Model theory of generalized Urysohn spaces

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Model Theoretic Motivation

Model theoretic classification theory uses combinatorial properties of first-order theories to find and characterize good structural behavior.

A recurring theme is that complicated or bad behavior is exemplified by either "order" or "randomness" (or both).

Theorem (Shelah)

An unstable theory has either the strict order property ("order") or the independence property ("randomness").

Significant structural results have been found for unstable theories *without the independence property* (unstable NIP theories).

For theories *without the strict order property*, progress has been limited to regions of fairly low complexity (e.g. simple theories).

However, there are many interesting examples in this region; in particular, homogeneous graphs and metric spaces.

The Rational Urysohn Space

Definition. The **rational Urysohn space**, $\mathcal{U}_{\mathcal{Q}}$, is the unique, countable, ultrahomogeneous, and universal metric space, with rational distances.

Explicitly: $\mathcal{U}_{\mathcal{Q}}$ is the Fraïssé limit of the class of finite metric spaces with rational distances.

Heuristic #1: In an appropriate relational language, the theory of $\mathcal{U}_{\mathcal{Q}}$ is as complicated as possible, without having the strict order property.

Definition. The **random graph** is the unique, countable, ultrahomogeneous, universal graph.

The random graph can be thought of as a simplified version of $\mathcal{U}_{\mathcal{Q}}$, in which only distances $\{0, 1, 2\}$ are allowed.

Heuristic #2: The theory of the random graph is the least complicated unstable theory without the strict order property.

Other Distance Sets

Fix a countable subset $S \subseteq \mathbb{R}^{\geq 0}$, with $0 \in S$.

Definition. An S**-Urysohn space** is a countable, ultrahomogeneous, and universal metric space with distances in S.

If an S-Urysohn space exists, then it is unique.

Theorem (Delhommé, Laflamme, Pouzet, Sauer)

The S-Urysohn space exists if and only if S satisfies the four-values condition.

Proposition (Sauer)

Assume S is closed under $u +_S v := \sup\{x \in S : x \leq u + v\}$. Then S satisfies the four-values condition if and only if $+_S$ is associative.

Distance monoids

Definition

(1) A structure $\mathcal{R} = (R, \oplus, \leq, 0)$ is a **distance monoid** if $(R, \oplus, 0)$ is a commutative monoid and $\leq$ is a total, translation-invariant order with least element 0.

(2) Let $\mathcal{R}$ be a countable distance monoid. The **$\mathcal{R}$-Urysohn space**, $\mathcal{U}_{\mathcal{R}}$, is the unique, countable, ultrahomogeneous $\mathcal{R}$-metric space.

Examples

(*i*) $\mathcal{Q} = (\mathbb{Q}^{\geq 0}, +, \leq, 0)$. $\mathcal{U}_{\mathcal{Q}}$ is the rational Urysohn space.

(*ii*) $\mathcal{S} = (S, +_S, \leq, 0)$ where $S \subseteq \mathbb{R}^{\geq 0}$ is countable, closed under $+_S$, and $+_S$ is associative.
If $S = \{0, 1, 2\}$, then $\mathcal{U}_{\mathcal{S}}$ is the random graph.

(*iii*) $\mathcal{R} = (R, \max, \leq, 0)$ where $(R, \leq, 0)$ is a countable linear order with least element 0. $\mathcal{U}_{\mathcal{R}}$ is an ultrametric Urysohn space.

Model theory of generalized Urysohn spaces

Fix a countable distance monoid $\mathcal{R} = (R, \oplus, \leq, 0)$.

Let $\text{Th}(\mathcal{U}_\mathcal{R})$ be the complete, first-order theory of $\mathcal{U}_\mathcal{R}$ in a relational language $\{d_r(x, y) : r \in R\}$, where $d_r(x, y)$ is interpreted "$d(x, y) \leq r$".

By compactness, saturated models of $\text{Th}(\mathcal{U}_\mathcal{R})$ may contain points with no interpretable "distance" in $\mathcal{R}$.

E.g. in models of $\text{Th}(\mathcal{U}_\mathcal{Q})$ there are points realizing $\{d(x, y) > 0\} \cup \{d(x, y) \leq r : r \in \mathbb{Q}^+\}$.

These new distances lead to interesting model theoretic phenomena:

(1) (Casanovas-Wagner) Infinitesimal distance in $\text{Th}(\mathcal{U}_\mathcal{Q})$ yields non-eliminable hyperimaginaries.

(2) (C.) Infinite distance in $\text{Th}(\mathcal{U}_\mathcal{Q})$ yields a strict independence relation for $(\text{Th}(\mathcal{U}_\mathcal{Q}))^{\text{eq}}$, which is distinct from thorn-forking.

New Distances

Goal: Construct a distance monoid extension $\mathcal{R}^*$ of $\mathcal{R}$ with the property that any model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is an $\mathcal{R}^*$-metric space.

Idea: Define distance monoid structure on space of quantifier-free 2-types consistent with $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

(1) Construct $(R^\epsilon, \leq, 0)$: if $r \in R$ is non-maximal and has no immediate successor in R, add an immediate successor r^+.

(2) Let $(R^*, \leq, 0)$ be the Dedekind-MacNeille completion of $(R^\epsilon, \leq, 0)$.

(3) Define $\alpha \oplus \beta = \inf\{r \oplus s : r, s \in R,\ \alpha \leq r,\ \beta \leq s\}$.

Proposition (C.)

For $\alpha \in R^*$, let $p_\alpha(x,y) = \{d(x,y) > r : r < \alpha\} \cup \{d(x,y) \leq r : \alpha \leq r\}$

(*a*) $\alpha \mapsto p_\alpha$ is a bijection from R^* to $S_2^{qf}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}}))$.

(*b*) $\alpha \oplus \beta = \sup\{\gamma \in R^* : p_\alpha(x,y) \cup p_\beta(y,z) \cup p_\gamma(x,z) \text{ is consistent}\}$.

Saturated Models and Quantifier Elimination

Theorem (C.)

Let $\mathcal{R}$ be a countable distance monoid.

(a) *Given $M \models \mathrm{Th}(\mathcal{U}_{\mathcal{R}})$, define $d_M : M^2 \longrightarrow R^*$ such that, for any $a, b \in M$, $d_M(a, b)$ is the unique $\alpha \in R^*$ such that $M \models p_\alpha(a, b)$. Then (M, d_M) is an $\mathcal{R}^*$-metric space.*

(b) *Any $\mathcal{R}^*$-metric space can be embedded some (sufficiently saturated) model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.*

(c) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *has quantifier elimination if and only if, for all $r \in R$, $x \mapsto x \oplus r$ is continuous from R^* to R^*.*

Examples with quantifier elimination

- $\mathcal{R}$ is finite
- $\mathcal{R} = (R, \max, \leq, 0)$ for a countable linear order $(R, \leq, 0)$
- $\mathcal{R}$ is the nonnegative part of a countable ordered abelian group

Neostability

Define the **archimedean complexity of** $\mathcal{R}$, $\text{arch}(\mathcal{R})$, to be the smallest integer n such that, for all $r_0 \leq r_1 \leq \ldots \leq r_n$ in R,

$$r_0 \oplus r_1 \oplus \ldots \oplus r_n = r_1 \oplus \ldots \oplus r_n.$$

If there is no such n, let $\text{arch}(\mathcal{R}) = \omega$.

Theorem (C.)

Assume $\text{Th}(\mathcal{U}_\mathcal{R})$ *has quantifier elimination.*

(a) $\text{Th}(\mathcal{U}_\mathcal{R})$ *does not have the strict order property.*

(b) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is stable if and only if* $\text{arch}(\mathcal{R}) \leq 1$,
i.e. $r_0 \oplus r_1 = \max\{r_0, r_1\}$ *for all* $r_0, r_1 \in R$, *so* $\mathcal{U}_\mathcal{R}$ *is an ultrametric space.*

(c) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is simple if and only if* $\text{arch}(\mathcal{R}) \leq 2$.

(d) *Given* $n \geq 3$, $\text{Th}(\mathcal{U}_\mathcal{R})$ *is* NSOP_n *if and only if* $\text{arch}(\mathcal{R}) < n$.

There are similar algebraic characterizations of superstability, supersimplicity, U-rank, weak elimination of imaginaries, etc....

Future Work

(1) Expansions of generalized Urysohn spaces, or generalized Urysohn spaces with "forbidden" subspaces.

(2) Dynamics of Isom($\mathcal{U}_{\mathcal{R}}$).

(3) Ramsey-type properties of $\mathcal{U}_{\mathcal{R}}$.

Future Work

(4) **Fact.** For any $S = \{0, s_1, \ldots, s_n\} \subseteq \mathbb{R}^{\geq 0}$, there is $S' = \{0, s'_1, \ldots, s'_n\} \subseteq \mathbb{N}$ with

$$(S, +_S, \leq, 0) \cong (S', +_{S'}, \leq, 0).$$

Conjecture. We may choose S' so that $2^k - 1 \leq s'_k \leq 2^n - 1$ for all $1 \leq k \leq n$.

(5) **Question.** Suppose $S \subseteq \mathbb{R}^{\geq 0}$ satisfies the four-values condition (i.e. the S-Urysohn space exists). Given a finite subset $A \subseteq S$, is there a *finite* subset $S' \subseteq S$ which contains A and satisfies the four-values condition?

thank you