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A Stability Criterion for Arbitrarily Switched Second Order LTI Systems

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***Abstract*— In this paper, an easily verifiable, necessary and sufficient condition is derived for stability of arbitrarily switched systems with two stable second order linear time-invariant subsystems.**

I. Introduction

The stability issues of switched systems, especially switched linear systems, have attracted considerable interest in the recent decade, see for example the survey papers [1],[2], the recent book [3] and the references cited therein. It is known that the stability of switched systems depends on not only the dynamics of the subsystems but also the properties of the switching signals. One of the basic problems for switched systems is to identify conditions that guarantee the stability of a switched system under all possible switching signals, or arbitrary switching. A popular way to deal with this problem is based on finding a common Lyapunov function. This approach is justified by the converse Lyapunov theorem proposed in [4] for arbitrary switching systems. However, most existing efforts, e.g. [5], [6], are based on or imply the existence of a common quadratic Lyapunov function (CQLF), which is known to be sufficient only. Therefore, the study of non-quadratic Lyapunov functions has been attracting more and more attentions, e.g. [7]. Nevertheless, these non-quadratic Lyapunov functions are not easy to determine in general.

In this paper, we aim to identify a necessary and sufficient condition for the stability of switched systems under arbitrary switching. In particular, we consider the following switched systems with a pair of second-order continuous-time LTI subsystems:

$$S_{ij}: \dot{x} = \sigma x, \quad \sigma \in \{A_i, B_j\} \tag{1}$$

where both A_i and B_j are Hurwitz, and $i, j \in \{1, 2, 3\}$ denote the types of A and B respectively. We classify a matrix $A \in \mathbb{R}^{2\times 2}$ into three types according to its eigenvalue and eigenstructure. Type I: A has real eigenvalues and diagonalizable; Type II: A has real eigenvalues but undiagonalizable; Type III: A has two complex eigenvalues.

The argument here is based on the characterization of the most unstable switching signal for the switched systems. The idea is very simple: if the switched system remains stable under the most "unstable" switching signal, then the switched system must be stable for all possible switching signals. Similar idea has been used in [8] to derive a verifiable necessary and sufficient condition for absolute stability of second-order systems. However, the condition proposed in [8] can not be applied to general cases of second order subsystems and the checking of this condition is not straightforward. In this paper, we characterize the worst case switching signal by analyzing system trajectory in polar coordinates and use it to derive an easily verifiable, necessary and sufficient condition.

The rest of the paper is organized as follows. In Section II, the trajectories of the switched system are analyzed under the polar coordinates. In Section III, the worst case switching signal is characterized. In section IV, the main result and its proofs are given. And the final section concludes the paper.

II. Polar coordinates presentation

Consider a second-order LTI system

$$\dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x \tag{2}$$

and define $x_1 = r cos\theta, \quad x_2 = r sin\theta$, it follows that

$$\frac{dr}{dt} = r[a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta] \tag{3}$$

$$\frac{d\theta}{dt} = a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta \tag{4}$$

Assume $\frac{d\theta}{dt} \neq 0$ [1], then

$$\frac{dr}{d\theta} = r\frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta} \tag{5}$$

Denote

$$f(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta} \tag{6}$$

we have

$$\frac{1}{r}dr = f(\theta)d\theta \tag{7}$$

Lemma 2.1: The trajectories of system (2) in r-θ coordinates, except the ones lie on the eigenvectors, can be expressed as

$$r(t) = C\exp\left(\int f(\theta(t))d\theta\right) \triangleq Cu(\theta(t)) \tag{8}$$

where C is a positive constant depending on initial state (r_0, θ_0) and $u(\theta)$ is positive definite.

[1] The case that $\frac{d\theta}{dt} = 0$ happens when the trajectory stays along an eigenvector corresponding to a real eigenvalue, which will be dealt with separately later.

Equation (8) can be readily shown by integrating both sides of (7). In addition, we are to obtain expression of $u(\theta)$. It follows from (6) that

$$f(\theta) = \frac{(a_{11}-a_{22})\cos 2\theta + (a_{12}+a_{21})\sin 2\theta + (a_{11}+a_{22})}{(a_{12}+a_{21})\cos 2\theta - (a_{11}-a_{22})\sin 2\theta + (a_{21}-a_{12})} \tag{9}$$

Denote $m = \sqrt{(a_{11}-a_{22})^2 + (a_{12}+a_{21})^2}$. If $m = 0$, we have $f(\theta) = \frac{a_{11}+a_{22}}{a_{21}-a_{12}}$ and

$$r = C\exp\left(\frac{a_{11}+a_{22}}{a_{21}-a_{12}}\theta\right) \tag{10}$$

If $m \neq 0$, denote $\cos\varphi = (a_{12}+a_{21})/m$, $\sin\varphi = (a_{11}-a_{22})/m$, $p = (a_{21}-a_{12})/m$ and $q = (a_{11}+a_{22})/m$. Hence, (7) becomes

$$\frac{1}{r}dr = \frac{\sin(2\theta+\varphi)+q}{\cos(2\theta+\varphi)+p}d\theta \tag{11}$$

1) A has two distinct real eigenvalues: $|p| < 1$

$$r = Cu_1(\theta)\left|\frac{tg(\theta+\frac{\varphi}{2})+h}{tg(\theta+\frac{\varphi}{2})-h}\right|^{\frac{qh}{2(p+1)}}$$

where $u_1(\theta) = |\cos(2\theta+\varphi)+p|^{-\frac{1}{2}}$ and $h = \sqrt{\left|\frac{1+p}{1-p}\right|}$.

2) A has two multiple real eigenvalues: $|p| = 1$

$$r = Cu_1(\theta)\exp\left[\frac{q}{2}tg(\theta+\frac{\varphi}{2})\right],\ \ p = 1$$

$$r = Cu_1(\theta)\exp\left[\frac{q}{2}ctg(\theta+\frac{\varphi}{2})\right],\ \ p = -1$$

3) A has a pair of complex eigenvalues: $|p| > 1$

Denote $\theta = \bar{\theta} + n\pi$, $\bar{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that

$$u(\theta) = u(\bar{\theta}+n\pi) = v(n)\bar{u}(\bar{\theta})$$

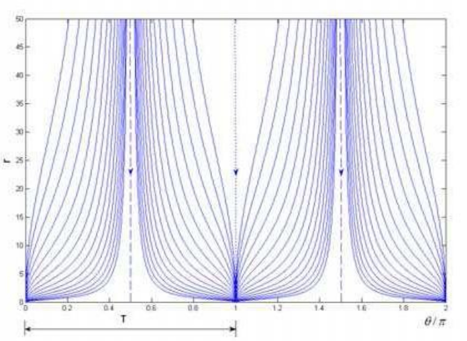
where $v(n) = \exp\left(\frac{qh}{p+1}n\pi\right)$ and

$$\bar{u}(\bar{\theta}) = u_1(\bar{\theta})\exp\left\{\frac{qh}{p+1}tg^{-1}\left[\frac{1}{h}tg(\bar{\theta}+\frac{\varphi}{2})\right]\right\}$$

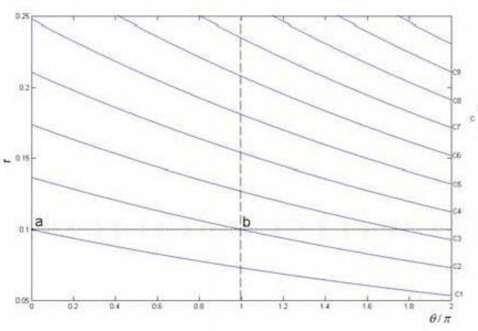
Comment 1: When A has real eigenvalues, $u(\theta)$ is a periodic function of θ with a period of π; When A has complex eigenvalues, $u(\theta)$ consists of a periodic part $\bar{u}(\bar{\theta})$ and a piecewise constant part $v(n)$. It implies that all the trajectories for $n \neq 0$ is just a scaling of the trajectories when $n = 0$. With reference to Fig.1, the behaviors of the trajectory starting from the state b is the same as that from the state a with a phase shift of π. Therefore, it is sufficient to analyze stability of a system with complex eigenvalues, as well as real eigenvalues, in a interval of $\theta \in [\theta_0, \theta_0+\pi)$ for any θ_0. Without loss of generality, we can assume $\theta_0 = -\frac{\pi}{2}$.

Comment 2: Geometrically, a larger C indicates an outer layer curve. For any given initial state, if no switching happens, the trajectory of the system will follow one of the curves (C remains constant), which is determined by the initial state, and converge to the origin for asymptotically stable system. Intuitively, if we can bring the state to outer layers, equivalently increase C, then it may be possible to make the system unstable.

Now, we proceed to analyze the switched system (1) and investigate how to orchestrate between these two subsystems



(a) A system with real eigenvalues



(b) A system with complex eigenvalues

Fig. 1. Phase diagrams of planar systems in polar coordinates.

to generate the most 'unstable' trajectory. The basic idea is to increase the value of C as much as possible at each switching. Denote the two subsystems of (1) as:

$$\Sigma_A : \dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x \tag{12}$$

$$\Sigma_B : \dot{x} = Bx = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} x \tag{13}$$

Follow the definition of $f(\theta)$ in equation (5), we define $f_A(\theta)$ and $f_B(\theta)$ for subsystems A and B respectively.

$$f_A(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta}$$

$$f_B(\theta) = \frac{b_{11}\cos^2\theta + b_{22}\sin^2\theta + (b_{12}+b_{21})\sin\theta\cos\theta}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22}-b_{11})\sin\theta\cos\theta}$$

It follows from Lemma 2.1 that

$$r = C_A\exp\left(\int f_A(\theta)d\theta\right) = C_A u_A(\theta) \tag{14}$$

$$r = C_B\exp\left(\int f_B(\theta)d\theta\right) = C_B u_B(\theta) \tag{15}$$

Combine (14) and (15), we obtain the piecewise solution of the switched system (1)

$$r = \begin{cases} C_A u_A(\theta), & when \quad \sigma = A \\ C_B u_B(\theta), & when \quad \sigma = B \end{cases} \tag{16}$$

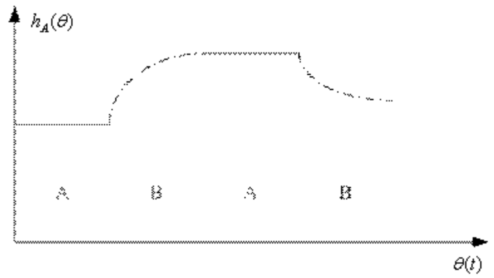


Fig. 2. The variations of h_A under switching

where

$$\left.\frac{dC_A}{dt}\right|_{\sigma=A} = 0, \quad \left.\frac{dC_B}{dt}\right|_{\sigma=B} = 0 \tag{17}$$

For any state (r, θ) in the phase plane, it can be described by either (14) or (15)

$$r = C_A u_A(\theta) = C_B u_B(\theta) \tag{18}$$

Hence, a solution of the switched system can be represented in a compact way as

$$r = h_A(\theta) u_A(\theta) \tag{19}$$

where

$$h_A(\theta) = \begin{cases} C_A, & \sigma = A \\ C_B \frac{u_B(\theta)}{u_A(\theta)}, & \sigma = B \end{cases} \tag{20}$$

or similarly

$$r = h_B(\theta) u_B(\theta) \tag{21}$$

where

$$h_B(\theta) = \begin{cases} C_A \frac{u_A(\theta)}{u_B(\theta)}, & \sigma = A \\ C_B, & \sigma = B \end{cases} \tag{22}$$

For convenience, we denote

$$H_A(\theta(t)) \triangleq \left.\frac{dh_A}{dt}\right|_{\sigma=B}, H_B(\theta(t)) \triangleq \left.\frac{dh_B}{dt}\right|_{\sigma=A} \tag{23}$$

Equation (19) indicates even when the actual trajectory follows Σ_B, we still describe the trajectory by the function of Σ_A with a varying h_A. Then, we can use the variations of h_A to describe the behavior of the switched system, as shown in Fig.2.

Since $u_A(\theta)$ converges to zero for a Hurwitz A, the only way to make r diverge to infinity is increasing h_A, which can only be done by Σ_B. Geometrically, the positive H_A (the increase of h_A) means that Σ_B helps Σ_A to bring states to the outer trajectories of Σ_A, which is necessary to make the system unstable.

Lemma 2.2: Switched systems (1) are stable *if* one of H_A and H_B is non-positive for all θ.

Proof: We consider the case when the switching does not stop. (If the switching stops, the switched system must be stable since both subsystems are stable.) If H_A is non-positive for all θ, then h_A is bounded. Furthermore, $u_A(\theta)$ is bounded for stable Σ_A, it follows from the equation (19) that the magnitude r of the switched system is bounded. Since the switched linear system under arbitrary switching is still a linear system, boundedness of the state implies the Lyapunov stability. Similarly, we can prove the case when H_B is non-positive.

Although the existence of positive H_A and H_B are necessary, it is not sufficient for divergence of the switched systems. Hence a comprehensive analysis is needed.

III. WORST CASE SWITCHING SIGNAL

In this section, based on the signs of H_A and H_B, we establish a criterion to evaluate which subsystem is worse. By choosing the worse subsystem for each θ, we are able to find a worst case switching signal (WCSS) for switched systems. Then the stability problem under arbitrary switching is transformed to the stability problem under WCSS.

First, we find the expressions of $H_A(\theta)$ and $H_B(\theta)$. It follows from equation (20) and (23) that

$$\begin{aligned} H_A(\theta) &= \left.\frac{dh_A}{dt}\right|_{\sigma=B} = C_B \left(\frac{u_B(\theta)}{u_A(\theta)}\right)' \\ &= -C_B \frac{u_B(\theta)}{u_A(\theta)} [f_A(\theta) - f_B(\theta)] \left.\frac{d\theta}{dt}\right|_{\sigma=B} \end{aligned} \tag{24}$$

In (24), C_B is a constant since $\sigma = B$. Similarly, we have

$$H_B(\theta) = C_A \frac{u_A(\theta)}{u_B(\theta)} [f_A(\theta) - f_B(\theta)] \left.\frac{d\theta}{dt}\right|_{\sigma=A} \tag{25}$$

Denote

$$K_A(\theta) = C_B \frac{u_B(\theta)}{u_A(\theta)}, \quad K_B(\theta) = C_A \frac{u_A(\theta)}{u_B(\theta)}$$

$$P(\theta) = f_A(\theta) - f_B(\theta)$$

$$Q_A(\theta) = \left.\frac{d\theta}{dt}\right|_{\sigma=A}, Q_B(\theta) = \left.\frac{d\theta}{dt}\right|_{\sigma=B}$$

then (24) and (25) become

$$H_A(\theta) = -K_B(\theta) P(\theta) Q_B(\theta) \tag{26}$$

$$H_B(\theta) = K_A(\theta) P(\theta) Q_A(\theta) \tag{27}$$

In (26) and (27), since both $K_A(\theta), K_B(\theta)$ are positive definite, $P(\theta)$ is the same, it follows that

1) If the signs of Q_A and Q_B are the same, then the signs of H_A and H_B are opposite.
2) If the signs of Q_A and Q_B are opposite, then the signs of H_A and H_B are the same.

The geometrical meaning of the signs of Q_A and Q_B is the trajectory direction. A positive Q_A implies a counter clockwise trajectory of Σ_A in x-y coordinates.

Since the interesting interval of θ is $[-\frac{\pi}{2}, \frac{\pi}{2})$, we denote $k = tg\theta$, then all above functions of θ can be transformed to the functions of k. Straightforward algebraic manipulation yields

$$Q_A(k) = -\frac{1}{k^2+1} D_A(k) \tag{28}$$

$$Q_B(k) = -\frac{1}{k^2+1} D_B(k) \tag{29}$$

$$H_A(k) = K_B(k) \frac{N(k)}{D_A(k)} \tag{30}$$

$$H_B(k) = -K_A(k) \frac{N(k)}{D_B(k)} \tag{31}$$

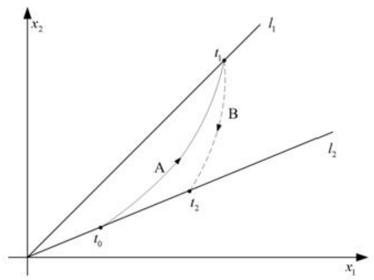


Fig. 3. The region where both H_A and H_B are positive

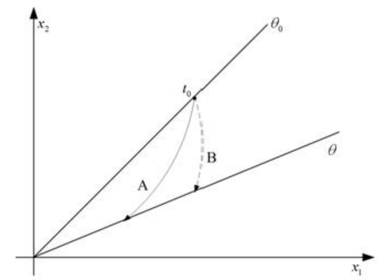


Fig. 4. The region where H_A is positive and H_B is negative

where

$$D_A(k) = a_{12}k^2 + (a_{11} - a_{22})k - a_{21} \quad (32)$$

$$D_B(k) = b_{12}k^2 + (b_{11} - b_{22})k - b_{21} \quad (33)$$

and

$$N(k) = p_2k^2 + p_1k + p_0 \quad (34)$$

where $p_2 = a_{12}b_{22} - a_{22}b_{12}$, $p_1 = a_{12}b_{21} + a_{11}b_{22} - a_{21}b_{12} - a_{22}b_{11}$ and $p_0 = a_{11}b_{21} - a_{21}b_{11}$.

It shows that the signs of equations (28)-(31) depend on the signs of $D_A(k)$, $D_B(k)$ and $N(k)$. Geometrically, if we describe vectors in x-y plane as $[1, k]^T$, then the vectors satisfying $D_A(k) = 0$ are the eigenvectors of A and similarly the vectors satisfying $D_B(k) = 0$ are the eigenvectors of B. In addition, if we denote two distinct real solutions of $N(k) = 0$ by k_1 and k_2, and assume $k_2 < k_1$, then $[1, k_1]^T$ and $[1, k_2]^T$ are two vectors on which $\frac{dr}{d\theta}$ of the two subsystems are the same.

Definition 3.1: A region of k is a continuous interval where the signs of (28)-(31) preserve for all k in the interval.

The boundaries of the regions are the vectors satisfying $D_A(k) = 0$, $D_B(k) = 0$ or $N(k) = 0$. These vectors divide x-y plane to several sectors.

Now we proceed to establish criteria to determine the worse subsystem for every θ based on the signs of H_A and H_B.

A. Both H_A and H_B are positive

Lemma 3.1: The switched system (1) is unstable if there is a region of k, $[k_l, k_u]$, where both H_A and H_B are positive.

Proof: With reference to Fig.3, define l_1 and l_2 as the lines where $x_2 = k_u x_1$ and $x_2 = k_l x_1$. Consider an initial state on l_2 at t_0, let trajectory follow Σ_A until it hits l_1 at t_1 and switch back to Σ_B until it returns to the line l_2 again at t_2. Define the states at t_0, t_1 and t_2 as (r_0, θ_0), (r_1, θ_1) and (r_2, θ_2) respectively. It follows that $r_0 = C_{A0}u_A(\theta_0) = C_{B0}u_B(\theta_0), r_1 = C_{A1}u_A(\theta_1) = C_{B1}u_B(\theta_1), r_2 = C_{A2}u_A(\theta_2) = C_{B2}u_B(\theta_2)$.

Since $H_A(k)$ and $H_B(k)$ are both positive, trajectories of two subsystems have opposite directions. It follows that $\theta_2 = \theta_0$ and $C_{A2} > C_{A1} = C_{A0}$. It can be shown that the increment $\Delta = C_{A2} - C_{A0}$ is a positive constant determined by k_l, k_u and the entries of A and B. By repeating the switching, we have

$$\lim_{t\to\infty} r(t) = \lim_{n\to\infty} r(t_0 + nT) = \lim_{n\to\infty} C_{A0}(1+\Delta)^n u(\theta_0) \to \infty$$

where $T = t_2 - t_0$ and n is the number of switching periods.

B. H_A is positive and H_B is negative

The worse subsystem is Σ_B. In this case, two trajectories have the same directions. With reference to Fig.4, consider an initial state with an angle θ_0 at t_0. Analyzing the magnitudes of states along different subsystems, we have

$$\begin{aligned} &\triangle r(\theta) = r_B(\theta) - r_A(\theta) = h_A(\theta)u_A(\theta) - C_A u_A(\theta) \\ &= \left(\int_{t_0}^{t} H_A dt\right) u_A(\theta) > 0 \end{aligned} \quad (35)$$

It shows that the trajectories of Σ_B always have a larger magnitude than the corresponding ones of Σ_A for all θ in this region.

C. H_A is negative and H_B is positive

Similarly, the worse subsystem is Σ_A.

D. Both H_A and H_B are negative

Based on Lemma 2.2, if the trajectory can not go out of this region, the system is stable in this region. Therefore, the worst case is the subsystem whose trajectory is able to go out of this region. If both trajectories can go out, then no matter which subsystem is chosen, the trajectories will leave this region and the stability of the system is not affected.

E. Both H_A and H_B are zeros

In this case, we can choose either one as the worse case switching signal.

F. On real eigenvectors

It can be readily shown that the worse subsystem is Σ_A if trajectory is on the eigenvector of B, and vice versa.

IV. Main Result

In this section, we express the entries of subsystems by their eigenvalues and eigenvectors. Assumptions are made to simplify the analysis and to obtain a compact result for system S_{ij} in (1) with different combinations of i, j.

Without loss of generality, we define the standard form J_i for different types of second order matrix.

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \; J_3 = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \quad (36)$$

where

$$\lambda_2 \le \lambda_1 < 0, \quad \lambda < 0, \quad \mu < 0, \omega > 0 \quad (37)$$

for Hurwitz matrix.

In system (1), we assume that one of subsystems is in its standard form $A_i = J_i$, and the other one can be expressed as $B_j = Q_j J_j Q_j^{-1}$ with $i \leq j$. If A_i and B_j do not share a real eigenvector (these cases will be proved separately), the transformation matrix Q_j can be taken as

$$Q_1 = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix} \quad (38)$$

There are totally six combinations of S_{ij} in terms of i, j, and additional assumptions are needed for individual combinations.

1) If $S_{ij} = S_{11}$, $\beta < 0$
2) If $S_{ij} = S_{12}$, $\alpha < 0$
3) If $S_{ij} = S_{13}$, $k_1, k_2 < 0$ (if exist)
4) If $S_{ij} = S_{33}$, $p_2 \neq 0$ (if $A \neq B$)
5) If $S_{ij} = S_{33}$, $p_2 < 0$ (if $N(k)$ has two distinct real roots)

Any given switched linear systems (1) can be transformed to satisfy these assumptions by coordinates transformation when necessarily while stability properties of the switched system preserve. The proofs of the feasibility of these assumptions are presented in the Appendix.

Theorem 4.1: Switched systems (1) is not stable for arbitrary switching signal if and only if $N(k)$ (34) has two distinct real roots, $k_2 < k_1$, satisfying

$$\begin{cases} N < k_2 < k_1 < M & if \det(Q_j) < 0 \\ \|\exp(B_j T_B) \exp(A_i T_A) x(0)\|_2 > \|x(0)\|_2 & if \det(Q_j) > 0 \end{cases} \quad (39)$$

where

$$\begin{cases} N = \alpha, M = 0 & S_{ij} = S_{11} \\ N \to -\infty, M \to +\infty & otherwise \end{cases} \quad (40)$$

$$T_A = \int_{\theta_2}^{\theta_1} \frac{1}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta} d\theta \quad (41)$$

$$T_B = \int_{\theta_1}^{\theta_2 + \pi} \frac{1}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22} - b_{11})\sin\theta\cos\theta} d\theta \quad (42)$$

where $\theta_1 = tg^{-1} k_1$, $\theta_2 = tg^{-1} k_2$ and $x(0) = [1, k_2]^T$.

A. Proof of the special cases

If A and B share a common real eigenvector, then A and B can be transformed to lower-triangular Hurwitz matrix $\bar{A}$ and $\bar{B}$ simultaneously by a nonsingular matrix whose second column is the common real eigenvector. It follows that $\bar{A}$, $\bar{B}$ share a CQLF and the switched system (1) is stable. The result can also be obtained by Theorem 4.1. Two lower-triangular Hurwitz matrix leads to $p_2 = 0$ in (34), which violates the condition: $N(k)$ has two roots. Therefore, Theorem 4.1 is valid for these special cases.

B. Proof of $S_{ij} = S_{13}$

In this case, two subsystems are expressed as

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix} \quad (43)$$

$$B_3 = Q_3 \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} Q_3^{-1} = \frac{\omega}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix} \quad (44)$$

From (37), we have

$$\lambda_{2a} \leq \lambda_{1a} < 0, \mu < 0, \omega > 0 \quad (45)$$

Denote

$$\lambda_{1a} = k_A \lambda_{2a}, \quad \xi = \frac{\mu}{\omega} \quad (46)$$

it follows that $k_A \in (0, 1]$ and $\xi < 0$. Substitute (43) and (44) to (32), (33) and (34), it yields that

$$D_A(k) = \lambda_{2a}(k_A - 1)k \quad (47)$$

$$D_B(k) = \frac{1}{\beta}[(k - \alpha)^2 + \beta^2] \quad (48)$$

$$N(k) = \frac{-\lambda_{2a} n}{\beta} \bar{N}(k) \quad (49)$$

where $\bar{N}(k)$ is a monic polynomial with the same roots as $N(k)$

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\xi + (k_A + 1)\alpha]k + k_A(\alpha^2 + \beta^2) \quad (50)$$

Hence

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k) \quad (51)$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k)) \quad (52)$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(k) \quad (53)$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta) \quad (54)$$

Based on (51)-(54), we discuss all the possibilities based on the signs of $\det(Q)$ (or β equivalently) and root conditions of $\bar{N}(k)$.

1) $\bar{N}(k)$ has no real root or has two multiple roots: In this case, $\bar{N}(k)$ is non-negative, as a result, H_B is non-positive $\forall k$. Based on Lemma 2.2, the switched system (1) is stable.

2) $\bar{N}(k)$ has two roots and $\det(Q_3) < 0$: In this case, $\beta > 0$, H_A and H_B are both positive $\forall k \in (k_2, k_1)$. Based on Lemma 3.1, the switched system (1) is unstable.

3) $\bar{N}(k)$ has two roots and $\det(Q_3) > 0$: In this case, $\beta < 0$, we need to determine the worst case switching signal (WCSS) for every region of k:

a) $k < k_2$: H_A is positive and H_B is negative, the WCSS is Σ_B.

b) $k_2 < k < k_1$: H_A is negative and H_B is positive, the WCSS is Σ_A.

c) $k = k_1$ and $k = k_2$: Both H_A and H_B are zero, without loss of generality, the WCSS is Σ_A.

d) $k_1 < k < 0$: H_A is positive and H_B is negative, the WCSS is Σ_B.

e) $k = 0$: On the real eigenvector of Σ_A, the WCSS is Σ_B.

f) $k > 0$: Both H_A and H_B are negative, the WCSS is Σ_B since only trajectory of Σ_B can go out of this region.

From above analysis, we conclude that the WCSS for all k is

$$\begin{cases} \sigma = A & k_2 \leq k \leq k_1, \\ \sigma = B & \text{otherwise.} \end{cases} \tag{55}$$

The simplest way to determine the stability of the system is to follow the WCSS originating from an initial state on the the line $x_2 = k_2 x_1$ until it returns to the same line again. Then the switched system (1) is not stable *if and only if* $\|\exp(B_3 T_B)\exp(A_1 T_A)x(0)\|_2 > \|x(0)\|_2$, which meet the second inequality in Theorem 4.1.

Due to space limitation, we only present the proof of the case $S_{ij} = S_{13}$ as an example of general cases, other cases of S_{ij} can be proved similarly.

C. An example

Consider a switched linear system with two LTI planar systems

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -8 \\ 1/8 & -1 \end{bmatrix} \tag{56}$$

This switched system has been proved to be stable under arbitrary switching although A and B do not share a CQLF in [4] to show that the existence of CQLF is only sufficient, but not necessary for stability under arbitrary switching. Now we check it by applying Theorem 4.1.

1) Step 1: Simple checking yields that both A and B have a pair of complex eigenvalues:$-1 \pm i$. So it is the case S_{33} and $J_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ is the standard form for both A and B.

2) Step 2: It is noticed that A is already in its standard form and the Assumption 4 and 5 for S_{33} are satisfied since $p_2 = a_{12}b_{22} - a_{22}b_{12} = -7$. So no further transformations are needed. Since B and the standard form of B are known, we can obtain $Q = \begin{bmatrix} 0 & 1 \\ -1/8 & 0 \end{bmatrix}$ whose structure is defined in (38).

3) Step 3: Substitute entries of A and B in (56) into (34) and we have $k_2 = -0.1019, k_1 = 1.2269$ and $\det(Q) = 1/8 > 0$. Therefore, we need to check the second inequality to determine the stability of the switched system.

4) Step 4: Substitute (56) to (41) and (42), we obtain $T_A = T_B = 0.9885$ and

$$\|\exp(BT_B)\exp(AT_A)x(0)\|_2 = 0.6761 < \|x(0)\|_2 = 1.0052$$

Therefore, the switched system is stable which is the same result as shown in [4].

V. Conclusion

In this paper, we analyzed the system trajectories of a switched planar systems consisting of a pair of stable LTI subsystems in the polar coordinates, and revealed in which cases unstable behaviors could be generated through switching. Based on the signs of H_A and H_B (23), we characterize the "worst case switching signal" and use it to derive an easily verifiable, necessary and sufficient condition to determine the stability of switched linear systems with two stable planar subsystems. Future work will consider extending the results to obtain corresponding results for discrete-time systems.

Appendix

Proof of Assumptions

Assumptions 1-3 can be satisfied by the transformation $\bar{x}_1 = -x_1$ when necessary.

Assumption 5 can be satisfied by similarity transformation with a unitary matrix $P = \begin{bmatrix} \gamma & -\eta \\ \eta & \gamma \end{bmatrix}$ when necessary, where $\det(P) = \sqrt{\gamma^2 + \eta^2} = 1$. Since P is unitary and real, $P^{-1} = P^T$. It follows that

$$\bar{A}_3 = P^{-1}A_3P = P^TA_3P = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}$$

where $\bar{a}_{11} = a_{11}\gamma^2 - (a_{12} + a_{21})\gamma\eta + a_{22}\eta^2$, $\bar{a}_{12} = a_{12}\gamma^2 + (a_{11} - a_{22})\gamma\eta - a_{21}\eta^2$, $\bar{a}_{21} = a_{21}\gamma^2 + (a_{11} - a_{22})\gamma\eta - a_{12}\eta^2$, $\bar{a}_{22} = a_{22}\gamma^2 + (a_{12} + a_{21})\gamma\eta + a_{11}\eta^2$.

It follows from $A_3 = J_3$ that $a_{11} = a_{22}$ and $a_{12} = -a_{21}$, hence $\bar{A}_3 = A_3$. Similarly, we have

$$\bar{B}_3 = P^{-1}B_3P = P^TB_3P = \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix}$$

where $\bar{b}_{11} = b_{11}\gamma^2 - (b_{12} + b_{21})\gamma\eta + b_{22}\eta^2$, $\bar{b}_{12} = b_{12}\gamma^2 + (b_{11} - b_{22})\gamma\eta - b_{21}\eta^2$, $\bar{b}_{21} = b_{21}\gamma^2 + (b_{11} - b_{22})\gamma\eta - b_{12}\eta^2$, $\bar{b}_{22} = b_{22}\gamma^2 + (b_{12} + b_{21})\gamma\eta + b_{11}\eta^2$.

Then we have

$$\begin{aligned} \bar{p}_2 &= \bar{a}_{12}\bar{b}_{22} - \bar{a}_{22}\bar{b}_{12} = a_{12}\bar{b}_{22} - a_{22}\bar{b}_{12} \\ &= \eta^2\left[p_2\left(\frac{\gamma}{\eta}\right)^2 + p_1\frac{\gamma}{\eta} + p_0\right] \end{aligned} \tag{57}$$

The polynomial inside the bracket in (57) has the same coefficients as $N(k)$! If $p_2 > 0$ and $N(k)$ has two roots $k_2 < k_1$, it is always possible to get a negative $\bar{p}_2$ by a pair of (γ, η) satisfying $k_2 < \frac{\gamma}{\eta} < k_1$.

By the same way, we can prove the Assumption 4. According to (57), if $p_2 = 0$ and $p_2^2 + p_1^2 + p_0^2 \neq 0$ ($A \neq B$), it is easy to find a pair of (γ, η) to make $\bar{p}_2 \neq 0$.

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