FORKING AND DIVIDING WITH FREE AMALGAMATION

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In this note, we show that forking and dividing are the same for complete types in *free amalgamation theories* with disintegrated algebraic closure. The canonical examples of such theories come from Fraïssé limits of free amalgamation classes in finite relational languages, and also the universal, existentially closed (K_n+K_3)-free graph in which one has free amalgamation over algebraically closed bases.

Let T be a complete first-order theory with monster model $\mathbb{M}$. We say that a small subset $A \subset \mathbb{M}$ is *closed* if $\operatorname{acl}(A) = A$. We use singleton letters $a, b, c, \ldots$ to denote tuples from $\mathbb{M}$ (which may be infinite). The following definition is from [4].

Definition 1. T is a **free amalgamation theory** if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$, defined on small subsets of $\mathbb{M}$, satisfying the following axioms.

(*i*) *(invariance)* For all A, B, C, if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $\sigma \in \operatorname{Aut}(\mathbb{M})$ then $\sigma(A) \mathop{\smile\!\!\!\!\!\!\mid}_{\sigma(C)} \sigma(B)$.

(*ii*) *(monotonicity)* For all A, B, C, if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $A_0 \subseteq A$, and $B_0 \subseteq B$, then $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$.

(*iii*) *(symmetry)* For all A, B, C, if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ then $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$.

(*iv*) *(full transitivity)* For all A and $D \subseteq C \subseteq B$, $A \mathop{\smile\!\!\!\!\!\!\mid}_D B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_D C$.

(*v*) *(full existence)* For all $B, C \subset \mathbb{M}$ and tuples $a \in \mathbb{M}$, if C is closed then there is $a' \equiv_C a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

(*vi*) *(stationarity)* For all closed $C \subset \mathbb{M}$ and closed tuples $a, a', b \in \mathbb{M}$, with $C \subseteq a \cap b$, if $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$, $a' \mathop{\smile\!\!\!\!\!\!\mid}_C b$, and $a' \equiv_C a$, then $ab \equiv_C a'b$.

(*vii*) *(freedom)* For all A, B, C, D, if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $C \cap AB \subseteq D \subseteq C$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_D B$.

(*viii*) *(closure)* For all closed A, B, C, if $C \subseteq A \cap B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ then AB is closed.

We will ultimately focus on the case when T has *disintegrated algebraic closure*, which is to say that the algebraic closure of any set $A \subset \mathbb{M}$ is the union of the algebraic closures of singleton elements in A. This is equivalent to the property that AB is closed for any closed $A, B \subset \mathbb{M}$. The following are the main motivational examples of free amalgamation theories with disintegrated algebraic closure.

Example 2.

(1) Let $\mathcal{L}$ be a finite relational language and let $\mathcal{K}$ be a Fraïssé class of finite $\mathcal{L}$-structures, which is closed under free amalgamation of $\mathcal{L}$-structures. Let T be the complete theory of the Fraïssé limit of $\mathcal{K}$. Then T is a free amalgamation theory, and $\operatorname{acl}(A) = A$ for any $A \subset \mathbb{M}$.

(2) Let $\mathcal{L}$ be the language of graphs and fix $n \geq 3$. There is a unique (up to isomorphism) countable, universal, and existentially complete $(K_n + K_3)$-free graph (where $K_n + K_3$ denotes the free amalgamation of K_n and K_3 over a single vertex). If T is the complete theory of this graph, then T is a free amalgamation theory with disintegrated algebraic closure.

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In the above examples, the desired ternary relation $\perp$ is free amalgamation of relational structures: given $A, B, C \subset \mathbb{M}$, $A \perp_C B$ if and only if, for any relation $R \in \mathcal{L}$ and tuple $x \in ABC$ (of appropriate length), if $R(x)$ holds then $x \in AC$ or $x \in BC$. The verification of the axioms of Definition 1 for these examples is sketched in [4]. In the first case, all axioms are immediate from classical Fraïssé theory (see, e.g., [5]). For the second case, the axioms rely on work of Cherlin, Shelah, and Shi [2], and Patel [6].

Let $\perp^f$ and $\perp^d$ denote the ternary relations on $\mathbb{M}$ given by nonforking independence and nondividing independence, respectively, for complete types. The following is our main result.

Theorem 3. *Let T be a free amalgamation theory with disintegrated algebraic closure. Given $A, B, C \subset \mathbb{M}$, we have $A \perp^f_C B$ if and only if $A \perp^d_C B$.*

In [3], the above result is shown for the special case that T is the theory of the generic K_n-free graph, for $n \geq 3$. Our proof of Theorem 3 generalizes the strategy from [3]. In particular, the main tool is the following "mixed transitivity" lemma.

Lemma 4. *Let T be a free amalgamation theory with disintegrated algebraic closure. Suppose $A, B, C, D \subset \mathbb{M}$ are such that $D \subseteq C \subseteq B$ and C, D are closed. Then*

$$A \perp^d_D C \text{ and } \operatorname{acl}(AC) \perp_C B \quad \Rightarrow \quad A \perp^d_D B.$$

Proof. Assume $A \perp^d_D C$ and $\operatorname{acl}(AC) \perp_C B$. Enumerate $B = b = (b_i : i \in I)$. Assume $I_0 \subseteq J$ are initial segments of I such that $D = (b_i : i \in I_0)$ and $C = c = (b_i : i \in J)$. Let $(b^n)_{n<\omega}$ be a D-indiscernible sequence, with $b^0 = \bar{b}$. Let a enumerate A. We want to find a' such that $a'b^n \equiv_D ab$ for all $n < \omega$.

For $n < \omega$, let $c^n = (b^n_i : i \in J)$, and note that $(c^n)_{n<\omega}$ is D-indiscernible with $c^0 = c$. There is some I_1 such that $I_0 \subseteq I_1 \subseteq J$ and, for all $m \leq n < \omega$ and $i \in J$, $c^m_i = c^n_i$ if and only if $i \in I_1$. If $D' = (b_i : i \in I_0)$, then D' is closed, $D \subseteq D' \subseteq C$ and so, by base monotonicity for $\perp^d$, we have $A \perp^d_{D'} C$. Note also that $(b^n)_{n<\omega}$ and $(c^n)_{n<\omega}$ are each D'-indiscernible. Altogether, we may assume without loss of generality that $I_1 = I_0$ and $D' = D$. Since $A \perp^d_D C$, there is a_* such that $a_* c^n \equiv_D ac$ for all $n < \omega$.

Set $C_* = \operatorname{acl}(c^{<\omega})$. By full existence for $\perp$, there is $a' \equiv_{C_*} a_*$ such that $\operatorname{acl}(a'C_*) \perp_{C_*} b^{<\omega}$. For each $n < \omega$, we have $a'c^n \equiv_D a_* c^n \equiv_D ac$. By monotonicity, $\operatorname{acl}(a'c^n) \perp_{C_*} b^n$ for all $n < \omega$.

Claim: For any $n < \omega$, $C^* \cap \operatorname{acl}(a'c^n)b^n = c^n$.

Proof: Since algebraic closure in T is disintegrated, we have $\operatorname{acl}(a'c^n) = \operatorname{acl}(a')c^n$ and $C^* = c^{<\omega}$. So it suffices to show $c^{<\omega} \cap \operatorname{acl}(a')b^n = c^n$. Fix some $x \in c^{<\omega} \cap \operatorname{acl}(a')b^n$. There is $m < \omega$ and $i \in J$ such that $x = b^m_i$. Suppose first that $x \in \operatorname{acl}(a')$. Then $b^m_i \in \operatorname{acl}(a') \cap c^m$, which means $b_i \in \operatorname{acl}(a) \cap c$. Since $A \perp^d_D C$, we have $\operatorname{acl}(a) \cap c \subseteq D$, and so $i \in I_0$. Thus $b^m_i = b^n_i \in c^n$. Finally, suppose $x \in b^n$. There is $j \in I$ such that $b^m_i = b^n_j$. It follows that $b^m_i = b^n_i$ (if $m = n$ this is trivial, and if $m \neq n$ use $b^m_i = b^n_j$ and indiscernibility). So $x = b^n_i \in c^n$. $\dashv_{\text{claim}}$

To finish the proof, we show $a'b^n \equiv_D ab$ for all $n < \omega$. So fix $n < \omega$, and let $\sigma \in \operatorname{Aut}(\mathbb{M}/D)$ be such that $\sigma(b^n) = b$ (note that $\sigma(c^n) = c$). By the claim and freedom, we have $\operatorname{acl}(a'c^n) \perp_{c^n} b^n$. So $\operatorname{acl}(\sigma(a')c) \perp_c b$ by invariance, and since $\sigma(\operatorname{acl}(a'c^n)) = \operatorname{acl}(\sigma(a')c)$. Also, we have $\sigma(a')c \equiv_D a'c^n \equiv_D ac$, and so $\sigma(a')c \equiv_c ac$. Therefore $\operatorname{acl}(\sigma(a')c) \equiv_c \operatorname{acl}(ac)$. So we may fix tuples e and e'

such that $\operatorname{acl}(ac) = ace$, $\operatorname{acl}(\sigma(a')c) = \sigma(a')ce'$, and $ace \equiv_c \sigma(a')ce'$. We have $\sigma(a')ce' \mathop{\smile\hskip -0.9em ^| \ }_c b$ and, by assumption, $ace \mathop{\smile\hskip -0.9em ^| \ }_c b$. Since $c \subseteq ace \cap b$, we may apply stationarity to conclude $aceb \equiv_c \sigma(a')ce'b$. In particular, $a'b^n \equiv_D \sigma(a')b \equiv_D ab$. □

The proof of the main result now follows rather quickly.

Proof of Theorem 3. It suffices to show $\mathop{\smile\hskip -0.9em ^| \ }^d$ satisfies extension, i.e, if $A \mathop{\smile\hskip -0.9em ^| \ }^d_C B$ and $\hat{B} \supseteq B$ then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\hskip -0.9em ^| \ }^d_C \hat{B}$ (see [1]). Recall also that, given $A, B, C \subset \mathbb{M}$, we have $A \mathop{\smile\hskip -0.9em ^| \ }^d_C B$ if and only if $\operatorname{acl}(AC) \mathop{\smile\hskip -0.9em ^| \ }^d_{\operatorname{acl}(C)} \operatorname{acl}(BC)$ (again, see [1]). Altogether, to prove the result it suffices to fix closed $A, B, \hat{B}, C \subset \mathbb{M}$ such that $C \subseteq B \subseteq \hat{B}$ and $A \mathop{\smile\hskip -0.9em ^| \ }^d_C B$, and find $A' \equiv_B A$ such that $A' \mathop{\smile\hskip -0.9em ^| \ }^d_C \hat{B}$.

Let $\mathop{\smile\hskip -0.9em ^| \ }$ witness that T is a free amalgamation theory. By full existence there is A' such that $A' \equiv_B A$ and $\operatorname{acl}(A'B) \mathop{\smile\hskip -0.9em ^| \ }_B \hat{B}$. By invariance of $\mathop{\smile\hskip -0.9em ^| \ }^d$, we have $A' \mathop{\smile\hskip -0.9em ^| \ }^d_C B$. By Lemma 4, $A' \mathop{\smile\hskip -0.9em ^| \ }^d_C \hat{B}$, as desired. □

Remark 5. In the case that $\operatorname{acl}(A) = A$ for all $A \subset \mathbb{M}$ (e.g. Example 2(1)), the statement of Lemma 4 is equivalent to: if $D \subseteq C \subseteq B$ then

$$A \mathop{\smile\hskip -0.9em ^| \ }^d_D C \text{ and } A \mathop{\smile\hskip -0.9em ^| \ }_C B \quad\Rightarrow\quad A \mathop{\smile\hskip -0.9em ^| \ }^d_D B.$$

Since $\mathop{\smile\hskip -0.9em ^| \ }$ implies $\mathop{\smile\hskip -0.9em ^| \ }^d$ (see [4]), this can be seen as a weakening of transitivity for $\mathop{\smile\hskip -0.9em ^| \ }^d$. It is worth noting that many examples of such theories are not simple (e.g. the theory of the generic K_n-free graph for $n \geq 3$), and so transitivity fails for $\mathop{\smile\hskip -0.9em ^| \ }^d$ in such examples.

Remark 6. Given $n \geq 3$, let T_n be the theory of the generic K_n-free graph. In [3], $\mathop{\smile\hskip -0.9em ^| \ }^d$ is characterized for T_n by purely combinatorial properties of graphs. It would be interesting to give similar descriptions of $\mathop{\smile\hskip -0.9em ^| \ }^d$ for other theories in Example 2. It is also shown in [3] that forking and dividing are not the same for formulas in T_n. Thus Theorem 3 cannot be strengthened to formulas.

Question 7. Does Theorem 3 hold without the assumption of disintegrated algebraic closure, or under the weaker assumption that algebraic closure is modular?

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