$ for all $w \in W$, $\alpha \in \Phi^+$. By definition of $\hat{\Lambda}$,

$$a_w \psi(w) + a_{ws_\alpha}\psi(ws_\alpha) \equiv (a_w + a_{ws_\alpha})\psi(w) \equiv 0 \pmod{\mathcal{S}^{(\alpha)}}$$

and so $\sum_{w \in W} a_w \psi(w) \in \cap_{\alpha \in \Phi^+} \mathcal{S}^{(\alpha)} = \mathcal{S}$. This shows that

$$\psi(\overline{h}) \subseteq \mathcal{S} \qquad \text{for all } h \in H' \text{ and } \psi \in \hat{\Lambda}. \tag{2}$$

Now we can give the proof of (a). By 3.3, $\overline{H} \subseteq H'$. Conversely, let $h \in H'$. Write $\overline{h} = \sum_w t_w b_w$ for some unique $b_w \in Q$. Then $\xi^w(\overline{h}) = b_w \in \mathcal{S}$ by (2), and so $\overline{h} \in H$ as required.

For part (b), the inclusion of the right hand side in the left is clear since δ_r (for $r \in S$) and $\mathcal{S}$ are contained in the intersection. Conversely, suppose $h = \sum_w q_w \delta_w \in H \cap \overline{H}$. Then for $w \in W$ and $\alpha \in \Phi^+$, one has $q_w \in \alpha^{-1}\mathcal{S}^{(\alpha)}$ and $q_w + q_{s_\alpha w} \in \mathcal{S}^{(\alpha)}$ (since $h \in H'$), and $w(q_{w^{-1}}) + (s_\alpha w)(q_{(s_\alpha w)^{-1}}) \in \mathcal{S}^{(\alpha)}$ (since $\overline{h} = \sum_{w \in W} w(q_{w^{-1}})\delta_w \in H'$). The last of these three sets of equations is equivalent to $q_w + s_\alpha(q_{s_\alpha w}) \in \mathcal{S}^{(\alpha)}$; together with the first and second sets, it implies $q_w \in \mathcal{S}^{(\alpha)}$, hence $q_w \in \cap_{\alpha \in \Phi^+} \mathcal{S}^{(\alpha)} = \mathcal{S}$. This completes the proof.

5.3. The following result is proved for finite Weyl groups in [2]; the proof there easily extends to the general situation, but instead we obtain it as a "dual" version of the lemma. We let either

(a) $B = \mathcal{S}$, $\mathbb{W} = \Lambda$

or, if our root system arises by extension of scalars from an integral root system,

(b) $B = \mathcal{S}_{\mathbb{Z}}$, $\mathbb{W} = \Lambda_{\mathbb{Z}}$.

In either case, let R denote the B-algebra of functions $W \to B$ with pointwise operations. By 3.8(2), 3.7(a) and 3.12, $\mathbb{W}$ can be naturally regarded as a B-subalgebra of R, and hence so can the subring $\mathbb{W}^J$ of W_J-invariant elements of $\mathbb{W}$ for any $J \subseteq S$.

Corollary. *The homogeneous component $\mathbb{W}^J_n$ of degree n is identified by the above with the functions $\psi: W \to B$ satisfying the conditions* (i)–(iii) *below;*
(i) *ψ is constant on each coset $W_J u$ with $u \in W$*
(ii) *$\psi(w) \equiv \psi(ws_\beta) \pmod{\beta B}$ if $w \in W$, $\beta \in \Phi^+$*
(iii) *$\psi(w) \in B_n$ for all $w \in W$.*

Proof. Since $\mathbb{W}^J = \{\, \psi \in \mathbb{W} \mid \psi(uw) = \psi(w) \text{ for all } u \in W_J \,\}$, we may assume $J = \emptyset$ without loss of generality. Let $\hat{\mathbb{W}}$ denote the set of functions $W \to B$ satisfying (ii)–(iii). By 3.8(2), 3.7(c) and 3.12(b), $\mathbb{W} \subseteq \hat{\mathbb{W}}$. For the converse, suppose $\psi \in \hat{\mathbb{W}}_n$. Write $\psi = \sum_{w\in W} \xi^w q_w$ for some $q_w \in Q$, possibly with infinitely many q_w non-zero.

Assume first that that we are in case (a) and moreover that the fundamental chamber for W on V is sufficiently large. By 5.2(2) and 5.2(a), $q_w = \psi(t_w) \in \mathcal{S}_{n-2l'(w)}$. In particular, $q_w \in \mathcal{S}$ and $q_w = 0$ for almost all w, so $\psi \in \Lambda$. Now in general (with $J = \emptyset$), Remark 1.4 says one can choose a $\mathbb{R}$-space $U \supseteq V$, with W-action from dual based root systems on U and V', so that the fundamental chamber of W on U is sufficiently large. Then in either situation (a) or (b), B may be naturally regarded as a subring of the symmetric algebra $\mathcal{S}'$ of U over $\mathbb{R}$, and one may naturally identify the dual nil Hecke ring of W on U with $\Lambda \otimes_B \mathcal{S}'$, as $\mathcal{S}'$-algebra. By the special case already considered, all $q_w \in \mathcal{S}'_{n-2l'(w)}$. If some $q_w \notin B$, take w of minimal length $l'(w)$ with this property. Then $q_v\xi^v(w) \in B$ for all $v <' w$ and $q_v\xi^v(w) = 0$ for $v \not\leq w$, so $q_w\xi^w(w) \in \psi(w) + B \subseteq B$. Since $\xi^w(w)$ is a product of (indivisible if $B = \mathcal{S}_{\mathbb{Z}}$) roots and $q_w \in \mathcal{S}'$, it follows that $q_w \in B$, a contradiction which completes the proof.

5.4. The next part of this section describes some variants and extensions of 5.1 and 5.2. In each of three specific (and closely related) situations we consider, there will be a commutative ring R and a family $\sigma := \{\sigma_x : U \to B\}_{x\in X}$ of distinct R-algebra homomorphisms from a commutative R-algebra U into a commutative domain (and R-algebra) B with quotient field K. In each case, one defines the coalgebra $K_\sigma \cong \oplus_{x\in X}\sigma_x K$ (consisting of certain functions $U \to K$) and its B-submodule B_σ of functions $U \to B$ as in 8.3. In particular,

$$B_\sigma := \{\, \psi \in K_\sigma \mid \psi(U) \subseteq B \,\} = \{\, \sum_{x\in X} \sigma_x a_x \mid a_x \in K,\ \sum_{x\in X} a_x \sigma_x(u) \in B \text{ for all } u \in U \,\}. \tag{1}$$

We provide in each of the three cases a "local" description (analogous to 5.1) and a B-basis of B_σ, and show B_σ may naturally be regarded as a coalgebra over B.

In each of the three cases, there will be a parabolic subgroup W_J of the Coxeter group W such that $X = W^J$ and a ring $\mathbb{W}$ on which W acts as a group of ring automorphisms, such that $U = \mathbb{W}^J$ (the ring of W_J-invariants on $\mathbb{W}$). There will be ring homomorphisms $\sigma'_x : \mathbb{W} \to B$ for all $x \in W$ (satisfying in fact $\sigma'_x|_{\mathbb{W}^J} = \sigma'_y|_{\mathbb{W}^J}$ whenever $W_Jx = W_Jy$), and for $x \in W^J$, σ_x will be defined as the restriction $\sigma_x = \sigma'_x|_{\mathbb{W}^J}$. Henceforward, we drop the notational distinction between σ and σ'. We now describe R, $\mathbb{W}$, B and the $\{\sigma_x\}_{x\in W}$ in each of the three cases (a)–(c) of interest here.

(a) In this case only, assume that W_J is finite and that the fundamental chamber of W on V is sufficiently large. Let $R = \mathbb{R}$, $\mathbb{W} = B = \mathcal{S}$ and for $x \in W$, let $\sigma_x : \mathcal{S} \to \mathcal{S}$ be the unique extension of the $\mathbb{R}$-linear map $x^{-1} : V \to V$ to a graded $\mathbb{R}$-algebra automorphism of $\mathcal{S}$. If $J = \emptyset$, the results to be given for this case reduce to 5.1 and 5.2(a).

(b) In this case and the next, W_J need not be finite. Set $\mathbb{W} = \Lambda$, $R = B = \mathcal{S}$. For $\chi \in \mathbb{W}$, $x \in W$ there is a unique $\sigma_x(\chi) \in B$ such that

$$\chi\xi^x \in \xi^x\sigma_x(\chi) + \sum_{y>'x} \xi^y B. \tag{2}$$

Then $\sigma_x: \mathbb{W} \to B$ is clearly a B-algebra homomorphism, and

$$\sigma_x(\xi^y) = P^x_{y,x} = (-1)^{l'(x)} x^{-1}(S^T_{x,y}) \tag{3}$$

By 3.6(g), the restriction of σ_x to Λ^J depends only on the coset $W_J x$ of x as claimed.

(c) In this third case, we suppose that the root system arises by extension of scalars from an integral root system. We take $R = B = \mathcal{S}_{\mathbb{Z}}$ and $\mathbb{W} = \Lambda_{\mathbb{Z}}$. Then we define the σ_x for $x \in W$ by the formula (2) again, and note that (3) still holds in this context.

5.5. In any of these three situations 5.4(a)–(c), we regard the elements of Φ^+ as pairwise non-associate, prime elements of B. For $\beta \in \Phi^+$, extend the earlier notation $\mathcal{S}^{(\beta)}$ by defining the localization $B^{(\beta)} := B[\alpha^{-1} \mid \alpha \in \Phi^+ \setminus \{\beta\}]$ of B. One has

$$B = \cap_{\beta \in \Phi^+} B^{(\beta)}. \tag{1}$$

The following result (and its analogue in the sequel to this paper) is fundamental in the study of the reprentation categories described in [21].

Theorem. *Let $\{a_x\}_{x \in W^J}$ be a family of elements of K, of which all but finitely many are zero. Then in any one of the three situations* 5.4(a)–(c) *above, the following two conditions are equivalent:*

(i) $\sum_{x \in W^J} a_x \sigma_x(u) \in B$ *for all* $u \in \mathbb{W}^J$

(ii) *for any* β *in* Φ^+ *and* $x, y \in W^J$ *with* $y \in W_J x s_\beta$, *one has*

$$\begin{cases} a_x \in \beta^{-1} B^{(\beta)} \text{ and } a_x + a_y \in B^{(\beta)} & \text{if } x \neq y \\ a_x \in B^{(\beta)} & \text{if } x = y. \end{cases}$$

Proof. Consider first the situation of 5.4(a). If $J = \emptyset$, the Theorem reduces to 5.1. Now consider arbitrary J with W_J finite. Extend the a_x for $x \in W^J$ to a W-indexed family $\{a_w\}_{w \in W}$ constant on cosets $W_J w$. Recall that $\mathbb{W} = B = \mathcal{S}$. Since the map $\mathbb{W} \to \mathbb{W}$ given by $u \mapsto \sum_{w \in W_J} \sigma_w(u)$ has $\mathbb{W}^J$ as image, it follows that (i) is equivalent to $\sum_{w \in W} a_w \sigma_w(u) \in B$ for all $u \in \mathbb{W}$, and this is equivalent in turn to (ii), by the equivalence for $J = \emptyset$ of (i) and (ii).

Now we prove the equivalence in the situation of 5.4(b). Fix $\beta \in \Phi^+$. For the remainder of this proof, for any $w \in W$, we denote the unique element of $W_J w s_\beta \cap W^J$ by w'. Define $\Delta^{(\beta)}_w \in K_\sigma$ for $w \in W^J$ by

$$\Delta^{(\beta)}_w = \begin{cases} (\sigma_w - \sigma_{w'}) \frac{1}{\beta} & \text{if } w' <' w \\ \sigma_w & \text{otherwise} \end{cases} \tag{2}$$

(note that $w' <' w$ iff $w s_\beta <' w$). It follows immediately from 3.7(c) that

$$\Delta^{(\beta)}_w(\mathbb{W}^J) \subseteq B \qquad \text{i.e. } \Delta^{(\beta)}_w \in B_\sigma. \tag{3}$$

We now prove the following claim:

(4) if elements $\{a_w\}_{w \in W^J}$ of K, almost all zero, satisfy $(\sum_{w \in W^J} \Delta^{(\beta)}_w a_w)(\chi) \in B^{(\beta)}$ for all $\chi \in \mathbb{W}^J$, then all $a_w \in B^{(\beta)}$.

For this, it suffices to show that if q is any prime element of B, not associate to any element of $\Phi^+ \setminus \{\beta\}$ and elements $\{b_w\}_{w\in W^J}$ of B, almost all zero, satisfy $(\sum_{w\in W^J} \Delta_w^{(p)} b_w)(\chi) \in qB$ for all $\chi \in \mathbb{W}^J$, then $q \mid b_w$ for all w. In fact, it will even be enough by (3) to show that in this situation, $q \mid b_v$ if $v \in W^J$ is of maximal length $l'(v)$ with $b_v \neq 0$. Fix any such v. There are unique $c_w \in B$ with $\sum_w \Delta_w^{(p)} p b_w = \sum_w \sigma_w c_w$. In fact,

$$c_w = \begin{cases} b_w & \text{if } w' <' w \\ pb_w - b_{w'} & \text{if } w' >' w \\ pb_w & \text{if } w' = w. \end{cases}$$

We have by 3.6(b) and 3.2(3) that

$$pq \mid \sum_w pb_w \Delta_w^{(p)}(\xi^v) = c_v d \qquad \text{where } d = \pm \prod_{\substack{\gamma\in\Phi^+ \\ vs_\gamma < \gamma}} \gamma.$$

Now we consider the following three cases ($\sim$ denotes the relation of being associate in the unique factorization domain B). If $v' <' v$ and $p \sim q$, then q divides d at most once, so $q \mid c_v = b_v$. If $v' <' v$ and $p \not\sim q$, then q doesn't divide d at all, so $q \mid c_v = b_v$ again. Finally, if $v' \geq' v$, then neither p nor q divides d since $vs_\beta >' v$, so $pq \mid c_v = pb_v$ and again $q \mid b_v$. This completes the proof of the claim (4).

Now to finish the proof of the theorem in the situation 5.4(b), note that by (1) and (4), condition (i) of the theorem is equivalent to the condition that for each $\beta \in \Phi^+$, $\sum_w \sigma_w a_w = \sum_w \Delta_w^{(\beta)} b_w^{(\beta)}$, for some $b_w^{(\beta)} \in B^{(\beta)}$. This in turn is easily seen to be just a restatement of condition (ii) of the theorem.

Finally, the proof of the theorem in the situation 5.4(c) is essentially the same as for 5.4(b), substituting a reference to 3.12(b) for the one to 3.7(c).

5.6. Now define elements $n_v \in K_\sigma$ for $v \in W^J$ in each situation 5.4(a)–(c) as follows:

(1) $$n_v := \sum_{y\in W^J} \sigma_y \Big(\sum_{u\in W_J} (uy)^{-1}(S^{\emptyset}_{uy,v}) \Big).$$

Clearly, $\{n_v\}_{v\in W^J}$ is a K-basis of K_σ. In fact, using 3.2(2) and 2.14, one checks

(2) $$\sigma_y = \sum_{x\in W^J} n_x (-1)^{l'(y)} y^{-1}(S^T_{y,x}).$$

From (1), (2), 3.2(2) again and 3.5, one easily sees that

(3) $$\Delta(n_v) = \sum_{x,y\in W^J} n_x \otimes n_y P^v_{x,y}$$

in the comultiplication defined as in 8.3 (note that the coefficients on the right of (2), (3) are in B). By 3.2(6), if W_J is finite, then

(4) $$n_v = \sum_{y\in W^J} \sigma_y y^{-1}\Big(S^{\emptyset}_{y,\omega_J v} \prod_{\alpha\in\Phi_J^+} \alpha \Big).$$

Proposition. *In each situation* 5.4(a)–(c), *the elements* $\{n_v\}_{v\in W^J}$ *form a B-module basis of the B-module* $B_\sigma = \{\,\psi \in K_\sigma \mid \psi(U) \subseteq B\,\}$.

Proof. First, consider the situation 5.4(a). As is well known (see 4.5(3)) the map $\overline{t_{\omega_J}}: \mathcal{S} \to \mathcal{S}$ is a surjection $\mathcal{S} \to \mathcal{S}^J$. Set $D = \prod_{\alpha\in\Phi_J^+} \alpha$. Using 4.5(1) and 3.2(2), one computes that for $\chi \in \mathcal{S}$,

$$\begin{aligned} n_v\overline{t_{\omega_J}}(\chi) &= \sum_{y\in W^J} y^{-1}\Big(S^{\emptyset}_{y,\omega_J v} D\Big) y^{-1}\Big(\frac{1}{D}\sum_{u\in W_J}(-1)^{l'(\omega_J)-l'(u)}u^{-1}(\chi)\Big)\\ &= (-1)^{l'(\omega_J)}\sum_{y\in W^J}\sum_{u\in W_J}(-1)^{l'(uy)}(uy)^{-1}(S^{\emptyset}_{uy,\omega_J v})(uy)^{-1}(\chi)\\ &= (-1)^{l'(\omega_J)}\overline{t_{\omega_J v}}(\chi).\end{aligned}$$

Suppose $n = \sum_{v\in W^J} n_v b_v \in K_\sigma$, with the $b_v \in K$. Then $n \in B_\sigma$ iff for all $\chi \in \mathcal{S}$, $\sum_{v\in W^J} b_v\overline{t_{\omega_J v}}(\chi) \in \mathcal{S}$ i.e. iff all $b_v \in \mathcal{S}$, by 5.2.

Now consider cases 5.4(b) and 5.4(c). For $w \in W^J$, 3.2(1) and 2.14 give

$$\begin{aligned} n_v(\xi^w) &= \sum_{y\in W^J}\Big(\sum_{u\in W_J}(uy)^{-1}(S^{\emptyset}_{uy,v})\Big)(-1)^{l'(y)}y^{-1}(S^T_{y,w})\\ &= \sum_{y\in W^J}\Big(\sum_{u\in W_J}(-1)^{l'(uy)}(uy)^{-1}(S^{\emptyset}_{uy,v}S^T_{uy,w})\Big) = \delta_{v,w}\end{aligned}$$

and the result follows easily (see 8.4(1)) since the ξ^w for $w \in W^J$ form a B-basis of $\mathbb{W}^J$ and the σ_v are B-linear.

Remark. In situation 5.4(a) (resp., (b), (c)), one may dualize B_σ as in 8.3 to obtain a B-algebra $B^*_\sigma = \mathrm{Hom}_B(B_\sigma, B)$. Then by (3), Λ^J (resp., $\Lambda^J_{\mathbb{Z}}$, Λ^J) may be identified naturally with a B-subalgebra of B^*_σ, identifying ξ^w, for $w \in W^J$ with the element n^*_w of B^*_σ defined by $n^*_w(n_v) = \delta_{v,w}$.

5.7. By 8.12, Theorem 5.5 reduces the calculation of certain Ext^1-groups of importance in studying the representation categories from [21] to the simultaneous solution of certain explicit congruences. The most important special case is the following (stated in [21] in cases 5.4(a) and (b), with a proof given fully only for $J = \emptyset$ in case 5.4(a)).

Define the ring $U_B = U \otimes_R B$; in 5.4(b)–(c), one has $U_B \cong U$ naturally since $R = B$. For $x \in W$, define the U_B-module (which we regard also as (U,B)-bimodule) B_x which is equal to B as right B-module and has left U-action given by $(u,b) \mapsto b\sigma_x(u)$ for $u \in U$ and $b \in B_x$. Then $B_x = B_y$ if $x \in W_J y$.

Corollary. *Let $x, y \in W$. Then as right B-modules,*
(a) $\mathrm{Hom}_{U_B}(B_x, B_y) \cong B$ *if $x \in W_J y$ and is zero otherwise.*
(b) *if $x \notin W_J y$, then* $\mathrm{Ext}^1_{U_B}(B_x, B_y) \cong B/\gamma B$ *if $y \in W_J x s_\gamma$ for some (necessarily unique) $\gamma \in \Phi^+$ and the* Ext*-group is zero otherwise.*

Proof. The result follows immediately from Theorem 5.5 and the general description of such Hom and Ext^1-groups in 8.6(1)–(2).

5.8. Finally in this section, we record a basic technical fact with applications to the study of the conjectural dualities in the representation categories from [21].

We define first in each case 5.4(a)–(c) a U_B-module (i.e. a (U,B)-bimodule) $\mathbb{M}$. In situation 5.4(a), we regard the Λ^J-module $\mathbb{M}=\Lambda_\Gamma$ as a $U_B=\mathcal{S}^J\otimes_{\mathbb{R}}\mathcal{S}$-module $\mathbb{M}$ by means of the $\mathcal{S}^J\otimes_{\mathbb{R}}\mathcal{S}$-algebra structure on Λ^J as in 4.6. Define the $U_B=\Lambda^J\otimes_{\mathcal{S}}\mathcal{S}=\Lambda^J$-module $\mathbb{M}=\Lambda_\Gamma$ in case 5.4(b), and the $U_B=\Lambda^J_{\mathbb{Z}}\otimes_{\mathcal{S}_{\mathbb{Z}}}\mathcal{S}_{\mathbb{Z}}=\Lambda^J_{\mathbb{Z}}$-module $\mathbb{M}=(\Lambda_\Gamma)_{\mathbb{Z}}$ in case 5.4(c). We write Q for the quotient field of B in each case 5.4(a)–(c) (this is contrary to our usual use of Q in case 5.4(c)).

Recall our spherical poset Γ and $L\subseteq S$ with $W_L\Gamma\subseteq\Gamma$, and take $J\subseteq L$. For a formal symbol δ^J, consider the topological right Q-module $K_Q^{\uparrow}(\Gamma^J,\delta^J)$ defined in 2.10. It has a stucture of topological left $U=\mathbb{W}^J$-module structure with $\psi(\sum^{\uparrow}\delta^J_w a_w)=\sum^{\uparrow}_{w\in\Gamma^J}\delta^J_w a_w\sigma_w(\psi)$ for $\psi\in\mathbb{W}^J$ and $a_w\in Q$ supported in a finitely generated coideal of Γ^J, and therefore it becomes a U_B-module by restriction.

Corollary. *In each case* 5.4(a)–(c)*, there is a continuous embedding*

$$\mathbb{M}^J\to K_Q^{\uparrow}(\Gamma^J,\delta^J)$$

of U_B-modules mapping $\eta^A_w\mapsto\sum^{\uparrow}\delta^J_v a_{v,w}$ where $a_{v,w}=v^{-1}(S^{T\setminus A}_{v,w}/S^{T\setminus A}_v)$.

Moreover, if $v\neq w\in\Gamma^J$ and $\beta\in\Phi^+$ are such that $w\in W_Jvs_\beta$, then $a_{v,w}\cong\frac{u}{\beta}$ (mod $B^{(\beta)}$) *for some unit $u\in B^{(\beta)}$.*

Proof. The case 5.4(a) follows just by restricting scalars from the result in case 5.4(b), so assume we are in case 5.4(b) or (c). Note that by construction of $\mathbb{M}$, there is certainly such an embedding in case $J=\emptyset$; take $\delta^\emptyset_y=(-1)^{l'(y)}D_y y^{-1}(S^{T\setminus A}_y)$ in 3.9(7). For general J, one easily sees it is enough to treat the special case with Γ^J finite. By 5.7 and 8.8, there is some embedding $\mathbb{M}^J\to K_Q^{\uparrow}(\Gamma^J,\delta^J)$ of (W_J,B)-bimodules mapping $\eta^A_w\mapsto\sum_{v\in\Gamma^J}\delta^J_v v^{-1}(s_{v,w})$ for some $s_{v,w}\in Q$ with $s_{w,w}=1$ and $s_{v,w}=0$ unless $w\leq_A v$. Now for any $\psi\in\mathbb{W}$, $w\in\Gamma$ one may write $\psi\eta^A_w=\sum^{\uparrow}_{u\in\Gamma}\eta^A_u\Omega_{w,u}(\psi)$ for (unique) $\Omega_{w,u}(\psi)\in B$. Since $\Omega_{w,u}(\psi)=0$ if $\psi\in\mathbb{W}^J$, $w\in\Gamma^J$ and $u\in\Gamma\setminus\Gamma^J$, and since the corollary holds for $J=\emptyset$, it follows by the discussion in 8.6 applied to $\mathbb{M}$ and $\mathbb{M}^J$ that $s_{v,w}=S^{T\setminus A}_{v,w}/S^{T\setminus A}_v$ for $v,w\in\Gamma^J$. Now the second claim follows using Lemma 3.4 and 3.4(2).

6. The Iwahori-Hecke algebra and bimodules for the dual nil Hecke ring

In this section, we construct a homomorphism between the Iwahori-Hecke algebra of W and a Grothendieck group of graded bimodules for the dual nil Hecke ring Λ under tensor product.

6.1. Recall the definition of the Iwahori-Hecke algebra $\mathcal{H}$ of W. Let v be an indeterminate. Then $\mathcal{H}$ is the $\mathbb{Z}[v,v^{-1}]$-algebra (with identity T_e) generated by elements $\{T_r\}_{r\in S}$ subject to quadratic relations

$$T_r^2=v^2T_e+(v^2-1)T_r,\ \text{for } r\in S \tag{1}$$

and the braid relations on the T_r for $r\in S$.

By the monoid lemma, for any $w\in W$, there is a well-defined element T_w of $\mathcal{H}$ such that $T_w=T_{r_1}T_{r_2}\dots T_{r_n}$ for any reduced expression $w=r_1r_2\dots r_n$ for w. It is known that the family $\{T_w\}_{w\in W}$ is a $\mathbb{Z}[v,v^{-1}]$-basis of $\mathcal{H}$. Another basis

$\{ C'_w \}_{w\in W}$ of $\mathcal{H}$ over $\mathbb{Z}[v, v^{-1}]$ is defined in [25]; we don't repeat the definition here in general.

6.2. Since the only order on W considered in this section will be Chevalley order, we denote it here by $\leq$, and the standard length function by l, contrary to our usual conventions. Throughout this section, let B be a $\mathbb{N}$-graded $\mathcal{S}$-algebra; the main cases of interest are $B = \mathcal{S}$ or $B = \mathbb{R}$, and so we assume for simplicity that $B_0 = \mathbb{R}$, and that $\dim_{\mathbb{R}} B_n$ is finite for all n. Let $\mathbb{W}^J = (\Lambda \otimes_{\mathcal{S}} B)^J = \Lambda^J \otimes_{\mathcal{S}} B$, regarded as a B-algebra. We do not distinguish notationally between $\psi \in \Lambda^J$ and $\psi \otimes 1_B \in \mathbb{W}^J$; in particular, we often write χ for the image of $\chi \in \mathcal{S}$ under the structural homomorphism $\mathcal{S} \to B$. In the following sections, tensor products are over $\mathbb{W}$ unless otherwise indicated.

For any $s \in \mathcal{S}$, define the graded $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{B}'_s = \mathbb{W} \bigotimes_{\mathbb{W}^s} \mathbb{W}\langle -1\rangle$. Let $\mathcal{D}$ denote the full subcategory of graded $(\mathbb{W}, \mathbb{W})$-bimodules consisting of all graded bimodules $\mathbb{B}$ isomorphic to $(\mathbb{W}, \mathbb{W})$-bimodule direct summands of finite direct sums of tensor products $\mathbb{B}'_{s_1} \otimes \ldots \otimes \mathbb{B}'_{s_n}\langle m\rangle$, for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and sequences $s_1, \ldots , s_n \in S$. Let $\mathcal{H}'$ denote the split Grothendieck group of $\mathcal{D}$; this is an abelian group with generators $[\mathbb{B}]$ for the (isomorphism classes of) objects $\mathbb{B}$ of $\mathcal{D}$ and a relation $[\mathbb{B}] = [\mathbb{B}'] + [\mathbb{B}'']$ for each split exact sequence $0 \to \mathbb{B}' \to \mathbb{B} \to \mathbb{B}'' \to 0$ (with maps homogeneous of degree 0) in $\mathcal{D}$. Then $\mathcal{H}'$ becomes an an algebra over $\mathbb{Z}[v, v^{-1}]$ with $[\mathbb{B}][\mathbb{B}'] = [\mathbb{B} \otimes_{\mathbb{W}} \mathbb{B}']$, identity $[\mathbb{W}]$, and $v^n[\mathbb{B}] = [\mathbb{B}\langle n\rangle]$.

Theorem. *Let $\mathcal{H}$ be the Iwahori-Hecke algebra of W over $\mathbb{Z}[v, v^{-1}]$. Then there is a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism $\mathcal{E}: \mathcal{H} \to \mathcal{H}'$ with $\mathcal{E}(C'_s) = [\mathbb{B}'_s]$ for all $s \in S$, where $C'_s = v^{-1}(T_s + T_e)$.*

Remarks. The proof and its corollaries will occupy the remainder of this section. The same statement is true with $\mathbb{W} = \mathcal{S}$ instead (see [32], [21]) and can be deduced easily from the theorem here. Over $\mathbb{W} = \mathcal{S}$, it is known from [21] as well that if the fundamental chamber for W on V is sufficiently large, there are graded indecomposable $(\mathbb{W}, \mathbb{W})$-bimodules $\mathbb{B}_w$ such that any direct summand of any $\mathbb{B}_{s_1} \otimes \ldots \mathbb{B}_{s_n}\langle m\rangle$ is a finite direct sum with uniquely determined multiplicities, of modules $\mathbb{B}_w\langle m\rangle$, and that $\mathcal{E}$ is an isomorphism. It may be conjectured that similar $\mathbb{B}_w$ exist for $\mathbb{W} = \Lambda \otimes_{\mathcal{S}} B$ as well, and may be chosen (in either case) so that the $\mathcal{E}(C'_w) = [\mathbb{B}_w]$ where $\{C'_w\}_{w\in W}$ is the Kazhdan-Lusztig basis of $\mathcal{H}$.

6.3. In subsequent subsections, we will define for each $J \subseteq S$ with $\sharp(J) \leq 2$ and each $w \in W_J$ a graded $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{B}_w$ with the following property; for $r \in S$ one has

$$\mathbb{B}_r \cong \mathbb{B}'_r, \qquad \mathbb{B}'_r \otimes \mathbb{B}'_r = \mathbb{B}'_r\langle 1\rangle \oplus \mathbb{B}'_r\langle -1\rangle \tag{1}$$

and for $w \in W_J$, $r \in J \subseteq S$ with $\sharp(J) \leq 2$ and $rw > w$, one has

$$\mathbb{B}_r \otimes \mathbb{B}_w = \oplus_{\substack{v\in W_J,\ rv<v \\ v\lessdot w \text{ or } w\lessdot v}} \mathbb{B}_v. \tag{2}$$

The theorem (and also the statement that $\mathcal{E}(C'_w) = [\mathbb{B}_w]$ for all w lying in any rank two standard parabolic subgroup of W) then follows from the lemma below.

Lemma. *As $\mathbb{Z}[v, v^{-1}]$-algebra, $\mathcal{H}$ is generated by generators C'_w, for $w \in W_J$ for some $J \subseteq S$ with $\sharp(J) \leq 2$, subject to the relations ($C'_e = \mathrm{Id}$ and)* (i) *and* (ii) *below:*

(i) ${C'_r}^2 = (v + v^{-1})C'_r$ *for* $r \in S$.
(ii) *for* $r \in J \subseteq S$ *and* $w \in W_J$ *with* $\sharp(J) \leq 2$ *and* $rw > w$, *one has* $C'_r C'_w = \sum_{\substack{v \in W_J,\ rv < v \\ v < w \text{ or } w < v}} C'_v$.

Proof. Since the non-zero Kazhdan-Lusztig polynomials for a dihedral group are all equal to 1, it follows from [25] that the elements C'_w of $\mathcal{H}$ for w in some W_J with $\sharp(J) \leq 2$ generate $\mathcal{H}$ and satisfy the relations (i)–(ii). Under replacement of C'_r by $v^{-1}(T_r + T_e)$, (i) is obviously equivalent to the quadratic relation for T_r, and it remains to show that (if $r \neq s$ and rs has finite order $n = n_{r,s}$) (i)–(ii) imply the braid relation on T_r and T_s.

Let $r_i = r$ for even i and $r_i = s$ for odd i. Then (i)–(ii) imply that for $m \leq n$, one has $C'_{r_1} \dots C'_{r_m} \in C'_{r_1 \dots r_m} + \sum_{1 \leq k < m} \mathbb{Z}[v, v^{-1}] C'_{r_1 \dots r_k}$ and that a similar formula holds for $C'_{r_0} \dots C'_{r_{m-1}}$. Hence relations (i)–(ii) imply

$$C'_{r_1 \dots r_n} = \sum_{1 \leq m \leq n} a_m C'_{r_1} \dots C'_{r_m} = \sum_{m \leq n} a_m C'_{r_0} \dots C'_{r_{m-1}}$$

for some $a_m \in \mathbb{Z}[v, v^{-1}]$ with $a_n = 1$ Multiplying the rightmost two terms of this equality by v^n, making the substitution $C_{r_i} = v^{-1}(T_{r_i} + T_e)$ and simplifying using just the quadratic relations on the T_{r_i} must give the braid relation $T_{r_1} \dots T_{r_n} = T_{r_0} \dots T_{r_{n-1}}$ or else one would have a contradiction to linear independence over $\mathbb{Z}[v, v^{-1}]$ of $\{T_w\}_{w \in W}$.

6.4. We introduce some notation to be used in the proof of the theorem. Let $\mathbb{Z}[[v]][v^{-1}]$ denote the ring of Laurent power series (with poles of finite order). It will be convenient to write $p(M) = \sum_{n \in \mathbb{Z}} \dim_{\mathbb{R}} M_n v^n \in \mathbb{Z}[[v]][v^{-1}]$ for the Poincaré series of a graded $\mathbb{R}$-vector space $M = \oplus_{n \in \mathbb{Z}} M_n$ with $M_n = 0$ for $n << 0$ and $\dim_{\mathbb{R}} M_n$ finite for all n. Also, set $p(y) = \sum_{x \leq y} v^{l(y) - 2l(x)}$ for any $y \in W$. For two elements f, g of $\mathbb{Z}[[v]][v^{-1}]$, we write $f \leq g$ if $g - f$ has non-negative coefficents.

Fix $s \in S$. We use frequently below without explicit mention the facts that $\xi^s s(\xi^s) \in \mathbb{W}^s$, $\xi^s + s(\xi^s) \in \mathbb{W}^s$. Let $e_s = 1 \otimes 1 \in \mathbb{B}'_s$ and $f_s = -\xi^s \otimes 1 + 1 \otimes s(\xi^s) = s(\xi^s) \otimes 1 - 1 \otimes \xi^s \in \mathbb{B}'_s$. Now 4.2(b) implies

(1) $\{e_s, f_s\}$ is a basis of $\mathbb{B}'_s$ as left $\mathbb{W}$-module, and also a basis as right $\mathbb{W}$-module. Moreover, one has

$$y f_s = f_s y \tag{2}$$

for all $y \in \mathbb{W}$, since this holds for $y = \xi^x$ with $x \in \mathbb{W}^s$ and for $y = \xi^s$. We also note the following characterization of $\mathbb{B}'_s$ for $s \in S$:

(3) $\mathbb{B}'_s$ is the graded $(\mathbb{W}, \mathbb{W})$-bimodule generated by an element $b = e_s$ of degree -1 subject to relations $\xi^x b = b \xi^x$ for $x \in \mathbb{W}^s$.

We can now prove the second assertion in 6.3(1). Indeed, let $\mathbb{B}$ (resp., $\mathbb{B}'$) denote the subbimodule of $\mathbb{B}'_s \otimes \mathbb{B}'_s$ generated by $e_s \otimes e_s$ (resp., by $e_s \otimes f_s$). A simple computation shows that in $\mathbb{B}'_s \otimes \mathbb{B}'_s$,

$$e_s \otimes f_s - f_s \otimes e_s = -s(\xi^s)(e_s \otimes e_s) + (e_s \otimes e_s) s(\xi^s),$$

$$f_s \otimes f_s = -\xi^s (e_s \otimes f_s) + (e_s \otimes f_s) s(\xi^s).$$

Now by (1),

(4) $\mathbb{B}'_s \otimes \mathbb{B}'_s$ is a free left $\mathbb{W}$ module of rank 4 (with basis consisting of the four elements $x \otimes y$ where $x, y \in \{e_s, f_s\}$).

It follows that $\mathbb{B}'_s \otimes \mathbb{B}'_s = \mathbb{B} + \mathbb{B}'$. Also, by (2) and (3), one has that $\mathbb{B}$ (resp., $\mathbb{B}'$) is isomorphic to a quotient of $\mathbb{B}'_s\langle -1\rangle$ (resp., of $\mathbb{B}'_s\langle 1\rangle$). But by (1) and (4), one has $p(\mathbb{B}'_s \otimes \mathbb{B}'_s) = (v + v^{-1})p(\mathbb{B}'_s)$. This implies that $\mathbb{B} \cong \mathbb{B}'_s\langle -1\rangle$, $\mathbb{B}' \cong \mathbb{B}'_s\langle 1\rangle$ and that $\mathbb{B}'_s \otimes \mathbb{B}'_s = \mathbb{B} \oplus \mathbb{B}' \cong \mathbb{B}'_s\langle -1\rangle \oplus \mathbb{B}'_s\langle 1\rangle$.

6.5. We now begin work on 6.3(2). Fix $J \subseteq S$ with $\sharp(J) = 2$, say $J = \{r, s\}$ where $r = s_\alpha$, $s = s_\beta$ with $\alpha, \beta \in \Pi$. For $i \in \mathbb{Z}$, define $\alpha_i = \beta$ for i odd, $\alpha_i = \alpha$ for i even and $s_i = s_{\alpha_i}$. Consider the polynomial algebra $\mathbb{W}[t]$ over $\mathbb{W}$ in the indeterminate t. Define elements $y_j^{(m)} \in \mathbb{W}$, for $j \in \mathbb{N}$, by

$$(1) \qquad (tx_m - 1)(tx_{m-1} - 1)\dots(tx_1 - 1) = \sum_j y_j^{(m)} t^j,$$

where $x_1 = \xi^r$ and $x_j = s_j s_{j-1} \dots s_2(\xi^r)$ for $j > 1$. Note that $y_j^{(m)}$ is homogeneous of degree $2j$. We claim that

$$(2) \qquad y_j^{(m)} \in \mathbb{W}^{s_m}.$$

To see this, note that the action of W as an automorphism group of $\mathbb{W}$ extends to one on $\mathbb{W}[t]$ with $w(t) = t$ for all $w \in W$. The claim (2) holds since if m is even, s_m permutes the factors on the left of (1) in pairs, while if m is odd, s_m fixes the last factor on the left of (1) and permutes the remaining factors in pairs.

6.6. From 3.6(a), for any $K \subseteq S$, the $\mathcal{S}$-module I_K spanned by the elements ξ^w with $w \notin W_K$ is an ideal of $\mathbb{W}$. It is easily seen that the quotient algebra $\mathbb{W}/I_K$ is naturally isomorphic to $\Lambda' \otimes_{\mathcal{S}} B$ where Λ' is the dual nil Hecke ring of (W_K, K) (in its reflection representation on V). Moreover, the elements $\xi^w + I_K$ ($w \in W_K$) of the quotient may be identified with the standard basis elements $\xi^w \otimes_{\mathcal{S}} 1_B$ with $w \in W_K$ for $\Lambda \otimes_{\mathcal{S}} B$. The automorphisms x (and operators $\bar{t}_x$) for $x \in W_K$ preserve the ideal I_K, and the induced operators on $\mathbb{W}/I_K$ may be identified with the corresponding operators on $\Lambda' \otimes_{\mathcal{S}} B$.

6.7. For the proof of 6.3(2), we need certain properties (8)–(9) below which hold in the special case in which (W, S) is dihedral; this gives some information for general $\mathbb{W}$ by means of 6.6. Suppose for this subsection only that $\Pi = \{\alpha, \beta\}$ where $r_\alpha = r$ and $r_\beta = s$. We assume to begin with that we are in the symmetric case $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle$. Set $\gamma = \langle \alpha, \beta^\vee \rangle$ and define a sequence $\{p_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ recursively by $p_0 = 0$, $p_1 = 1$ and $p_n = \gamma p_{n-1} - p_{n-2}$ for $n \geq 2$ Set $p_{-n} = -p_n$ so that $p_n = \gamma p_{n-1} - p_{n-2}$ for all $n \in \mathbb{Z}$. By induction, for all integers n, k, one has

$$(1) \quad p_n p_{n+k} - p_{n-1} p_{n+k+1} = p_{k+1} \quad \text{or, equivalently,} \quad p_k p_n - p_{k-1} p_{n-1} = p_{n+k-1}.$$

It is easily seen ([13]) that

$$(2) \qquad s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{n-1}}(\alpha_n) = p_n \alpha_1 + p_{n-1} \alpha_2 \qquad \text{for } n > 0,$$

$$(3) \qquad p_n = 0 \text{ if } n = n_{r,s} < \infty, \quad p_n > 0 \text{ if } 1 \leq n < n_{r,s}.$$

By 3.10(f), $r(\xi^r) = -\alpha\xi^e - \xi^r + \gamma\xi^s$ and $r(\xi^s) = \xi^s$. Using this, one verifies by induction that

$$(4) \quad x_m = s_m s_{m-1} \dots s_2(\xi^r) = -(p_1\alpha_2 + \dots + p_{m-1}\alpha_m) - p_{m-1}\xi^{s_m} + p_m\xi^{s_{m-1}}.$$

By 3.10(g), for $n < n_{r,s}$, one has

$$(5) \qquad \xi^r\xi^{s_2\dots s_m} = (p_1\alpha_2 + \dots + p_{m-1}\alpha_m)\xi^{s_2\dots s_m} + p_m\xi^{s_2\dots s_{m+1}}$$

$$(6) \qquad \xi^r\xi^{s_1\dots s_m} = (p_1\alpha_2 + \dots + p_{m-1}\alpha_m)\xi^{s_1\dots s_m} + \xi^{s_0\dots s_m} + p_m\xi^{s_1\dots s_{m+1}}.$$

By induction using (5) and (6) one has for finite m with $1 \leq m \leq n_{r,s}$ that

$$(7) \qquad x_1x_2\dots x_m = p_1p_2\dots p_m\xi^{s_{m+1}\dots s_2}.$$

In particular, the above results give that
(8)
$$x_1x_2\dots x_m \in \mathbb{R}^\bullet\xi^{s_{m+1}\dots s_2}, \qquad x_m \in \mathbb{R}^\bullet\xi^{s_{m-1}} + \mathbb{R}\xi^{s_m} + V\xi^e \quad \text{for } 1 \leq m < n_{r,s}.$$

Note also that (5), (6) imply that

(9) the ideal of $\mathbb{W}$ generated by ξ^x has $\{\, \xi^y \mid y \geq x \,\}$ as B-basis.

Now consider the case of dihedral W without the symmetry assumption $\langle\alpha, \beta^\vee\rangle = \langle\beta, \alpha^\vee\rangle$. Replacing α by $c\alpha$ and $\alpha^\vee$ by $c^{-1}\alpha^\vee$ for some $c \in \mathbb{R}_{>0}$, leaves the W-action on V and hence the dual nil Hecke ring Λ unchanged, whereas the basis elements ξ^w remain the same only up to multiplication by non-zero elements of $\mathbb{R}$. Choosing c appropriately reduces us to the symmetric case considered earlier, so we may conclude that (8) and (9) hold in the general dihedral case.

6.8. We now return to 6.3(2) for general W, keeping notation from 6.5.

Let $n = n_{r,s}$. Define a graded $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{B}_w$ for $w = s_m \dots s_2s_1 \in W_J$ with $0 \leq m = l(w) \leq n$ (and m finite, of course) as follows. First, if $m = 0$, set $B_e = \mathbb{W}$. If $m > 0$, let $\mathbb{B}_w$ denote the graded $(\mathbb{W}, \mathbb{W})$-bimodule generated by an element b of degree $-m$ subject to the relations

$$(1) \qquad \sum_{j=0}^{m} y_j^{(m)} b(\xi^r)^{m-j} = 0, \qquad \xi^x b = b\xi^x, \quad \text{for } x \in \mathbb{W}^J$$

Note that if $m > 0$, then setting $b' = x_m b - b\xi^r$, one has

$$(2) \qquad \sum_{j=0}^{m-1} y_j^{(m-1)} b'(\xi^r)^{m-1-j} = 0, \qquad \xi^x b' = b'\xi^x \text{ for } x \in W^J,$$

$$(3) \qquad \sum_{j=0}^{m+1} y_j^{(m+1)} b(\xi^r)^{m+1-j} = 0.$$

The rest of this subsection is devoted to the proof of the following.

Lemma. *Let notation be as above. Then*
(a) $\mathbb{B}_w$ *is a graded free right* $\mathbb{W}$*-module with (homogeneous) basis* $\{\xi^x b\}_{x\leq w}$.
(b) *if* $m<n$, $t=s_{m+1}$ *and* $w=s_m\dots s_1$ *then*

$$\mathbb{B}'_t\otimes\mathbb{B}_w\cong\oplus_{\substack{w'\in W_J,\ tw'<w'\\ w'\lessdot w \text{ or } w\lessdot w'}}\mathbb{B}_{w'}.$$

Proof. To begin, we show that the elements $\xi^x b$ with $x\leq w$ span $\mathbb{B}_w$ as right $\mathbb{W}$-module, implying

$$p(\mathbb{B}_w)\leq p(w)p(\mathbb{W}). \tag{4}$$

Firstly, by 4.2(b) and the second relations in (1), the elements $\xi^x b$ with $x\in W_J$ span $\mathbb{B}_w$ as right $\mathbb{W}$-module. Also, by 6.7(8), 6.6 and 4.2(2), the first relation in (1) is equivalent (in the presence of the second relations in (1)) to a relation of the form

$$\xi^y b-\sum_{x<y}\xi^x bX_x=0,\qquad \text{where } y=s_{m+1}\dots s_2 \tag{5}$$

for certain $X_x\in\mathbb{W}$. Suppose that $y'\in W_J$ with $y'\geq y$. By 6.7(9), 6.6 and 4.2(2) again, one may choose a homogeneous $z\in\sum_{v\in W_J}\xi^v B$ such that $z\xi^y\in\xi^{y'}+\sum_{x<y'}\xi^{x'}\mathbb{W}^J$. Now for $x<y$, one has $z\xi^x\in\sum_{x'<y'}\xi^{x'}\mathbb{W}^J$ by degree considerations; hence multiplying (5) by z and using the second relations in (1), one obtains a relation of the same form as (5) with y replaced by y'. The claim at the start of the proof of this lemma follows. Moreover, if $m=1$, the above implies that $p(\mathbb{B}_s)\leq(v^{-1}+v)p(\mathbb{W})$. But by 6.4(3), $\mathbb{B}'_s=\mathbb{W}\otimes_{\mathbb{W}^s}\mathbb{W}\langle -1\rangle$ is a quotient of $\mathbb{B}_s$, so these two modules can be identified. Note that this proves the first assertion in 6.3(1) and establishes that (a) holds if $m=1$.

Assume now that $1<m<n_{r,s}$. Let $\mathbb{B}$ (resp., $\mathbb{B}'$) denote the subbimodule of $\mathbb{B}_t\otimes\mathbb{B}_w$ generated by $e_t\otimes b$ (resp., $e_t\otimes b'$ where $b'=x_m b-b\xi^r$). Then by 6.7(8), $\mathbb{B}+\mathbb{B}'$ is also the subbimodule generated by $e_t\otimes b$ and $e_t\xi^t\otimes b$, so it contains the elements $\xi^x e_t\xi^y\otimes b\xi^u$ and $\xi^x e_t\xi^t\xi^y\otimes b\xi^u$ with $x,u\in W$ and $y\in W^t$. Therefore by 4.2(b) we get

$$\mathbb{B}+\mathbb{B}'=\mathbb{B}_t\otimes\mathbb{B}_w. \tag{6}$$

We now claim that

(7) $\mathbb{B}$ (respectively, $\mathbb{B}'$) is a quotient bimodule of $\mathbb{B}_{tw}$ (respectively, of $\mathbb{B}_{w'}$ where $w'=s_{m-1}\dots s_2s_1$).

We give the proof for $\mathbb{B}$. Clearly, if $x\in W^J$ then $\xi^x(e_t\otimes b)=(e_t\otimes b)\xi^x$. Now using 6.5(2) and 6.8(3),

$$\sum_{j=0}^{m+1}y_j^{(m+1)}(e_t\otimes b)(\xi^r)^{m+1-j}=\sum_{j=0}^{m+1}e_t\otimes y_j^{(m+1)}b(\xi^r)^{m+1-j}=0.$$

Since $e_t\otimes b\in\mathbb{B}_{-m-1}$, (7) follows for $\mathbb{B}$ by definition of $\mathbb{B}_{tw}$. The proof of (7) for $\mathbb{B}'$ is similar.

Now if $m = 1$ in the above, then $b' = 0$ so $\mathbb{B} = \mathbb{B}_t \otimes \mathbb{B}_w$. This implies that $p(\mathbb{B}_t \otimes \mathbb{B}_w) = p(t)p(w)p(\mathbb{W}) = p(tw)p(\mathbb{W})$. But we've already shown that $\mathbb{B}_{tw}$ is spanned as right $\mathbb{W}$-module by the elements $\xi^x b$ with $x \leq tw$, and $p(\mathbb{B}_{tw}) \leq p(tw)p(\mathbb{W})$. It follows from (7) that $\mathbb{B}_t \otimes \mathbb{B}_w \cong \mathbb{B}_{tw}$, proving (b) for $m = 1$. Moreover, one has (still with $m = 1$)

$$p(\mathbb{B}_{tw}) = p(tw)p(\mathbb{W}), \tag{8}$$

so the elements $\xi^x b$, $x \leq tw$ are in fact a basis of $\mathbb{B}_{tw}$ as right $\mathbb{W}$-module, proving (a) with $m = 2$.

We can now prove (a) and (b) together by induction. Assume inductively that (b) holds for all $\hat{w} = s_{m'} \dots s_2 s_1 \in W_J$ with $m' < m$ where $2 \leq m < n_{r,s}$ and (a) holds for all such $\hat{w}$ with $m' \leq m$. Then the inductive assumption, (4), (7) and (6) give

$$\begin{aligned} p(\mathbb{B}_t \otimes \mathbb{B}_w) &= p(t)p(w)p(\mathbb{W}) = (p(tw) + p(w'))p(\mathbb{W}) \\ &\geq p(\mathbb{B}_{tw}) + p(\mathbb{B}_{w'}) \geq p(\mathbb{B}) + p(\mathbb{B}') \geq p(\mathbb{B}_t \otimes \mathbb{B}_w) \end{aligned}$$

where w' is as in (7). It follows from this that $\mathbb{B} \cong \mathbb{B}_{tw}$, that $\mathbb{B}' \cong \mathbb{B}_{w'}$ and that the sum in (6) is direct. This establishes that (b) holds for $w = s_m \dots s_2 s_1$. Moreover, this also proves (8), and therefore that (a) holds for $w = s_{m+1} \dots s_2 s_1$, completing the inductive proof of (a) and (b).

As a corollary, we observe that

(9) if W_J is finite (of rank at most two) with longest element ω_J, then $\mathbb{B}_{\omega_J}$ is the $(\mathbb{W}, \mathbb{W})$-bimodule generated by an element b of degree $-l(\omega_J)$ subject to relations $\xi^x b = b\xi^x$ for $x \in W^J$.

Indeed, this is trivial or known for $\sharp(J) \leq 1$. For rank two, note using 4.2(b) that the bimodule with presentation in (9) is spanned as right $\mathbb{W}$-module by the elements $\xi^x b$, $x \in W_J$, and it has $\mathbb{B}_{\omega_J}$ as a quotient.

6.9. We now complete the proof of 6.3(2) and hence Theorem 6.2. For any w contained in some W_J with $\sharp(J) \leq 2$, one has a graded $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{B}_w$ defined by $\mathbb{B}_e = \mathbb{W}$ if $l(w) = 0$ and by the presentation 6.8(1) (with $J = \{r, s\}$, $w = s_m \dots s_2 s_1$, $m \leq \mathrm{ord}(rs)$) otherwise. Note that $\mathbb{B}_w$ is well-defined by 6.8(9). Finally, the properties 6.3(1)–(2) have been previously established.

6.10. For a finite sequence $X\colon r_1, \dots, r_n$ in S, let $\mathbb{W}_X$ denote the subbimodule of the $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{B}_X := \mathbb{B}_{r_1} \otimes \dots \otimes \mathbb{B}_{r_n}$ generated by $b_X = e_{r_1} \otimes \dots \otimes e_{r_n}$ (by convention, $\mathbb{W}_\emptyset = \mathbb{W}$ where $\emptyset$ denotes the empty sequence). One has the following corollary of the above proof.

Corollary. *For each $x \in W$, there exists a $(\mathbb{W}, \mathbb{W})$-bimodule $\mathbb{W}_x$ such that if $x = r_1 \dots r_n$ is a reduced expression, then $\mathbb{W}_x \cong \mathbb{W}_X$ for $X\colon r_1, \dots, r_n$.*

Proof. Suppose $w = xyz$ where $x, z \in W$, y is the longest element of a rank two finite standard parabolic subgroup W_J of W and $l(w) = l(x) + l(y) + l(z)$. Write $J = \{r, s\}$, $m = n_{r,s}$. Set $t_i = r$ if $i \in \mathbb{N}$ is odd, and $t_i = s$ for even $i \in \mathbb{N}$, and let Y (resp., Y') denote the sequence $t_1, \dots, t_m$ (resp., $t_2, \dots, t_{m+1}$). Choose sequences $X\colon r_1, \dots, r_n$, $Z\colon s_1 \dots s_p$ with $x = r_1 \dots r_n$, $n = l(x)$ and $z = s_1 \dots s_p$, $p = l(z)$.

Now from the proof of 6.2, it is clear that $\mathbb{B}_Y = \mathbb{W}_X \oplus \mathbb{B}$ for some subbimodule $\mathbb{B}$ of $\mathbb{B}_Y$, where $\mathbb{W}_X \cong \mathbb{B}_y$. Indicate the operation of concatenation of sequences by "$\cdot$". Now $\mathbb{W}_{X\cdot Y\cdot Z}$ is the subbimodule of $\mathbb{B}_{X\cdot Y\cdot Z} \cong \mathbb{B}_X \otimes \mathbb{B}_Y \otimes \mathbb{B}_Z$ generated by $b_X \otimes b_Y \otimes b_Z$, so is isomorphic to the sub-bimodule of $\mathbb{B}_X \otimes \mathbb{B}_y \otimes \mathbb{B}_Z$ generated by $b_X \otimes b \otimes b_Z$ for any non-zero $b \in (\mathbb{B}_y)_{-l(y)}$. By symmetry, this last subbimodule is isomorphic to $\mathbb{W}_{X\cdot Y'\cdot Z}$, hence $\mathbb{W}_{X\cdot Y\cdot Z} \cong \mathbb{W}_{X\cdot Y'\cdot Z}$. The corollary now follows from 1.3(ii).

Remark. Consider an arbitrary finite sequence $X\colon r_1, \dots, r_n$ in S. By 6.4(1), the elements $x_1 \otimes \dots \otimes x_n$ with each $x_i \in \{e_{r_i}, f_{r_i}\}$ form a $\mathbb{W}$-basis of $\mathbb{B}_X$ as left $\mathbb{W}$-module (and as right $\mathbb{W}$-module). Fix $x_1, \dots, x_n$ and write $\{\, i \mid x_i = e_{r_i}, 1 \leq i \leq n \,\} = \{i_1, \dots, i_m\}$ where $1 \leq i_1 < \dots < i_m \leq n$. It is well known that $\{\, s_1 \dots s_m \mid$ each $s_j = 1$ or $s_j = r_{i_j} \,\}$ has a unique maximal element in Chevalley order, say y. An argument similar to that in the proof of the corollary above shows that the subbimodule of $\mathbb{B}_X$ generated by $x_1 \otimes \dots \otimes x_n$ is isomorphic to $\mathbb{W}_y\langle l(y) + n - 2m\rangle$.

7. Polyhedral cones

This section briefly sketches the analogues for polyhedral cones of some of the preceeding results in sections 2–5. As a general reference, one has for instance [8].

7.1. Consider a real Euclidean space V; we denote the inner product by $\langle \cdot | \cdot \rangle$. Now fix a polyhedral cone $\mathcal{C} \subseteq V$; that is, $\mathcal{C}$ is the intersection of finitely many closed half-spaces $\mathcal{C} = \cap_{i=1}^n H_{\alpha_i}$ for $\alpha_i \in V \setminus \{0\}$, where for $\alpha \in V$, $H_\alpha := \{\, v \in V \mid \langle v, \alpha \rangle \geq 0 \,\}$. The dual polyhedral cone $\mathcal{C}^\vee$ is defined by

$$(1) \qquad \mathcal{C}^\vee := \{\, \alpha \in V \mid \langle \alpha, v \rangle \geq 0 \text{ for all } v \in \mathcal{C} \,\}.$$

A face of $\mathcal{C}$ is by definition a subset $F \subseteq \mathcal{C}$ of the form $F = \{\, v \in \mathcal{C} \mid \langle v, \alpha \rangle = 0 \,\}$ for some α in $\mathcal{C}^\vee$. It is known that the map $F \to F' := \mathcal{C}^\vee \cap F^\perp$ is an inclusion reversing bijection between the faces F of $\mathcal{C}$ and F' of $\mathcal{C}^\vee$, with $F = \mathcal{C} \cap (F')^\perp$.

Fix a set Γ with a given bijection $x \mapsto F_x$ between X and the set of faces of $\mathcal{C}$. We order Γ by setting $x \leq y$ iff $F_x \subseteq F_y$; Γ is then a finite lattice, called the face lattice of Γ. For $x, y \in \Gamma$, we define $l(x)$ to be the dimension of the linear subspace $\langle F_x \rangle$ of V spanned by F_x. We write $x \lessdot y$ to indicate that $x < y$ and there is no $z \in \Gamma$ with $x < z < y$; this holds iff $x \leq y$ and $l(y) = l(x) + 1$.

7.2. For $x \in \Gamma$, let σ'_x denote the orthogonal reflection in $F_x^\perp$ i.e. the unique $\mathbb{R}$-linear map $V \to V$ which fixes $F_x^\perp$ pointwise and acts by multiplication by -1 on F_x.

Let $\mathcal{S}$ denote the symmetric algebra of V graded as usual so $\mathcal{S}_0 = \mathbb{R}$, and $\mathcal{S}_2 = V$, and extend σ'_x to a graded $\mathbb{R}$-algebra automorphism σ_x of $\mathcal{S}$.

For $x \neq y$ in Γ, define $d_{x,y} \in V \subseteq \mathcal{S}$ as follows. If $x \lessdot y$, let $d_{x,y}$ be the unique unit vector α in $\langle F_y \rangle \cap \langle F_x \rangle^\perp$ such that $\langle \alpha, v \rangle \geq 0$ for all $v \in F_y$, and set $d_{y,x} = d_{x,y}$. If $x \neq y$ and neither $x \lessdot y$ nor $y \lessdot x$, set $d_{x,y} = 1$. It is not hard to see (details are left to the reader) that

(1) for $x \neq y$ in X, $d_{x,y}$ is a greatest common divisor in $\mathcal{S}$ for the elements of $(\sigma_x - \sigma_y)(\mathcal{S})$.

7.3. Let $\Psi = \{ d_{x,y} \mid x \lessdot y \in \Gamma \}$. For each $\alpha \in \Psi$, let $\mathcal{S}^{(\alpha)}$ be the localization $\mathcal{S}^{(\alpha)} := \mathcal{S}[\beta^{-1} \mid \beta \in \Psi \setminus \{\pm\alpha\}]$ of $\mathcal{S}$, regarded as a subring of the quotient field K of $\mathcal{S}$.

Proposition. *For a family $\{q_x\}_{x\in\Gamma}$ of elements of K, one has $\sum_x q_x\sigma_x(\chi) \in \mathcal{S}$ for all $\chi \in \mathcal{S}$ iff the following two conditions hold for all $\gamma \in \Psi$ and $x \in \Gamma$:*
(i) *if there is no $y \in \Gamma$ with $d_{x,y} = \pm\gamma$, then $q_x \in \mathcal{S}^{(\gamma)}$*
(ii) *if $y \neq x$ satisfies $d_{x,y} = \pm\gamma$, then $q_x \in \gamma^{-1}\mathcal{S}^{(\gamma)}$ and $q_x + q_y \in \mathcal{S}^{(\gamma)}$.*

Proof. Note that there can be at most one $y \in \Gamma$ satisfying $d_{x,y} = \pm\gamma$ (for fixed x and γ). The proof using 7.2(1) and 8.2 is very similar to that of 5.1, and details are therefore omitted.

7.4. We recall from [20] the definition of a graded $(\mathcal{S},\mathcal{S})$-bimodule $\mathbb{M} = \mathbb{M}_{\mathcal{C}}$ associated to the polyhedral cone $\mathcal{C}$ and the inner product on V (we regard $\mathbb{M}$ equivalently as a $(\mathcal{S},\mathcal{S})$ bimodule when convenient, via $\chi m\chi' = (\chi \otimes \chi')m$ for $\chi, \chi' \in \mathcal{S}$ and $m \in \mathbb{M}$). First, $\mathbb{M}$ has a graded basis $\{m_x\}_{x\in\Gamma}$ as right $\mathcal{S}$-module, with $m_x \in \mathbb{M}_{2l(x)}$. The left $\mathcal{S}$-action is given by

$$\chi m_x = m_x\sigma_x(\chi) - \sum_{y\in\Gamma: x\lessdot y} \langle \chi \mid d^\vee_{x,y} \rangle m_y \qquad \text{for } \chi \in V \tag{1}$$

where $d^\vee_{x,y} = 2d_{x,y}$ for $x \lessdot y$.

Denote by m'_x be the standard basis element in $\mathbb{M}_{\mathcal{C}^\vee}$ corresponding to the face F'_x of $\mathcal{C}^\vee$. and let $\{m^*_x\}_{x\in X}$ denote the dual right $\mathcal{S}$-basis of the graded right $\mathcal{S}$-dual $\mathbb{M}^* := \mathrm{Hom}_{\mathbb{R}\otimes\mathcal{S}}(\mathbb{M},\mathcal{S})$ defined by $m^*_x(m_y) = \delta_{x,y}$. Give $\mathbb{M}^*$ the non-standard $(\mathcal{S},\mathcal{S})$-bimodule stucture with $(\chi f\chi')(m) = -f(\chi m\chi')$ for $\chi \in V$, $\chi' \in \mathcal{S}$, $f \in M^*$ and $m \in M$. Then one can check that there is an isomorphism of $(\mathcal{S},\mathcal{S})$-bimodules

$$\mathbb{M}^* \cong \mathbb{M}_{\mathcal{C}^\vee}\langle -2\dim(V)\rangle \tag{2}$$

mapping m^*_x to $(-1)^{l(x)}m'_x$.

7.5. We may naturally regard $\mathbb{M}$ as a $(\mathcal{S},\mathcal{S})$-subbimodule of the $(\mathcal{S},K)$-bimodule $\mathbb{M}' := \mathbb{M} \otimes_{\mathcal{S}} K$. Now by 7.2(1), 8.6 and 8.8, there is a basis δ_x, for $x \in \Gamma$, of M' as right K-space with the following properties;
(i) $\chi\delta_x = \delta_x\sigma_x(\chi)$ for $\chi \in \mathcal{S}$
(ii) $m_x = \sum_{y\in\Gamma} \delta_y a_{x,y}$ for some $a_{x,y} \in K$ with $a_{x,x} = 1$ and $a_{x,y} = 0$ unless $x \leq y$.

By 8.10, the elements δ_x are uniquely determined by these conditions. One may also write $\delta_x = \sum_{y\in X} m_y b_{x,y}$ for some $b_{x,y} \in K$ with $b_{x,x} = 1$ and $b_{x,y} = 0$ unless $x \leq y$.

For $x \leq z \leq y$ in Γ, set $I_z(x,y) := \{ d_{z,z'} \neq 1 \mid x \leq z' \leq y,\ z' \neq z \}$. Then (see [18]) if $x \leq z \leq y$ $I_z(z,y)$ (resp., $I_z(x,z)$) is the set of unit vectors of V lying in the extreme rays (i.e. one-dimensional faces) of the polyhedral cone $F_y \cap F_z^\perp$ (resp., $F_x^\perp \cap F_z$).

Proposition. *For any $x \leq y$ in Γ, one may write $a_{x,y} = f_{x,y}/(\prod_{\alpha\in I_y(x,y)} \alpha)$ (respectively., $b_{x,y} = (-1)^{l(y)-l(x)} g_{xy}/(\prod_{\alpha\in I_x(x,y)} \alpha)$ for some non-zero, homogeneous element $f_{x,y} \in \mathcal{S}_{2(\sharp(I_y(x,y))-l(y)+l(x))}$ (resp., $g_{x,y} \in \mathcal{S}_{2(\sharp(I_y(x,y))-l(y)+l(x))}$) which is* *expressible as a linear combination with non-negative coefficients of products of elements of $I_y(x,y)$ (resp., $I_x(x,y)$).*

Proof. The proof is a refinement of that of 8.10. For $x < y$, 7.4(1) gives

$$a_{x,y}(\sigma_y(\chi) - \sigma_x(\chi)) = -\sum_{z\in\Gamma: x\lessdot z} \langle\, \chi \mid d^{\vee}_{x,z}\,\rangle a_{z,y} \tag{1}$$

for $\chi \in V$. Assume inductively that the result for $a_{x,y}$ holds with x replaced by any x' with $x < x' \leq y$. Choose $\chi = d_{u,y}$ for some $x \leq u \lessdot y$. Then (see [18]) $\langle\, \chi, d^{\vee}_{x,z}\,\rangle = 0$ if $x \lessdot z \leq u$ while $\langle\, \chi, d^{\vee}_{x,z}\,\rangle > 0$ for $x \lessdot z \leq y$ with $z \not\leq u$. Also, $\sigma_y(\chi) = -\sigma_x(\chi) = -d_{u,y}$. The result for $a_{x,y}$ follows immediately by induction, and then that for $b_{x,y}$ follows by 7.4(2).

7.6. For $x \in \Gamma$, define the $(\mathcal{S},\mathcal{S})$-bimodule $\mathcal{S}_x$ which is equal to $\mathcal{S}$ as right $\mathcal{S}$-module, and with left $\mathcal{S}$-module structure $\chi m = m\sigma_x(\chi)$ for $m \in \mathcal{S}_x$, $\chi \in \mathcal{S}$. The following result is the analogue in this context of 5.7-5.8.

Corollary. *Let $x, y \in \Gamma$. Then as right $\mathcal{S}$-modules,*
(a) $\mathrm{Hom}_{\mathcal{S}\otimes_{\mathbb{R}}\mathcal{S}}(\mathcal{S}_x, \mathcal{S}_y) \cong \mathcal{S}$ *if $x = y$ and is zero otherwise.*
(b) *if $x \neq y$, then* $\mathrm{Ext}^1_{\mathcal{S}\otimes_{\mathbb{R}}\mathcal{S}}(\mathcal{S}_x, \mathcal{S}_y) \cong \mathcal{S}/d_{x,y}\mathcal{S}$.
(c) *If $d_{x,y} \neq 1$ with, say, $x \leq y$, then $a_{x,y} = -b_{x,y} = d^{-1}_{x,y}$.*

8. Generalities from commutative algebra

In this section, we make some mostly trivial remarks concerning certain rings defined by extending one definition of the nil Hecke ring to the general context of a family of ring homomorphisms $U \to B$ between commutative rings U and B. We also describe some general features of certain (U,B)-bimodules associated to a suitable ordered family of such ring homomorphisms.

8.1. Throughout this section, U denotes a commutative ring and B denotes a commutative domain with quotient field K. We suppose given a family $\sigma = \{\sigma_x: U \to B\}_{x\in\Gamma}$ of pairwise distinct ring homomorphisms, where Γ is now any set. If $\{c_x\}_{x\in\Gamma}$ is a family of elements of K, almost all zero, we define the function $\sum_{x\in\Gamma}\sigma_x c_x: U \to K$ by $u \mapsto \sum_{x\in\Gamma} c_x\sigma_x(u)$. The set K_σ of all functions $U \to K$ so arising has a natural (U,K)-bimodule structure given by $(ufk)(u') = kf(uu')$ for $u, u \in U$, $k \in K$, $f \in K_\sigma$; one has

$$u(\sum_x \sigma_x c_x)k = \sum_x \sigma_x c_x k \sigma_x(u).$$

By Dedekind's lemma, the elements σ_x for $x \in \Gamma$ form a right K-module basis of K_σ. In this subsection and the next, we give some useful extensions of this fact.

For an ideal $\mathfrak{b}$ of B and $x, y \in \Gamma$, define the equivalence relation $\equiv_{\mathfrak{b}}$ on Γ by setting $x \equiv_{\mathfrak{b}} y$ if $(\sigma_x - \sigma_y)(U) \subseteq \mathfrak{b}$.

Lemma. *Suppose that B is a local ring with maximal ideal $\mathfrak{m}$, and that $\mathfrak{a}$ is any B-submodule of K. Then for a family $\{c_x\}_{x\in\Gamma}$ of elements of K, almost all zero, the following conditions* (i)–(ii) *are equivalent:*

(i) $(\sum_{x\in\Gamma}\sigma_x c_x)(U)\subseteq\mathfrak{a}$
(ii) *for each* $y\in\Gamma$, $(\sum_{\substack{x\in\Gamma\\ x\equiv_{\mathfrak{m}} y}}\sigma_x c_x)(U)\subseteq\mathfrak{a}$.

Proof. Obviously (ii) implies (i). For (i) implies (ii), suppose that $\Gamma=\{1,\dots,n\}$ with $\sigma_i\not\equiv_{\mathfrak{m}}\sigma_j$ for $i\le p$ and $j>p$, where $p\ge 1$. Assuming $(\sum_{i=1}^n\sigma_i c_i)(U)\subseteq\mathfrak{a}$, it is enough to show $(\sum_{i=p+1}^n\sigma_i c_i)(U)\subseteq\mathfrak{a}$. If $p=n$ this is certainly true, so suppose that $p<n$. Choose $y\in U$ so $(\sigma_1-\sigma_n)(y)\notin\mathfrak{m}$. For any $u\in U$, we have $(\sum_{i=1}^n\sigma_i c_i)(uy)\in\mathfrak{a}$ and $\big(\sum_{i=1}^n c_i\sigma_i(u)\big)\sigma_n(y)\in\mathfrak{a}$. Subtracting gives $(\sum_{i=1}^{n-1}\sigma_i c_i d_i)(U)\subseteq\mathfrak{a}$ where $d_i=(\sigma_i-\sigma_n)(y)\in B$ and d_1 is a unit in B. Repeating this argument $n-p-1$ more times gives $(\sum_{i=1}^p\sigma_i c_i e_i)(U)\subseteq\mathfrak{a}$ for some elements e_i of B with $e_1\in B^\bullet$. Multiplying this relation by e_1^{-1} and subtracting from $(\sum_{i=1}^n\sigma_i c_i)(U)\subseteq\mathfrak{a}$ gives a relation $(\sum_{i=2}^n\sigma_i c_i')(U)\subseteq\mathfrak{a}$ for some $c_i'\in K$ ($i=2,\dots,n$) with $c_i'=c_i$ for $i>p$. Repeating the entire argument $p-1$ more times shows that $(\sum_{i=p+1}^n\sigma_i c_i)(U)\subseteq\mathfrak{a}$ as required.

8.2. We maintain the general notation U, B etc from 8.1, but do not assume that B is local.

Corollary. *Suppose that $\mathfrak{a}$ ia a B-submodule of K and that P is a set of prime ideals of B such that $\mathfrak{a}=\cap_{\mathfrak{p}\in P}\mathfrak{a}_{\mathfrak{p}}$ as B-submodules of K ($\mathfrak{a}_{\mathfrak{p}}$ denotes the localization of $\mathfrak{a}$ at $\mathfrak{p}$, regarded as a subset of K). Then for a family $\{c_x\}_{x\in\Gamma}$ of elements of K, almost all zero, the following conditions are equivalent:*
(i) $(\sum_{x\in\Gamma}\sigma_x c_x)(U)\subseteq\mathfrak{a}$
(ii) *for each* $y\in\Gamma$ *and* $\mathfrak{p}\in P$, $(\sum_{\substack{x\in\Gamma\\ x\equiv_{\mathfrak{p}} y}}\sigma_x c_x)(U)\subseteq\mathfrak{a}_{\mathfrak{p}}$.

Proof. The proof is immediate from the assumption $\mathfrak{a}=\cap_{\mathfrak{p}\in P}\mathfrak{a}_{\mathfrak{p}}$ and Lemma 8.1, on noting that $(\sigma_x-\sigma_y)(U)\subseteq\mathfrak{p}B_{\mathfrak{p}}$ iff $x\equiv_{\mathfrak{p}} y$.

Remark. The hypotheses of the corollary hold in the following situations (a)–(b):
(a) P is the set of maximal ideals of B and $\mathfrak{a}$ is a principal fractional ideal i.e. $\mathfrak{a}=Bk$ for some $k\in K$.
(b) B is a Krull domain, $\mathfrak{a}$ is (zero or) a divisorial fractional ideal in K and P is the set of height one prime ideals of B. This includes the case where B is a UFD (then the divisorial fractional ideals are the principal fractional ideals Bk for $k\in K^\bullet$, and the height one primes are the ideals Bp for p a prime element of B).

8.3. For the remainder of this section, we assume that both U and B are algebras over a commutative ring R, and that Γ is a set of pairwise distinct R-algebra homomorphisms $U\to B$ (e.g. $R=\mathbb{Z}$). Then K_σ is naturally a $U\otimes_R K$-module. We make K_σ into a coalgebra over K, with the standard K-basis elements $\sigma_x\in\sigma$ of K_σ grouplike i.e. the comultiplication $\Delta\colon K_\sigma\to K_\sigma\otimes_K K_\sigma$ is given by $\Delta(\sum\sigma_x c_x)=\sum\sigma_x\otimes\sigma_x c_x$ and the counit $\epsilon\colon K_\sigma\to K$ given by $\epsilon(\sum_x\sigma_x c_x)=\sum_x c_x$. Define the $U_B:=U\otimes_R B$-submodule $B_\sigma:=\{\,f\in K_\sigma\mid f(U)\subseteq B\,\}$ of K_σ, and consider the following condition:

(i) B_σ is a free (or projective) B-module and, under the natural inclusion $B_\sigma\otimes_B B_\sigma\to K_\sigma\otimes_K K_\sigma$, one has $\Delta(B_\sigma)\subseteq B_\sigma\otimes_B B_\sigma$.

Suppose that (i) holds. Then B_σ is naturally a coalgebra over B and we denote by $B_\sigma^*:=\operatorname{Hom}_B(B_\sigma,B)$ the dual B-algebra. In fact, the U_B-module structure on B_σ gives B_σ^* a natural U_B-algebra structure, and one can consider the further condition

(ii) the U_B-algebra structural homomorphism $U_B \to B_\sigma^*$ is surjective.

The condition (i) (and sometimes (ii)) holds in a number of specific situations associated to Coxeter groups studied in this paper and its sequel.

8.4. This subsection and the next lists some observations concerning conditions 8.3(i)–(ii); the observations are not used essentially in this paper (or the sequel), and proofs are left to the interested reader. Maintain the notation from the preceeding subsection.

(1) If there is a K-basis $\{e_x\}_{x\in\Gamma}$ of K_σ with all $e_x \in B_\sigma$ and elements $\{u_x\}_{x\in\Gamma}$ of U such that $e_x(u_y) = \delta_{x,y}$, then 8.3(i)–(ii) both hold (using "free" in (i)).

(2) If Γ is finite and B is a discrete valuation ring, then (1) holds. Hence 8.3(i)–(ii) both hold (using "projective" in (i)) if Γ is finite and B is a Dedekind domain.

8.5. In this subsection, we discuss a special case of the situation considered above. Suppose that B is a commutative R-algebra which is a domain with quotient field K, and G is a group of R-algebra automorphisms of B. For any finite subgroup H of G and $g \in G$, let g_H denote the restriction of g to a R-algebra homomorphism $B^H \to B$; then $g_H = g'_H$ iff $gH = g'H$. We let $\sigma_H := \{ g_H \mid g \in \Gamma_H \}$ where $\Gamma_H := G/H$ denotes a set of coset representatives of H in G.

One may define B_{σ_H} and the $B^H \otimes_{B^G} B$-algebra $B^*_{\sigma_H}$ as above (assuming that the family σ_H of homomorphisms $B^H \to B$ satisfies 8.3(i); we also assume this for $\sigma := \sigma_{\{e\}}$). Then B_σ is naturally a ring (under composition of functions) and B_σ^* therefore acquires a natural stucture of right B_σ-module. In particular, there is a natural action of G as a group of algebra automorphisms of B_σ^* defined by $(gf)(b) = f(g^{-1}b)$ for $g \in G$, $f \in B_\sigma^*$ and $b \in B_\sigma$. One can therefore form the B-algebra $(B_\sigma^*)^H$ of H-invariant elements of B_σ^*. In a number of situations associated to Coxeter groups considered in this paper and its sequel, one (or more) of the following related conditions holds for certain finite subgroups $H' \subseteq H$ of G;

(i) the structural homomorphism $B^H \otimes_{B^G} B \to B^*_{\sigma_H}$ is an isomorphism of rings

(ii) $B^*_{\sigma_H} \cong (B_\sigma^*)^H$ as $B^H \otimes_{B^G} B$-algebras

(iii) $B^*_{\sigma_{H'}}$ is a free (or projective) $B^*_{\sigma_H}$-module

(iv) $\mathrm{Hom}_{B^*_{\sigma_H}}(B^*_{\sigma_{H'}}, B^*_{\sigma_H}) \cong B^*_{\sigma_{H'}}$ as $B^*_{\sigma_{H'}}$-module.

Using 8.4(2), one can show that

(1) if G is finite and B, B^G are both Dedekind domains, then (i)–(iii) hold for any subgroups $H' \subseteq H \subseteq G$ (using "projective" in (iii)).

Another situation in which the study of the conditions in 8.3 and this subsection might be of interest is mentioned in 10.5.

8.6. Maintain notation from 8.3. Identify the category of (U, B)-bimodules (with left and right R action coinciding) with the category of left $U_B := U \otimes_R B$-modules. For $x \in \Gamma$, let B_x denote the U_B-module which is equal to B as right B-module, with left U-action given by $\chi m = m\sigma_x(\chi)$ for $m \in B_x$ and $\chi \in U$. If $y \in \Gamma$ as well, then (see [21])

$$\text{(1)} \qquad \operatorname{Hom}_{U_B}(B_x, B_y) \cong \begin{cases} B & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{(2)} \quad \operatorname{Ext}^1_{U_B}(B_x, B_y) \cong \{\, k \in K \mid k(\sigma_x - \sigma_y)(u) \in B \text{ for all } u \in U \,\}/B \qquad \text{if } x \neq y$$

as right B module. Regarding $\sigma_x : U \to B$ as a homomorphism $U \to K$, one can identify the U_K-bimodule K_x with $B_x \otimes_B K$. Note that in general, (2) gives

$$\text{(3)} \qquad \operatorname{Ext}^1_{U_K}(K_x, K_y) \cong 0 \qquad \text{for } x \neq y$$

8.7. For the remainder of this section, we assume that Γ is endowed with a fixed partial ordering $\leq$ such that the (pairwise distinct) R-algebra homomorphisms $\{\sigma_x : U \to B\}_{x \in \Gamma}$ satisfy the condition

$$\text{(1)} \qquad \operatorname{Ext}^1_{U_B}(B_x, B_y) = 0 \qquad \text{if } x \not< y \text{ and } y \not\leq x.$$

We also assume (just for simplicity) that Γ is finite.

Remark. If B is Noetherian, a representation category can be associated to such data as in [20]. The general questions considered in this section are of potential interest in situations where there are "natural" choices satisfying (1) of the partial order on Γ, since for these it is possible the representation category itself may be of interest.

8.8. Consider a U_B-module M equipped with a fixed finite basis $\{m_i\}_{i \in I}$ as right B-module, indexed by a finite set I (disjoint from Γ) with a function $i \to \bar{i} : I \to \Gamma$ such that

$$\text{(2)} \qquad \chi m_i \in m_i \sigma_{\bar{i}}(x) + \sum_{\substack{j \in I \\ \bar{j} > \bar{i}}} m_j B$$

for $\chi \in U$. For a coideal Y in Γ , $M(Y) := \sum_{j : \bar{j} \in Y} m_j B$ is then a U_B submodule of M. Any sequence $\Gamma = Y_m \supseteq \ldots \supseteq Y_0 = \emptyset$ of coideals of Γ for which $Y_j \setminus Y_{j-1} = \{y_j\}$ has one element for each j, gives a filtration $M = M(Y_m) \supseteq \ldots \supseteq M(Y_0) = 0$ of M with each successive subquotient $M(Y_j)/M(Y_{j-1}))$ isomorphic to a finite direct sum of U_B-modules B_{y_j}. By 8.6(3), $M' := M \otimes_B K \cong \oplus_j \big(M(Y_j)/M(Y_{j-1}) \otimes_B K\big)$ as U_K-module, where $M(Y_j)\big/M(Y_{j-1}) \otimes_B K$ is a direct sum of finitely many copies of K_{y_j}. Since M is free as right B-module, M may be naturally identified with the U_B-submodule $M \otimes_B B$ of M'. It follows from this that there is a basis of M' as right K-vector space with the following two properties:

(i) $\chi \delta_i = \delta_i \sigma_{\bar{i}}(\chi)$ for $i \in I$ and $\chi \in U$

(ii) $m_i = \sum_{j \in I} \delta_j a_{ij}$ for some $a_{ij} \in K$ satisfying $a_{i,i} \neq 0$, and $a_{i,j} = 0$ unless $i = j$ or $\bar{j} > \bar{i}$.

In fact, one may take all $a_{i,i} = 1$, but we do not necessarily assume that this is done.

8.9. Conversely, suppose given a U_K-module M' with basis $\{\delta_i\}_{i\in I}$ as right K-space satisfying 8.8(i) for some function $i \to \bar{i}\colon I \to \Gamma$. Let m_i be defined as in 8.9(ii) for some constants $a_{i,j}$ satisfying the conditions there, and let M be the right B-submodule of M' spanned by the m_i. We will determine the condition on the $a_{i,j}$ for M to be a U_B-submodule of M' (it will then necessarily be of the type considered in 8.8).

Since the matrix $(a_{i,j})$ is upper triangular with non-zero diagonal entries, with respect to a suitable ordering of I, it is invertible, so the m_i form a K-basis of M' as well. Hence

(1) $\delta_i = \sum_{j\in I} m_j b_{i,j}$ for some $b_{i,j} \in K$ satisfying $b_{i,i} \neq 0$, $b_{i,j} = 0$ unless $i = j$ or $\bar{j} > \bar{i}$.

Then for $\chi \in U$, $\chi m_i = \sum_j \delta_j \sigma_{\bar{j}}(\chi) a_{i,j} = \sum_k m_k \big(\sum_j a_{i,j} b_{j,k} \sigma_{\bar{j}}(\chi)\big)$. Thus,

(2) M is a U_B-submodule of M' iff for all $i, k \in I$, the element $\Omega_{i,k}$ of K_σ defined by $\Omega_{i,k} = \sum_j \delta_{\bar{j}} a_{i,j} b_{j,k}$ satisfies $\Omega_{i,k}(U) \subseteq B$ i.e. iff $\Omega_{i,k} \in B_\sigma$ as defined in 8.2.

8.10. Suppose M is as in 8.8. The functions $\Omega_{i,k}$ defined by 8.9(2) satisfy

(1) $$\chi m_i = \sum_j m_j \Omega_{i,j}(\chi) \qquad \text{for } \chi \in U.$$

Replacing the basis elements m_i by their expressions in terms of the δ_i, recalling $\chi\delta_i = \delta_i \sigma_{\bar{i}}(\chi)$ and taking the coefficient of δ_i on both sides gives

(2) $$\big(\sigma_{\bar{j}} - \sigma_{\bar{i}})(\chi)\big) a_{i,j} = \sum_{\substack{k\in I\\ \bar{i}<\bar{k}\leq\bar{j}}} \Omega_{i,k}(\chi) a_{k,j}.$$

It follows that all the $a_{i,j}$ are recursively determined by the structure constants $\Omega_{i,k}(\chi)$ and the $a_{i,i}$. For specified values of $a_{i,i} \neq 0$, the K-basis elements δ_i of M' satisfying 8.8(i)–(ii) are therefore uniquely determined by the B-basis elements m_i of M.

8.11. The functions $\Omega_{i,k}$ also arise also arise naturally as follows. Note that M' above naturally becomes a right comodule for $C := K_\sigma$, with comodule structure map $\Delta\colon M' \to M' \otimes_K C$ given by $\Delta(\sum \delta_i c_i) = \sum \delta_{\bar{i}} \otimes \delta_i c_i$ (this comodule structure is independent of the choice of K-basis δ_i for M' satisfying 8.8(i)–(ii)). Since the m_i are also a K-basis of M', there are unique elements $\Omega_{i,k} \in C$ such that

(1) $$\Delta(m_i) = \sum_k \Omega'_{i,k} \otimes m_k$$

The definitions immediately give that

(2) $$\Omega'_{i,k} = \sum_j \delta_{\bar{j}} a_{i,j} b_{j,k} = \Omega_{i,k}.$$

Note also that from (1), (2)

(3) $$\Delta(\Omega_{i,k}) = \sum_j \Omega_{i,j} \otimes \Omega_{j,k}$$

or equivalently $\Omega_{i,k}(\chi\chi') = \sum_j \Omega_{i,j}(\chi)\Omega_{j,k}(\chi')$ for $\chi, \chi \in U$.

8.12. Consider a U_B-module M with B basis $\{m_i\}_{i\in I}$ as in 8.8. Write $\chi m_i = \sum_j m_j \Omega_{i,j}(\chi)$ as in 8.10(1). Suppose there is some non-empty coideal $Y \subseteq \Gamma$ such that $M(Y) = 0$, and fix $x \in Y$. Let K^I denote the B-module consisting of I-tuples $(c_i)_{i\in I}$ of elements of K and E_x be the B-submodule of K^I consisting of those I-tuples $(c_i)_{i\in I}$ with

$$(1) \qquad c_i(\sigma_x - \sigma_{\bar{i}}) - \sum_{\substack{j\in I\\ \bar{j}>\bar{i}}} \Omega_{ij} c_j \in B_\sigma \qquad \text{for all } i \in I.$$

The following result generalizes 8.6(2).

Lemma. *As B-module, $\mathrm{Ext}^1_{U_B}(M, B_x) \cong E_x/B^I$ where B^I is the submodule of K^I consisting of I-tuples of elements of B and E_x is as defined above.*

Proof. Choose in the module B_x a basis element δ as right B-module. Note $(M \otimes_B K) \oplus K_x$ is a U_K-module. For $c = (c_i)_{i\in I}$ in K^I, let $\hat{M} \subseteq (M \otimes_B K) \oplus K_x$ be the B-submodule spanned by elements $\hat{m}_i = m_i + \delta c_i$ for $i \in I$, together with δ. Then by 8.9(2), $\hat{M}$ is a U_B-module (i.e. is stable under the U-action) iff $c \in E_x$. In that case, there is an obvious exact sequence $0 \to B_x \to \hat{M} \to M \to 0$ of U_B-modules which is seen to split iff $c \in B^I$. This gives an injective homomorphism of B-modules $E_x/B^I \to \mathrm{Ext}^1_{U_B}(M, B_x)$, which is surjective as follows from 8.8.

8.13. The following trivial observations are sometimes useful. Consider a U_B-module M with B basis $\{m_i\}_{i\in I}$ as in 8.8, and the U_K-module $M' := M \otimes_B K$. The dual K-vector space $M'^* := \mathrm{Hom}_K(M', K)$ has a natural U_K-module structure induced by that on M'; one has $\chi\delta_i^* = \delta_i^*\sigma_{\bar{i}}(\chi)$ for $\chi \in U$, where the δ_i^* are the basis of M'^* dual to the δ_i. Define $m_i^* = \sum_j \delta_j^* b_{j,i} \in M'^*$, so also $\delta_i^* = \sum_j m_j^* a_{j,i}$. The m_i^* are of course the basis of $M^* := \mathrm{Hom}_B(M, B)$ dual to the basis m_i of M (under the obvious identification of M^* with a U_B-submodule of M'^*). One has $\chi m_j^* = \sum_i m_i^* \Omega_{i,j}(\chi)$. Note that the right B-basis m_i^* of M^* satisfies the same conditions as assumed in 8.8 for the basis m_i of M, but for the opposite poset Γ^{op} instead of Γ.

8.14. Here, we assume that B is a UFD. Note then that the B-module on the right hand side of the formula 8.6(1) is isomorphic to $B/Bd_{x,y}$ where $d_{x,y}$ is a greatest common divisor for the elements $(\sigma_x - \sigma_y)(u)$ of B for $u \in U$ (or for u in a set of R-algebra generators for U).

Lemma. *Let M be as in 8.8, and choose bases $\{m_i\}_{i\in I}$ for M and $\{\delta_i\}_{i\in I}$ for $M \otimes_B K$ as in that subsection, related by the equations $m_i = \sum_{j\in I} \delta_j a_{i,j}$ and $\delta_i = \sum_{j\in I} m_j b_{i,j}$. Then for any $i, j \in I$ with $\bar{i} < \bar{j}$*

(a) $a_{i,j} \in a_{j,j}\left(\prod_{\bar{i}\le y<\bar{j}} d_{y,\bar{j}}\right)^{-1} B$

(b) $b_{i,j} \in b_{i,i}\left(\prod_{\bar{i}<y\le\bar{j}} d_{\bar{i},y}\right)^{-1} B.$

(c) $\Omega_{i,j} \in \sum_{x:\bar{i}\le x\le\bar{j}} \; \delta_x\left(\frac{1}{\prod_{y\ne x:\bar{i}\le y\le\bar{j}} d_{x,y}} B\right).$

Proof. By 8.13, part (b) follows from (a). Also, (c) follows from (a) and (b). To prove (a), suppose $\bar{i} < \bar{j}$ and assume inductively that (a) is true with i replaced by any $i' \in I$ with $\bar{j} > \bar{i'} > \bar{i}$. Using induction to give a common denominator, 8.10(2) can be written as

$$a_{i,j}(\sigma_{\bar{j}} - \sigma_{\bar{i}})(\chi) = \frac{f_\chi a_{j,j}}{\prod_{\substack{y \in X \\ \bar{i} < y < \bar{j}}} d_{y,\bar{j}}}$$

for some $f_\chi \in B$. Now (a) follows on recalling the definition of $d_{\bar{i},\bar{j}}$.

9. Appendix: A monoid lemma for "mixed" braid relations

This appendix discusses a variant of Matsumoto's monoid lemma for Coxeter groups which can in special cases be used to give alternative constructions of the modules considered in Section 2 of this paper.

9.1. Note that there is an action of W on $\mathcal{P}(T)$ defined by $w \cdot A = N(w) + wAw^{-1}$ for $w \in W$, $A \subseteq T$ (see 1.3(e)). We consider the following condition on a subset A of T;

(i) For each finite rank two parabolic subgroup W' of W, there exists an element $w' \in W'$ with $N(w') \cap W' = A \cap W'$.

Since we assume the condition in (1) for all dihedral parabolic subgroups (not just the standard ones), it follows that

(1) if $A \subseteq T$ satisfies (i), so does $w \cdot A$ for any $w \in W$.

Indeed, it is enough to check this for $w \in S$, which is easy. The condition (i) is equivalent to the corresponding condition with "rank two" omitted from the statement, though we shall not need this fact, and holds if A is any initial section of a reflection order on T.

Fix for now two distinct simple reflections $r, s \in S$ whose product has finite order $m = n_{r,s}$. Set $r_i = r$ for i odd and $r_i = s$ for i even. Let M be a monoid and $x_r = (r^+, r^-) \in M \times M$, $x_s = (s^+, s^-) \in M \times M$ be two ordered pairs of elements of M. We say that x_r, x_s satisfy the mixed braid relations for r and s if the following condition holds:

(ii) for any sequence of symbols $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ from $\{+, -\}$ with at most one pair of unequal consecutive terms (i.e. $\epsilon_i \neq \epsilon_{i+1}$ for at most one $1 \leq i \leq m-1$),

$$r_m^{\epsilon_1} r_{m-1}^{\epsilon_2} \ldots r_1^{\epsilon_m} = r_{m+1}^{\epsilon_m} r_m^{\epsilon_{m-1}} \ldots r_2^{\epsilon_1}.$$

In particular (ii) implies that the elements r^+ (resp., r^-) satisfy the ordinary braid relations of (W, S).

Lemma. *Let M be any monoid and $x_r = (r^+, r^-) \in M \times M$, for $r \in S$, be pairs of elements of M satisfying the mixed braid relations (i.e. x_r and x_s satisfy the mixed braid relations for r, s whenever r, s in S with $1 < n_{r,s} < \infty$).*

Then for any $A \subseteq T$ satisfying (i)*, there exist unique elements $x_A \in M$ for $x \in W$ such that e_A is the identity element of M and if $r \in S$, $x \in W$ with $l'(rx) > l'(x)$, then $(rx)_A = r^\epsilon x_A$ where ϵ denotes "$-$" if $r \in x \cdot A$ and "$+$" otherwise.*

Proof. If the x_A exist, then for any reduced expression $x = r_1 \dots r_n$, one has

$$x_A = r_1^{\epsilon_1} \dots r_n^{\epsilon_n} \tag{2}$$

where ϵ_i denotes "$-$" if $r_i \in r_{i+1} \dots r_n \cdot A$ (i.e. if $r_n \dots r_i \dots r_n \in A$) and denotes "$+$" otherwise. To prove the lemma, it will suffice to show the right hand side of (2) is independent of the choice of reduced expression for x. By 1.3(ii), it is enough to prove the right hand side of (2) does not change when the reduced expression for x is changed by a braid relation. By the definitions, the proof of this immediately reduces to the case when x is the longest element of a dihedral standard parabolic subgroup $W' = W_{r,s}$ of W, with $m = n_{r,s} < \infty$. By (1), $A \cap W' = N(w) \cap W'$ for some $w \in W'$ with $l'(w) = p$. If $p < m$, interchange r and s if necessary and suppose without loss of generality that $l'(rw) > l'(w)$. Then the equality of the expression (2) for the two reduced expressions of x is just the mixed braid relation (ii) for a sequence $\epsilon_1, \dots, \epsilon_m$ of p "$-$" signs followed by $m - p$ "$+$" signs.

9.2. Obviously, if $r^+ = r^-$ for all $r \in S$, one just has the monoid lemma. We now discuss another example. The group B generated by elements $\dot{r}$ for $r \in S$, subject only to the braid relations of (W, S), is called the braid group of W. There is a natural surjective homomorphism $B \to W$, with $\dot{r} \mapsto r$ for all $r \in S$.

Lemma. *Suppose that G is a group and that there are elements $\dot{r} \in G$, for $r \in R$, satisfying the braid relations for W (e.g. $G = B$, the braid group of W). Then the pairs $x_r = (\dot{r}, \dot{r}^{-1}) \in G \times G$ satisfy the mixed braid relations for W.*

Proof. Each of the mixed braid relations for (x_r, x_s), for $r \neq s$ in S, is equivalent to the usual braid relation on $\dot{r}$ and $\dot{s}$, in this case.

9.3. In this subsection, we discuss the relevance of 9.2 to construction of the elements m_w^A in the Q_W-module $\tilde{Q}_W$ from Section (ii), in the situation 2.2(ii). In this case, the invertible elements t_r of Q_W, for $r \in S$, satisfy the braid relations. Acting on the left on a completely arbitrary element $h \in \tilde{Q}_W$ by the elements in Q_W whose existence is given by 9.1, one gets elements $\tilde{m}_w \in \tilde{Q}_W$ for $w \in W$, with $m_e = h$, satisfying

$$t_r \tilde{m}_w = \begin{cases} \tilde{m}_{rw} & \text{if } rw >_A w \\ X^2 \tilde{m}_{rw} & \text{otherwise.} \end{cases}$$

Write $m_w = \sum_{x \in W} \delta_x x^{-1}(s_{x,w}^A)$ with $s_{x,w}^A \in Q$ (this is a possibly infinite formal sum, with no conditions on supports). The $s_{x,w}^A$ satisfy the same recurrence equation 2.5(5) as $S_{x,w}^A$, and the initial values $s_{x,e}^A$ for $x \in W$ could be arbitrarily prescribed. Hence if $s_{e,e}^A = 1$ and $s_{x,w}^A = 0$ for $x \not\leq_A w$, it would follow follow that $S_{x,w}^A = s_{x,w}^A$ for any $[x, w] \in \mathcal{P}_A$. It may be conjectured that there exists a choice of initial values $s_{x,e}^A$ for $x \in W$ (i.e. effectively, a choice of the element $h \in \tilde{Q}^W$) so that $s_{e,e}^A = 1$ and $s_{x,w}^A = 0$ for $x \not\leq_A w$. If the whole of W is spherical in the order $\leq_A$, the conjecture is true (for unique elements $s_{x,e}^A$). In general there may be many choices of the $s_{x,e}^A$ to satisfy the conjecture, and it seems unlikely that there is any "natural" choice in general (an interesting question, however, is whether there is such a natural choice in the special case of the standard order on the alcoves of an affine Weyl group, see [29]). Although the need for proof of this conjecture has been circumvented in this paper, there are closely analogous situations in the sequel where corresponding difficulties remain unresolved except in special cases.

9.4. The remainder of this subsection describes some additional matters related to 9.1, none of which are used anywhere else in this paper. Suppose $G = B$ is the braid group of W in Lemma 9.2. For $A \subseteq T$ satisfying 9.1(i) and for any reduced expression $y = r_1 \dots r_n$, one has

$$y_A = \dot{r}_1^{\epsilon_1} \dots \dot{r}_n^{\epsilon_n} \tag{1}$$

where ϵ_i denotes -1 if $r_n \dots r_i \dots r_n \in A$ and denotes $+1$ otherwise. It can be shown from (1) (cf. [9]) that

(2) if $v, w \in W$ and A, C are two subsets of T satisfying 9.1(i), then $v_A = w_C$ iff $v = w$ and $N(v^{-1}) \cap A = N(w^{-1}) \cap C$.

Let us write $\dot{y} := y_\emptyset = \dot{r}_1 \dots \dot{r}_n$. From the definitions, one sees

$$(xy^{-1})_{N(y)} = \dot{x}\dot{y}^{-1} \text{ for } y \in W. \tag{3}$$

Now any finite $A \subseteq T$ satisfying 9.1(i) is of the form $A = N(y)$ for some $y \in W$. Hence if W is finite, (3) determines the w_A (for all such A satisfying 9.1(1) and all $w \in W$) in terms of the usual cross-section $\{\dot{w}\}_{w \in W}$ of W in the braid group.

Remark. If W is a finite Weyl group, combining (2) and (3) gives another formulation of the necessary and sufficient condition $\dot{x}\dot{y}^{-1} = \dot{z}\dot{w}^{-1}$ for isomorphism of certain principal series modules $I(x, y)$ and $I(z, w)$ for an associated semisimple complex Lie group (see [9]).

9.5. Next, we apply 9.1 to the the Iwahori-Hecke algebra $\mathcal{H}$ of (W, S) over the ring $\mathbb{Z}[v, v^{-1}]$ of integral Laurent polynomials in an indeterminate v, to give the following minor refinement (and alternative proof) of [16, (4.1)].

Proposition. *For any initial section $A \subseteq T$ of a reflection order , there exists a $\mathbb{Z}[v, v^{-1}]$-basis $\{\tilde{T}_{w,A}\}_{w \in W}$ of $\mathcal{H}$ with $\tilde{T}_{e,A} = T_e$, such that for $r \in S$ and $w \in W$,*

$$\tilde{T}_r \tilde{T}_{w,A} = \begin{cases} \tilde{T}_{rw,A} & \text{if } rw >_A w \\ \tilde{T}_{rw,A} + (v - v^{-1})\tilde{T}_{w,A} & \text{otherwise.} \end{cases}$$

Proof. Take $x_r = (v^{-1}T_r, vT_r^{-1})$ (elements of the unit group of $\mathcal{H}$) in 9.2. These elements satisfy the mixed braid relations so we obtain elements $x_A \in \mathcal{H}$ for $x \in W$ which we denote by $\tilde{T}_{x,A} = x_A$. These satisfy the multiplication formula in the proposition. Moreover, $\tilde{T}_w := \tilde{T}_{w,\emptyset} = v^{-l'(w)} T_w$ where the T_w are the standard basis elements of $\mathcal{H}$ over $\mathbb{Z}[v, v^{-1}]$. From 9.4(1), it follows that $T_{w,A} \in \tilde{T}_w + \sum_{l'(v) < l'(w)} \mathbb{Z}[v, v^{-1}]\tilde{T}_v$, so the $\tilde{T}_{w,A}$ form a basis of $\mathcal{H}$ as well.

9.6. Finally, we mention one other general class of elements satisfying mixed braid relations.

Proposition. *If Δ_r, for $r \in S$, are idempotent elements satisfying the braid relations in a unital ring R, then the pairs $x_r = (\Delta_r, 1 - \Delta_r)$ satisfy the mixed braid relations in the multiplicative monoid underlying R.*

Proof. In the Iwahori-Hecke algebra $\mathcal{H}$ of W over $\mathbb{Z}[v]$, the elements $x_r = (T_r, T_r - (v^2 - 1))$ satisfy the mixed braid relations for W, since (using 9.2) the pairs $(T_r, v^2 T_r^{-1})$ satisfy the mixed braid relations in $\mathcal{H} \otimes_{\mathbb{Z}[v]} \mathbb{Z}[v, v^{-1}]$. The result follows on applying the ring homomorphism $\mathcal{H} \to R$ determined by $v \mapsto 1$ and $T_r \mapsto -\Delta_r$ for $r \in S$.

10. Appendix: Polynomial invariants of pseudoreflection groups

In this section we obtain some results in the invariant theory of finite pseudoreflection groups (rederived for Coxeter groups in Section 4) as specializations of general facts from commutative algebra. As general references for the (well known) facts quoted in this section, one has [3] or [7]. The section finishes with some observations suggesting that some of the questions from Section 8 studied in this paper for Coxeter groups may also be of interest more generally e.g. for finite complex pseudoreflection groups.

10.1. Let K be a field of characteristic $p \geq 0$, and V be a finite-dimensional vector space over K. An element $g \in \mathrm{GL}(V)$ is said to be a pseudoreflection if the linear map $g - \mathrm{Id}_V \colon V \to V$ has one-dimensional image. Let $\mathcal{S}$ denote the symmetric algebra of V over K, given the grading $\mathcal{S} = \oplus_{n \in \mathbb{N}} \mathcal{S}_n$ with $\mathcal{S}_0 = K$ and $\mathcal{S}_2 = V$. Thus, $\mathcal{S}$ is non-canonically isomorphic to the graded polynomial ring in $\dim V$ indeterminates of degree 2. For any subgroup G of $\mathrm{GL}(V)$, , the G-action on V extends uniquely to an action of G as a group of graded K-algebra automorphisms of $\mathcal{S}$. Let

$$\mathcal{S}^G := \{\, f \in \mathcal{S} \mid g(f) = f \text{ for all } g \in G \,\} \tag{1}$$

denote the subalgebra of G-invariants of $\mathcal{S}$. Recall the following well-known theorem of Shephard and Todd, Chevalley and Serre.

Theorem. *Suppose that G is a finite pseudoreflection group i.e. G is finite and generated by pseudoreflections. Then the following conditions* (i)–(ii) *are equivalent, and hold in coprime characteristic (i.e. if either $p = 0$, or $p > 0$ and the order of G is coprime to p).*
(i) *$\mathcal{S}$ is graded free of finite rank as a module over $\mathcal{S}^G$*
(ii) *$\mathcal{S}^G$ is a graded polynomial ring in $n = \dim V$ indeterminates i.e. it is generated over K by n algebraically independent homogeneous elements $x_1, \ldots, x_n$ of positive degree.*

Let n_G denote the number of pseudoreflections in G. It is also known that in coprime characteristic,

$$2n_G = \sum_{i=1}^{n} \Big(\deg(x_i) - 2\Big), \qquad \mathrm{rank}_{\mathcal{S}^G}(\mathcal{S}) = \sharp(G) = \prod_{i=1}^{n} \Big(\deg(x_i)/2\Big). \tag{2}$$

10.2. In general, let $B = \oplus_{n \in \mathbb{N}} B_n$ be a positively graded algebra over a field K, with $B_0 = K$. Assume that B is generated as a K-algebra by finitely many homogeneous elements of positive degree. Let n denote the Krull dimension of B i.e. the maximum number of elements of B which are algebraically independent over K. We use below the following general fact.

Proposition. *In the above situation, the following two conditions are equivalent:*
(a) *there exist homogeneous elements $\theta_1, \dots, \theta_n$ of B generating a polynomial subring $K[\theta_1, \dots, \theta_n]$ of B over which B is a finitely generated free module*
(b) *whenever $\theta_1, \dots, \theta_n$ are homogeneous elements of B generating a graded polynomial subalgebra $K[\theta_1, \dots, \theta_n]$ of B over which B is a finitely generated module, B is actually a free $K[\theta_1, \dots, \theta_n]$-module.*

If (a)–(b) *hold, B is called a graded Cohen-Macaulay K-algebra. There is then a graded B-module $\omega(B)$ (the "canonical module for B") such that for any graded polynomial subring $C = K[\theta_1, \dots, \theta_n]$ of B over which B is a finitely-generated module, there is an isomorphism*

$$\omega(B) \cong \mathrm{Hom}_C(B, C) \ \langle \sum_{i=1}^{n} (\deg(\theta_i) - 2) \rangle$$

of graded B-modules.

10.3. Now we have the following.

Proposition. *Let G be a finite pseudoreflection group acting faithfully on the K vector space V in coprime characteristic, and let H be any subgroup of G which is also generated by reflections. Then*
(a) *$\mathcal{S}^H$ is a graded free $\mathcal{S}^G$-module of rank $[G : H]$*
(b) $\mathrm{Hom}_{\mathcal{S}^G}(\mathcal{S}^H, \mathcal{S}^G) \cong \mathcal{S}^H \langle -2(n_G - n_H) \rangle$ *as graded $\mathcal{S}^H$-modules.*

Proof. First, $B = \mathcal{S}^H$ is a graded polynomial ring $B = K[y_1, \dots, y_n]$ by 10.1; in particular, it is Cohen-Macaulay by 10.2(a). Of course, $A = \mathcal{S}^G$ is also a graded polynomial ring. Recalling 10.1(2), (a) is now immediate from 10.2(b), and (b) follows on applying the last statement in 10.2 with $C = B$ and $C = A$ in turn.

10.4. For a finite subgroup G of $\mathrm{GL}(V)$, define the element $J_G \in \mathcal{S}$ as follows. For each pseudoreflection $g \in G$, choose $\phi_g \in V^*$ and $a_g \in V$ such that $g(v) = v + \langle \phi_g, v \rangle a_g$ for all $v \in V$. Then define J_G as the product of the elements a_g for all pseudoreflections g in G. We can now give the following more explicit version of 10.3(b) (with a different proof).

Proposition. *Assume G is a finite pseudoreflection group on V in coprime characteristic, and let H be a subgroup of G generated by pseudoreflections. Then the map*

$$\theta \colon \mathcal{S}^H \to \mathrm{Hom}_{\mathcal{S}^G}(\mathcal{S}^H, \mathcal{S}^G) \langle 2(n_G - n_H) \rangle$$

with $\theta(b)(b') = J_G^{-1} \sum_{g \in G/H} \det(g) g(b J_H b')$ (where the sum is over a set of coset representatives of H in G) is an isomorphism of graded $\mathcal{S}^H$-modules.

Proof. In general, let B/A be a finite extension of normal domains (so B is finitely generated, in particular integral, as an A-module) with quotient fields F/E. For example, if B is a normal domain, finitely generated as algebra over a Noetherian ring R, one could take $A = B^G$ for any finite group G of R-algebra automorphisms of B, in which case F/E is Galois with Galois group G. In general, we denote by $\mathrm{Tr}_{F/E}$ or $\mathrm{Tr}_{B/A}$ the trace map $F \to E$ for the (finite) field extension F/E.

Define as usual the inverse different $\mathcal{D}_{B/A}^{-1}$ and different $\mathcal{D}_{B/A}$ by

$$(1) \quad \mathcal{D}_{B/A}^{-1} := \{\, f \in F \mid \mathrm{Tr}_{F/E}(fB) \subseteq A \,\}, \qquad \mathcal{D}_{B/A} := \{\, f \in F \mid f\mathcal{D}_{B/A}^{-1} \subseteq B \,\}.$$

Note that any A-module homomorphism $B \to A$ is the restriction of a unique E-linear map $F \to E$; since the trace form of F/E is non-degenerate if F/E is separable,

(2) If F/E is separable, the map $\psi\colon \mathcal{D}_{B/A}^{-1} \to \mathrm{Hom}_A(B, A)$ given by $\psi(f)(b) = \mathrm{Tr}_{F/E}(bf)$ is an isomorphism of B-modules.

The different and inverse different are known to be divisorial ideals of K; in particular,

(3) if B is a unique factorization domain (UFD), one has $\mathcal{D}_{B/A}^{-1} = d^{-1}B$ for some $0 \neq d \in B$.

We apply below the following special case of the general transitivity property for the inverse different,

(4) If $C/B/A$ are finite extensions of Noetherian UFD's, with corresponding quotient fields $L/F/E$, and L/E is separable, then $\mathcal{D}_{C/A}^{-1} = \mathcal{D}_{C/B}^{-1}\mathcal{D}_{B/A}^{-1}$.

(if we did not have UFD's, the right hand side would be replaced by its divisorialization).

Now return to the situation of the proposition. One has from [3, 7] that

$$(5) \qquad \mathcal{D}_{\mathcal{S}/\mathcal{S}^G} = J_G\mathcal{S} = \{\, b \in \mathcal{S} \mid g(b) = \det(g^{-1})b \text{ for all } g \in G \,\}$$

and of course the analogous statements for H. By (3)–(4) and 10.1,

$$(6) \qquad \mathcal{D}_{\mathcal{S}^H/\mathcal{S}^G}^{-1} = (J_G/J_H)^{-1}\mathcal{S}^H.$$

Hence multiplication by $d := (J_G/J_H)^{-1}$ gives an isomorphism of $\mathcal{S}^H$-modules $\psi'\colon \mathcal{S}^H \to \mathcal{D}_{\mathcal{S}^H/\mathcal{S}^G}^{-1}$. We claim that as a map of ungraded S^H-modules, $\theta = \psi \circ \psi'$ where $\psi\colon \mathcal{D}_{\mathcal{S}^H/\mathcal{S}^G}^{-1} \to \mathrm{Hom}_{\mathcal{S}^G}(\mathcal{S}^H, \mathcal{S}^G)$ is the isomorphism given by (2). For let $b, b' \in \mathcal{S}^H$. Using (1) for both H and G, one has

$$(\psi \circ \psi')(b)(b') = \mathrm{Tr}_{\mathcal{S}^H/\mathcal{S}^G}(bdb') = \frac{1}{\sharp(H)} \sum_{g \in G} g(bdb') = \frac{1}{J_G} \sum_{g \in G/H} \det(g)g(bJ_Hb').$$

It now follows that θ is an isomorphism of ungraded modules; as a map from the graded module $\mathcal{S}^H$ to $\mathrm{Hom}_{\mathcal{S}^G}(\mathcal{S}^H, \mathcal{S}^G)\langle 2(n_G - n_H)\rangle$ it is homogeneous of degree zero, hence a graded isomorphism.

10.5. Finally, we make some suggestive observations concerning particular cases of some questions in Section 8.

Let V and V' be finite-dimensional vector spaces over a field k and let $\mathcal{S}$, $\mathcal{S}'$ be their respective symmetric algebras graded as usual in this paper. For $\Gamma \subseteq$

$\mathrm{Hom}_k(V, V')$ and for $x \in \Gamma$, denote by $\sigma_x: \mathcal{S} \to \mathcal{S}'$ the graded $\mathbb{R}$-algebra homorphism restricting to x as a map $V \to V'$ on the homogeneous components of degree 2. Let $U = \mathcal{S}$, $B = \mathcal{S}'$, $R = \mathbb{R}$ and $\sigma := \{\sigma_x\}_{x\in\Gamma}$. Define B_σ and the equivalence realtions $\cong_{\mathfrak{a}}$ as in 8.1-8.5. We claim that the following statements hold;

(1) For a prime element p of $\mathcal{S}'$ and $x, y \in \Gamma$, one has $x \cong_{p\mathcal{S}'} y$ iff $p \in V'$ and $(x - y)(V) \subseteq kp$.

(2) Suppose that Γ is finite and there is $0 \neq \alpha \in V' \subseteq \mathcal{S}'$ such that $x \cong_{\alpha\mathcal{S}'} y$ for all $x, y \in \Gamma$. Then the condition in 8.4(1) holds, and hence so do 8.3(i)–(ii).

(3) Suppose that $V = V'$ (so $\mathcal{S} = \mathcal{S}'$) and that Γ is a subgroup G of $\mathrm{GL}(V)$. Let N be the (normal) subgroup of G generated by the pseudoreflections in G. Let $\sigma' := \{\sigma_x\}_{x\in N}$; one may naturally regard $B_{\sigma'}$ as a subring of B_σ. Then as right $\mathcal{S}$-modules, $B_\sigma = \oplus_{g\in N\backslash G} B_{\sigma'}\sigma_g$ where the sum is over a set of coset representatives for N in G; in particular, B_σ is a free left $B_{\sigma'}$-module and 8.3(i) holds for the family σ iff it holds for σ'.

Indeed, (1) is obvious (cf. the start of the proof of 5.1), and (3) follows by (1) and 8.2. The remaining fact (2) is not used in any essential way below, and its proof is left to the reader. In the situation (3), the results (1), 8.2 and (2) imply that 8.3(i) holds "after localization at any height one prime ideal of $\mathcal{S}$" for any finite G in any characteristic. I don't know for which G 8.3(i) itself holds, except for the following very special case.

Proposition. *Make the assumptions as in* (3) *above, and also assume that G is finite and $K = \mathbb{R}$. Then* 8.3(i) *holds.*

Proof. By (3), one may assume without loss of generality that $G = N$. So by the classification of finite real reflection groups, G is a finite Coxeter group in a reflection representation on V. One may choose a system of simple roots for G; the fundamental chamber on V is automatically sufficiently large, and so the Proposition follows from [26] (or 5.2, 2.12 and 3.7(a) of this paper).

Remark. It would be interesting to know, in the situation (3) with $K = \mathbb{C}$, if 8.3(i) and 8.5(i) hold for G, H finite complex pseudoreflection groups on V.

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