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Abstract We consider the problem of state-feedback stabilization for multi-channel systems in the differential-game theoretic framework where the class of admissible strategies for the players is induced from a solution set of the individual objective functions that are associated with certain dissipativity inequality properties. In such a framework, we characterize the feedback Nash equilibria over an infinite-time horizon via a set of non-fragile stabilizing state-feedback solutions corresponding to the constrained dissipativity problems. Moreover, we show that the existence of a weak-optimal solution to the constrained dissipativity problem is a sufficient condition for the existence of a feedback Nash equilibrium, whereas the set of non-fragile stabilizing state-feedback solutions is fully described in terms of a set of dilated linear matrix inequalities.

Keywords Differential games · Dissipativity inequalities · Multi-channel system · Nash strategy · Dilated linear matrix inequality · Robust stabilization

1 Introduction

In this paper, we consider a multi-channel system governed by several players (*or decision makers*) where the stability of the overall closed-loop system is a common objective while each player aims to minimize different types of objective functions. In such a scenario, Nash strategy offers a suitable framework to study an inherent robustness or non-fragile property of the strategies under a family of information structures, since no player can improve his payoff by deviating unilaterally from the Nash strategy once the equilibrium is attained (e.g., see references [14], [21], [22], [15], [29], [4], [5], [11]).

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In the past, several theoretical results have been established to characterize control related problems in the context of Nash equilibria via a game theoretic interpretation [27], [23], [25], [30], [1], [31], [4] and [32]. For example, the existence of open-loop Nash strategies for linear-quadratic games over a finite time-horizon, assuming that all strategies lie in compact subsets of an admissible strategy space, has been addressed in [33], [21] and [28]; the existence of Nash Strategies for linear-quadratic differential games over an infinite-horizon has been studied in detail in [27], [25], [1], [4] and [31]. We also note that some of these works have discussed the uniqueness of the optimal strategies for linear-quadratic games with structured uncertainties, where the bound for the objective function is based on the existence of a set of solutions for appropriately parameterized Riccati equations. Moreover, in the area of multiobjective H_2/H_∞ control theory, the concept of differential games has been applied by interpreting uncertainty (or neglected dynamics) as a fictitious player while the model of the system is supposed to be well known; where the fictitious player is usually introduced in the criteria through a weighting matrix (e.g., see references [10], [19], [3], [30], [7] and [5]).

On the other hand, the use of different simplified models of the same system has been employed for capturing certain information structures, models or objective functions that individual players may hold about the overall system. Thus, the resulting problem can be best described by nonzero-sum differential games where the individual players are allowed to minimize different types of objective functions (e.g., see references [27], [8], [17], [26]). An extensive survey on the area of noncooperative dynamic games is provided in the book by Başar and Olsder [4].

Our main focus in this paper is to take this line of approach, where individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity inequality property of the multi-channel system, where the optimality concept is that of Nash equilibrium. We characterize the feedback Nash equilibria over an infinite-time horizon via a set of stabilizing state-feedback solutions corresponding to a family of perturbed multi-channel systems with dissipativity inequality properties (see [34], [35], [37] and references therein for a review of systems with dissipative properties). We further show that the existence of a weak-optimal solution to the constrained dissipativity problem is a sufficient condition for the existence of a feedback Nash equilibrium, with the latter also having a nice property of strong time consistency.

The rest of the paper is organized as follows. In Section 2, we present a verifiable stability condition for a multi-channel system in terms of a set of dilated linear matrix inequalities (LMIs), with a certain dissipativity inequality property being used to extend the stability condition when there is a model perturbation in the system. Section 3 presents the main results, where we provide a sufficient condition for the existence of Nash equilibria via weak-optimal solutions of the differential game corresponding to the dissipativity inequality property of the system. Finally, Section 4 provides some concluding remarks.

Throughout the paper, we use the following notations. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{He}\,(A)$ denotes a hermitian matrix defined by $\mathrm{He}\,(A) = A + A^T$, where A^T is the transpose of A. For a matrix $B \in \mathbb{R}^{n\times p}$, $B^\perp \in \mathbb{R}^{(n-r)\times n}$ denotes an orthogonal complement of B, which is a matrix satisfies $B^\perp B = 0$ and $B^\perp B^{\perp T} > 0$, with $r = \mathrm{rank}\, B$. $\mathbb{R}_+$ denotes the set of non-negative real numbers, which is $\mathbb{R}_+ \triangleq \{x \in \mathbb{R} \,|\, x \geq 0\}$. $\mathcal{S}_+^n$ denotes the set of strictly positive definite real matrices. $\mathcal{U}_\rho$ denotes a compact uncertainty set.

2 Preliminaries

Consider a continuous-time N-channel system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t), \quad x(0) = x_0 \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times r_i}$, $x(t) \in \mathbb{R}^n$ is the state of the system, and $u_i(t) \in \mathbb{R}^{r_i}$ is a control input to the ith-channel of the system.

For this system, consider the set of all stabilizing state-feedback gains

$$\mathcal{K}_N = \left\{ (K_1, K_2, \ldots, K_N) \;\middle|\; \left(A + \sum_{i=1}^{N} B_i K_i \right) \text{ is a Hurwitz matrix} \right\} \tag{2}$$

where $K_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$.

Then, we characterize the set $\mathcal{K}_N$ in terms of a set of dilated LMIs as follows.

Theorem 1 *The set of all stabilizing state-feedback gains $\mathcal{K}_N$ in* (2) *is well defined if there exist $X \in \mathcal{S}_+^n$, $U \in \mathbb{R}^{n\times n}$, $\epsilon > 0$, $W_i \in \mathbb{R}^{n\times n}$ and $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$ such that*

$$\begin{bmatrix} 0 & X\tilde{E} \\ \tilde{E}^T X & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \tilde{A} \\ -\tilde{W}_D \end{bmatrix} \begin{bmatrix} \tilde{E}^T & \epsilon I \end{bmatrix} \right) \prec 0 \tag{3}$$

with $\tilde{A} = [\, AU \; B_1L_1 \; B_2L_2 \cdots B_NL_N \,]$, $\tilde{W}_D = \text{block diag}\{U, W_1, W_2, \ldots, W_N\}$ *and* $\tilde{E} = [\, I \; I \; I \cdots I \,]$.

Once this condition is satisfied, then the state-feedback gains that achieve stabilization are given by

$$K_i = L_i W_i^{-1} \tag{4}$$

with non-singular solutions U and W_i for $i = 1, 2, \ldots, N$.[1]

Proof Sufficiency: Note that

$$\begin{bmatrix} \tilde{A} \\ -\tilde{W}_D \end{bmatrix}^{\perp} = \begin{bmatrix} I & \tilde{A}\tilde{W}_D^{-1} \end{bmatrix}, \qquad \begin{bmatrix} \tilde{E} \\ \epsilon I \end{bmatrix}^{\perp} = \begin{bmatrix} \epsilon I & -\tilde{E} \end{bmatrix}. \tag{5}$$

Then, eliminating $\tilde{W}_D$ from (3) by using these matrices, we have two inequalities

$$\begin{bmatrix} I & \tilde{A}\tilde{W}_D^{-1} \end{bmatrix} \begin{bmatrix} 0 & X\tilde{E} \\ \tilde{E}^T X & 0 \end{bmatrix} \begin{bmatrix} I \\ (\tilde{W}_D^{-1})^T \tilde{A}^T \end{bmatrix} = \mathrm{He}\left((A + \sum_{i=1}^{N} B_i K_i) X \right) \prec 0 \tag{6}$$

$$\begin{bmatrix} \epsilon I & -\tilde{E} \end{bmatrix} \begin{bmatrix} 0 & X\tilde{E} \\ \tilde{E}^T X & 0 \end{bmatrix} \begin{bmatrix} \epsilon I \\ -\tilde{E}^T \end{bmatrix} = -2\epsilon(N+1)X \prec 0. \tag{7}$$

Hence, we see that (6) and (7) state exactly the Lyapunov stability condition with $X \in \mathcal{S}_+^n$ and state-feedback gains $K_i = L_i W_i^{-1}$ for $i = 1, 2, \ldots, N$.

[1] Recently, a similar dilated LMIs condition has been investigated by Fujisaki and Befekadu [12] in the context of reliable decentralized stabilization for multi-channel systems.

Necessity: Suppose the system in (1) is stable with state-feedback gains $K_i = L_i W_i^{-1}$ for $i = 1, 2, \ldots, N$. Then, there exists a sufficiently small $\epsilon > 0$ that satisfies

$$\mathrm{He}\,\Big(\,(A + \sum_{i=1}^{N} B_i K_i)X\,\Big) + \frac{1}{2}\epsilon \tilde{A}\tilde{X}_D\tilde{A}^T \prec 0. \tag{8}$$

with $\tilde{X}_D = \text{block diag}\,\{\,X, X, \ldots, X\,\}$.

Note that $\tilde{X}_D \succ 0$ and $\tilde{X}_D\tilde{E}^T = \tilde{E}^T X$, employing the Schur complement for (8), then we have

$$\begin{aligned}
&\begin{bmatrix} \mathrm{He}\,\big(\,(A + \sum_{i=1}^{N} B_i K_i)X\,\big) & \epsilon\tilde{A}\tilde{X}_D \\ \epsilon\tilde{X}_D\tilde{A}^T & -2\epsilon\tilde{X}_D \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{He}(\tilde{A}\tilde{W}_D^{-1}\tilde{E}^T X) & \epsilon\tilde{A}\tilde{W}_D^{-1}\tilde{X}_D + X\tilde{E} - \tilde{E}\tilde{X}_D \\ \epsilon\tilde{X}_D(\tilde{A}\tilde{W}_D^{-1})^T + \tilde{E}^T X - \tilde{X}_D\tilde{E}^T & -2\epsilon\tilde{X}_D \end{bmatrix} \\
&= \begin{bmatrix} 0 & X\tilde{E} \\ \tilde{E}^T X & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \tilde{A}\tilde{W}_D^{-1} \\ -I \end{bmatrix} \tilde{X}_D \begin{bmatrix} \tilde{E}^T & \epsilon I \end{bmatrix}\right) \\
&\prec 0.
\end{aligned}$$

This means that (3) holds with $\tilde{W}_D = \tilde{X}_D$. □

Remark 1 In this paper, we assume that the pair $(A, [\,B_1\ B_2\ \cdots\ B_N\,])$ is stabilizable.

Consider next a multi-channel system with a perturbation term, i.e.,

$$\dot{x}(t) = (A + u_\rho \varDelta A)x(t) + \sum_{i=1}^{N} B_i u_i(t) \tag{9}$$

where $u_\rho \in [-\rho, \rho]$, $\rho \in \mathbb{R}_+$ is the uncertainty level and $\varDelta A \in \mathbb{R}^{n\times n}$ is the perturbation term in the system. Here we assume that the perturbed matrix $(A + u_\rho \varDelta A)$ lies in a compact uncertainty set $\mathcal{U}_\rho$.[2]

In what follows, we assume there exits a set of stabilizing state-feedback gains $\mathcal{K}_N$ that maintains the stability of the system in (1) and this set is completely characterized via a solution of (3). Then, we will estimate an upper bound $\hat{\rho} \in \mathbb{R}_+$ on the uncertainty level for which the state-feedback gains preserve robust (*or non-fragile*) stability property of the perturbed multi-channel system.

Theorem 2 *Suppose $X \in \mathcal{S}_+^n$, $U \in \mathbb{R}^{n\times n}$, $\epsilon > 0$, $W_i \in \mathbb{R}^{n\times n}$ and $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$ that characterize the set of stabilizing state-feedback gains in Theorem 1 are given. For a given $\alpha > 0$, $\beta \geq 1$ and $Z \in \mathcal{S}_+^n$, if there exist $Y \in \mathcal{S}_+^n$ and an upper bound $\hat{\rho} \in \mathbb{R}_+$ that satisfy*

$$\beta^{-1} Z \preceq Y \preceq Z \tag{10}$$

$$\begin{bmatrix} I \\ \tilde{W}_D^{-1}\tilde{E}^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}}\,\mathrm{He}(\varDelta A^T Y) & Y\tilde{A} \\ \tilde{A}^T Y & 0 \end{bmatrix} \begin{bmatrix} I \\ \tilde{W}_D^{-1}\tilde{E}^T \end{bmatrix} \preceq -\alpha Z. \tag{11}$$

Then, the perturbed multi-channel system in (9) *is stable for all instances of perturbation $u_{\hat{\rho}} \in [-\hat{\rho}, \hat{\rho}]$ in the system.*

[2] Note that the existence of a solution for state trajectories is well-defined and it is always upper semicontinuous in x_0 (e.g., see references [9] and [18]).

Proof To prove the above theorem, we require the following system

$$\begin{aligned}\dot{x}(t) &= (A + u_\rho \Delta A + \sum_{i=1}^{N} B_i K_i) x(t) + 0_{n\times 1}\tilde{u}(t)\\ \tilde{y}(t) &= x(t) + 0_{n\times 1}\tilde{u}(t)\end{aligned} \tag{12}$$

to satisfy certain dissipativity inequality property for all instances of perturbation in the system.

Define the following supply rate

$$w(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix} \tag{13}$$

with $Z \in \mathcal{S}_+^n$ and $\alpha > 0$. We clearly see that if the system in (12) is stable for all instances of perturbation. Then, the following *dissipation inequality* will hold

$$V(x(0)) + \int_0^t w(\tilde{y}(t), \tilde{u}(t))dt \geq V(x(t)) \tag{14}$$

for all $t \geq 0$ with non-negative quadratic storage function $V(x(t)) = x(t)^T Y x(t)$, $Y \in \mathcal{S}_+^n$ that satisfies $V(0) = 0$.

Condition (14) with (13) further implies the following

$$\mathrm{He}\left((A + u_\rho \Delta A + \sum_{i=1}^{N} B_i K_i)^T Y \right) \preceq -\alpha Z. \tag{15}$$

Therefore, there exists an upper bound $\hat{\rho} \in \mathbb{R}_+$ for which the dissipativity condition in (15) will hold true for all instances of perturbation in the system.

Then, we have the following result

$$\begin{aligned}&\mathrm{He}\left((\tilde{A}\tilde{W}_D^{-1}\tilde{E}^T + u_{\hat{\rho}}\Delta A)^T Y \right) = \\ &\begin{bmatrix} I \\ \tilde{W}_D^{-1}\tilde{E}^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}}\,\mathrm{He}(\Delta A^T Y) & Y\tilde{A} \\ \tilde{A}^T Y & 0 \end{bmatrix} \begin{bmatrix} I \\ \tilde{W}_D^{-1}\tilde{E}^T \end{bmatrix} \preceq -\alpha Z\end{aligned} \tag{16}$$

with $u_{\hat{\rho}} \in [-\hat{\rho},\ \hat{\rho}]$.[3]

On the other hand, let us define the following matrix interval

$$\mathcal{I}_Y = \left\{ Y \;\middle|\; \beta^{-1} Z \preceq Y \preceq Z \right\} \tag{17}$$

where $Z \in \mathcal{S}_+^n$ and $\beta \geq 1$ are assumed to be known *a priori*. Suppose that Y satisfies the conditions in (10) and (11), then the trajectories of the perturbed closed-loop system

$$\dot{x}(t) = (A + u_\rho \Delta A + \sum_{i=1}^{N} B_i K_i) x(t)$$

[3] Note that the upper bound $\hat{\rho}$ continuously depends (*in the weak sense*) on x_0 and K_i, $i = 1, 2, \ldots, N$.

satisfy

$$\frac{d}{dt}(x^T(t)Yx(t)) = x^T(t)\,\mathrm{He}\,\Big(\,(A + u_\rho \Delta A + \sum_{i=1}^{N} B_i K_i)^T Y\,\Big)\,x(t)$$
$$\le -\alpha x^T(t)Zx(t)$$
$$\le -\alpha x^T(t)Yx(t). \tag{18}$$

Note that condition (18) further implies the following two conditions

$$x^T(t)Yx(t) \le \exp(-\alpha t)\,x^T(0)Zx(0) \tag{19}$$

and

$$x^T(t)Zx(t) \le \beta\exp(-\alpha t)\,x^T(0)Zx(0). \tag{20}$$

Hence, conditions (18), (19) and (20) stating, equivalently, that $Y \in \mathcal{I}_Y$ is a dissipativity certificate with supply rate (13) for all instances of perturbation in (12) (e.g., see references [2], [6]). □

Remark 2 We remark that if there exists a solution set X for Theorem 2 that gives a minimum distance between X and the set $\mathcal{I}_Y = \{Y \mid \beta^{-1}Z \preceq Y \preceq Z\}$, i.e., $\inf_{Y\in\mathcal{I}_Y}\|X-Y\|$, then we essentially have a weak-optimal solution. This solution is unique since $\mathcal{I}_Y$ is a convex and compact set [20]. Moreover, finding an upper bound $\hat{\rho} \in \mathbb{R}_+$ and Y from a closed and convex set $\mathcal{I}_Y$ is equivalent to solving the verification problem, i.e., the constrained dissipativity control problem (e.g., see reference [13]).

In the next section, we will see that such additional information structure, i.e., the dissipativity inequality property, about the system is indeed useful in the context of differential games.

3 Main results

In this section, we establish an equivalence result between the set of non-fragile state-feedback gains corresponding to constrained dissipativity problem and the feedback Nash equilibria. we specifically provide a game-theoretic interpretation in which several players (or decision makers) are influencing cooperatively the overall system, where the individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity inequalities, of the following systems

$$\dot{x}(t) = (A + u_{\rho_j}\Delta A_j + \sum_{i=1}^{N} B_i K_i)x(t) + 0_{n\times 1}\tilde{u}(t)$$
$$\tilde{y}(t) = x(t) + 0_{n\times 1}\tilde{u}(t) \tag{21}$$

where $u_{\rho_j} \in [-\rho_j, \rho_j]$, $\rho_j \in \mathbb{R}_+$ and $\Delta A_j \in \mathbb{R}^{n\times n}$ are the uncertainty levels and the perturbation terms associated with the jth-player, respectively. We further assume that each perturbed system matrix $(A + u_{\rho_j}\Delta A_j)$ lies in a compact uncertainty set $\mathcal{U}_{\rho_j}$ for $j = 1, 2, \ldots, N$ and $(K_1, K_2, \ldots,\ K_N) \in \mathcal{K}_N$.

Introduce the following continuous mappings

$$J_j(x_0, u_{\rho_j}, K_{(-j)}) \colon \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathcal{K}_N \to \mathbb{R}_+ \tag{22}$$

for $j = 1, 2, \ldots, N$.[4]

Moreover, an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$, (i.e., $K^* \triangleq (K_1^*, K_2^*, \ldots, K_N^*)$) is called a feedback Nash equilibrium if the following inequality holds

$$J_j(x_0, u_{\hat{\rho}_j}, K_{(-j)}^*) \le J_j(x_0, u_{\hat{\rho}_j}, K^*) \tag{23}$$

for each $x_0 \in \mathbb{R}^n$ and all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ and state-feedback matrix $\Gamma_j \in \mathbb{R}^{r_j \times n}$ for $j = 1, 2, \ldots, N$ such that

$$K_{(-j)}^* \triangleq (K_1^*, \ldots, K_{j-1}^*, \Gamma_j, K_{j+1}^*, \ldots, K_N^*) \in \mathcal{K}_N. \tag{24}$$

In the following, we assume that the strategy space for each player is restricted to linear time-invariant state-feedback gains, and the resulting multi-channel closed-loop system is also assumed to be stable for all (or some) initial conditions $x_0 \in \mathbb{R}^n$.

Introduce the following set of supply rate functions

$$\mathcal{W}_{(\tilde{y}(t),\tilde{u}(t))} = \left\{ w_j(\tilde{y}(t), \tilde{u}(t)) \;\middle|\; w_j(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha_j Z_j & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}, \right.$$
$$\left. \text{for } j = 1, 2, \ldots, N \right\} \tag{25}$$

and a matrix interval set $\mathcal{I}_{Y_j}$

$$\mathcal{I}_{Y_j} = \left\{ Y_j \;\middle|\; \beta_j^{-1} Z_j \preceq Y_j \preceq Z_j \right\} \tag{26}$$

with $\alpha_j > 0$, $\beta_j \ge 1$ and $Z_j \in \mathcal{S}_+^n$ for $j = 1, 2, \ldots, N$ that are assumed to be known $a\ priori$.

In light of Theorem 2 and above discussion, we have the following theorem which provides a sufficient condition for the existence of feedback Nash equilibria.

Theorem 3 *Suppose* $X_j \in \mathcal{S}_+^n$, $U_j \in \mathbb{R}^{n \times n}$, $W_j' \in \mathbb{R}^{n \times n}$, $L_j' \in \mathbb{R}^{r_j \times n}$ *and* $\epsilon_j > 0$ *for* $j = 1, 2, \ldots, N$ *and* $W_i \in \mathbb{R}^{n \times n}$, $L_i \in \mathbb{R}^{r_i \times n}$ *for* $i = 1, 2, \ldots, N$ *that characterize the set of stabilizing state-feedback gains in* (2) *are given by the solution of the following dilated LMIs condition*

$$\begin{bmatrix} 0 & X_j \tilde{E} \\ \tilde{E}^T X_j & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \tilde{A}_{(-j)} \\ -\tilde{W}_{D(-j)} \end{bmatrix} \begin{bmatrix} \tilde{E}^T & \epsilon_j I \end{bmatrix} \right) \prec 0 \tag{27}$$

where $\tilde{A}_{(-j)} = [A U_j \; B_1 L_1 \; \cdots \; B_{j-1} L_{j-1} \,\vdots\, B_j L_j' \,\vdots\, B_{j+1} L_{j+1} \; \cdots \; B_N L_N]$ *and* $\tilde{W}_{D(-j)} = \operatorname{block\,diag}\{U_j, W_1, \ldots, W_{j-1}, W_j', W_{j+1}, \ldots, W_N\}$ *with* $K_j = L_j W_j^{-1}$ *and* $\Gamma_j = L_j' W_j'^{-1}$.

[4] In this paper, the game is essentially defined in the framework of an incomplete information, since the jth-player's objective function involves different uncertainty information, i.e., u_{ρ_j}, about the system. However, we remark that the jth-player decides his own strategy by solving the optimization problem with the opponents' strategies $(K_1, \ldots, K_{j-1}, K_{j+1}, \ldots, K_N)$ fixed.

Then, for a given $\alpha_j > 0$, $\beta_j \geq 1$ and $Z_j \in \mathcal{S}_+^n$, there exists $Y_j \in \mathcal{I}_{Y_j}$ that satisfies

$$\sup_{(x_0, u_{\rho_j}, \Gamma_j) \in \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\rho_j}, K_{(-j)}) = \hat{\rho}_j. \tag{28}$$

Furthermore, the closed-loop systems in (21) *are stable for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ with $K_j^* \in \arg\sup_{\Gamma_j \in \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\hat{\rho}_j}, K_{(-j)}^*)$ for all $j \in \{1, 2, \ldots, N\}$.*[5]

Proof Suppose all the perturbed systems in (21) satisfy the following dissipativity inequalities

$$V_j(x(0)) + \int_0^t w_j(\tilde{y}(t), \tilde{u}(t))dt \geq V_j(x(t)) \tag{29}$$

for all $t \geq 0$ with non-negative quadratic storage functions $V_j(x(t)) = x(t)^T Y_j x(t)$ and $Y_j \in \mathcal{I}_{Y_j}$ that satisfy $V_j(0) = 0$ for $j = 1, 2, \ldots, N$.

Thus, the trajectories of each perturbed closed-loop system (i.e., for $j = 1, 2, \ldots, N$)

$$\dot{x}(t) = (A + u_{\rho_j} \Delta A_j + \sum_{i=1}^N B_i K_i^*) x(t)$$

satisfy

$$\begin{aligned}
\frac{d}{dt}(x^T(t) Y_j x(t)) &= x^T(t) \operatorname{He}\left(\, (A + u_{\rho_j} \Delta A_j + \sum_{i=1}^N B_i K_i^*)^T Y_j \,\right) x(t) \\
&\leq -\alpha_j x^T(t) Z_j x(t) \\
&\leq -\alpha_j x^T(t) Y_j x(t).
\end{aligned} \tag{30}$$

for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ in the system.

Then, the rest of the proof follows the same lines as that of Theorem 1. In fact, replacing the following

$$\tilde{A} = \tilde{A}_{(-j)} (\tilde{W}_{D(-j)})^{-1}, \quad \tilde{W}_D = \tilde{W}_{D(-j)} \;\; \text{and} \quad X = X_j$$

with $\tilde{A}_{(-j)} = [AU_j \; B_1 L_1 \; \cdots \; B_{j-1} L_{j-1} \vdots B_j L_j' \vdots B_{j+1} L_{j+1} \; \cdots \; B_N L_N]$ and $\tilde{W}_{D(-j)} = \operatorname{block\,diag}\{U_j, W_1, \ldots, W_{j-1}, W_j', W_{j+1}, \ldots, W_N\}$ in Theorem 1 immediately gives the condition in (27) of Theorem 3. Note that K_j^* and Γ_j are given by

$$K_j^* = L_j W_j^{-1} \;\; \text{and} \quad \Gamma_j = L_j' W_j'^{-1}$$

for $j = 1, 2, \ldots, N$.

Moreover, the N-tuple $(Y_1, Y_2, \cdots, Y_N) \in \prod_{j=1}^N \mathcal{I}_{Y_j}$ is a collection of dissipativity certificates corresponding to a set of supply rates (25) for all instances of perturbation in (21). □

[5] In general, simultaneously solving a set of optimization problems, i.e., solving (28) together with (27), is not easy since it is a non-convex optimization problem which involves finding a solution satisfying at the intersection of a set of constrained quadratic functionals [38] (c.f. Remark 2, Section 2 above).

We next present a more realistic game-theoretic interpretation in terms of the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for all $j \in \{1, 2, \ldots, N\}$ that describe the N-tuple uncertainty set $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N}[-\hat{\rho}_j, \hat{\rho}_j]$ together with the existence of stabilizing state-feedback gains that provide a sufficient condition for obtaining a set of feedback Nash equilibria.

Hence, we have the following equivalent statements:

(i). $\exists K^* \in \mathcal{K}_N,\ \forall x_0,\ \forall u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j],\ \forall K^*_{(-j)} \in \mathcal{K}_N,\ \forall j \in \{1, 2, \ldots, N\}$ such that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{(-j)}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*). \tag{31}$$

(ii). The dilated LMIs condition in (27) and the dissipativity inequalities of (29) with a set of supply rates $\mathcal{W}_{(\tilde{y}(t), \tilde{u}(t))}$ in (25) fully describes the set of non-fragile stabilizing state-feedback gains.

The equivalence between (i) and (ii) leads to characterization of feedback Nash equilibria over an infinite-time horizon in terms of stabilizing solutions of a set of dilated LMIs.

Furthermore, the exact characterization of the feedback Nash equilibria is given by the following two theorems.

Theorem 4 *Let $X_j \in \mathcal{S}^n_+$, $U_j \in \mathbb{R}^{n\times n}$, $W'_j \in \mathbb{R}^{n\times n}$, $L'_j \in \mathbb{R}^{r_j\times n}$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$ and $W_i \in \mathbb{R}^{n\times n}$, $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$ be a solution set for the dilated LMIs condition in* (27). *Then, there exists an N-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}_N$ feedback Nash equilibrium with respect to the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for $j = 1, 2 \ldots, N$ of* (28).

Proof The first part of this theorem is already provided in Theorem 3, i.e., from the standard argument of the stabilizability of the pair $(A, [\,B_1\ B_2\ \cdots\ B_N\,])$, we can always find an N-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}_N$ and Γ_j for all $j \in \{1, 2 \ldots, N\}$ such that (27) holds. Applying (28) of Theorem 3 together with the dissipativity certificates $Y_j \in \mathcal{I}_{Y_j}$ and a set of supply rates $\mathcal{W}_{(\tilde{y}(t), \tilde{u}(t))}$ (25) for all instances of perturbation in (21), we will then obtain an upper bound $\hat{\rho}_j \in \mathbb{R}_+$ for a fixed $(x_0, K^*) \in \mathbb{R}^n \times \mathcal{K}_N$ so that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{(-j)}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*)$$

for all $j \in \{1, 2, \ldots, N\}$.

Hence, we immediately see that the N-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}_N$ satisfies the feedback Nash equilibrium. □

Remark 3 The class of admissible strategies for all players are generated through a set of individual objective functions that are induced from dissipativity inequalities of (29) with a set of supply rates (25).

Theorem 5 *If the N-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}_N$ is a feedback Nash equilibrium with respect to the objective function values of* (28)*, then there exists a solution set $X_j \in \mathcal{S}^n_+$, $U_j \in \mathbb{R}^{n\times n}$, $W'_j \in \mathbb{R}^{n\times n}$, $L'_j \in \mathbb{R}^{r_j\times n}$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$ and $W_i \in \mathbb{R}^{n\times n}$, $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$ that satisfies the dilated LMIs condition of* (27).

Proof Suppose the N-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}_N$ is a feedback Nash equilibrium such that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{(-j)}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*)$$

where the value for the continuous mapping $J_j(x_0, u_{\rho_j}, K)\colon \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathcal{K}_N \to \mathbb{R}_+$ is claimed as

$$\sup_{(x_0, u_{\rho_j}, \Gamma_j) \in \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\rho_j}, K_{(-j)}) = \hat{\rho}_j$$

with $K_j^* \in \arg\sup_{\Gamma_j \in \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\hat{\rho}_j}, K_{(-j)}^*)$ for all $j \in \{1, 2, \ldots, N\}$.

Then, we can always find a solution set that satisfies the dilated LMIs condition in (27) for which the closed-loop systems in (21) are robustly stable for all instances of perturbations $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N} [-\hat{\rho}_j, \hat{\rho}_j]$. □

Remark 4 Note that all closed-loop systems in (21) satisfy the dissipative inequality properties of (29) with a set of supply rates (25) for all $j \in \{1, 2, \ldots, N\}$ and instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$.

Finally, the feedback Nash equilibrium has a strong time consistency property. This fact corresponds to the information structure that is associated with the dissipative inequalities of the system where the equilibrium trajectory $x_{eq}(t)$ (or the equilibrium point $x(0) = x_0$) of the system if it is truncated part in the time interval $[T, \infty)$, where $T > 0$, asymptotically represents an equilibrium (c.f. references [36], [39]).[6] This further implies any (sub-)game starting at $t = T$, does not depend on the initial condition $x_{eq}(T)$ (e.g., see references [24], [16]). Moreover, the differential game, where the class of admissible strategies for all players is induced from a solution set of the individual objective functions (22), is an infinite-time horizon game. Thus, this game has non-unique feedback Nash equilibrium solutions that are associated with a set of non-fragile stabilizing state-feedback gains of Theorem 3.

Remark 5 Note that the equivalence between (i) and (ii) (i.e., Theorem 4: (ii) ⇒ (i) and Theorem 5: (i) ⇒ (ii)) leads exactly to characterization of the feedback Nash equilibrium via a set of non-fragile stabilizing state-feedback solutions of the dilated LMIs.

4 Concluding remarks

In this paper, we have looked the problem of state-feedback stabilization for a multi-channel system in the framework of differential game, where the class of admissible strategies for the players is induced from a solution set of the objective functionals that are realized through certain dissipativity inequalities. In such a scenario, we characterized the feedback Nash equilibria over an infinite-time horizon via a set of non-fragile stabilizing state-feedback gains corresponding to constrained dissipativity problems. Moreover, we showed that the existence of a weak-optimal solution to the constrained dissipativity problem is a sufficient condition for the existence of a feedback Nash equilibrium, with the latter having a nice property of strong time consistency.

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[6] Note that the stability behavior is considered here over an infinite-time horizon.

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