

# Model theory of the Urysohn sphere

## part 1

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September 24, 2013  
UIC Logic Seminar  
(joint work with Caroline Terry)

# A Discrete Mathematician's Guide to Continuous Logic

- Structures are complete metric spaces.
- Formulas are uniformly continuous functions  $\varphi(\bar{x}) : M^n \rightarrow [0, 1]$  (a continuum of truth values).
- $\varphi(\bar{a}) = 0$  is “like”  $\varphi(\bar{a})$  being true.
- $\sup_x$  and  $\inf_x$  act as universal and existential quantifiers, respectively.
- We have notions of compactness, types, saturation, indiscernibility, etc...

# The $n$ -Strong Order Property

## Definition (Classical Logic)

Given  $n \geq 3$ , a theory has the  $n$ -strong order property,  $SOP_n$ , if there is a formula  $\varphi(x, y)$  and a sequence  $(a_i)_{i < \omega}$  such that

- (i)  $\mathbb{M} \models \varphi(a_i, a_j)$  for all  $i < j$ ,
- (ii)  $\{\varphi(x_1, x_2), \dots, \varphi(x_{n-1}, x_n), \varphi(x_n, x_1)\}$  is unsatisfiable.

## Definition (Continuous Logic)

Given  $n \geq 3$ , a theory has the  $n$ -strong order property,  $SOP_n$ , if there is a formula  $\varphi(x, y)$  and a sequence  $(a_i)_{i < \omega}$  such that

- (i)  $\varphi(a_i, a_j) = 0$  for all  $i < j$ ,
- (ii)  $\{\varphi(x_1, x_2) = 0, \dots, \varphi(x_{n-1}, x_n) = 0, \varphi(x_n, x_1) = 0\}$  is unsatisfiable, i.e.

$$\inf_{x_1, \dots, x_n} \max\{\varphi(x_1, x_2), \dots, \varphi(x_{n-1}, x_n), \varphi(x_n, x_1)\} > 0.$$

# Indiscernible Sequences

## Definition

Given an indiscernible sequence  $(a_i)_{i < \omega}$ , let  $p(x, y) = \text{tp}(a_0, a_1)$ . For  $n \geq 3$ ,  $(a_i)_{i < \omega}$  is  **$n$ -cyclic** if

$$p(x_1, x_2) \cup \dots \cup p(x_{n-1}, x_n) \cup p(x_n, x_1)$$

is satisfiable.

## Proposition

A theory is  $\text{NSOP}_n$  if and only if every indiscernible sequence is  $n$ -cyclic.

# All the Finite Metric Spaces!

The Urysohn sphere is the unique *countably universal and homogeneous separable metric space* (with distances bounded by 1).

- Every separable metric space (with distances bounded by 1) can be isometrically embedded into it.
- Every isometry between finite subspaces can be extended to an isometry of the whole space.

We can consider the Urysohn sphere as a metric structure in the empty language (containing only the distance function  $d : M^2 \rightarrow [0, 1]$ ).

The theory has quantifier elimination in this language and so types are completely determined by distances:

$$\begin{aligned} \text{tp}(\bar{a}/C) \equiv & \{d(x_i, x_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \\ & \cup \{d(x_i, c) = d(a_i, c) : 1 \leq i \leq n, c \in C\}. \end{aligned}$$

# SOP<sub>n</sub>

Let  $\mathbb{U}$  be a  $\kappa$ -saturated model of the theory of the Urysohn sphere, for some large  $\kappa$ .

## Theorem

$\text{Th}(\mathbb{U})$  has SOP<sub>n</sub> for all  $n \geq 3$ .

Given  $n \geq 3$ , we construct an indiscernible sequence that is not  $n$ -cyclic.

Construct  $(\bar{a}^k)_{k < \omega}$ , with  $l(\bar{a}^0) = n$ , such that given  $k < l < \omega$ ,

$$d(a_i^k, a_j^l) = \begin{cases} 2^{j-n} & \text{if } i \leq j \\ 2^{i-n} - 2^{j-n} & \text{if } i > j \end{cases},$$

and given  $k < \omega$ ,  $1 \leq i < j \leq n$ ,

$$d(a_i^k, a_j^k) = 2^{j-n}.$$

# Requisite Soul-Crushing Check of Triangles

In order to make sure  $(\bar{a}^k)_{k < \omega}$  actually exists in  $\mathbb{U}$ , we have to verify that the definition satisfies the triangle inequality.

So pick  $a_i^r, a_j^s, a_k^t$ , with  $r \leq s \leq t$  and set

$$d_{i,j} = 2^n d(a_i^r, a_j^s), \quad d_{i,k} = 2^n d(a_i^r, a_k^t), \quad \text{and} \quad d_{j,k} = 2^n d(a_j^s, a_k^t).$$

We want to show

- (a)  $d_{i,j} \leq d_{i,k} + d_{j,k}$ ,
- (b)  $d_{i,k} \leq d_{i,j} + d_{j,k}$ ,
- (c)  $d_{j,k} \leq d_{i,j} + d_{i,k}$ .

:(

There are six cases to consider.

(1)  $i \leq j \leq k$ ,

(4)  $j < i \leq k$  and  $r < s \leq t$ ,

(2)  $i \leq k < j$  and  $r \leq s < t$ ,

(5)  $j \leq k < i$  and  $r < s \leq t$ ,

(3)  $k < i \leq j$  and  $r \leq s < t$ ,

(6)  $k < j < i$  and  $r < s < t$ .

Case 3:  $d_{i,j} = 2^j$ ,  $d_{i,k} = 2^i - 2^k$ , and  $d_{j,k} = 2^j - 2^k$ .

Then  $d_{i,j} \leq d_{i,k} + d_{j,k}$  becomes

$$2^k + 2^k \leq 2^j.$$

This is true since  $k < j$ .



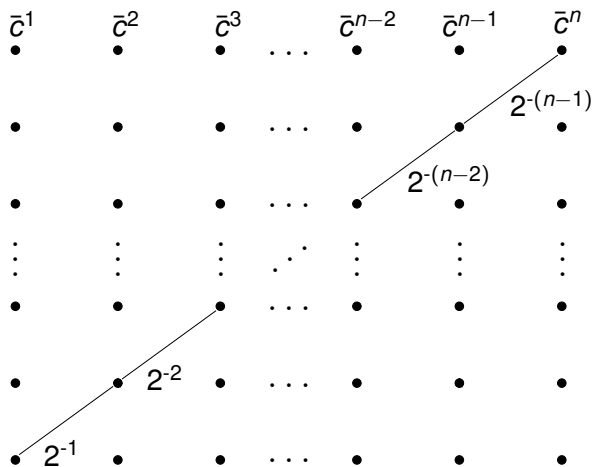
We have left to show that  $(\bar{a}^k)_{k < \omega}$  is not  $n$ -cyclic.

Suppose  $p(\bar{x}, \bar{y}) = \text{tp}(\bar{a}^0, \bar{a}^1)$  and  $(\bar{c}^1, \dots, \bar{c}^n)$  realizes

$$p(\bar{x}^1, \bar{x}^2) \cup \dots \cup p(\bar{x}^{n-1}, \bar{x}^n) \cup p(\bar{x}^n, \bar{x}^1).$$

Recall that for  $1 \leq i < n$ , we have

$$d(a_{i+1}^0, a_i^1) = 2^{i+1-n} - 2^{i-n} = 2^{i-n}.$$



$$d(c_n^1, c_1^n) \leq \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} = \frac{2^n - 2}{2^n} < 1$$

But  $d(a_1^0, a_n^1) = 2^{n-n} = 1$  so " $d(c_1^n, c_n^1) = 1$ "  $\in p(\bar{c}^n, \bar{c}^1)$ .

# Partial Semimetric Spaces

## Definition

Given a set  $X$ , a partial function  $f : \text{dom}(f) \subseteq X \times X \rightarrow [0, 1]$  is a **semimetric** if it is

- (i) reflexive, i.e., for all  $(x, y) \in \text{dom}(f)$ ,  $f(x, y) = 0$  if and only if  $x = y$ ,
- (ii) symmetric, i.e., for all  $x, y \in X$  if  $(x, y) \in \text{dom}(f)$  then  $(y, x) \in \text{dom}(f)$  and  $f(x, y) = f(y, x)$ .

We call  $(X, f)$  a **partial semimetric space**.  $(X, f)$  is **consistent** if  $f$  can be extended to a metric on  $X$ .

For the rest of the talk, we will only consider partial semimetrics  $f$  such that  $\Delta(X) \subseteq f$ .

## Take it to the max

Suppose  $(X, f)$  is partial semimetric space. Given  $x, y \in X$  we define  $D(x, y) = \{z \in X : (x, z), (z, y) \in \text{dom}(f)\}$  and set

$$f_{\max}(x, y) = \inf_{z \in D(x, y)} f(x, z) + f(z, y).$$

By convention, we set  $\inf \emptyset = 1$ .

One way to interpret this value is that if  $f$  were able to be extended to a metric on  $X$ , then  $f_{\max}(x, y)$  is an upper bound for  $f(x, y)$ .

However, even if  $f$  can be extended to a metric on  $X$ , we may still not be able to realize  $f(x, y) = f_{\max}(x, y)$ .

## $d_f$ Construction

Suppose  $(X, f)$  is a finite partial semimetric space. We extend  $f$  to a total semimetric  $d_f : X \times X \rightarrow [0, 1]$  via the following construction.

Build sequence of partial semimetrics  $f = f^0 \subsetneq f^1 \subsetneq \dots \subsetneq f^{k_f} = d_f$ .

To obtain  $f^{k+1}$  from  $f^k$ , we define

$$\gamma_{k+1} = \min_{(x,y) \notin \text{dom}(f^k)} f_{\max}^k(x, y).$$

Build  $f^{k+1}$  to include  $\{(x, y) \notin \text{dom}(f^k) : f_{\max}^k(x, y) = \gamma_{k+1}\}$ , and set  $f^{k+1}(x, y) = \gamma_{k+1}$ .

Eventually we reach some  $k_f$  where  $\text{dom}(f^{k_f}) = X \times X$  and we let  $d_f = f^{k_f}$ .

## Definition

Given a partial semimetric space  $(X, f)$  and  $n \in \mathbb{N}$ ,  $f$  is  **$n$ -transitive** if for all  $x_0, \dots, x_n \in X$ , with  $(x_0, x_1), \dots, (x_{n-1}, x_n)$ , and  $(x_0, x_n) \in \text{dom}(f)$ , we have

$$f(x_0, x_n) \leq f(x_0, x_1) + f(x_1, x_2) + \dots + f(x_{n-1}, x_n).$$

# Characterization of Consistent Finite Partial Semimetric Spaces

## Theorem

Let  $(X, f)$  be a finite partial semimetric space. The following are equivalent:

- (i)  $(X, f)$  is consistent;
- (ii)  $d_f$  is a metric on  $X$ ;
- (iii)  $f$  is  $n$ -transitive for all  $n \in \mathbb{N}$ ;
- (iv)  $f$  is  $2^{k_f+1}$ -transitive.

## Remarks

- (1) If  $0 < k < k_f$  then  $\gamma_k < \gamma_{k+1}$ , so  $k_f = |d_f((X \times X) \setminus \text{dom}(f))|$ .
- (2)  $(|X| - 1)$ -transitivity implies  $n$ -transitivity for all  $n$ .

# Proof of the Theorem

(ii)  $\Rightarrow$  (i): Trivial.

(i)  $\Rightarrow$  (iii): If  $(X, f)$  is consistent then there is a metric extending it, which must be  $n$ -transitive for all  $n$ . So  $f$  is  $n$ -transitive for all  $n$ .

(iii)  $\Rightarrow$  (iv): Trivial.

(iv)  $\Rightarrow$  (ii):

**Lemma:** If  $f^k$  is  $2^n$ -transitive for for some  $n \geq 1$  then  $f^{k+1}$  is  $2^{n-1}$ -transitive.

So if  $f = f^0$  is  $2^{k_f+1}$ -transitive then after repeated application of the Lemma we get that  $d_f = f^{k_f}$  is 2-transitive, i.e. satisfies the triangle inequality.



# Free Amalgamation of Finite Metric Spaces

If  $(X, f)$  is consistent then  $d_f$  can be regarded as the “most free” metric extending  $f$ .

## Proposition

Let  $(X, f)$  be a finite partial semimetric space. Then for any metric  $d$  on  $X$  extending  $f$ , we have  $d(x, y) \leq d_f(x, y)$  for all  $x, y \in X$ .

There is a notion of *free amalgamation of finite metric spaces*,  $X *_A Y$ , where  $A = X \cap Y$ . We let  $X$  and  $Y$  keep their own distances and, for  $x \in X \setminus A$  and  $y \in Y \setminus A$ , assign

$$d(x, y) = \min_{a \in A} d_X(x, a) + d_Y(a, y).$$

Considering  $(X \cup Y, f = d_X \cup d_Y)$  as a finite partial semimetric space,  $d_f$  is this metric on  $X *_A Y$ .

# The Strong Order Property

## Definition

A theory has the **strong order property**, SOP, if there is a formula  $\varphi(x, y)$  and a sequence  $(a_i)_{i < \omega}$  such that for all  $n \geq 2$ ,

- (i)  $\varphi(a_i, a_j) = 0$  for all  $i < j$ ,
- (ii)  $\{\varphi(x_1, x_2) = 0, \dots, \varphi(x_{n-1}, x_n) = 0, \varphi(x_n, x_1) = 0\}$  is unsatisfiable, i.e.

$$\inf_{x_1, \dots, x_n} \max\{\varphi(x_1, x_2), \dots, \varphi(x_{n-1}, x_n), \varphi(x_n, x_1)\} > 0.$$

## Proposition

If  $T$  has SOP then there is an indiscernible sequence that is not  $n$ -cyclic for any  $n \geq 2$ .

# SOP in the Urysohn Sphere

## Theorem

$\text{Th}(\mathbb{U})$  is NSOP.

## Theorem

Given  $n \geq 2$ , if  $(\bar{a}^k)_{k < \omega}$  is an indiscernible sequence in  $\mathbb{U}$ , with  $I(\bar{a}^0) = n - 1$ , then  $(\bar{a}^k)_{k < \omega}$  is  $n$ -cyclic.

Fix  $n \geq 2$  and an indiscernible sequence  $(\bar{a}^k)_{k < \omega}$ , with  $l(\bar{a}^0) = n - 1$ . We construct the partial semimetric space  $(C, f)$  as follows.

$$C = \{x_i^k : 1 \leq k \leq n, 1 \leq i < n\}$$

$$\begin{aligned} \text{dom}(f) = & \{(x_i^k, x_j^k) : 1 \leq i \leq j < n, 1 \leq k \leq n\} \\ & \cup \{(x_i^k, x_j^{k+1}) : 1 \leq i, j < n, 1 \leq k < n\} \\ & \cup \{(x_i^1, x_j^n) : 1 \leq i, j < n\} \end{aligned}$$

- (i)  $f(x_i^k, x_j^k) = d(a_i^0, a_j^0)$  for all  $1 \leq i \leq j < n, 1 \leq k \leq n$
- (ii)  $f(x_i^k, x_j^{k+1}) = d(a_i^0, a_j^1)$  for all  $1 \leq i, j < n, 1 \leq k < n$
- (iii)  $f(x_i^n, x_j^1) = d(a_i^0, a_j^1)$  for all  $1 \leq i, j < n$

Note that  $(\bar{a}^k)_{k < \omega}$  is  $n$ -cyclic if and only if  $(C, f)$  is consistent.

We show by induction that  $f$  is  $m$ -transitive for all  $m \geq 2$ .

The base case is 2-transitive, which is equivalent to saying that  $f$  satisfies the triangle inequality wherever it is defined.

The cases when  $n = 2$  or  $n = 3$  must be done separately.

We assume  $n \geq 4$ . Any triangle in  $\text{dom}(f)$  is either already in the indiscernible sequence, or uses  $(x_i^1, x_j^n)$  as a side. But if  $n \geq 4$  then the third point must be of the form  $x_k^1$  or  $x_k^n$ . Therefore the triangle is also in the indiscernible sequence:

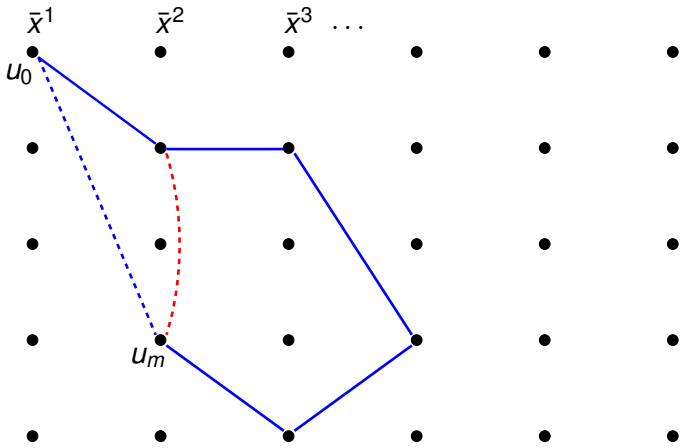
$$\{x_i^1, x_j^n, x_k^1\} \longrightarrow \{a_i^1, a_j^0, a_k^1\}$$

$$\{x_i^1, x_j^n, x_k^n\} \longrightarrow \{a_i^1, a_j^0, a_k^0\}$$

Assume  $f$  is  $m'$ -transitive for all  $2 \leq m' < m$ . Fix a sequence  $\bar{u} = (u_0, \dots, u_m)$  such that  $(u_0, u_1), \dots, (u_{m-1}, u_m)$  and  $(u_0, u_m)$  are in  $\text{dom}(f)$ . We want to show

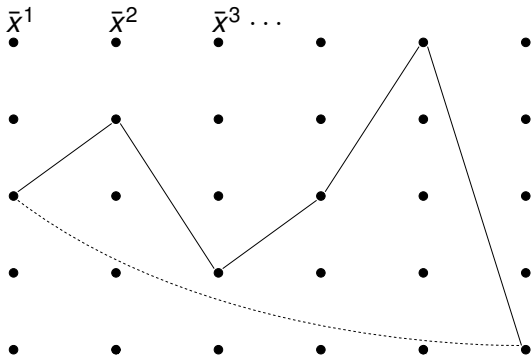
$$f(u_0, u_m) \leq f(u_0, u_1) + \dots + f(u_{m-1}, u_m).$$

Suppose  $u_t \in \bar{x}^{l_t}$ . We can use the induction hypothesis to reduce to the case where  $l_s \neq l_t$  for all  $s \neq t$ .



$\bar{u} = (u_0, \dots, u_m)$  and  $u_t \in \bar{x}^{l_t}$ . We have  $l_s \neq l_t$  for all  $s \neq t$ .

Using “rotational symmetry” of  $(C, f)$ , we may assume  $m = n - 1$  and  $\bar{u} = (x_{i_1}^1, \dots, x_{i_n}^n)$  for some  $i_1, \dots, i_n \in \{1, \dots, n - 1\}$ .



It follows that there are  $1 \leq s < t \leq n$  such that  $i_s = i_t$ .



For  $1 \leq i, j < n$ , let  $\epsilon_{i,j} = d(a_i^0, a_j^1)$ .

## Proposition

For any  $r \geq 1$  and  $j_0, \dots, j_r \in \{1, \dots, n-1\}$ ,

$$\epsilon_{j_0, j_r} \leq \epsilon_{j_0, j_1} + \epsilon_{j_1, j_2} + \dots + \epsilon_{j_{r-1}, j_r}.$$

## Proof.

$$\begin{aligned} \epsilon_{j_0, j_r} &= d(a_{j_0}^0, a_{j_r}^1) \\ &= d(a_{j_0}^0, a_{j_r}^r) \\ &\leq d(a_{j_0}^0, a_{j_1}^1) + d(a_{j_1}^1, a_{j_2}^2) + \dots + d(a_{j_{r-1}}^{r-1}, a_{j_r}^r) \\ &= \epsilon_{j_0, j_1} + \epsilon_{j_1, j_2} + \dots + \epsilon_{j_{r-1}, j_r}. \end{aligned}$$



Recall we have  $\bar{u} = (x_{i_1}^1, \dots, x_{i_n}^n)$  and we have  $1 \leq s < t \leq n$  such that  $i_s = i_t$ . We want to show

$$f(x_{i_1}^1, x_{i_n}^n) \leq f[\bar{u}] := f(x_{i_1}^1, x_{i_2}^2) + \dots + f(x_{i_{n-1}}^{n-1}, x_{i_n}^n).$$

Assume  $1 < s < t < n$  and define sequences

$$\bar{v} = (x_{i_1}^1, \dots, x_{i_s}^s)$$

$$\bar{w} = (x_{i_s}^s, \dots, x_{i_t}^t)$$

$$\bar{y} = (x_{i_t}^t, \dots, x_{i_n}^n)$$

Note that  $f[\bar{u}] = f[\bar{v}] + f[\bar{w}] + f[\bar{y}]$ .

$$\bar{v} = (x_{i_1}^1, \dots, x_{i_s}^s)$$

$$\bar{w} = (x_{i_s}^s, \dots, x_{i_t}^t)$$

$$\bar{y} = (x_{i_t}^t, \dots, x_{i_n}^n)$$

By indiscernibility and the previous proposition,

$$\begin{aligned} f(x_{i_1}^1, x_{i_n}^n) &= \epsilon_{i_n, i_1} \\ &= d(a_{i_n}^1, a_{i_1}^2) \\ &\leq d(a_{i_n}^1, a_{i_s}^0) + d(a_{i_s}^0, a_{i_s}^3) + d(a_{i_s}^3, a_{i_1}^2) \\ &= \epsilon_{i_s, i_n} + \epsilon_{i_s, i_s} + \epsilon_{i_1, i_s} \\ &= \epsilon_{i_t, i_n} + \epsilon_{i_s, i_t} + \epsilon_{i_1, i_s} \\ &\leq f[\bar{y}] + f[\bar{w}] + f[\bar{v}] \\ &= f[\bar{u}]. \end{aligned}$$

# The Urysohn Sphere in Classical Logic

## Theorem

$\text{Th}(\mathbb{U})$  has  $\text{SOP}_n$  for all  $n \geq 3$ , but is NSOP.

The theory of the Urysohn sphere can be considered in classical logic, either by using distance relations, or with a distance function and a sort for  $[0, 1]$ .

Quantifier elimination results lead to the same situation where complete types are determined by distances.

The previous arguments go through in this setting, so the above theorem can be stated for the classical Urysohn sphere also.

thank you