A list of statements/theorems that you should be able to prove, together with the main idea of the proof for some of them.

1. Let $A, A_k$ be elementary subsets of $[0, 1] \times [0, 1]$, such that

$$A \subseteq \bigcup_{k=1}^{\infty} A_k.$$ 

Then

$$\tilde{m}(A) \leq \sum_{k=1}^{\infty} \tilde{m}(A_k).$$

[Replace $A$ by a slightly smaller closed set, and enlarge each $A_k$ a bit to get open sets. Then use compactness of the new $A$ to reduce to a finite union.]

2. For every $A \subseteq [0, 1] \times [0, 1]$ we have

$$\mu^*(A) \leq \mu^*(A).$$

[Otherwise we would have $\mu^*(A) + \mu^*(E \setminus A) < 1$, where $E = [0, 1] \times [0, 1]$, and we could get too small a cover of $E$, contradicting the previous theorem.]

3. Suppose that $A, A_k \subseteq [0, 1] \times [0, 1]$, and $A \subseteq \bigcup_{k=1}^{\infty} A_k$. Then

$$\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k).$$

[By definition of $\mu^*$ each $A_k$ can be covered by rectangles whose areas sum to slightly more than $\mu^*(A_k)$. The collection of all these rectangles cover $A$, giving an upper bound on $\mu^*(A)$.]

4. If $A \subseteq [0, 1] \times [0, 1]$ is an elementary set, then $A$ is measurable and $\mu(A) = \tilde{m}(A)$.

[\(\tilde{m}(A) \leq \mu^*(A)\) follows from first theorem, and $\mu^*(A) \leq \tilde{m}(A)$ follows from writing $A$ as a disjoint union of rectangles. Apply the same also to the complement of $A$.]

5. The union, intersection, difference and symmetric difference of two measurable subsets of $[0, 1] \times [0, 1]$ is measurable.

[Use the fact that $A$ is measurable if and only if for every $\epsilon > 0$ there is an elementary set $B$ such that $\mu^*(A \Delta B) < \epsilon$.]

6. Suppose that $A_1, A_2$ are disjoint measurable subsets of $[0, 1] \times [0, 1]$. Then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

[Approximate $A_1$ and $A_2$ with elementary sets, and use the additivity of $\tilde{m}$. Note that the approximating sets may not be disjoint.]

7. The union of a countable collection of measurable sets is measurable.

[It’s enough to do this for disjoint unions. Approximate the countable union with a finite union, and approximate the finite union with an elementary set.]

8. If $A = \bigcup_k A_k$ is a disjoint union of a countable collection of measurable sets, then

$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k).$$

[We know that $\mu(A) \leq \sum_1^{\infty} \mu(A_k)$. For the converse inequality use finite additivity.]
9. Let \( f_n : X \to \mathbb{R} \) be measurable, such that the limit \( f(x) = \lim_{n \to \infty} f_n(x) \) exists for all \( x \). Then \( f \) is measurable. 
[Express \( f^{-1}(-\infty, c) \) in terms of sets of the form \( f_n^{-1}(-\infty, d) \) using countably many unions / intersections]

10. If \( f : X \to \mathbb{R} \) is measurable, and \( g : \mathbb{R} \to \mathbb{R} \) is continuous, then \( g \circ f \) is measurable.

11. A function \( f : X \to \mathbb{R} \) is measurable if and only if \( f \) is a uniform limit of simple functions. 
[For any \( k > 1 \) define a simple function \( g : X \to \mathbb{R} \) by letting \( g(x) = \frac{m}{k} \) if \( \frac{m}{k} \leq f(x) < \frac{m+1}{k} \) for an integer \( m \). Then \( g \) is simple, and \(|f - g| \leq \frac{1}{k}|\)]

12. Suppose \( f, g : X \to \mathbb{R} \) are measurable and \( c \in \mathbb{R} \). Then \( f + g, cf, fg, f/g \) are measurable if \( g \) is nowhere vanishing in the case of \( f/g \).

13. Let \( f, g : [0, 1] \to \mathbb{R} \) be continuous such that \( f(x) = g(x) \) for almost every \( x \) (with respect to Lebesgue measure). Then \( f(x) = g(x) \) for all \( x \).

14. (Egorov’s theorem) Let \( f_n : X \to \mathbb{R} \) be a sequence of measurable functions, converging almost everywhere to \( f : X \to \mathbb{R} \). For any \( \delta > 0 \) there exists a set \( Y \subset X \) such that \( \mu(X \setminus Y) < \delta \) and \( f_n \to f \) uniformly on \( Y \).
[For \( m, n > 0 \) let \( E_n^m \) be the set of \( x \) such that \( |f_i(x) - f(x)| < 1/m \) for all \( i > n \). For almost every \( x \) we have \( x \in \bigcup_n E_n^m \). Use this to show that there is an \( N_m \) such that \( \mu(E_n^{m,N_m}) > \mu(X) - 2^{-m}\delta \). Finally define \( Y = \bigcap_n E_n^{m,N_m} \).]

15. Suppose that \( \phi : A \to \mathbb{R} \) is integrable and \( f : A \to \mathbb{R} \) satisfies \(|f(x)| \leq \phi(x)\) for all \( x \in A \). Then \( f \) is integrable and 
\[
\left| \int_A f(x) \mu \right| \leq \int_A \phi(x) \mu.
\]
[First assume that \( f, \phi \) are simple functions. Then use approximation to extend to the general case.]

16. (Chebyshev’s inequality) If \( f : A \to \mathbb{R} \) is integrable and \( f(x) \geq 0 \) for all \( x \in A \), then
\[
\mu\{x; x \in A, f(x) \geq c\} \leq \frac{1}{c} \int_A f(x) d\mu,
\]
for all \( c > 0 \).
[Split the integral over \( A \) into two parts, over the sets where \( f \geq c \) and where \( f < c \).]

17. Let \( f : A \to \mathbb{R} \) be integrable. For any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that 
\[
\left| \int_E f(x) d\mu \right| \leq \epsilon,
\]
whenever \( E \subset A \) satisfies \( \mu(E) < \delta \). 
[First find an \( N \) such that the integral of \(|f|\) on the set where \(|f| > N\) is at most \( \epsilon/2 \). Then the integral of \(|f|\) on a set \( E \) will be at most \( \epsilon/2 + N\mu(E) \).]
18. (Bounded convergence theorem) Let $f_n \to f$ almost everywhere on $A$, and let $\phi : A \to \mathbb{R}$ be an integrable function such that $|f_n(x)| \leq \phi(x)$ for almost every $x \in A$. Then $f$ is integrable, and

$$\int_A f(x) \, d\mu = \lim_{n \to \infty} \int_A f_n(x) \, d\mu.$$  

[By Egorov’s theorem $f_n \to f$ uniformly outside of a small set $C \subset A$. If $C$ has sufficiently small measure, then we can make the integral of $\phi$ on $C$ as small as needed.]

19. (Monotone convergence theorem) Suppose that $f_1(x) \leq f_2(x) \leq \ldots$ for all $x \in A$, each $f_n$ is integrable, and

$$\int_A f_n \, d\mu \leq M,$$

for some constant $M$. Then $f(x) = \lim_{n \to \infty} f_n(x)$ is defined almost everywhere on $A$, $f$ is integrable, and

$$\int_A f \, d\mu = \lim_{n \to \infty} \int_A f_n \, d\mu.$$  

[First show that for almost every $x$ the sequence $f_n(x)$ is bounded, by looking at the measure of the set of $x$ for which $f_n(x) > C$ for large $C$. Now you can define $f(x)$ almost everywhere. To control its integral, find a simple function bigger than $f$, and show that it’s integrable, then apply the bounded convergence theorem.]

20. (Fatou’s theorem) Let $f_n \geq 0$ be integrable on $A$, such that for some $M > 0$ we have

$$\int_A f_n \, d\mu \leq M,$$

and $\lim_{n \to \infty} f_n(x) = f(x)$ for almost every $x \in A$. Then $f$ is integrable, and

$$\int_A f \, d\mu \leq M.$$  

[Let $\phi_n = \inf_{k \geq n} f_k$, and apply the monotone convergence theorem to $\phi_n$.]

21. The space $L^1(X, \mu)$ is complete.

[If $f_n$ is a Cauchy sequence in $L^1$, then a subsequence $f_{n_k}$ satisfies $\|f_{n_{k+1}} - f_{n_k}\|_1 < 2^{-k}$. Construct the limit of this subsequence as a telescoping sum, and use the monotone convergence theorem to give the existence of the limit.]

22. The space $L^2(X, \mu)$ is complete.

[Show that a Cauchy sequence in $L^2$ is also a Cauchy sequence in $L^1$, and then use the completeness of $L^1$.]

23. (Lusin’s Theorem) Let $f : [a, b] \to \mathbb{R}$ be measurable, with respect to the Lebesgue measure. For every $\epsilon > 0$ there is a set $E \subset [a, b]$ with $\mu([a, b] \setminus E) < \epsilon$ such that the restriction of $f$ to $E$ is continuous.

[Use the density of continuous functions in $L^1$ together with Egorov’s theorem]
24. (Riesz representation theorem) For every bounded linear functional \( f : H \rightarrow \mathbb{C} \) on a Hilbert space \( H \), there is an element \( y \in H \) such that

\[
f(x) = \langle x, y \rangle, \text{ for all } x \in H.
\]

[Let \( y = \|\hat{y}\|^{-2}\hat{y} \), where \( \hat{y} \) is the closest point to the origin in \( f^{-1}(0) \).]

25. If \( A : E \rightarrow F \) is a bounded linear operator between Banach spaces, then the adjoint \( A^* \) is bounded, and \( \|A^*\| = \|A\| \).

[To get a lower bound on \( \|A^*\| \), you need to use the Hahn-Banach theorem to write \( \|Ax\| = |g(Ax)| \) for some \( g \in F^* \).]

26. The set of invertible elements in a Banach algebra with unit is open.

[Use that if \( \|a\| < 1 \), then \( e - a \) is invertible.]

27. Any maximal ideal in a Banach algebra with unit is closed.

[Use the fact that the invertible elements form an open set to show that the closure of a proper ideal is closed.]

28. If \( a \in A \) is an element in a Banach algebra with unit, then the spectral radius \( \nu(a) \) satisfies

\[
\nu(a) \leq \|a\|.
\]

[Use that \( e - x \) is invertible if \( \|x\| < 1 \).]

29. If \( A \) is a Banach algebra where every non-zero element is invertible, then \( A \cong \mathbb{C} \).

[The spectrum of every element is non-empty.]

30. If \( a \in A \) is normal in a \( C^* \)-algebra \( A \), then \( \nu(a) = \|a\| \).

[Use the spectral radius formula.]

31. If \( A \) is a commutative Banach algebra with unit, then there is a bijection between maximal ideals of \( A \) and non-zero homomorphisms \( A \rightarrow \mathbb{C} \).

[The bijection is given by identifying a homomorphism with its kernel.]

32. For a commutative Banach algebra \( A \) with unit, the spectrum \( \sigma(a) \) of any element is the range of its Gelfand transform \( \hat{a} \).

[Use that an element \( x \in A \) is invertible if and only if \( \phi(x) \neq 0 \) for all non-zero homomorphisms \( \phi \).]