Identification and control of quantum systems

Zhengui Xue[1], Hai Lin[2†] and Tong Heng Lee[1,2]
[1]NUS Graduate School for Integrative Sciences and Technology, National University of Singapore, Singapore 117456, Singapore
Email: zhenguixue@nus.edu.sg
[2]Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117576, Singapore
Email: elelh@nus.edu.sg, eleleeth@nus.edu.sg

***Abstract*— This paper aims at finding the real-valued dynamics that is equivalent to the Schrödinger equation and then implementing quantum control by making use of the well developed classical control theory. Firstly, pure state identification approaches are presented for two-level, three-level and n-level systems, respectively. Secondly, based on the discussions on the pure state identification, real-valued dynamics that are equivalent to the Schrödinger equations are deduced for both two-level and three-level systems. Finally, a control strategy based on Lyapunov approach is proposed by making use of the obtained real-valued dynamics. Different from the existing Lyapunov control based on the Schrödinger equation, the proposed control strategy can achieve state convergence to its goal state without any constraints on the internal Hamiltonian. Simulation results are included to demonstrate the effectiveness of the approach.**

***Index Terms*— Pure state identification, quantum control, Lyapunov control**

I. INTRODUCTION

The last three decades have seen the growth of quantum control theory from the initial controllability topic to a broad variety of control problems. For closed quantum systems, the developed control strategies include Lyapunov control [1], [2], optimal control [3], [4], variable structure control [5], control based on Lie group decompositions [6] and measurement based control [7], [8], etc. Existing control schemes for open quantum systems include error correcting codes [11], [12], feedback control [17], decoherence suppression by optimal control [15], [16], the use of decoherence-free subspaces [9], quantum Zeno effects [10] and quantum dynamical decoupling [13], [14], etc (see [18] and the references therein). Here, it is difficult to enumerate all of the control approaches. We can refer to [19]- [21] for a more completed understanding about the quantum control approaches. A general principle for these control approaches is that most of them were implemented by combining the well developed classical control theory with the unique characteristics of quantum systems.

Different from classical systems, a quantum system is described by the Schrödinger equation that is a complex-valued dynamics due to the physical properties of quantum states. To exploit the well developed results from classical control theory into quantum control, one of the inspiring strategies is to derive a real-valued dynamics to replace the complex-valued Schrödinger equation. This paper aims at finding the real-valued dynamics so as to facilitate quantum control. As it is known, quantum state tomography is a method to determine the state of a quantum system via a series of measurements, since the measurement of an observable gives a certain result with a probability depending on the state [26]. In [23], the authors studied the minimal informationally complete measurements for pure states. They discussed the pure state reconstruction based on measurement outcomes.

In this paper, to obtain the real-valued dynamics that is equivalent to the Schrödinger equation, we will reexplain and present the pure state identification strategy from a different point of view. We will show that a pure state is characterized by a set of real value parameters and thus can be uniquely determined by a series of measurements. Based on the discussion on the pure state identification, we will propose a strategy to obtain the real-valued dynamics. Specially, we will deduce the dynamics for two-level and three-level systems.

With the obtained real-valued dynamics, many well developed classical control approaches can be applied to achieve control specifications. In this paper, the state transfer problem for a two-level system will be considered. A control strategy based on Lyapunov approach will be proposed. Compared with the existing Lyapunov control approaches based on the Schrödinger equation as introduced in [1] and [2], the proposed control strategy has the following advantages: 1) the goal state in this paper can be an arbitrary state, while the existing Lyapunov control mainly focuses on the state transfer to an eigenstate of the internal Hamiltonian; and 2) the exponential convergence of the state to the goal state can be achieved in this paper, while most of the existing Lyapunov control approaches can guarantee that the final state converges to a neighborhood of the goal state or to the goal state with some constrained conditions on the quantum systems.

The rest of this paper is organized as follows: In Section II, the pure state identification method is presented. In Section III, the real-valued dynamics of two-level and three-level systems are deduced. In Section IV, the control strategy is proposed based on the obtained real-valued dynamics for two-level systems. Section V concludes the paper.

† Corresponding author

II. Pure State Identification Based on System Outputs

The pure state identification approach presented in this section is essentially equivalent to the one discussed in [23]. For the sake of completeness and clearness, we will reexplain and present the identification procedure in detail from a different point of view.

A. Identification of Pure States for Two-Level Systems

Without loss of generality, we consider the identification of two-level states described in the following form

$$\begin{aligned} |\psi\rangle &= a_0|0\rangle + a_1|1\rangle \\ &\triangleq a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{1}$$

For the two-level system, we define the outputs as the average values of observables C_k

$$y_k = \langle\psi|C_k|\psi\rangle, \ k = 1, 2, 3 \tag{2}$$

where

$$C_k = \begin{bmatrix} c_{k11} & c_{k12R} + ic_{k12I} \\ c_{k12R} - ic_{k12I} & 0 \end{bmatrix}. \tag{3}$$

with c_{k11}, c_{k12R} and c_{k12I} being real numbers. We can obtain

$$y_k = c_{k11}|a_0|^2 + 2c_{k12R}Re\{a_0^* a_1\} - 2c_{k12I}Im\{a_0^* a_1\}. \tag{4}$$

Denote

$$\theta_1 = |a_0|^2, \ \theta_2 = Re\{a_0^* a_1\}, \ \theta_3 = Im\{a_0^* a_1\}. \tag{5}$$

(4) can be written as

$$y_k = c_{k11}\theta_1 + 2c_{k12R}\theta_2 - 2c_{k12I}\theta_3, \tag{6}$$

i.e.,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{7}$$

where

$$A = \begin{bmatrix} c_{111} & 2c_{112R} & -2c_{112I} \\ c_{211} & 2c_{212R} & -2c_{212I} \\ c_{311} & 2c_{312R} & -2c_{312I} \end{bmatrix}. \tag{8}$$

It is obvious that, if A is nonsingular, i.e.,

$$rank\left\{ \begin{bmatrix} c_{k11} \\ 2c_{k12R} \\ -2c_{k12I} \end{bmatrix} \right\} = 3, \tag{9}$$

$\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T$ can be uniquely determined. Therefore, $|a_0|$ and $a_0^* a_1$ can be obtained. Furthermore, we can deduce $|a_1|$ and the relative phase between a_1 and a_0, $\angle\{a_0^* a_1\}$. Hence, the pure state can be reconstructed in the sense that global phase has no physical meaning.

Remark 1: The above discussion assumes $a_0 \neq 0$. In practical experiments, if all of the three outputs $y_k = 0$, then we can draw the conclusion $a_0 = 0$ and thus $|\psi\rangle = |1\rangle$.

Remark 2: The reason why we choose such kind of C_k: In the outputs y_k, c_{k11} is related to $|a_0|$, and $c_{k12} = c_{k12R}+ic_{k12I}$ is related to the coherence between the eigenstates $|0\rangle$ and $|1\rangle$. In this way, the quantum states can be uniquely determined.

Remark 3: The reason why we choose three outputs: The rank condition (9) is equivalent to the set $\left\{ \begin{bmatrix} c_{k11} \\ 2c_{k12R} \\ -2c_{k12I} \end{bmatrix} \right\}$ compose a persistent excitation (PE) signal. The PE condition plays a significant role in identification problems, since it is related to the convergence of estimate parameters. The PE condition requires that the signal rotates sufficiently in the space [25]. Therefore, given three estimate parameters, we choose three outputs to guarantee the convergence of the estimate parameters.

B. Identification of Pure States for Three-Level Systems

Consider the identification of the following three-level state

$$\begin{aligned} |\psi\rangle &= a_0|0\rangle + a_1|1\rangle + a_2|2\rangle \\ &\triangleq a_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{10}$$

Define the outputs

$$y_k = \langle\psi|C_k|\psi\rangle, \ k = 1, 2, \ldots, 5 \tag{11}$$

where

$$C_k = \begin{bmatrix} c_{k11} & c_{k12R}+ic_{k12I} & c_{k13R}+ic_{k13I} \\ c_{k12R}-ic_{k12I} & 0 & 0 \\ c_{k13R}-ic_{k13I} & 0 & 0 \end{bmatrix} \tag{12}$$

with c_{k11}, c_{k12R}, c_{k12I}, c_{k13R} and c_{k13I} being real numbers. We can obtain

$$\begin{aligned} y_k = \ & c_{k11}|a_0|^2 + 2c_{k12R}Re\{a_0^* a_1\} - 2c_{k12I}Im\{a_0^* a_1\} \\ & + 2c_{k13R}Re\{a_0^* a_2\} - 2c_{k13I}Im\{a_0^* a_2\}. \end{aligned} \tag{13}$$

Denote

$$\begin{aligned} & \theta_1 = |a_0|^2, \ \theta_2 = Re\{a_0^* a_1\}, \ \theta_3 = Im\{a_0^* a_1\}, \\ & \theta_4 = Re\{a_0^* a_2\}, \ \theta_5 = Im\{a_0^* a_2\}. \end{aligned} \tag{14}$$

(13) can be written as

$$y_k = c_{k11}\theta_1 + 2c_{k12R}\theta_2 - 2c_{k12I}\theta_3 + 2c_{k13R}\theta_4 - 2c_{k13I}\theta_5, \tag{15}$$

i.e.,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = B \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \tag{16}$$

where

$$B = \begin{bmatrix} c_{111} & 2c_{112R} & -2c_{112I} & 2c_{113R} & -2c_{113I} \\ c_{211} & 2c_{212R} & -2c_{212I} & 2c_{213R} & -2c_{213I} \\ c_{311} & 2c_{312R} & -2c_{312I} & 2c_{313R} & -2c_{313I} \\ c_{411} & 2c_{412R} & -2c_{412I} & 2c_{413R} & -2c_{413I} \\ c_{511} & 2c_{512R} & -2c_{512I} & 2c_{513R} & -2c_{513I} \end{bmatrix}. \tag{17}$$

Similar to the case of two-level systems, if

$$rank\left\{\begin{bmatrix} c_{k11} \\ 2c_{k12R} \\ -2c_{k12I} \\ 2c_{k13R} \\ -2c_{k13I} \end{bmatrix}\right\} = 5, \tag{18}$$

$\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{bmatrix}^T$ can be uniquely determined, i.e., $|a_0|$, $a_0^*a_1$ and $a_0^*a_2$ can be obtained. Furthermore, we can deduce $|a_1|$, $|a_2|$ and the relative phases between a_1 and a_0 and between a_2 and a_0, $\angle\{a_0^*a_1\}$ and $\angle\{a_0^*a_2\}$. Therefore, the state $|\psi\rangle$ can be reconstructed in the sense that global phase has no physical meaning.

Remark 4: The above discussion assumes $a_0 \neq 0$. In practical experiments, we can firstly measure the observable

$$C_1 = \begin{bmatrix} c_{111} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

If we obtain $y_1 = 0$, we have $a_0 = 0$. In the subsequent procedure, the three-level state identification can be resolved by treating it as a two-level state identification problem. Specifically, we can reconstruct the state by measuring the following three observables

$$C_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{k22} & c_{k23R}+ic_{k23I} \\ 0 & c_{k23R}-ic_{k23I} & 0 \end{bmatrix}, \ k = 2,3,4 \tag{20}$$

where c_{k22}, c_{k23R} and c_{k23I} are real numbers.

C. Identification of Pure States for N-Level Systems

Based on the discussions in section II-A and section II-B, we can deduce the following identification procedure for the n-level pure state

$$\begin{aligned} |\psi\rangle &= a_0|0\rangle + a_1|1\rangle + \cdots + a_{n-1}|n-1\rangle \\ &\triangleq a_0\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_1\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{n-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned} \tag{21}$$

Step 1: Measure the following observable C_1

$$C_1 = \begin{bmatrix} c_{111} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{22}$$

If $y_1 = \langle\psi|C_1|\psi\rangle \neq 0$, then $l = 1$, go to step 2; Otherwise, we can obtain $a_0 = 0$ and continue measuring the following observable C_2

$$C_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & c_{222} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{23}$$

If $y_2 = \langle\psi|C_2|\psi\rangle \neq 0$, then $l = 2$, go to step 2; Otherwise, we can obtain $a_1 = 0$ and continue measuring the observable with only one diagonal element until $y_l \neq 0$.

Step 2: Obtain the remaining $2(n-l)$ outputs by measuring the following observables

$$C_k = \begin{bmatrix} 0 & & 0 & & \\ & c_{kll} & c_{kll+1R}+ic_{kll+1I} & \cdots & c_{klnR}+ic_{klnI} \\ & c_{kll+1R}-ic_{kll+1I} & 0 & \cdots & 0 \\ 0 & \vdots & \vdots & \cdots & \vdots \\ & c_{klnR}-ic_{klnI} & 0 & \cdots & 0 \end{bmatrix}, \quad k = l+1, \ldots, 2n-l \tag{24}$$

with

$$rank\{D\} = 2(n-l)+1 \tag{25}$$

where

$$D = \begin{bmatrix} c_{lll} & 0 & 0 & \cdots & 0 & 0 \\ c_{l+1ll} & 2c_{l+1ll+1R} & -2c_{l+1ll+1I} & \cdots & 2c_{l+1lnR} & -2c_{l+1lnI} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{2n-lll} & 2c_{2n-lll+1R} & -2c_{2n-lll+1I} & \cdots & 2c_{2n-llnR} & -2c_{2n-llnI} \end{bmatrix}.$$

Step 3: Denote

$$\theta_l = |a_{l-1}|^2, \ \theta_{l+1} = Re\left\{a_{l-1}^*a_l\right\}, \ \theta_{l+2} = Im\left\{a_{l-1}^*a_l\right\}, \ \ldots$$
$$\theta_{2n-l-1} = Re\left\{a_{l-1}^*a_{n-1}\right\}, \theta_{2n-l} = Im\left\{a_{l-1}^*a_{n-1}\right\}. \tag{26}$$

We have

$$\begin{bmatrix} y_l \\ y_{l+1} \\ y_{l+2} \\ \vdots \\ y_{2n-l} \end{bmatrix} = D\begin{bmatrix} \theta_l \\ \theta_{l+1} \\ \theta_{l+2} \\ \vdots \\ \theta_{2n-l} \end{bmatrix}. \tag{27}$$

Therefore, we can determine $\theta_l, \theta_{l+1}, \ldots, \theta_{2n-l}$. Furthermore, we can obtain

$$\begin{aligned} &|a_{l-1}|, \ |a_l|, \ \ldots, \ |a_{n-1}|, \\ &\angle\{a_{l-1}^*a_l\}, \ \angle\{a_{l-1}^*a_{l+1}\}, \ \ldots, \ \angle\{a_{l-1}^*a_{n-1}\}. \end{aligned} \tag{28}$$

We can reconstruct the n-level pure state based on the obtained result (28).

III. Real-valued Dynamics Formulation

A. Dynamics of Two-Level Systems

Consider a two-level system described by

$$i\frac{d|\psi\rangle}{dt} = [H_0 + \sigma_x u_x(t) + \sigma_y u_y(t)]|\psi\rangle \tag{29}$$

where

$$H_0 = \frac{1}{2}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{30}$$

is the internal Hamiltonian with λ_1 and λ_2 representing the possible values for the energy. The interaction Hamiltonians σ_x and σ_y are the pauli matrices and they play the role of transferring the state populations between the two eigenstates. The Pauli matrices are

$$\sigma_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{31}$$

For a two-level particle, the density matrix ρ can be decomposed as

$$\rho = \frac{1}{2}I + m_x\sigma_x + m_y\sigma_y + m_z\sigma_z. \tag{32}$$

Considering the system (29), the density matrix ρ should satisfy the Liouville's equation

$$i\frac{d\rho}{dt} = [H_0 + \sigma_x u_x(t) + \sigma_y u_y(t),\ \rho]. \tag{33}$$

From (32) and (33), we can obtain the following Bloch equation (see [26] and [27] for the interpretations of Bloch sphere and Bloch equation)

$$\begin{aligned} \frac{dm_x}{dt} &= -\frac{1}{2}[m_y(\lambda_1 - \lambda_2) - 2m_z u_y(t)] \\ \frac{dm_y}{dt} &= \frac{1}{2}[m_x(\lambda_1 - \lambda_2) - 2m_z u_x(t)] \\ \frac{dm_z}{dt} &= m_y u_x(t) - m_x u_y(t). \end{aligned} \tag{34}$$

Furthermore, according to the discussion on the pure state identification, a two-level state can be uniquely determined by the three parameters θ_1, θ_2 and θ_3 defined as (5). Hence, the density matrix for the two-level system (29) can be expressed as

$$\rho = \begin{bmatrix} \theta_1 & \theta_2 - i\theta_3 \\ \theta_2 + i\theta_3 & 1 - \theta_1 \end{bmatrix}. \tag{35}$$

From (32) and (35), we can obtain

$$m_z = 2\theta_1 - 1,\ m_x = 2\theta_2,\ m_y = 2\theta_3. \tag{36}$$

It can be checked that m_x, m_y and m_z satisfy the following normalization property

$$m_x^2 + m_y^2 + m_z^2 = 1. \tag{37}$$

This normalization property implies that the state $m = \begin{bmatrix} m_x & m_y & m_z \end{bmatrix}^T$ evolves on the unit sphere.

B. Dynamics of Three-Level Systems

Consider the system

$$\begin{aligned} i\frac{d|\psi\rangle}{dt} = {} & [H_0 + H_1u_1(t) + H_2u_2(t) + H_3u_3(t) + \\ & H_4u_4(t) + H_5u_5(t) + H_6u_6(t)]\,|\psi\rangle \end{aligned} \tag{38}$$

where

$$H_0 = \frac{1}{2}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{39}$$

represents the internal Hamiltonian. The interactive Hamiltonians are in the forms of

$$\begin{aligned} H_1 &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ H_3 &= \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ H_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \\ H_5 &= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ H_6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \end{aligned} \tag{40}$$

where $H_1\&H_2$, $H_3\&H_4$ and $H_5\&H_6$ play the role of transferring the populations between the eigenstates $|0\rangle\&|1\rangle$, $|0\rangle\&|2\rangle$ and $|1\rangle\&|2\rangle$, respectively. Hence the system is pure sate controllable.

Without loss of generality, consider the case where the state can be uniquely determined by the coherent terms with the ground state $|0\rangle$, i.e., the case where a_0 of the goal state satisfies $a_0 \neq 0$.

The density matrix for the pure sate $|\psi\rangle$ is

$$\rho = \begin{bmatrix} |a_0|^2 & a_0a_1^* & a_0a_2^* \\ a_0^*a_1 & |a_1|^2 & a_1a_2^* \\ a_0^*a_2 & a_1^*a_2 & |a_2|^2 \end{bmatrix}. \tag{41}$$

Based on the discussion on the state identification, the state can be uniquely determined by θ_1, θ_2, θ_3, θ_4 and θ_5 defined as (14). Therefore, the density matrix can be expressed as

$$\rho = \begin{bmatrix} \theta_1 & \theta_2 - i\theta_3 & \theta_4 - i\theta_5 \\ \theta_2 + i\theta_3 & \frac{(\theta_2)^2 + (\theta_3)^2}{\theta_1} & \frac{(\theta_4\theta_2 + \theta_3\theta_5) + i(\theta_3\theta_4 - \theta_5\theta_2)}{\theta_1} \\ \theta_4 + i\theta_5 & \frac{(\theta_4\theta_2 + \theta_3\theta_5) - i(\theta_3\theta_4 - \theta_5\theta_2)}{\theta_1} & \frac{(\theta_4)^2 + (\theta_5)^2}{\theta_1} \end{bmatrix}. \tag{42}$$

The density matrix ρ should satisfy the Liouville's equation

$$\begin{aligned} i\frac{d\rho}{dt} = {} & [H_0 + H_1u_1(t) + H_2u_2(t) + H_3u_3(t) + \\ & H_4u_4(t) + H_5u_5(t) + H_6u_6(t),\ \rho]. \end{aligned} \tag{43}$$

From (42) and (43), we can obtain

$$\begin{aligned} \frac{d\theta_1}{dt} = {} & \theta_3u_1(t) - \theta_2u_2(t) + \theta_5u_3(t) - \theta_4u_4(t) \\ \frac{d\theta_2}{dt} = {} & \frac{1}{2}\Bigg[(\lambda_2 - \lambda_1)\theta_3 + \left(\theta_1 - \frac{\theta_2^2 + \theta_3^2}{\theta_1}\right)u_2(t) - \frac{\theta_3\theta_4 - \theta_5\theta_2}{\theta_1}u_3(t) \\ & - \frac{\theta_4\theta_2 + \theta_3\theta_5}{\theta_1}u_4(t) + \theta_5u_5(t) - \theta_4u_6(t)\Bigg] \end{aligned}$$

$$\begin{aligned} \frac{d\theta_3}{dt} = {} & \frac{1}{2}\Bigg[(\lambda_1 - \lambda_2)\theta_2 + \left(\frac{\theta_2^2 + \theta_3^2}{\theta_1} - \theta_1\right)u_1(t) + \frac{\theta_4\theta_2 + \theta_3\theta_5}{\theta_1}u_3(t) \\ & - \frac{\theta_3\theta_4 - \theta_5\theta_2}{\theta_1}u_4(t) - \theta_4u_5(t) - \theta_5u_6(t)\Bigg] \end{aligned}$$

$$\begin{aligned} \frac{d\theta_4}{dt} = {} & \frac{1}{2}\Bigg[(\lambda_3 - \lambda_1)\,\theta_5 - \frac{\theta_3\theta_4 - \theta_5\theta_2}{\theta_1}u_1(t) - \frac{\theta_4\theta_2 + \theta_3\theta_5}{\theta_1}u_2(t) \\ & + \left(\theta_1 - \frac{\theta_4^2 + \theta_5^2}{\theta_1}\right)u_4(t) + \theta_3u_5(t) + \theta_2u_6(t)\Bigg] \end{aligned}$$

$$\frac{d\theta_5}{dt} = \frac{1}{2}\left[(\lambda_1-\lambda_3)\theta_4+\frac{\theta_4\theta_2+\theta_3\theta_5}{\theta_1}u_1(t)+\frac{\theta_3\theta_4-\theta_5\theta_2}{\theta_1}u_2(t)\right.$$
$$\left.+\left(\frac{\theta_4^2+\theta_5^2}{\theta_1}-\theta_1\right)u_3(t)-\theta_2u_5(t)+\theta_3u_6(t)\right]. \quad (44)$$

Define

$$m_1=2\theta_1-1,\ m_2=2\theta_2,\ m_3=2\theta_3,\ m_4=2\theta_4,\ m_5=2\theta_5. \quad (45)$$

Similar to the Bloch equation for the two-level system, we have the following real-valued dynamics for the three-level system

$$\begin{aligned}
\frac{dm_1}{dt}&=m_3u_1(t)-m_2u_2(t)+m_5u_3(t)-m_4u_4(t)\\
\frac{dm_2}{dt}&=\frac{1}{2}\Big[-\lambda_1m_3+\lambda_2m_3-\frac{m_2^2+m_3^2}{1+m_1}u_2(t)+(1+m_1)u_2(t)-\\
&\quad\frac{m_3m_4-m_5m_2}{1+m_1}u_3(t)-\frac{m_4m_2+m_3m_5}{1+m_1}u_4(t)+\\
&\quad m_5u_5(t)-m_4u_6(t)\Big]\\
\frac{dm_3}{dt}&=\frac{1}{2}\Big[\lambda_1m_2-\lambda_2m_2+\frac{m_2^2+m_3^2}{1+m_1}u_1(t)-(1+m_1)u_1(t)+\\
&\quad\frac{m_4m_2+m_3m_5}{1+m_1}u_3(t)-\frac{m_3m_4-m_5m_2}{1+m_1}u_4(t)-\\
&\quad m_4u_5(t)-m_5u_6(t)\Big]\\
\frac{dm_4}{dt}&=\frac{1}{2}\Big[-\lambda_1m_5+\lambda_3m_5-\frac{m_3m_4-m_5m_2}{1+m_1}u_1(t)-\\
&\quad\frac{m_4m_2+m_3m_5}{1+m_1}u_2(t)-\frac{m_4^2+m_5^2}{1+m_1}u_4(t)+\\
&\quad(1+m_1)u_4(t)+m_3u_5(t)+m_2u_6(t)\Big]\\
\frac{dm_5}{dt}&=\frac{1}{2}\Big[\lambda_1m_4-\lambda_3m_4+\frac{m_4m_2+m_3m_5}{1+m_1}u_1(t)+\\
&\quad\frac{m_3m_4-m_5m_2}{1+m_1}u_2(t)+\frac{m_4^2+m_5^2}{1+m_1}u_3(t)-\\
&\quad(1+m_1)u_3(t)-m_2u_5(t)+m_3u_6(t)\Big]. \quad (46)
\end{aligned}$$

It can be checked that the following normalization property is satisfied

$$m_1^2+m_2^2+m_3^2+m_4^2+m_5^2=1. \quad (47)$$

This normalization property implies that the state $m = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5\end{bmatrix}^T$ evolves on the $5-$dimensional unit sphere.

IV. Control of Two-Level Systems

A. Control Design

Consider the two-level system (29) with the Bloch equation (34). Our objective is to develop a controller such that the state starting from arbitrary initial state can be driven to the expected state denoted as $\begin{bmatrix} m_{xf} & m_{yf} & m_{zf}\end{bmatrix}^T = \begin{bmatrix} x & y & z\end{bmatrix}^T$. The following theorem indicates how the control objective can be achieved.

Theorem 1: Consider the two-level system (29) with the Bloch equation (34). The state $m = \begin{bmatrix} m_x & m_y & m_z\end{bmatrix}^T$ starting from any initial state $m_0 = \begin{bmatrix} m_{x0} & m_{y0} & m_{z0}\end{bmatrix}^T$ converges exponentially to the goal state $m_f = \begin{bmatrix} x & y & z\end{bmatrix}^T$ with the control signals $u_x(t)$ and $u_y(t)$ designed according to the following principle:

1) If $|ym_z-zm_y| \geq |zm_x-xm_z|$, choose

$$u_x(t) = \frac{\frac{1}{2}e_1m_y(\lambda_1-\lambda_2)-\frac{1}{2}e_2m_x(\lambda_1-\lambda_2)-K_x(e_1^2+e_2^2+e_3^2)}{ym_z-zm_y},$$
$$u_y(t) = 0; \quad (48)$$

2) If $|zm_x-xm_z| > |ym_z-zm_y|$, choose

$$u_x(t)=0,$$
$$u_y(t)=\frac{\frac{1}{2}e_1m_y(\lambda_1-\lambda_2)-\frac{1}{2}e_2m_x(\lambda_1-\lambda_2)-K_y(e_1^2+e_2^2+e_3^2)}{zm_x-xm_z} \quad (49)$$

where $K_x, K_y > 0$ are the control gains.

Proof: Define the state errors as

$$e_1 = m_x - x,\ e_2 = m_y - y,\ e_3 = m_z - z. \quad (50)$$

Choose

$$J = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2. \quad (51)$$

The derivative of J is

$$\frac{dJ}{dt} = e_1\frac{dm_x}{dt} + e_2\frac{dm_y}{dt} + e_3\frac{dm_z}{dt}. \quad (52)$$

By plugging (34) into (52), we have

$$\begin{aligned}\frac{dJ}{dt} &= -\frac{1}{2}e_1m_y(\lambda_1-\lambda_2)+\frac{1}{2}e_2m_x(\lambda_1-\lambda_2)+\\ &\quad (ym_z-zm_y)u_x(t)+(zm_x-xm_z)u_y(t). \quad (53)\end{aligned}$$

It is obvious that the coefficients of $u_x(t)$ and $u_y(t)$ in (53) are simultaneously zero if and only if

$$m_x = x,\ m_y = y,\ m_z = z, \quad (54)$$

or

$$m_x = -x,\ m_y = -y,\ m_z = -z, \quad (55)$$

since the normalization property (37) should be satisfied. Graphically, with the condition (55), the state is on the opposite point of the expected state on the sphere and thus $J = max\{J(m_x, m_y, m_z)\}$. Therefore, if the initial state satisfies (55), we can add arbitrary control effort to disturb the initial state such that $J < max\{J(m_x, m_y, m_z)\}$. The subsequent task is to design control laws to decrease J such that the state can be driven to the goal state.

With the control signals u_x and u_y in the forms of (48) and (49), we can obtain

$$\begin{aligned}\frac{dJ}{dt} &= -K_{x,y}(e_1^2+e_2^2+e_3^2)\\ &= -2K_{x,y}J. \quad (56)\end{aligned}$$

Hence, the errors e_1, e_2 and e_3 converge exponentially to zero. ■

Remark 5: We choose the control signals in the forms of (48) and (49) in order to obtain small control signals and avoid the singular problem in the case that the coefficient of $u_x(t)$ or $u_y(t)$ is zero. However, due to switch between the two control signals, this control design may put forward high requirements to control lasers or electromagnetic control fields, or other control equipments.

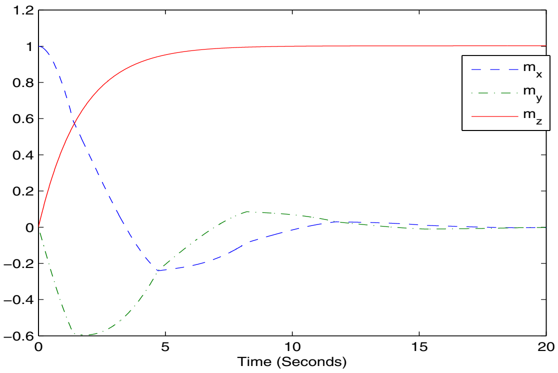

Fig. 1. State convergence

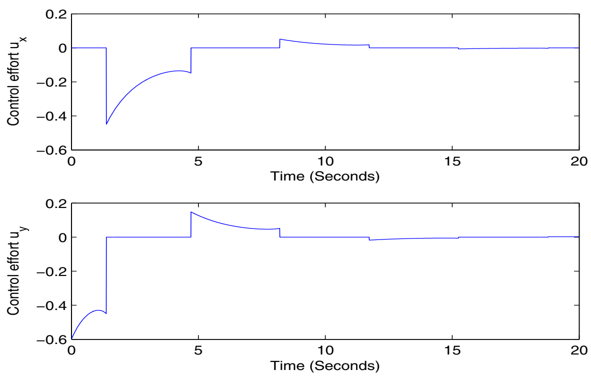

Fig. 2. Control efforts for the state convergence

Remark 6: Unlike the results obtained based on the Schrödinger equation in [2], the goal state here can be an arbitrary state instead of an eigenstate of the internal Hamiltonian. Moreover, the exponential convergence of the state to the goal state can be achieved without the requirement that the internal Hamiltonian should be nondegenerated.

B. Simulation Study

In this simulation, the parameters of the internal Hamiltonian are given as $\lambda_1 = 1$ and $\lambda_2 = 2$. The initial state is $\begin{bmatrix} m_{x0} & m_{y0} & m_{z0} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. We choose the expected state as $\begin{bmatrix} m_{xf} & m_{yf} & m_{zf} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The control gains are $K_x = 0.3$ and $K_y = 0.3$.

From Fig. 1, we can see that the quantum state is driven to the goal state. Fig. 2 shows the control effort to achieve this objective.

V. Conclusions

In this paper, we have reexplained and presented a pure state identification strategy. Based on the discussions on the pure state identification, the real-valued dynamics for two-level and three-level systems have been discussed in detail. A control scheme based on Lyapunov approach has been proposed for the obtained real-valued dynamics. The control scheme has great advantages over the Lyapunov control based on the Schrödinger equation in the sense that the state convergence to an arbitrary state can be achieved without any requirements on the internal Hamiltonian. As to higher level quantum systems, the derivation of real-valued dynamics and the control method are quite similar to the ones presented in this paper. In this case, the results obtained in this paper provide a systematic strategy for the control of quantum systems.

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