Model-Based Dissipative Control of Nonlinear Discrete-Time Systems over Networks

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Abstract—The problem of output feedback control of nonlinear discrete-time systems connected over a network is studied in this paper. The Model-Based Networked Control Systems (MB-NCS) scheme is used to reduce communication to free up network resources for other applications. This is done by implementing an approximate model of the plant at the controller node to predict plant output values between sensor measurements. Communication is further reduced by considering an aperiodic event-triggered communication scheme that transmits data only when the error in the output exceeds a specified threshold. With the model and aperiodic updates, the plant is able to operate in open-loop for relatively large time intervals while still maintaining a desired level of accuracy in the control signal. When control systems are allowed to operate in open-loop like this, they often become sensitive to unmodeled dynamics. This paper considers model mismatch between the plant and model as well as bounded disturbances that may cause performance issues. In this paper, the model-based network architecture is represented as a standard negative feedback design problem for analysis purposes. Dissipative theory is applied to the feedback system for stability analysis and control synthesis. The results provide average squared boundedness with a constructive bound of the system output despite the presence of aperiodic updates, nonlinear dynamics, model uncertainties, and external disturbances.

I. INTRODUCTION

In Networked Control Systems (NCS) a digital communication network is used to transfer information among the components of a control system including actuators, controllers, and sensors. This type of implementation differs significantly from classical control systems where all system components are attached directly to the control plant exchanging information using dedicated wiring [1].

One of the main problems in NCS is the design of control schemes accounting for the absence of feedback measurements for possibly long intervals of time. Reducing the amount of communication between sensor and controller nodes without compromising the stability of the control system has been discussed by different authors [2]-[6]. In particular, Walsh, *et al.* [2] introduced a network control protocol Try-Once-Discard (TOD) to allocate network resources to the different nodes in a Networked

Control System, all of which may access the network at any time assuming each access occurs before the Maximum Allowable Transfer Time (MATI). The work in [5]-[6] more efficiently uses the packet structure, that is, a reduction on communication is obtained by sending packets of information using all data bits available (excluding overhead) in the structure of the packet.

A different way to address reduction of communication in a feedback control network is by maximizing the time intervals between updates from the sensor node to the control node. An important framework that is able to reduce sampling rate and that also considers model uncertainties is called Model-Based Networked Control Systems (MB-NCS) [7]-[9]. The existence on plant-model mismatch is present in most control system applications. In NCS, it is especially important to design controllers and feedback schedulers that account for this difference between plant and model dynamics since even small uncertainties may produce undesired behavior and cause instability in the absence of feedback measurements for extended periods of time.

The framework in [10] extends the MB-NCS approach to consider aperiodic update intervals. The update instants in this case are based on error events. The use of event-triggered techniques in NCS has been increasingly used in the last years [11]-[17] in order to reduce network communication and update the controller only when it is necessary.

Montestruque [22] provided, separately, sufficient conditions for stability using the MB-NCS framework with periodic updates for two classes of nonlinear continuous-time systems:

$$\dot{x} = f(x) + g(u)$$
$$\dot{x} = f(x) + g(x,u).$$

This nonlinear MB-NCS scheme has also been used by Conte *et. al.* [23] and successfully applied for control of underwater Remotely Operated Vehicles (ROV). The work by Liu [24] also provided stability results for continuous-time nonlinear systems using the MB-NCS framework. A more general class of nonlinear systems was considered:

$$\dot{x} = f(t,x,u) + r(t,x,u)$$

where $r(t,x,u)$ represents an additive uncertainty. With respect to the network properties, the author focuses on local area control networks with high data rate and considers random but bounded time delays. For vanishing perturbations this work provides conditions for exponential stability and conditions for uniformly ultimately boundedness in the case of non-vanishing perturbations.

In relation to nonlinear systems and MB-NCS, Polushin *et al.* [25] considered a sampled version of a nonlinear continuous-time-varying system controlled by an approximate discrete-time model and proposed a communication protocol that considers network induced communication constraints such as irregularity of transfer intervals, existence of time-varying communication delays, and possibility of packet dropouts. Stability results were given that depend on the integration step parameter; that is, it is assumed that the mismatch between plant and model arises only from the approximate integration of the nonlinear dynamics and by making the parameter small (close to zero) it is possible to recover an exact model of the system. This assumption may be unrealistic in many real problems and may lead to results that lack robustness to parameter uncertainty. The work in [26] studies different control choices for nonlinear systems with sensor data losses which include: zero control, last available control, and open-loop control. The last one is equivalent to the MB-NCS approach in which the nominal nonlinear model is used to estimate the state of the plant and that estimate is used to compute the control input when sensor measurements are not available. The authors do not consider model uncertainties and the only difference between the model and the plant is that the plant is perturbed by external bounded disturbances.

Existing results concerning nonlinear MB-NCS only address the state feedback case. In contrast, the present paper considers the stabilization of nonlinear discrete-time systems subject to external disturbances and does not assume that the entire state of the system is available for measurement but only an output of the system can be measured. This approach can also be used for control of nonlinear continuous-time systems where the model represents an approximate discretization of the nonlinear dynamics and the mismatch between the plant and the model is due to both modeling and discretization errors.

This paper focuses on dissipative systems theory in the MB-NCS framework for nonlinear systems. Dissipativity is an energy-based property of dynamical systems. Roughly speaking, a dissipative system stores and dissipates energy supplied by its environment without generating its own energy. The notion of energy is a generalized energy represented by an energy storage function and an energy supply rate. The concept of dissipativity was formalized in [18] and [19]. Specifically the notion of QSR dissipativity [20]-[21] is used in this paper. The tools that dissipativity theory provides are powerful for analyzing nonlinear systems. For example, stability of a single system or of feedback loops can be analyzed when considering the dissipative rates. Additionally, the theory provides guidelines for designing stabilizing feedback controllers.

The main contribution of this paper is in proposing an alternative approach for designing MB-NCS with aperiodic updates that can be applied to nonlinear systems with uncertain models and ℓ_2 disturbances. This approach recasts the problem as a standard negative feedback design problem for analysis purposes. At this point, dissipativity theory can be applied for designing a feedback controller for the original MB-NCS. These results will be given in Section III. In order to present these results, background material will be provided on MB-NCS as well as dissipativity theory in Section II. Examples to illustrate this approach are given in Section IV. The paper is concluded in Section V.

II. BACKGROUND MATERIAL

A. Model-Based Networked Control Systems

One of the main problems in NCS which is studied in this paper is the design of control schemes accounting for the absence of feedback measurements for possibly long intervals of time. Model uncertainties are especially important to be considered under this situation. One of the attractive properties of a classical closed loop system with continuous feedback is that the appropriate design of closed loop controllers reduces sensitivity to model uncertainties. Naturally, this property is lost as feedback measurements are no longer received at the controller node. The MB-NCS approach represents an important framework that considers model uncertainties in the absence of continuous feedback. MB-NCS were introduced by Montestruque and Antsaklis [7]-[9]; this configuration makes use of an explicit model of the plant which is added to the actuator/controller node to compute the control input based on the state of the model rather than on the plant state.

In contrast to previous work in MB-NCS, the work in this paper does not assume that the entire state vector is available for measurement but only the output of the system. It is assumed that the dynamics of each system in Fig. 1 are decoupled. Without loss of generality the focus will be on a particular system/model pair. In MB-NCS the actuator/controller node with output feedback can be represented as in Fig. 2.

This work considers Single-Input Single-Output (SISO) uncertain and unstable nonlinear discrete-time systems that can be described by:

$$y(k) = f_{io}(y(k-1),...,y(k-n),u(k),...,u(k-m)) \tag{1}$$

and the dynamics of the model are given by:

$$\hat{y}(k) = \hat{f}_{io}(\hat{y}(k-1),...,\hat{y}(k-n),u(k),...,u(k-m)) \tag{2}$$

where the nonlinear function $\hat{f}_{io}(\bullet)$ represents the available model of the system function $f_{io}(\bullet)$.

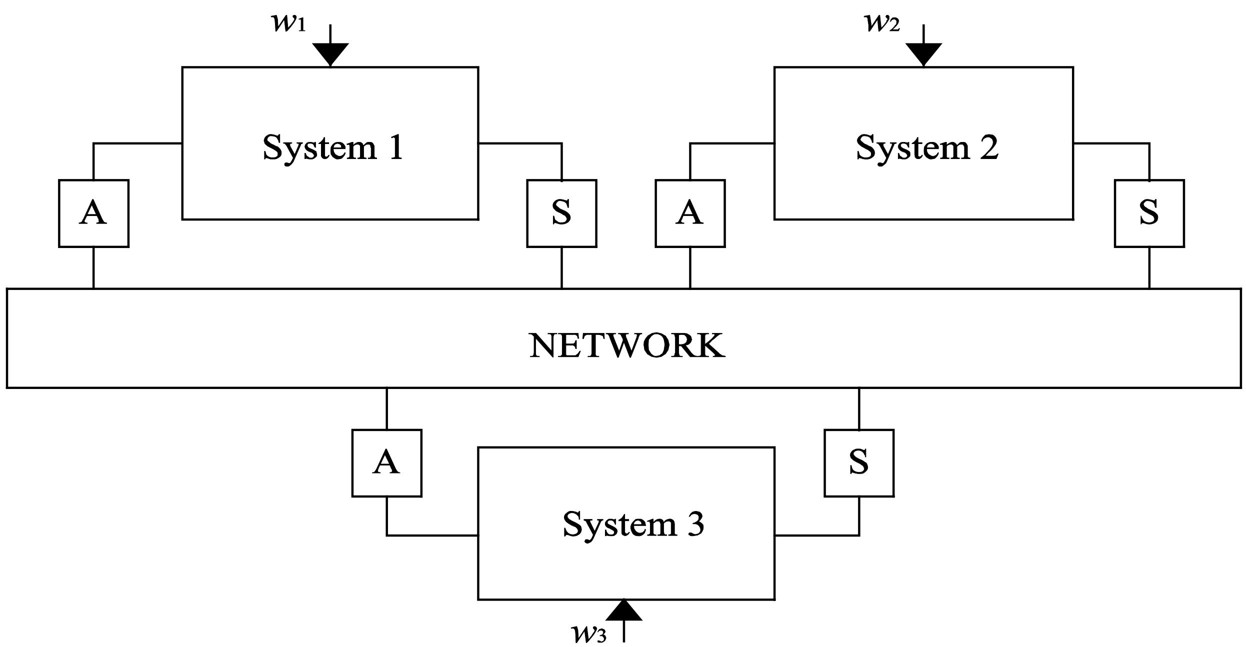

Fig. 1. Representation of Networked Control Systems. A: actuator nodes. S: sensor nodes. w_i represent external disturbances.

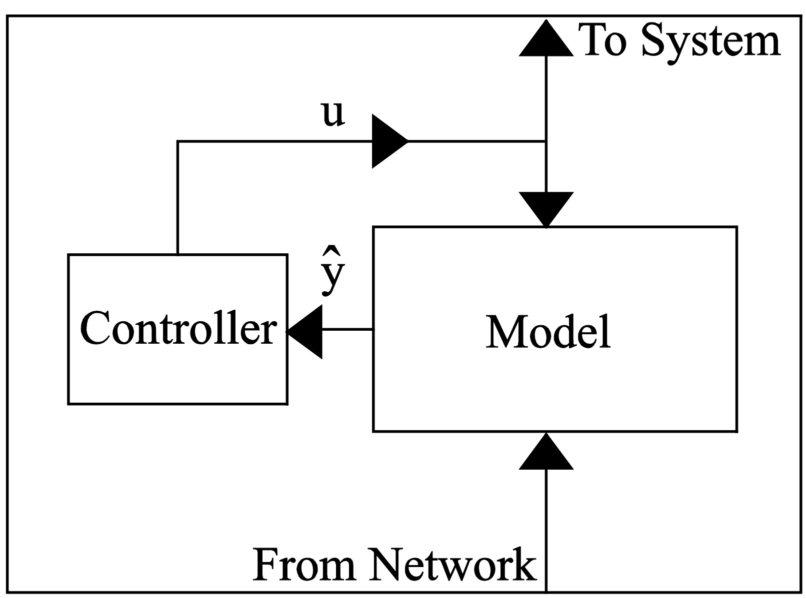

Fig. 2. Model-Based Networked Control System actuator node containing the model and controller.

The aim using this configuration is to operate in open-loop mode for as long as possible while maintaining desirable boundedness properties. This is done by using the estimated outputs $\hat{y}(k),\ldots,\hat{y}(k-n)$ provided by the model to generate the control input u. The system output measurements are used directly to update the current and past output variables of the model without need of implementing a state observer.

In the MB-NCS literature the update measurements are implemented in periodic fashion. The work in this paper discards the periodicity assumption for updating the model. Instead, a non-periodic approach is used that is based on events as in [10]. The estimate of the output given by the model of the plant is compared with the actual output. The sensor then transmits the current output of the plant and previous n output measurements in a single packet if the error is above some predefined tolerance. These measurements are used to update the internal variables of the model in the controller. At the same time the sensor uses exactly the same measurements to update its own copy of the model. The sensor contains a copy of the model and the controller so it can have access to the model output. It continuously measures the actual output and computes the model-plant output error, defined by:

$$e(k) = \hat{y}(k) - y(k). \tag{3}$$

The sensor also compares the norm of the error to a predefined threshold α, and it broadcasts the plant output to update the model state if the error is greater than the threshold. It is clear that while $\|e\| \le \alpha$ the plant is operating in open loop mode based on the model outputs.

It is important to note the reduction in network traffic by using this approach compared to the case in which a measurement of the current output $y(k)$ is sent at every sampling instant even in the case that n is large compared to the inter-update intervals. This case requires the transmission of n measurements of the output at every update instant, the current one and the past n-1 measurements. It has been shown that packet-based control [5]-[6] is able to significantly reduce data transmission by more efficiently using the packet structure, that is, reduction of communication is obtained by sending packets of information using all data bits available (excluding overhead) in the structure of the packet. The work in [5]-[6] focuses in the transmission of control input sequences. In the present paper a similar approach is taken but it is applied to the transmission of output measurements. Instead of sending a single output value in one data packet at every sampling instant, the packet structure is used more efficiently in order to include past measurements in the same packet as well. Packet size is a new feature in NCS compared to point-to-point architectures and it is a variable that depends on the protocol or type of network being implemented. For example, the minimum effective load in an Ethernet packet is 46 bytes and, if 2 bytes are used to represent a sensed quantity, which is accurate enough for most applications since those 16 bits can encode $2^{16} = 65536$ different levels of sensed signals, then it is possible to send at least 23 sensed signals in such a packet. Other networks that specialize in control applications require a considerably smaller minimum size packet. For instance, the Controller Area Network (CAN) protocol is optimized for small messages. With an overhead of 47 bits (minimum packet size), and a maximum data load of 8 bytes encourages designers to use all bits available to send different sensed data. In general, it is possible to decrease network traffic by reducing the number of packets sent by the sensor node since a high percentage of bits transmitted over the network is related to the large number of bits that are used as the packets overheads.

B. Dissipative System Theory

The approach for nonlinear MB-NCS used in this paper relies on dissipativity theory. Dissipativity is an energy-based property of dynamical systems. This property relates energy stored in a system to the energy supplied to the system. Dissipativity can be seen as an extension of Lyapunov stability theory to systems with an input-output representation. The energy stored in the system is defined by a positive definite energy storage function. The energy supplied to the system is a function of the system input *u* and output *y*. A system can be considered dissipative if it only stores and dissipates energy with respect to the specific energy supply rate and does not generate energy on its own. Let us consider nonlinear discrete-time systems represented by:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k), u(k)). \end{aligned} \tag{4}$$

Definition 1. *Consider a nonlinear discrete-time system in the form (4). This system is dissipative with respect to the energy supply rate ω(y,u) if there exists a positive definite energy storage function V(x) such that the following inequality holds, for all times k_1 and k_2 such that $k_1 \le k_2$,*

$$\sum_{k=k_1}^{k_2} \omega(y,u) \geq V(x(k_2)) - V(x(k_1)). \qquad (5)$$

This form of input-output dissipativity is very general but it is difficult to determine a supply rate. A more tractable form of dissipativity is the quadratic form in QSR dissipativity.

Definition 2. *A nonlinear discrete-time system (4) is QSR dissipative if it is dissipative with respect to the supply rate*

$$\omega(y,u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \qquad (6)$$

where $Q = Q^T$ *and* $R = R^T$.

The QSR dissipative framework generalizes many well-known areas of nonlinear systems analysis. The property of passivity can be captured when $Q = R = 0$ and $S = 1/2\,I$, where I is the identity matrix. Systems that are finite-gain l_2 stable can be represented by $S = 0$, $Q = 1/\gamma\, I$, and $R = \gamma I$ where γ is the gain of the system. The following theorems give stability results for single QSR dissipative systems as well as feedback interconnections of dissipative systems.

Theorem 1. *A discrete-time system is finite-gain l_2 stable if it is QSR dissipative with Q<0.*

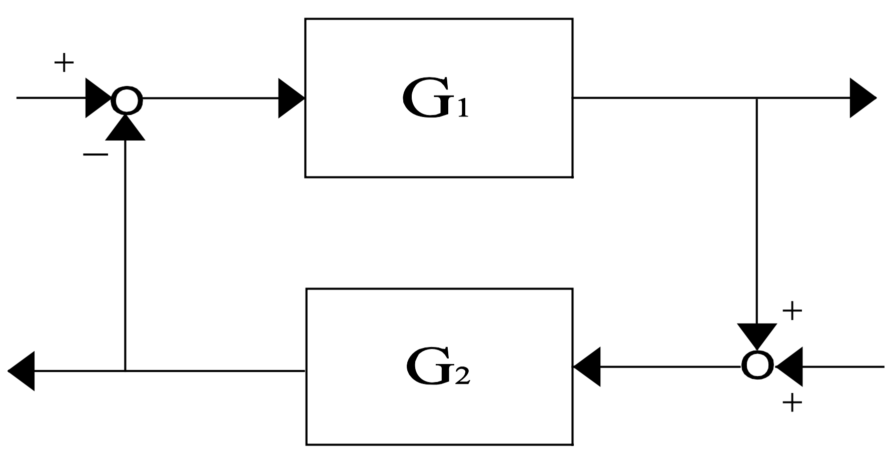

Fig. 3. The negative feedback interconnection of two nonlinear systems G_1 and G_2.

Theorem 2. *Consider the feedback interconnection of two QSR dissipative nonlinear systems. System G_1 is dissipative with respect to Q_1, S_1, R_1 and system G_2 is dissipative with respect to Q_2, S_2, R_2. The feedback interconnection of these two systems is ℓ_2 stable if there exists a positive constant a such that the following matrix is negative definite,*

$$\begin{bmatrix} Q_1 + aR_2 & aS_2{}^T - S_1 \\ aS_2 - S_1{}^T & R_1 + aQ_2 \end{bmatrix} < 0. \tag{7}$$

While this paper considers dissipativity for general nonlinear systems, the linear time-invariant (LTI) case is important for many applications. A discrete-time LTI system is given by the model,

$$\begin{aligned} x(k+1) &= Ax(k) + Bx(k) \\ y(k) &= Cx(k) + Dx(k). \end{aligned} \tag{8}$$

A LTI system is dissipative if and only if there exists a quadratic storage function,

$$V(x) = x^T Px, \tag{9}$$

where $P = P^T > 0$, to satisfy the dissipativity inequality. The matrix P can be found computationally using linear matrix inequality (LMI) methods. An optimization problem can be formulated to find a positive definite matrix P. The problem is feasible when P can be found to satisfy the following matrix inequality,

$$\begin{bmatrix} A^T PA - P - C^T QC & A^T PB - C^T QD - C^T S \\ B^T PA - D^T QC - S^T C & B^T PB - D^T QD - D^T S - S^T D - R \end{bmatrix} \le 0, \tag{10}$$

For general nonlinear systems, there doesn't exist a computational method of finding storage functions or proving dissipativity.

III. MAIN RESULTS

The main results of this paper are presented in this section. The goal of the first subsection III.A is to reconcile the two models discussed up to this point of the paper. The first model covered is the nonlinear input-output representation of a system, and the second is the nonlinear state space model. The second subsection III.B covers the notion of boundedness used in this paper and proves boundedness under appropriate assumptions.

A. Proposed Equivalent Representation of a MB-NCS

The type of nonlinear systems represented by (1) can also be described by the state representation (4). A straightforward way to obtain a state space representation of the input-output system (1) is as follows:

Define the state space variables:

$$\begin{aligned} x_1(k) &= y(k-1) \\ x_2(k) &= y(k-2) \\ &\vdots \\ x_n(k) &= y(k-n) \\ x_{n+1}(k) &= u(k-1) \\ x_{n+2}(k) &= u(k-2) \\ &\vdots \\ x_{n+m}(k) &= u(k-m) \end{aligned} \tag{11.a}$$

and the state space vector

$$x(k) = [x_1(k) \quad x_2(k) \ldots x_{n+m}(k)]^T. \tag{11.b}$$

Then a system of the form (1) can be represented as a state space dynamical system as follows:

$$x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \\ x_{n+1}(k+1) \\ x_{n+2}(k+1) \\ \vdots \\ x_{n+m}(k+1) \end{bmatrix} = \begin{bmatrix} f_{io}(x_1(k),...,x_n(k),u(k),x_{n+1}(k)...,x_{n+m}(k)) \\ x_1(k) \\ \vdots \\ x_{n-1}(k) \\ u(k) \\ x_{n+1}(k) \\ \vdots \\ x_{n+m-1}(k) \end{bmatrix} = f(x(k),u(k)) \tag{12}$$

$$y(k) = f_{io}(x_1(k),...,x_n(k),u(k),x_{n+1}(k)...,x_{n+m}(k)) = h(x(k),u(k)).$$

Following a similar procedure it is possible to obtain a state space representation of the model:

$$\begin{aligned} \hat{x}(k+1) &= \hat{f}(\hat{x}(k),u(k)) \\ \hat{y}(k) &= \hat{h}(\hat{x}(k),u(k)) \end{aligned} \tag{13}$$

The structure of the state space model is the same as that of the real plant. The uncertainty in the plant is only due to the output equation and to the first term in the state equation of (12), which is repeated in the output equation as well; those expressions contain the nonlinear uncertain dynamics given by (1) with respect to the nominal model (2).

By using the model parameters in (13) it is possible to follow the QSR dissipative analysis described in the previous section in order to design QSR dissipative and stabilizing controllers.

Now, the Model-Based Event-Triggered (MB-ET) architecture described in section II.A can be represented in the negative feedback interconnection suitable for dissipativity analysis as in Fig. 3.

Fig. 4 shows an equivalent representation of a MB-ET control system in which the updates of the state of the model are implicit in the model/controller block. From (3) it is clear that

$$\hat{y}(k) = y(k) + e(k). \tag{14}$$

The input to the controller is the output of the model $\hat{y}(k)$; this can be easily represented by (14) as in Fig. 4 where the output of the model is the result of the contribution of two terms: the output of the system and the output error. Although the actual implementation of the control system is not in this form, this form is useful for the analysis and design of stabilizing controllers using dissipative techniques.

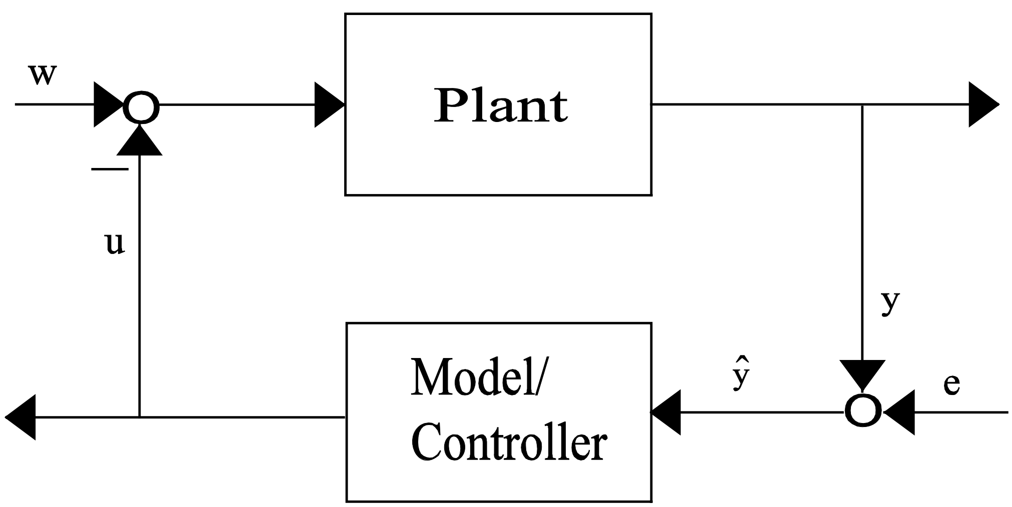

Fig. 4. This figure shows an equivalent feedback loop for the Model-Based Event-Triggered control system used for analysis.

By using a MB-ET implementation [10] the output error can be bounded by the appropriate design of a stabilizing threshold. The error (3) can be seen as piece-wise bounded external disturbance and at the communication update instants k_i , $e(k_i)=0$ because the model and plant output variables are equal.

B. Stabilization of networked discrete-time systems using output feedback event-triggered measurements

This section considers discrete-time nonlinear systems and models of the form (1)-(2) that are interconnected using a model-based implementation as described in Section II.A. The analysis and design of the stabilizing controller can be performed as shown in the previous section using the equivalent representation (12)-(13). Since it is only possible to measure the output of the system and not the whole state it is not possible to implement a state-space representation of the model directly. Instead, the model in both the controller and the sensor nodes is implemented as an equivalent nonlinear difference equation which represents the same input-output behavior as the state-space model. When the sensor decides that a measurement update needs to be sent according to the current output error, then it sends the current and *n* past output measurements which are used to update the model in the controller. At the same time the sensor uses exactly the same measurements to update its own copy of the model.

In order to make a decision as to whether or not it is necessary to send a measurement to update the model, the absolute value of the output error (3) is compared to a fixed positive threshold α. When the relation $\|e(k)\| > \alpha$ holds, the sensor transmits a measurement update. Assuming negligible delay the model is updated using the current and past *n* system output values at the same update instant k_i. At this point, the output error (3) is set to zero, since the model output is made equal to the real output of the system at time *k*. Therefore, the output error is bounded by:

$$\|e(k)\| \le \alpha. \tag{15}$$

Before the main results will be provided, the notion of stability must be covered. For uncertain systems that are perturbed by an external disturbance and that are operated open loop for some time periods, the notion of stability must be relaxed. While asymptotic stability is appealing, it simply is not achievable. When the system runs open loop, the state may diverge due to unstable dynamics that are not known exactly. A more reasonable notion of stability is in a form of boundedness such as uniform ultimate boundedness [27]. For systems that are ultimately bounded, as time goes to infinity the state is bounded by a known constant.

Instead of internal state stability, input-output stability is considered in this paper. For input-output stability, a notion such as ℓ_2 stability must be relaxed to a bound on the output as time goes to infinity. In general, a uniform bound on the output may not be tight. The output can be a function of internal state and system input. Since the input may not be predictable, a uniform bound on the output may be quite large. Instead a tighter bound on the average output amplitude may be found. In this paper the notion considered is *average output squared boundedness*.

Definition 3. *A nonlinear system is average output squared bounded if there exist a time* $\bar{k}$ *and a constant* b *such that the following bound on the output holds for all times* k_1 *and* k_2 *such that* $\bar{k} \leq k_1 < k_2$,

$$\frac{1}{(k_2 - k_1)} \sum_{k=k_1}^{k_2-1} y^T(k) y(k) \leq b.$$

This form of boundedness is a practical form of stability on the system output. While the output doesn't necessarily converge to zero, it is bounded on average with a known bound as time goes to infinity. Although this concept may not be useful for an arbitrarily large bound *b*, the concept is very informative for a small bound. The notion should be restricted to being used in the case when the bound is constructive and preferably when the bound can be made arbitrarily small by adjusting system parameters.

For the following theorem, assume that the plant and the model of the plant are QSR dissipative with respect to parameters Q_p, S_P, and R_P. Although the plant dynamics are not known exactly, experimental testing can be done to verify that the dissipative rate is at least a bound on the actual dissipative behavior of the system. The data taken to verify that the parameters hold to a specified level of certainty is similar data taken to identify system parameters. It is also assumed that a model-stabilizing dissipative controller has been designed such (7) is satisfied with Q_C, S_C, and R_C representing the QSR parameters of the controller. Next, we provide conditions under which we are able to stabilize uncertain unstable systems with limited feedback.

Theorem 3. *Consider the networked system (1) with uncertain dynamics, event-triggered aperiodic updates, and model-based output error (3). This feedback system, from external disturbance* w *to output* y*, is average output bounded stable if there exists a positive constant* a *such that the following matrix* $\tilde{Q}$ *is negative definite,*

$$\tilde{Q} = \begin{bmatrix} Q_p + aR_C & a{S_C}^T - S_p \\ aS_C - {S_P}^T & R_p + aQ_C \end{bmatrix} < 0.$$

Proof. The plant being QSR dissipative implies the existence of a positive definite storage function V_P, bounded above and below by class-*K* functions,

$$\underline{\alpha_P}(|x_P|) \leq V_P(x_P) \leq \overline{\alpha_P}(|x_P|),$$

such that the following inequality holds,

$$\Delta V_P(x_P) \leq \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q_P & S_P \\ {S_P}^T & R_P \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}.$$

The same applies for the controller being QSR dissipative, i.e.

$$\underline{\alpha_C}(|x_C|) \leq V_C(x_C) \leq \overline{\alpha_C}(|x_C|), \text{ and}$$

$$\Delta V_C(x_C) \leq \begin{bmatrix} u_c \\ \hat{y} \end{bmatrix}^T \begin{bmatrix} Q_C & S_C \\ {S_C}^T & R_C \end{bmatrix} \begin{bmatrix} u_c \\ \hat{y} \end{bmatrix}.$$

A total energy storage function can be defined, $V(x) = V_P(x_P) + aV_C(x_C)$, where $x = \begin{bmatrix} x_P \\ x_C \end{bmatrix}$. The total energy storage function has the dissipative property,

$$\Delta V(x) \leq \begin{bmatrix} y \\ u_c \\ w \\ e \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} y \\ u_c \\ w \\ e \end{bmatrix},$$

where

$$\tilde{Q} = \begin{bmatrix} Q_p + aR_C & aS_C{}^T - S_p \\ aS_C - S_P{}^T & R_p + aQ_C \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} S_P & aR_C \\ -R_P & aS_C \end{bmatrix}, \text{ and } \tilde{R} = \begin{bmatrix} R_P & 0 \\ 0 & aR_C \end{bmatrix}.$$

By assumption, $\tilde{Q}$ is negative definite and can be bounded above by a constant q, $\tilde{Q} \leq -qI$. The other two matrices can be bounded above, $\tilde{S} \leq sI$ and $\tilde{R} \leq rI$. This yields the following bound on ΔV

$$\Delta V(x) \leq -q[y^T y + u_c{}^T u_c] + 2s[y^T w + u_c{}^T e] + r[w^T w + e^T e].$$

A completing the square approach can be applied to remove the cross term leaving the following bound

$$\Delta V(x) \leq -\frac{q}{2}[y^T y + u_c{}^T u_c] + \frac{(4s^2 + 2qr)}{2q}[w^T w + e^T e].$$

Summing this inequality over a time interval from k_1 to k_2 yields the following evolution of the storage function.

$$V\big(x(k_2)\big) \leq V\big(x(k_1)\big) - \frac{q}{2}\sum_{k=k_1}^{k_2-1}(y^T y + u_c{}^T u_c) + \frac{(4s^2 + 2qr)}{2q}\sum_{k=k_1}^{k_2-1}(w^T w + e^T e)$$

The effect of the continuous ℓ_2 disturbance w can be bounded by some value $\varepsilon_w > 0$ after some time $\bar{k}$,

$$\sqrt{\frac{4s^2 + 2qr}{2q}}|w(k)| \leq \varepsilon_w.$$

Using the previous two equations, the following bound on the squared output can be found

$$\sum_{k=k_1}^{k_2-1} y^T y \leq \frac{2}{q}\left[V\big(x(k_1)\big) + \varepsilon_w{}^2\right] + \frac{(4s^2 + 2qr)}{q^2}\sum_{k=k_1}^{k_2-1} e^T e$$

This bound is made up of two quantities that are constant and the final summation that may increase with time.

The rest of this proof is done by breaking the analysis in two cases. Fixing the time $k_1 \geq \bar{k}$, for each time k_2 one of the following is true.

1. In the time range of k_1 to k_2 and for δ in the range $0 < \delta < 1$, the squared output is on average bounded by the following expression,

$$\frac{1}{(k_2 - k_1)} \sum_{k=k_1}^{k_2-1} y^T y \leq \frac{(4s^2 + 2qr)\alpha^2}{q^2(1-\delta)}$$

2. If the previous bound does not hold, the following holds

$$\frac{1}{(k_2 - k_1)} \sum_{k=k_1}^{k_2-1} y^T y > \frac{(4s^2 + 2qr)\alpha^2}{q^2(1-\delta)}$$

   This quantity can be used to bound the squared error accumulated over time. It can be shown that the following bound on the squared output holds.

$$\sum_{k=k_1}^{k_2-1} y^T y \leq \frac{2}{\delta q}[\varepsilon_w{}^2 + V(x(k_1))]$$

   With this bound on the total of the output squared, it is clear that the average value of the squared output is bounded. Since this bound is independent of time, it is fixed for arbitrarily large k_2 and any δ. This means that this bound would imply that the average of the output squared goes to zero as k_2 goes to infinity.

For either of the above cases, the squared output is bounded on average by a constant bound that is independent on time. Since both bounds hold, the maximum of the two is always at least a loose bound on the average of the squared output. Since the second average bound goes to zero over time, the first bound is the more relevant one on the infinite time horizon.

Remark 1. One important takeaway from the proof is that the average squared output is bounded by a constructive bound. These bounds can be made smaller by adjusting the values of control parameters. The bounds depend on q, s, and r which involve parameters of the plant that cannot be changed and parameters of the controller which may change. The first bound, the more relevant one, depends on the value of the state error threshold α. The output squared bound can be made arbitrarily small by making the error threshold smaller. Lastly, the bounds depend on δ and ε_w, which are constructed to analyze the behavior of the disturbance after time $\bar{k}$. The bounds can be made tighter by considering larger $\bar{k}$. Additionally, the parameter δ may be changed to adjust the relative magnitude of the two bounds. By picking an appropriate δ for each time k_2, the bounds may be chosen to be tighter.

Remark 2. The selection of the constant threshold α is made by considering the following tradeoff. A small threshold results in a smaller bound on the system's average output but, in general, increases the communication rate by sending measurement updates more frequently. A reduction on network usage can be achieved by increasing the threshold at the cost of a larger average output of the system.

In the case of linear systems it is possible to estimate the set of admissible uncertain plants that can be stabilized given a model and a controller. Using the same QSR parameters Q_p, S_P, and R_P that were used for the model and assuming that the real parameters contain additive uncertainties with respect to the nominal model parameters, i.e. $A = \hat{A} + \Delta_A$, $B = \hat{B} + \Delta_B$, $C = \hat{C} + \Delta_C$, $D = \hat{D} + \Delta_D$, then the following can be solved

$$\begin{bmatrix} X & W \\ W^T & Y \end{bmatrix} \leq 0$$

for P and using different values of $\Delta_A, \Delta_B, \Delta_C, \Delta_D$, where

$X = (\hat{A} + \Delta_A)^T P(\hat{A} + \Delta_A) - P - (\hat{C} + \Delta_C)^T Q(\hat{C} + \Delta_C)$

$Y = (\hat{B} + \Delta_B)^T P(\hat{B} + \Delta_B) - (\hat{D} + \Delta_D)^T Q(\hat{D} + \Delta_D) - (\hat{D} + \Delta_D)^T S - S^T(\hat{D} + \Delta_D) - R$

and $W = (\hat{A} + \Delta_A)^T P(\hat{B} + \Delta_B) - (\hat{C} + \Delta_C)^T Q(\hat{D} + \Delta_D) - (\hat{C} + \Delta_C)^T S$

When the above problem is feasible for given choice of uncertainties $\Delta_A, \Delta_B, \Delta_C, \Delta_D$ then the uncertain system *A*, *B*, *C*, *D* is an element of the set of admissible uncertain plants.

IV. EXAMPLES

The following examples demonstrate how the MB-NCS design method introduced in this paper may be used. The examples were chosen to be LTI for simplicity and ease of understanding. The theory applies to general nonlinear systems.

Example 1. Consider a model of an unstable system given by:

$$\hat{A} = \begin{bmatrix} -0.81 & 0.37 \\ 0.88 & 0.21 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \hat{D} = 1. \tag{16}$$

It can be shown that the model is QSR dissipative with respect to Q_P=0.5, S_P=0.5, and R_P=0.1, by using the storage function:

$$\hat{V}(\hat{x}) = \hat{x}^T \begin{bmatrix} 0.8 & 0.87 \\ 0.87 & 1.28 \end{bmatrix} \hat{x}.$$

A stabilizing controller is given by

$$A_c = 0.5, \quad B_c = 0.3, \quad C_c = 1, \quad D_c = 1. \tag{17}$$

This controller is passive and QSR dissipative with respect to Q_C=-0.2, S_C=0.5, and R_C=-0.6 which can be shown using the storage function $V_c(x_u) = 1.23 x_u^2$.

The controller can be shown to stabilize the model by evaluating (7).

$$\tilde{Q} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix} < 0. \tag{18}$$

For this example, an actual uncertain plant is given by:

$$A = \begin{bmatrix} -0.71 & 0.55 \\ 0.95 & 0.35 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.75 & 2.3 \end{bmatrix}, \quad D = 1.1. \tag{19}$$

The plant is also dissipative with respect to the same choice of QSR parameters Q_P=0.5, S_P=0.5, and R_P=0.1, This can be verified using the storage function

$$V(x) = x^T \begin{bmatrix} 1.21 & 1.13 \\ 1.13 & 1.78 \end{bmatrix} x.$$

Since the QSR parameters for the plant and the model are the same, then the controller (17) also stabilizes the plant and satisfies the inequality (7) with (18). Simulations of the model-based networked system that is also affected by an (ℓ_2) external disturbance $w(k)$ are shown in Fig. 5 using a threshold value α=0.02. The network communication signal $n_c(k)$ in Fig. 6 represents the time instants at which output measurements are sent from the sensor node to the controller node. The rest of the time the networked system operates in open-loop mode.

$$n_c(k) = \begin{cases} 1 & \text{if measurements are sent at time } k \\ 0 & \text{if measurements are not sent at time } k \end{cases} \tag{20}$$

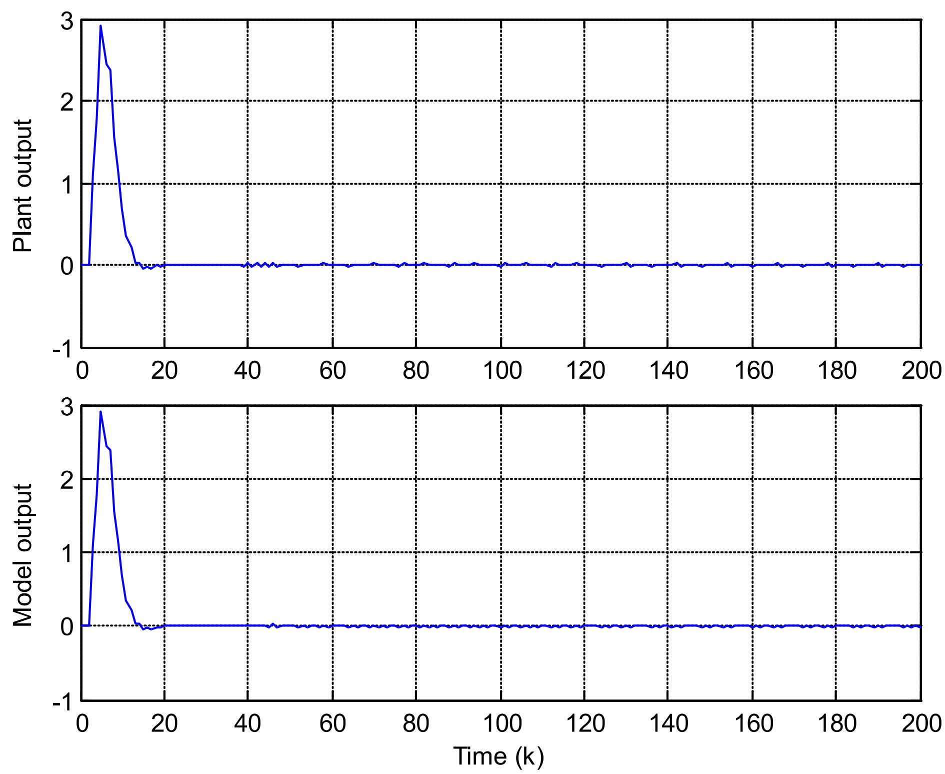

Fig. 5. Outputs of the plant (top) and the model (bottom) for example 1.

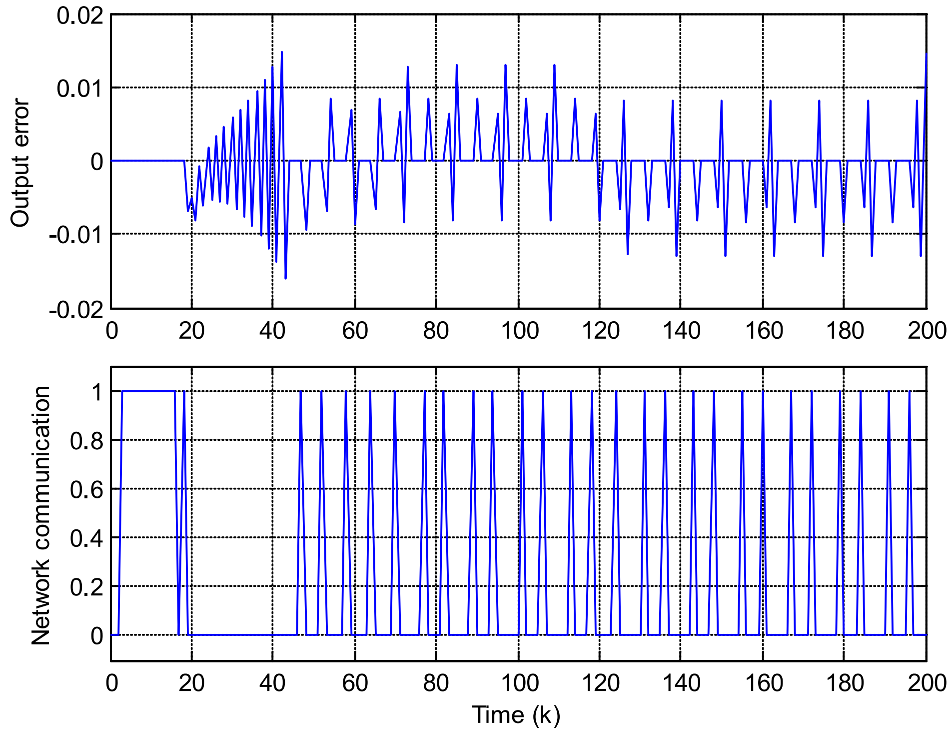

Fig. 6. Output error (top) and network communication instants for example 1.

Fig. 5 shows that the outputs of the model-based networked uncertain system are bounded, as expected. Although the parameters of the nominal model differ significantly from those of the real plant, it is still possible to stabilize the system. A larger reduction of network communication can be achieved by using a more accurate model that is QSR dissipative using the same choice of QSR parameters and that it also reflects more accurately the dynamics of the plant.

Example 2. Consider the same plant dynamics (19) and the following model parameters:

$$\hat{A} = \begin{bmatrix} -0.7 & 0.52 \\ 0.88 & 0.4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0.73 & 2.2 \end{bmatrix}, \quad \hat{D} = 1.2. \tag{21}$$

The results of simulations to the same external disturbance using the same threshold and the new model are shown in Fig. 7 and Fig. 8.

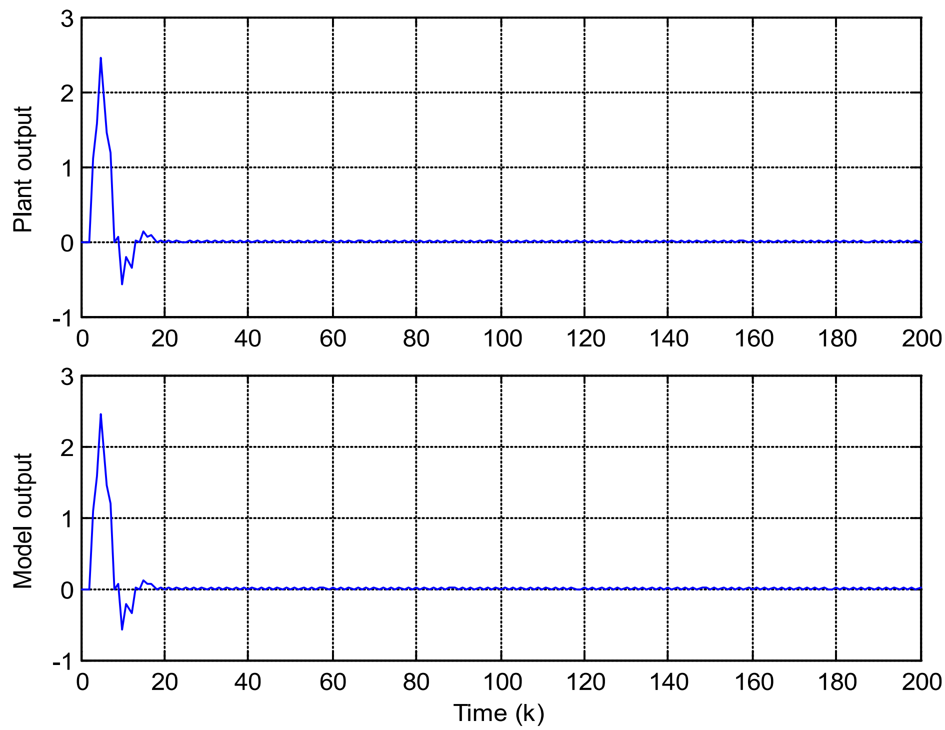

Fig. 7. Outputs of the plant (top) and the model (bottom) for Example 2.

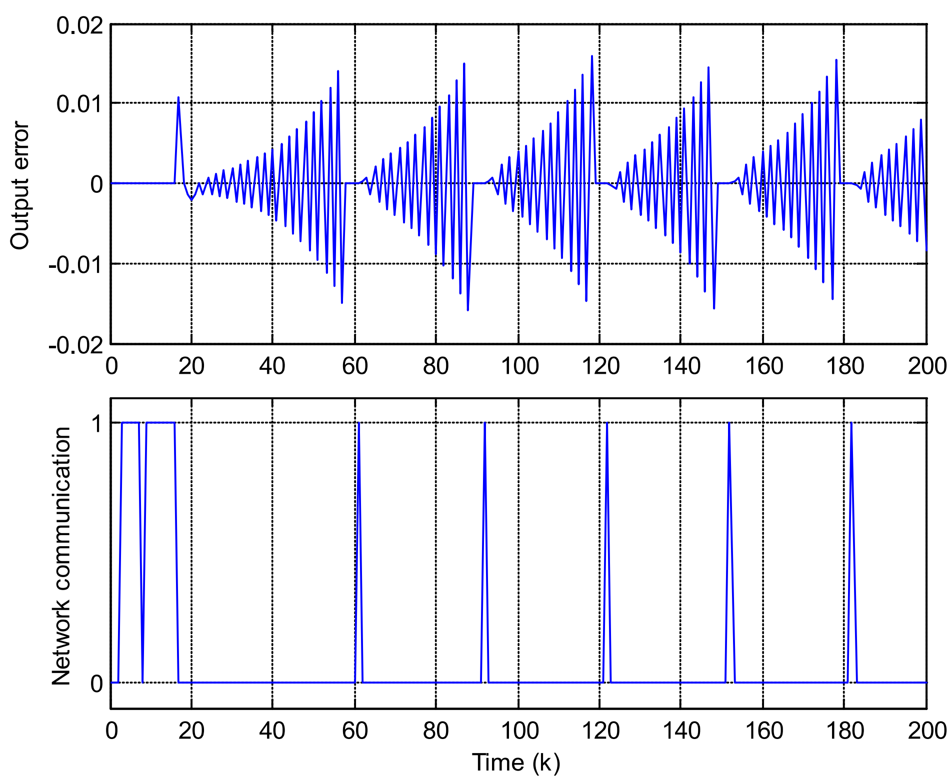

Fig. 8. Output error (top) and network communication instants for Example 2.

V. CONCLUSIONS

The stabilization of networked nonlinear systems using the MB-NCS approach has received little attention compared to the linear systems problem. Furthermore, previous results that consider nonlinear systems only address the state feedback case. This paper provides a method for designing stabilizing controllers based on dissipativity theory and the results can be applied to a general class of nonlinear systems with model uncertainties, with disturbances, and with states that cannot be measured directly as long as the output can be measured. The design of dissipative controllers for MB-NCS is made possible by modeling the model-based networked architecture as a standard negative feedback interconnection and by implementing the model as a difference (input-output) equation which can be updated using the system's output measurements directly without need of state observers. Communication rates are reduced significantly with the MB-NCS framework and then further reduced by implementing aperiodic event-triggered communication. The main result of the paper demonstrates boundedness of the average output squared with a constructive bound. This bound can be made quite small by varying the design parameters of the controller and varying the acceptable error threshold. Two examples demonstrate how communication can be significantly reduced while still maintaining a desirable bound on the output.

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