Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Implicit Lyapunov control of closed quantum systems

Shouwei Zhao, Hai Lin, Jitao Sun and Zhengui Xue

Abstract— **In this paper, we investigate the state convergence problem for closed quantum systems under degenerate cases. An implicit Lyapunov-based control strategy is proposed for the convergence analysis of finite dimensional bilinear Schrödinger equations. The degenerate cases that the systems do not satisfy the strong regular condition [17] and the condition $\langle\phi_i|H_1|\phi_j\rangle \neq 0,\ i,j \neq k$ for eigenstates $\phi_i,\ \phi_j$ of H_0 different from target state ϕ_k, are considered. First the Lyapunov function is defined by the implicit function and the existence is guaranteed by a fixed point theorem. Then the convergence analysis is investigated by the LaSalle invariance principle. Finally, an example is provided to show the effectiveness of proposed results.**

I. INTRODUCTION

Driven by scientific inquiry and the demands of advancing technology, the past decades have seen increasing theoretical and experimental research towards control of quantum systems. Control of quantum phenomena is essential in successful applications of quantum systems in a wide variety of areas such as quantum computation, quantum chemistry, nano-scale materials, NMR and Bose-Einstein condensates. Hence, recent years have seen a great deal of research efforts in the development of quantum control theory and many results about controllability and control methods have been obtained, see [1], [2] and references therein.

Quantum control can be roughly divided into two categories. The first category falls into the open-loop control scheme for which many control strategies utilize some forms of model-based feedback, both geometry based and optimization based [3]-[4]. This method is relatively simple to implement and many applications of open-loop Hamiltonian engineering in diverse areas from quantum chemistry to quantum information processing. But the question of when, i.e., for which systems and objectives, the method is effective and when it is not, has not been answered satisfactorily. The second category is of closed-loop control, for example, state reduction and stabilization using feedback from measurement is applied through a combination of geometric control and classical probabilistic techniques[5]-[10]. This method is somehow more exact and intuitive to control the given quantum systems than the open-loop control. However, closed-loop feedback control is a nontrivial problem as feedback requires measurements and any observation of a quantum system generally disturbs its state, and often results in a loss of coherence that can reduce the systems to mostly classical behavior. In order to mitigate this backaction, measurement and feedback in quantum systems lead to much more complicated models and dynamics than the Schrödinger equations.

In this paper, we consider Lyapunov open-loop control, where a Lyapunov function is defined and feedback from a model is used to generate controls to minimize its value. Lyapunov control has been widely used in feedback control to analyze the stability of closed-loop systems. Several recent papers have proposed the application of Lyapunov control designs to quantum systems [11]-[19]. Since the quantum measurement and feedback would lead to more complicated model than Schrödinger equations [15] and the super-short control time required by some quantum dynamics restricts the application of observation and feedback [11], currently the open loop Lyapunov control remains dominant [11]. Such open loop control needs to be first simulated. From simulation one obtains a control signal which is then, in practice, applied in open loop control. Lyapunov open loop control has proved to be a simple and effective method in achieving ideal control performance. Several papers on Lyapunov control for quantum systems only considered the control of quantum systems with target states that are eigenstates of the free Hamiltonian H_0, and therefore fixed points of the dynamical system [11],[12]. While in the degenerate cases that target states are not eigenstates of H_0, i.e., evolve with time, the issue of convergence analysis of such systems is investigated by (implicit) Lyapunov technique and the LaSalle invariance principle [20]. This means that the problem is reformulated to asymptotic convergence of the system's actual trajectory to that of the time-dependent target state [13]-[16]. Moreover, some authors considered the Lyapunov control for mixed-state quantum systems in the notion of orbit convergence or the trajectory tracking problem [17]-[19].

It is well-known that Lyapunov functions based on the the average value of an imaginary mechanical quantity is of great significance in Lyapunov control of bilinear Schrödinger equations [11]-[13]. In [11], the authors considered the convergence analysis of Schrödinger equations based on three kinds of functions. But the invariant set is generally large, the invariant principle is not sufficient to conclude the asymptotic convergence. Particularly, the authors utilized Lyapunov function based on the average value of the imaginary mechanical quantity to give a generalized

Shouwei Zhao is with Dept. of Mathematics, Tongji University, 200092, China and Dept. of Electrical and Computer Engineering, National University of Singapore, Singapore. zhaoshouwei@gmail.com

Hai Lin is with Dept. of Electrical and Computer Engineering, National University of Singapore, Singapore. elelh@nus.edu.sg

Jitao Sun is with Dept. of Mathematics, Tongji University, 200092, China. sunjt@sh163.net

Zhengui Xue is with NUS Graduate School for Integrative Sciences and Engineering, National University of Singapore, 117456, Singapore. zhenguixue@nus.edu.sg

The financial supports from Singapore Ministry of Education's AcRF Tier 1 funding, TDSI, and TL and the National Natural Science Foundation of China under Grant 60874027 are gratefully acknowledged.

978-1-4244-3872-3/09/$25.00 ©2009 IEEE

theorem(Theorem 6) on the largest invariant set of the closed-loop system where the eigenvalue of H_0 corresponding to the eigenvector $|\lambda_i\rangle$ must satisfy the following strong regular condition

$$\omega_{ij} \neq \omega_{lm},\ (i,j) \neq (l,m),$$

with $\omega_{ij} = \lambda_i - \lambda_j$. It should be noticed that this strong regular condition may fail to satisfy for many controllable systems [17]. On the other hand, from Theorem 6 in [11], it can be seen that if $\langle\lambda_i|H_1|\lambda_j\rangle \neq 0$ for $i \neq j \neq k$, the systems will be asymptotically stable using the proposed control. But if this condition does not hold, then it is difficult to drive the systems to the goal state using the control law designed in that paper. This motivates us to pursue another strategy for stability analysis and design under the above degenerate cases. Therefore, in the current paper, we investigate the asymptotic convergence of Schrödinger equations by implicit Lyapunov techniques and the LaSalle invariant principle under the above-mentioned degenerate cases. The main difficult lies in how to apply the LaSalle invariant principle combined with proper assumptions to drive systems asymptotically converge to the target state in the degenerate cases.

The rest of this paper is organized as follows. Some preliminaries are presented in Section II. The feedback control law is designed using implicit Lyapunov techniques, in which the Lyapunov function is based on the average value of the imaginary mechanical quantity and convergence analysis is derived by LaSalle invariance principle in Section III. A numerical example and simulation studies are discussed to demonstrate the effectiveness of the proposed method in Section IV. Some concluding remarks are drawn in Section V.

II. Preliminaries

Consider the following bilinear Schrödinger equation

$$\begin{aligned} i\frac{d}{dt}\Psi &= (H_0 + u(t)H_1)\Psi, \\ \Psi|_{t=0} &= \Psi_0,\ ||\Psi_0|| = 1, \end{aligned} \tag{1}$$

where H_0 is the free Hamiltonian, and H_1 is the interaction Hamiltonian; furthermore, both of them are Hermitian matrices. The state of the system verifies the conservation of probability: $||\Psi(t)|| = 1, \forall t \geq 0$, which means that the state is on the unit sphere of $\mathbb{C}^N : \mathbb{S} = \{x \in \mathbb{C}^N : ||x|| = 1\}$.

In this paper, the main objective is to consider the asymptotic convergence of closed quantum systems under the above degenerate cases using implicit Lyapunov techniques. We choose the Lyapunov function based on the average value of the imaginary mechanical quantity. First, for $r \in \mathbb{R}$, denote by $(\lambda_{k,r})_{1\leq k\leq N}$ the eigenvalues of the operator $H_0 + rH_1$, with $\lambda_{1,r} \leq \cdots \leq \lambda_{N,r}$ and by $(\phi_{k,r})_{1\leq k\leq N}$ the associated normalized eigenvectors:

$$(H_0 + rH_1)\phi_{k,r} = \lambda_{k,r}\phi_{k,r}.$$

We assume that for any small $r \neq 0$, $\langle\phi_{i,r}|H_1|\phi_{j,r}\rangle \neq 0$ for $i \neq j \neq k$. Let $\phi_{k,r}$ be the time-varying target state instead of ϕ_k. For simplicity, the kth eigenvector $\phi_k = \phi$ for some $k \in \{1, 2, \cdots, N\}$ is denoted to be the goal state, then we assume that the kth eigenspace of the free Hamiltonian H_0 is of dimension 1, so that the target state (i.e., kth eigenstate of the system) is defined without any ambiguity.

Then the control strategy based on implicit Lyapunov method is investigated under the degenerate cases. By introducing the feedback controllers, the state of controlled systems is steered to a moving target state $\phi_{k,r(t)}$ instead of ϕ_k, where $r(t)$ is defined implicitly by the state of the systems. The goal is to make $\phi_{k,r(t)}$ converge slowly to ϕ_k and at the same time, using the feedback controller to stabilize the system state as fast as possible around the vector function $\phi_{k,r(t)}$. The basic idea is shown by a direct way in Fig. 1 in [15].

III. The Lyapunov method based on the average value of an imaginary mechanical quantity

In this section, we first recall the Lyapunov function in the original reference [12] and develop the idea of the controller design. Suppose that the Hermitian operator P is a mechanical quantity of the quantum system. According to quantum theory, if the system is in an eigenstate of P, then the average value of P is the eigenvalue corresponding to the eigenstate of P. From this point of view, it is reasonable to consider the average value of P as a Lyapunov function.

$$V_0(\Psi) := \langle\Psi|P|\Psi\rangle \tag{2}$$

In [12], S. Grivopoulos and B. Bamieh proved the following important lemma in control theory via variational calculus.

Lemma 1: [12] With the constraint condition $\langle\Psi|\Psi\rangle = 1$, the set of critical points of the Lyapunov function $V_0(\Psi) = \langle\Psi|P|\Psi\rangle$ is given by the normalized eigenvectors of P. The eigenvectors with the largest eigenvalue are the maxima of V_0, the eigenvectors with the smallest eigenvalue are the minima and all others are saddle points.

According to Lemma 1, if the goal state ϕ_k corresponds to the smallest eigenvalue l_k of P, then $V_0(\Psi) = \langle\Psi|P|\Psi\rangle$ is equal to l_k at $|\Psi\rangle = \phi_k$. Thus, when the designed control fields make V_0 decrease continually to l_k, the state of the system will be possibly driven to ϕ_k, that is, the goal state. This idea will be used to design the control fields and construct the imaginary mechanical quantity P.

A. Controller design

Corresponding to the target state $\phi_{k,r}$ for any $r \in (0, r^*]$, we construct the analytic matrix P_r such that $\phi_{k,r}$ is the eigenvector of P_r and the corresponding eigenvalue $l_{k,r}$ is the smallest one of all the eigenvalues of P_r. The process to find such matrices can be found in [11].

Now we define the following Lyapunov function based on the average value of an imaginary mechanical quantity

$$V_1(\Psi) := \langle\Psi|P_r|\Psi\rangle, \tag{3}$$

And the function $\Psi \mapsto r(\Psi)$ is implicitly defined as follows,

$$r(\Psi(t)) = r(\Psi) := \theta(\langle\Psi|P_r|\Psi\rangle - l_k), \tag{4}$$

for a slowly varying real function θ. Noting that under the assumption of non-degeneracy for the kth eigenstate of H_0+rH_1 for $r\in[0,r^*]$, $\phi_{k,r}$ and P_r are analytic mappings of the parameter $r\in[0,r^*]$ [16]. In particular, we can consider the derivative of the map $r\mapsto P_r$ at least in the interval $[0,r^*]$. Denote by $\frac{dP_r}{dr}|_{r_0}$ the derivative of this map at the point r_0. Furthermore, as the dependence of $\phi_{k,r}$ with respect to r is analytic, $\frac{dP_r}{dr}$ is bounded on $[0,r^*]$ and thus

$$C:=\max\{||\frac{dP_r}{dr}|_{r_0}||;\ r_0\in[0,r^*]\}<\infty$$

A simple computation yields that

$$\begin{aligned}\frac{d}{dr}\theta(V_1(\Psi)-l_k)=&\frac{d}{dr}\theta(\langle\Psi|P_r|\Psi\rangle-l_k)\\=&\theta'\cdot(\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle).\end{aligned}\tag{5}$$

Choosing the function θ such that $||\theta'||_\infty$ is small enough and due to the fact that $C<\infty$, the function

$$\alpha\in[0,r^*]\mapsto\theta(V_1(\Psi)-l_k):=\theta(\langle\Psi|P_\alpha|\Psi\rangle-l_k)$$

will be contraction for fixed $\Psi\in\mathbb{S}$. Thus, for any fixed point $\Psi\in\mathbb{S}$, there exists a unique $r(\Psi)\in[0,r^*]$ such that (4) is satisfied.

Let us explain the one-to-one correspondence, $\Psi\in\mathbb{S}\mapsto r(\Psi)\in[0,r^*]$ by the implicit function theorem. Consider the following function

$$F(r,\Psi):=r-\theta(V_1(\Psi)-l_k).$$

F is regular with respect to r and Ψ, and for a fixed $\Psi\in\mathbb{S}$ we have $F(r(\Psi),\Psi)=0$; furthermore, we have

$$\frac{d}{dr}F(r,\Psi)=1-\theta'(\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle),$$

which is non-zero for θ which ensures $||\theta'||_\infty$ to be small enough. Thus, with the implicit function theorem and the uniqueness of the application $\Psi\mapsto r(\Psi)$, we have the following existence result:

Lemma 2: Let $\theta\in C^\infty(\mathbb{S};[0,r^*])$ be such that $\theta(0)=0$, $\theta(s)>0$, $\forall s>0$, $||\theta'||_\infty<\frac{1}{C^*}$ where $C^*:=1+\max\{||\frac{dP_r}{dr}|_{r_0}||;\ r_0\in[0,r^*]\}<\infty$. Then there exists a unique map $r\in C^\infty(\mathbb{S};[0,r^*])$ such that for every $\Psi\in\mathbb{S}$, $r(\Psi)=\theta(\langle\Psi|P_r|\Psi\rangle-l_k)$, with $r(\phi_k)=0$.

Assumption 1: There exists a r^* such that for every $r\in(0,r^*]$, we have $\lambda_{1,r}<\cdots<\lambda_{N,r}$ and the Hamiltonian H_0+rH_1 is not $\lambda_{k,r}$-degenerate. Let $\phi_{k,r}$ be an eigenstate of $H_0+r(\Psi)H_1$ and be also the goal state. We assume that all the eigenstates of $H_0+r(\Psi)H_1$ satisfying $\langle\phi_{j,r}|H_1|\phi_{i,r}\rangle\neq0$, $i,\ j\in\{1,2,\cdots,N\}$ $i,j\neq k$.

In the sequel, we assume that $\theta\in C^\infty(\mathbb{S};[0,r^*])$ and $||\theta'||_\infty<\frac{1}{2C^*}$. According to the controller to be designed with $u(\Psi(t))=r(\Psi(t))+v(\Psi(t))$, for simplicity, $u(t)=r(t)+v(t)$, the system (1) evolves as follows

$$i\frac{d}{dt}\Psi=(H_0+(r(t)+v(t))H_1)\Psi.$$

Differentiating V_1 with respect to t yields that

$$\begin{aligned}\frac{d}{dt}V_1(\Psi(t))=&\langle\dot\Psi|P_r|\Psi\rangle+\langle\Psi|P_r|\dot\Psi\rangle+\dot r(t)\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle\\=&i\langle\Psi|[H_0+r(t)H_1,P_r]|\Psi\rangle\\&+i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t)\\&+\dot r(t)\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle.\end{aligned}\tag{6}$$

And

$$\begin{aligned}\dot r(t)=&\theta'(V_1-l_k)\{i\langle\Psi|[H_0+r(t)H_1,P_r]|\Psi\rangle\\&+i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t)+\dot r(t)\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle\}.\end{aligned}\tag{7}$$

Let us denote by

$$K(t):=\theta'(V_1-l_k)(\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle).$$

From the assumption that $||\theta'||_\infty<\frac{1}{2C^*}$, $|K(t)|\leq\frac{1}{2}$ for any $t\in[0,+\infty)$. According to (6) and (7) and the condition that $[H_0+r(t)H_1,P_r]=0$, we have

$$(1-K(t))\dot r(t)=\theta'(V_1-l_k)i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t),$$

which means that $\dot r(t)=\frac{\theta'(V_1-l_k)}{1-K(t)}i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t)$. Now, we rewrite (6) as follows:

$$\begin{aligned}\frac{d}{dt}V_1(\Psi(t))&=i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t)\\&\quad+\frac{\theta'(V_1-l_k)}{1-K(t)}i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t)\langle\Psi|\frac{dP_r}{dr}|\Psi\rangle\\&=(1+\frac{K(t)}{1-K(t)})i\langle\Psi|[H_1,P_r]|\Psi\rangle v(t),\end{aligned}\tag{8}$$

where $1+\frac{K(t)}{1-K(t)}>0$ for every $t\geq0$. Thus, design a feedback law as follows

$$v(t)=v(\Psi(t)):=-cf(i\langle\Psi|[H_1,P_r]|\Psi\rangle)\tag{9}$$

with a positive constant c, where the image of function $y=f(x)$ passes the origin of plane $x-y$ monotonically and lies in quadrant I or III. It is clear that with the above controller we have $\frac{dV_1}{dt}\leq0$.

The main purpose of the next section is to provide the convergence analysis of this feedback design under some suitable assumptions. Characterization of the ω-limit set for the closed-loop system will be proposed by the LaSalle invariance principle.

B. Convergence analysis

In this section, we use the LaSalle invariance principle to analyze the convergence of the system (1) with the implicit feedback function. First, let us recall the LaSalle invariance principle [20]:

Lemma 3: [20] For an autonomous dynamical system, $\dot x=f(x)$, let $V(x)$ be a Lyapunov function on the phase space $\Omega=\{x\}$, satisfying $V(x)>0$ for all $x\neq x_0$ and $\dot V(x)\leq0$, and let $\mathcal{O}(x(t))$ be the orbit of $x(t)$ in the phase space. Then the invariant set $E = \{\mathcal{O}|\dot{V}(x(t)) = 0\}$ contains the positive limiting sets of all bounded solutions, i.e., any bounded solution converges to E as $t \to +\infty$.

Based on the controller design above, the convergence analysis of the controlled quantum system is presented.

Theorem 1: Consider the system (1) with the feedback design $u(\Psi(t)) := r(\Psi(t)) + v(\Psi(t))$ where $r(\Psi)$ is given by Lemma 2 and $v(\Psi(t)) := -cf(i\langle\Psi|[H_1, P_r]|\Psi\rangle)$ with a positive constant c. Moreover, let $\theta \in C^\infty(\mathbb{S}; [0, r^*])$ be such that the conditions in Lemma 2 are satisfied. Let us also suppose that

(i) $[H_0 + r(t)H_1, P_r] = 0$,
(ii) $\omega_{ij}^r \neq \omega_{lm}^r,\ (i,j) \neq (l,m)$,
(iii) $l_{r,i} \neq l_{r,j},\ i \neq j$,

where $\omega_{ij}^r = \lambda_i^r - \lambda_j^r$, $\lambda_i^r (i = 1, 2, \cdots, N)$ is the eigenvalue of $H_0 + r(t)H_1$ corresponding to the eigenvector $\phi_{i,r}$. Then the solution of system (1) converges toward $\mathbb{S} := \{\phi_k e^{i\theta}; \theta \in \mathbb{R}\}$ under Assumption 1 in the sense of $\lim_{t\to\infty} dist(\Psi(t), \mathbb{S}) = 0$.

Proof: From the Lyapunov function $V_1(\Psi)$ defined in the previous section and the feedback controller $u(\Psi(t)) = r(\Psi(t)) + v(\Psi(t))$ designed in the Theorem, we have $\frac{dV_1}{dt} \leq 0$. From Lemma 3, the trajectories of the closed-loop system converge to the largest invariant set contained in $\frac{dV_1}{dt} = 0$.

Let us characterize this invariant set. Suppose that Ψ is a solution of the system (1) such that $\frac{dV_1}{dt} = 0$. Then there exists a constant $\bar{V}$ such that $V_1(\Psi) = \bar{V}$. This implies that $r(\Psi)$ is a constant denoted by $r(\Psi) = \bar{r}$ where $\bar{r} := \theta(\bar{V})$. The equation $\frac{dV_1}{dt} = 0$ satisfies if and only if

$$v(\Psi(t)) := -cf(i\langle\Psi|[H_1, P_r]|\Psi\rangle) = 0. \tag{10}$$

Thus the controlled system can be represented by

$$\begin{aligned} i\frac{d}{dt}\Psi &= (H_0 + \bar{r}H_1)\Psi, \\ \Psi|_{t=0} &= \Psi_0,\ ||\Psi_0|| = 1. \end{aligned} \tag{11}$$

There are two cases to be considered.
(i) $\bar{r} = 0$, then $\theta(\bar{V}) = 0$, which means that $\bar{V} = 0$, so we have $\Psi \in \mathbb{S}$ from Lemma 1 and the definition of $V_1(\Psi)$. Then we complete the proof in this case.
(ii) $\bar{r} \neq 0$, which means that $0 < \bar{r} < r^*$. Without loss of generality, we assume that when $t = t_0$, (10) is satisfied. Now to verify that $\Psi(t_0)$ is the point in the invariant set of the close-loop system, we only need to verify that for any $\Delta t \in (0, +\infty)$, $v(\Psi(t_0 + \Delta t)) = v(t_0 + \Delta t) = 0$.

Denote the state of the system at t_0 by

$$\Psi(t_0) = \sum_{i=1}^{N} c_i(t_0)\phi_{i,\bar{r}}, \tag{12}$$

where $\phi_{i,\bar{r}}$ is the ith eigenvector of $H_0 + \bar{r}(\Psi)H_1$. By the property of invariance, $\Psi(t_0 + \Delta t)$ should also satisfy (10) i.e.,

$$\langle\Psi(t_0 + \Delta t)|[H_1, P_{\bar{r}}]|\Psi(t_0 + \Delta t)\rangle = 0. \tag{13}$$

Since Ψ solves the equation (11), we can derive $\Psi(t_0 + \Delta t)$ as follows:

$$\begin{aligned} \Psi(t_0 + \Delta t) &= e^{-i(H_0 + \bar{r}H_1)\Delta t}\Psi(t_0) \\ &= \sum_{i=1}^{N} c_i(t_0)e^{-i(H_0+\bar{r}H_1)\Delta t}\phi_{i,\bar{r}} \end{aligned} \tag{14}$$

Using (13), it follows that

$$\sum_{i,j=1}^{N} (l_{\bar{r},i} - l_{\bar{r},j})c_i(t_0)c_j^*(t_0)e^{i\omega_{ji}^{\bar{r}}\Delta t}\langle\phi_{j,\bar{r}}|H_1|\phi_{i,\bar{r}}\rangle = 0, \tag{15}$$

where $\omega_{ij}^{\bar{r}} = \lambda_i^{\bar{r}} - \lambda_j^{\bar{r}}$, $\lambda_i^{\bar{r}} (i = 1, 2, \cdots, N)$ is the eigenvalue of $H_0 + \bar{r}(t)H_1$ corresponding to the eigenvector $\phi_{i,\bar{r}}$. Furthermore, it can be written in the following simple form

$$\sum_{i,j=1}^{N} (l_{\bar{r},i} - l_{\bar{r},j})\langle\phi_{j,\bar{r}}|\rho(t_0)|\phi_{i,\bar{r}}\rangle\langle\phi_{j,\bar{r}}|H_1|\phi_{i,\bar{r}}\rangle e^{i\omega_{ji}^{\bar{r}}\Delta t} = 0 \tag{16}$$

where $\rho(t_0) := |\Psi(t_0)\rangle\langle\Psi(t_0)|$. From condition (ii) of the Theorem, by the arbitrary of Δt, $e^{i\omega_{ji}^{\bar{r}}\Delta t}$ are linear independent with each other. Then for the eigenvectors $\phi_{i,\bar{r}},\ \phi_{j,\bar{r}}$ such that $\langle\phi_{j,\bar{r}}|H_1|\phi_{i,\bar{r}}\rangle \neq 0$, we only need to construct the matrix $P_{\bar{r}}$ such that $l_{\bar{r},i} \neq l_{\bar{r},j}$. Thus equation (16) can be expressed by

$$\langle\phi_{j,\bar{r}}|\rho(t_0)|\phi_{i,\bar{r}}\rangle\langle\phi_{j,\bar{r}}|H_1|\phi_{i,\bar{r}}\rangle = 0,\ i,j \in \{1, 2, \cdots, N\} \tag{17}$$

If the system satisfies Assumption 1, then $\Psi \in \mathbb{S}$ and we finish the proof of the theorem. ∎

IV. SIMULATIONS

Consider the following system with H_0 and H_1

$$H_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, H_1 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let the second eigenstate of H_0, $\phi_2 = (0\ 1\ 0)^T$ be the target state. And we can see that this system does not satisfy the strong regular condition. If we apply the implicit Lyapunov technique based on the state distance using the feedback control law proposed in [15], simulaitons in Fig. 1 show that the systems can not be stabilized when $c = 1$ and the initial state $\Psi_0 = \frac{1}{\sqrt{3}}(1\ 1\ 1)^T$.

Now we adopt the Lyapunov method based on the average value of the imaginary mechanical quantity. Simple computation yields that Assumption 1 holds and all the conditions in Theorem 1 are satisfied. The first part of the control field $r(\Psi(t))$ is defined implicitly by (4). In order to find this function at each time step, we use a fixed point algorithm by computing iteratively the value of $\theta(V_1)$ and the function $\theta(s)$ is chosen to be $\theta(s) = s/20$. The second part of feedback law is given by (9) with $v(\Psi(t)) = -i(\langle\Psi|[H_1, P_r]|\Psi\rangle)$. The simulations in Fig.2 illustrate the performance of this approach. And for real quantum systems, we use the above control signals from the closed-loop simulations to achieve the state steering.

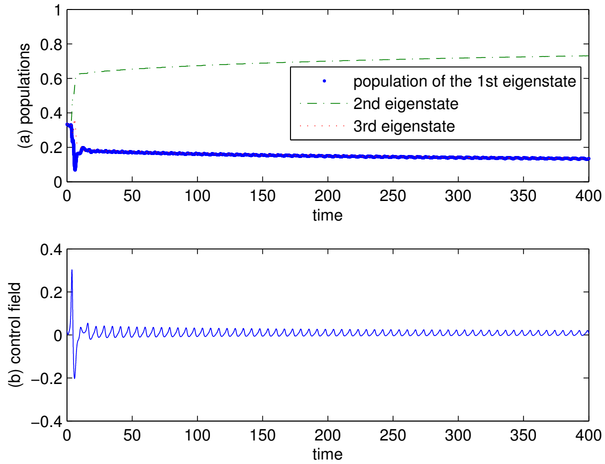

Fig. 1: (a): The population of the system trajectory $\Psi(t)$ solution of the system (1) with feedback design (9) and (15) in [15]. It can be seen that the system can not converge to ϕ_2 the second eigenstate of the internal Hamiltonian as $t \rightarrow +\infty$; (b): the control field $\gamma(\Psi) + v(\Psi)$.

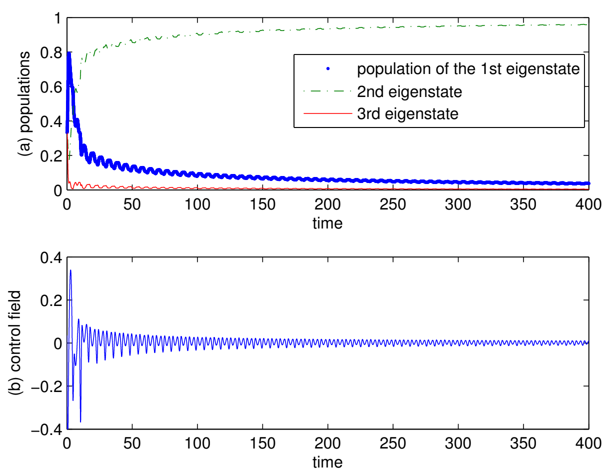

Fig. 2: (a): The population of the system trajectory $\Psi(t)$ solution of the system (1) with feedback design $u(\Psi) = r(\Psi) + v(\Psi)$. It can be seen that the system reaches ϕ_2 the second eigenstate of the internal Hamiltonian; (b): the control field $r(\Psi(t)) + v(\Psi(t))$.

V. Conclusions

A stabilization method for finite dimensional quantum systems and its convergence analysis have been proposed under the degenerate cases that the strong regular condition as well as the condition $\langle\phi_i|H_1|\phi_j\rangle \neq 0,\ i,j \neq k$ do not hold. By adopting the Lyapunov function based on the average value of the imaginary mechanical quantity, feedback control laws based on implicit Lyapunov functions have been designed. Moreover, convergence analysis has been investigated via the LaSalle invariance principle. Additionally, simulation studies have been provided to show the effectiveness of the proposed results.

References

[1] M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge; 2000.

[2] D. D'Alessandro, *Introduction to quantum control and dynamics*, Chapman & Hall/CRC; 2007.

[3] P. Vettori, "On the convergence of a feedback control strategy for multilevel quantum systems", *in Proceedings of the MTNS Conference*, Notre Dame, USA, 2002.

[4] S. Shi, A. Woody and H. Rabitz, "Optimal control of selective vibrational excitation in harmonic linear chain molecules", *J. Chem. Phys.*, vol. 88, 1988, pp 6870-6883.

[5] H.M. Wiseman and A.C. Doherty, "Optimal unravellings for feedback control in linear quantum systems", *Phys. Rev. Lett.*, vol. 94, 2005, 070405.

[6] Z.R. Xi and G.S. Jin, "Classical and quantum control of a simple quantum system", *International Journal of Quantum Information*, vol. 5, 2007, pp 857-884.

[7] M. Mirrahimi and R. Van Handel, "Stabilizing feedback controls for quantum systems", *SIAM J. Cont. Opt.*, vol. 46, 2007, pp 445-467.

[8] R. van Handel, J.K. Stockton and H. Mabuchi, "Feedback control of quantum state reduction", *IEEE Trans. Auto. contr.*, vol. 50, 2005, pp 768-780.

[9] H.M. Wiseman, "Quantum theory of continuous feedback", *Phys. Rev. A*, vol. 49, 1994, pp 2133-2150.

[10] K. Jacobs and A.P. Lund, "Feedback control of nonlinear quantum systems: A rule of thumb", *Phys. Rev. Lett.*, vol. 99, 2007, 020501.

[11] S. Kuang and S. Cong, "Lyapunov control methods of closed quantum systems", *Automatica*, vol. 44, 2008, pp 98-108.

[12] S. Grivopoulos and B. Bamieh, "Lyapunov-based control of quantum systems", *in Proc. 42th IEEE Conf. Decision and Control, 1,* 434-438, 2003.

[13] M. Mirrahimi and P. Rouchon, "Trajectory generation for quantum systems based on Lyapunov techniques", *in Proceedings of IFAC symposium NOLCOS*, Stuttgart, Germany, 2004.

[14] M. Mirrahimi and P. Rouchon, "Trajectory tracking for quantum systems: A Lyapunov approach", *in Proceedings of the international symposium MTNS*, Leuven, Belgium, 2004.

[15] K. Beaucharda, J.M. Coron, M. Mirrahimi and P. Rouchon, "Implicit Lyapunov control of finite dimensional Schrödinger equations", *System Control Lett.*, vol. 56, 2007, pp 388-395.

[16] M. Mirrahimi, P. Rouchon and G. Turinici, "Lyapunov control of bilinear Schrödinger equations", *Automatica*, vol. 41, 2005, pp 1987-1994.

[17] X.T. Wang and S.G. Schirmer, "Analysis of Lyapunov method for control of quantum states", arxiv: 0801.0702v1.

[18] C. Altafini, "Feedback stabilization of quantum ensembles: a global convergence analysis on complex flag manifolds", *in Proc. 45th IEEE Conf. Decision and Control*, San Diego, USA, 2471-2476, 2006.

[19] C. Altafini, "Feedback control of spin systems", *Quantum Information Processing* , vol. 6, 2007, pp 9-36.

[20] J. LaSalle and S. Lefschetz, *Stability by Liapunov's direct method with applications*, Academic Press, New York; 1961.