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Technical Report of the ISIS Group
at the University of Notre Dame
ISIS-2012-009
November, 2012

Interdisciplinary Studies in Intelligent Systems – Technical Report

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Version 1.0 August 31, 2012.

Noname manuscript No.
(will be inserted by the editor)

On characterization of robust feedback Nash equilibrium for generalized multi-channel systems

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Received: date / Accepted: date

Abstract In this paper, we consider the problem of robust state-feedback stabilization for a multi-channel system in a game-theoretic framework. Specifically, we provide a sufficient condition for the existence of a robust feedback Nash equilibrium when each agent aims to optimize different type of objective function which is linked-up with a certain dissipativity property of the multi-channel system. Furthermore, we assume that the agents may be unaware of all the aspects or the structure of the game. In such a scenario, we characterize the robust feedback Nash equilibria via a set of extended linear matrix inequalities and set-valued mappings.

Keywords Dissipativity properties · Game theory · Multi-channel system · Nash equilibrium · Extended linear matrix inequality · Robust stabilization

1 Introduction

We consider the problem of robust state-feedback stabilization for multi-channel systems in a game-theoretic framework. Specifically, we assume that each agent aims to optimize different type of objective function and when some of the agents may be unaware of all the aspects/structure of the game. In such a scenario, as is well known, Nash strategy (aka *Nash equilibrium*) provides a framework to study an inherent robustness property of the agents' strategies under a family of information structures, since no agent can improve his payoff by deviating unilaterally from the Nash strategy once the equilibrium is attained (e.g., see [16], [17], [19], [2], [5]).

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In the past decades, several theoretical results that arise from control related problem have been characterized in the context of Nash strategy using a game-theoretic interpretation (e.g., see [21], [15], [18], [22], [1] and [23] and the references therein). For example, the existence of open-loop Nash strategies for linear-quadratic games over a finite time-horizon, assuming that all strategies lie in compact subsets of an admissible strategy space, has been addressed in [24]; while the existence of Nash strategies for linear-quadratic differential games over an infinite-horizon has been studied in detail in [18], [1] and [5]. Some of these works have also discussed the uniqueness of the optimal strategies for linear-quadratic games with structured uncertainties, where the bound on the objective function is based on the existence of a set of solutions for appropriately parameterized Riccati equations. In the area of multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control theory, the concept of differential games has been applied by interpreting uncertainty (or neglected dynamics) as a fictitious agent which is typically introduced in the cost criteria through weighting matrices (e.g., see [14], [4], [8] and [22]).

On the other hand, the use of different simplified models of the same system has also been employed for capturing certain information structures or objective functions that individual agents may hold about the overall system. Thus, the resulting problem can be best described by nonzero-sum differential games where the individual agents are allowed to minimize different types of objective functions (e.g., see [21], [9], [13] and [20]). An extensive survey on the area of noncooperative dynamic games is also provided in [5].

In this paper, we follow this approach by assuming that individual agents may have different types of objective functions that are linked-up with the dissipativity property of the multi-channel system (e.g., see [26], [25] and references therein for a review of systems with dissipative properties) – and where the optimality concept is that of robust feedback Nash equilibrium. In particular, in this paper, we consider two fundamental problems: (i) we first provide a condition guaranteeing that the agents' strategy space is *sufficiently* decentralized to make the game-theoretic interpretation meaningful (i.e., a strategy space that is independently accessible by each agent), and (ii) then, we provide a sufficient condition for the existence of robust feedback Nash equilibrium. Furthermore, we show that the existence of a weak-optimal solution to a suitably defined dissipativity constrained problem is a sufficient condition for the existence of the feedback Nash equilibrium for the multi-channel system.

This paper is organized as follows. In Section 2, we state the problem of robust state-feedback stabilization for a multi-channel system using a game-theoretic framework. Section 3 provides a sufficient condition for the existence of a robust feedback Nash equilibrium using a concept from set-valued mappings. In Section 4, we present the main results of the paper, i.e., we characterize the robust feedback Nash equilibria via a set of solutions that corresponds to a set of extended linear matrix inequalities (LMIs) and dissipativity properties of the system. Finally, Section 5 provides some concluding remarks.

Notation. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{He}\,(A)$ denotes a hermitian matrix defined by $\mathrm{He}\,(A) \overset{\text{def}}{=} (A + A^T)$, where A^T is the transpose of A. For a matrix $B \in \mathbb{R}^{n\times p}$ with $r = \operatorname{rank} B$, $B^\perp \in \mathbb{R}^{(n-r)\times n}$ denotes the orthogonal complement of B, which is a matrix that satisfies $B^{\perp}B = 0$ and $B^{\perp}B^{\perp T} \succ 0$. $\mathbb{S}_{+}^{n}$ denotes the set of strictly positive definite $n \times n$ real matrices and $\mathbb{C}^{-}$ denotes the set of complex numbers with negative real parts, that is $\mathbb{C}^{-} \overset{\text{def}}{=} \{s \in \mathbb{C} \,|\, \mathrm{Re}\{s\} < 0\}$. $\mathrm{Sp}(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$, i.e., $\mathrm{Sp}(A) \overset{\text{def}}{=} \{\lambda \in \mathbb{C} \,|\, \mathrm{rank}(A - \lambda I) < n\}$, $\mathcal{U}_{\rho}$ denotes a compact uncertainty set in $\mathbb{R}^{n\times n}$ and $\mathrm{GL}_n(\mathbb{R})$ denotes the general linear group consisting of all $n \times n$ real nonsingular matrices.

2 Preliminaries and problem formulation

Consider the following finite-dimensional generalized multi-channel system

$$\dot{x}(t) = Ax(t) + \sum_{j\in\mathcal{N}} B_j u_j(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u_j(t) \in \mathbb{R}^{r_j}$ is the control input to the jth-channel of the system, $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times r_j}$ and $\mathcal{N} \overset{\text{def}}{=} \{1, 2, \ldots, N\}$ represents the set of controllers (or agents).

Let K_j be the jth-agent's strategy selected from a well defined strategy space $\mathcal{K}_j \subset \mathbb{R}^{r_j\times n}$. Also, let us introduce some additional notation that will be useful in the sequel

$$r \overset{\text{def}}{=} \sum_{i\in\mathcal{N}} r_i, \qquad K \overset{\text{def}}{=} (K_j)_{j\in\mathcal{N}} \in \mathcal{K}, \qquad \mathcal{K} \overset{\text{def}}{=} \prod_{j\in\mathcal{N}} \mathcal{K}_j \subseteq \prod_{j\in\mathcal{N}} \mathbb{R}^{r_j\times n},$$

$$r_{\neg j} \overset{\text{def}}{=} \sum_{i\in\mathcal{N}_{\neg j}} r_i, \quad K_{\neg j} \overset{\text{def}}{=} (K_i)_{i\in\mathcal{N}_{\neg j}} \in \mathcal{K}_{\neg j}, \quad \mathcal{K}_{\neg j} \overset{\text{def}}{=} \prod_{i\in\mathcal{N}_{\neg j}} \mathcal{K}_i \subseteq \prod_{i\in\mathcal{N}_{\neg j}} \mathbb{R}^{r_{\neg i}\times n},$$

where the sets $\mathcal{N}_{\neg j}$ are defined by $\mathcal{N}_{\neg j} \overset{\text{def}}{=} \mathcal{N} \setminus \{j\}$ for $j = 1, 2, \ldots, N$.

For the above multi-channel system, we restrict the set $\mathcal{K}$ to be the set of all linear, time-invariant stabilizing state-feedback controllers that satisfies

$$\mathcal{K} \subseteq \left\{ (K_1, K_2, \ldots, K_N) \in \prod_{j\in\mathcal{N}} \mathcal{K}_j \;\middle|\; \mathrm{Sp}\left(A + \sum_{j\in\mathcal{N}} B_j K_j \right) \subset \mathbb{C}^{-} \right\}. \tag{2}$$

Let us introduce the following matrices that will be used later.

$$E = \big[\underbrace{I_{n\times n}\; I_{n\times n}\; \cdots\; I_{n\times n}}_{(N+1)\ times}\big], \qquad \langle X, \widetilde{X}\rangle = \mathrm{block\,diag}\{\underbrace{X, \overbrace{X, X, \ldots, X}^{=\widetilde{X}}}_{(N+1)\ times}\},$$

$$[A, B]_{U,\widetilde{L}} = [\underbrace{AU\; B_1L_1\; B_2L_2\; \ldots\; B_NL_N}_{(N+1)\ times}], \langle U, \widetilde{W}\rangle = \mathrm{block\,diag}\{\underbrace{U, \overbrace{W_1, W_2, \ldots, W_N}^{=\widetilde{W}}}_{(N+1)\ times}\}.$$

where $X \in \mathbb{S}_{+}^{n}$, $U \in \mathrm{GL}_n(\mathbb{R})$, $W_i \in \mathrm{GL}_n(\mathbb{R})$ and $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$.

The following lemma (whose proof is given in [6]) characterizes the set of all stabilizing feedback gains $\mathcal{K}$. Note that we will later use the result of this lemma together with dissipativity properties of the multi-channel system for describing the strategy space for each agent.

Lemma 1 *Suppose the pair* $\big(A, [\, B_1 \; B_2 \cdots B_N \,]\big)$ *is stabilizable. Then, there exist* $X \in \mathbb{S}_+^n$, $\epsilon > 0$, $U \in \mathrm{GL}_n(\mathbb{R})$, $W_j \in \mathrm{GL}_n(\mathbb{R})$ *and* $L_j \in \mathbb{R}^{r_j \times n}$ *for* $j = 1, 2, \ldots, N$ *such that*

$$\begin{bmatrix} 0_{n\times n} & E\langle X, \widetilde{X}\rangle \\ \langle X, \widetilde{X}\rangle E^T & 0_{(N+1)n\times(N+1)n} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} [A, B]_{U,\widetilde{L}} \\ -\langle U, \widetilde{W}\rangle \end{bmatrix} \begin{bmatrix} E^T & \epsilon I_{(N+1)n\times(N+1)n} \end{bmatrix} \right) \prec 0, \tag{3}$$

For any family of N-tuples $(\, L_1, \, L_2, \, \ldots, \, L_N \,)$ *and* $(\, W_1, \, W_2, \, \ldots, \, W_N \,)$ *as above, if we set* $K_j = L_j W_j^{-1}$ *for each* $j = 1, 2, \ldots, N$*, then the matrix* $\big(A + \sum_{j\in\mathcal{N}} B_j K_j \big)$ *is a Hurwitz, i.e.,* $\mathrm{Sp}\big(A + \sum_{j\in\mathcal{N}} B_j K_j \big) \in \mathbb{C}^-$.

In the following, it will be convenient to identify each objective function $J_j \colon \mathbb{R}^n \times \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+$ using the following related function

$$\mathbb{R}^n \times \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+ \colon (x_0, K_j, K_{\neg j}) \mapsto J_j(x_0, K_j, K_{\neg j}), \tag{4}$$

for all $j = 1, 2, \ldots, N$.[1]

For a *complete information* game, the jth-agent decides his own strategy by solving the following optimization problem

$$\sup_{K_j \in \mathcal{K}_j} J_j(K_j, K_{\neg j}), \tag{5}$$

for some initial conditions $x_0 \in \mathbb{R}^n$; while the opponents' strategy $K_{\neg j} \in \mathcal{K}_{\neg j}$ are held fixed. Hence, for every agent $j \in \mathcal{N}$, the N-tuple $(K_1^*, \, K_2^*, \, \ldots \, K_N^*) \in \mathcal{K}$ that satisfies

$$K_j^* \in \arg \sup_{K_j \in \mathcal{K}_j} J_j(K_j, K_{\neg j}^*), \tag{6}$$

is called a *feedback Nash equilibrium*. That is, if every agent $j \in \mathcal{N}$ chooses a strategy K_j^*, then no agent has an incentive to change his own strategy from the feedback Nash equilibrium. Note that the feedback Nash equilibrium is well defined only when every agent can estimate his opponents' strategies and evaluate his own objective function exactly (e.g., [16], [17], [19]).

However, a more realistic model must include the possibility that any information may contain uncertainty corresponding to, for example, observation or estimation errors. In the following, we focus on a game with uncertainty (i.e., a game with an incomplete information). In this context, we introduce the following uncertainty sets:

[1] In Section 4, we provide the exact formulation of the objective function (cf. Equations (8)) and (15).

- $\mathcal{U}_\rho$ where $(A+u_\rho A^\delta) \in \mathcal{U}_\rho \subset \mathbb{R}^{n\times n}$ with $u_\rho \in [-\rho, \rho]$, $\rho \in \mathbb{R}_+$ is an uncertainty level and $A^\delta \in \mathbb{R}^{n\times n}$ is a perturbation term that is associated with the nominal system matrix A, and
- $\hat{\mathcal{K}}_{\neg j} \subset \mathcal{K}_{\neg j}$ where $\hat{K}_{\neg j} = (K_{\neg j} + K^\delta_{\neg j})$ with $K^\delta_{\neg j} \in \mathcal{K}^\delta_{\neg j} \subseteq \mathbb{R}^{r_{\neg j}\times n}$ is an uncertainty term which is associated with jth-agent's observation about his opponents strategies.

Assumption 1 *We assume the following statements about each agent $j \in \mathcal{N}$:*

(A1) *The objective function for the jth-agent involves an unknown parameter $u_{\rho_j} \in [-\rho_j, \rho_j]$ and can be expressed as*

$$\mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+,$$

such that

$$(x_0, u_{\rho_j}, K_j, K_{\neg j}) \mapsto \bar{J}_j(x_0, u_{\rho_j}, K_j, K_{\neg j}) \overset{\text{def}}{=} J_j^{u_{\rho_j}}(K_j, K_{\neg j}).$$

Here the jth-agent does not know the exact value of u_{ρ_j}; however, this agent can estimate that $u_{\hat{\rho}_j}$ from a nonempty set $[-\hat{\rho}_j, \hat{\rho}_j]$.

(A2) *The jth-agent may not know exactly his opponents' strategies $K_{\neg j}$; however, he can estimate his opponents' strategies from a nonempty compact strategy space $\hat{\mathcal{K}}_{\neg j} \subseteq \mathcal{K}_{\neg j}$.*

Thus, in this case, every agent is required to address a family of problems involving uncertainty terms $u_{\hat{\rho}_j}$ and $\hat{\mathcal{K}}_{\neg j}$ (or $\mathcal{K}^\delta_{\neg j}$)

$$\sup_{K_j \in \mathcal{K}_j} J_j^{u_{\hat{\rho}_j}}(K_j, \hat{K}_{\neg j}), \tag{7}$$

where $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ and $\hat{K}_{\neg j} \in \hat{\mathcal{K}}_{\neg j} \subseteq \mathcal{K}_{\neg j}$.

To solve this family of problems, we assume that each agent chooses a strategy according to a worst-case criterion. Thus, the jth-agent tries to maximize his worst-case objective function under Assumptions (A1) and (A2), i.e., each agent considers the worst-case objective function $\tilde{J}_j : \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+$ defined by

$$\tilde{J}_j(K_j, K_{\neg j}) \overset{\text{def}}{=} \inf\left\{ J_j^{u_{\rho_j}}(K_j, \hat{K}_{\neg j}) \;\middle|\; u_{\rho_j} \in [-\rho_j, \rho_j] \text{ and } \hat{K}_{\neg j} \in \hat{\mathcal{K}}_{\neg j} \right\}, \tag{8}$$

and solves the worst-case optimization problem (e.g., see [11], [7])

$$\sup_{K_j \in \mathcal{K}_j} \tilde{J}_j(K_j, K_{\neg j}). \tag{9}$$

Remark 1 Notice that the above problem formulation can be interpreted as a complete information game with objective functions $\tilde{J}_j(K_j, K_{\neg j})$ for all $j \in \mathcal{N}$.

Now we can define the robust feedback Nash equilibrium for the multi-channel system as follows:

Definition 1 Let $\tilde{J}_j(K_j, K_{\neg j})$ for all $j \in \mathcal{N}$ be defined by (8). Then the N-tuple $(K_1^*, K_2^*, \ldots K_N^*) \in \mathcal{K}$ is called a robust feedback Nash equilibrium for the game in (7), if $K_j^* \in \arg\sup_{K_j \in \mathcal{K}_j} \tilde{J}_j(K_j, K_{\neg j}^*)$ for all $j \in \mathcal{N}$ (i.e., if it is a feedback Nash equilibrium for the game in (9)).

Hence, the problem of finding a sufficient condition for the existence of a feedback Nash equilibrium solution for the game in (9) is called the problem of *robust feedback Nash equilibrium* for the multi-channel system in (1).

3 Existence of robust feedback Nash equilibrium

In this section, we provide a sufficient condition for the existence of a robust feedback Nash equilibrium for the multi-channel system in (1). Note that the set $\hat{\mathcal{K}}_{\neg j} \subseteq \mathcal{K}_{\neg j}$ (as defined in Assumption (A2)) can be considered as a set-valued mapping $\hat{\mathcal{K}}_{\neg j} : \mathcal{K}_{\neg j} \ni K_{\neg j} \to \hat{\mathcal{K}}_{\neg j}$ (e.g., see [3] and references therein for the review of set-valued mappings).

Further, we make the following assumptions.

Assumption 2 *The following statements hold true for each agent $j \in \mathcal{N}$:*

(A3) *The function $J_j^{u_{\hat{\rho}_j}} : \mathcal{K}_j \times \hat{\mathcal{K}}_{\neg j} \to \mathbb{R}_+$ is a continuous objective function.*

(A4) *The set $\hat{\mathcal{K}}_{\neg j}$ is a non-empty and compact set for any $K_{\neg j} \in \mathcal{K}_{\neg j}$.*

(A5) *The set $\mathcal{U}_\rho$ is non-empty and compact.*

(A6) *The function $\tilde{J}_j(., \mathcal{K}_{\neg j}) : \mathcal{K}_j \to \mathbb{R}_+$ is concave on $\mathcal{K}_j$ for any fixed $K_{\neg j}$ and $\hat{u}_{\rho_j}$.*

Remark 2 Assumptions (A3)–(A6) above imply that the function $\tilde{J}_j(.,.)$ in (8) has the following properties:

(P1) $\tilde{J}_j$ is continuous and finite at any $(K_j, K_{\neg j}) \in \mathcal{K}$.

(P2) For any fixed $K_{\neg j} \in \mathcal{K}_{\neg j}$, the objective function $\tilde{J}_j(., K_{\neg j}) : \mathcal{K}_j \to \mathbb{R}_+$ is concave on $\mathcal{K}_j$.

The following lemma is a well-known result for N-person noncooperative games (e.g., see [2]).

Lemma 2 *Suppose that the following statements hold for every agent $j \in \mathcal{N}$: (i) the strategy set $\mathcal{K}_j$ is a non-empty and compact set, (ii) the objective function $\tilde{J}_j : \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+$ is continuous, and (iii) $\tilde{J}_j(., K_{\neg j})$ is a concave function for any $K_{\neg j} \in \mathcal{K}_{\neg j}$. Then, the game in* (9) *has at least one feedback Nash equilibrium.*

Based on this lemma, we can obtain the following theorem for the existence of a robust feedback Nash equilibrium in (7).

Theorem 1 *Let Assumptions (A3)–(A6) hold true. Then the game in* (7) *has at least one robust feedback Nash equilibrium.*

Proof From Assumptions (A3)–(A6), for any $j \in \mathcal{N}$, the worst-case objective function $\tilde{J}_j(K_j, K_{\neg j})$ is continuous and finite at any $(K_j, K_{\neg j}) \in \mathcal{K}_j \times \mathcal{K}_{\neg j}$. Moreover, the function $\tilde{J}_j(., K_{\neg j})$ is convex on $\mathcal{K}_j$ for any $K_{\neg j} \in \mathcal{K}_{\neg j}$ for $j \in \mathcal{N}$. Therefore, from Lemma 2, the game in (9) has a feedback Nash equilibrium, that by definition implies that the game in (7) has a robust feedback Nash equilibrium. □

4 Main results

In this section, we consider the following closed-loop system, where we link the different types of the objective functions with a certain dissipativity property of the multi-channel system

$$\begin{aligned} \dot{x}(t) &= \left(\left(A + u_{\hat{\rho}_j} A_j^{\delta}\right) + B_j K_j + \sum_{i \in \mathcal{N}_{\neg j}} B_i K_i \right) x(t) + 0_{n\times 1}\tilde{u}(t), \\ \tilde{y}(t) &= x(t) + 0_{n\times 1}\tilde{u}(t), \qquad x(0) = x_0, \end{aligned} \tag{10}$$

where $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$, $\hat{\rho}_j \in \mathbb{R}_+$ and A_j^{δ} are an uncertainty level, an upper uncertainty bound and a base-perturbation term, respectively, that are associated with the jth-agent.

In the following, the above closed-loop system is assumed to be stable for some initial conditions $x_0 \in \mathbb{R}^n$.

Introduce the following set of supply rate functions

$$\mathcal{W} = \left\{ \prod_{j=1}^{N} w_{[\alpha_j, Z_j]}\big(\tilde{y}(t), \tilde{u}(t)\big) \right\}, \tag{11}$$

where, for $j = 1, 2, \ldots, N$, the supply rate functions $w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t))$ are given by

$$\big(\tilde{y}(t), \tilde{u}(t)\big) \mapsto w_{[\alpha_j, Z_j]}\big(\tilde{y}(t), \tilde{u}(t)\big) \overset{\text{def}}{=} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha_j Z_j & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}, \tag{12}$$

and a matrix interval set $\mathcal{I}_{[\beta_j, Z_j]} \in \mathbb{S}_+^n$

$$\mathcal{I}_{[\beta_j, Z_j]} = \left\{ Y_j \;\middle|\; \beta_j^{-1} Z_j \preceq Y_j \preceq Z_j \right\}, \tag{13}$$

where $\alpha_j > 0$, $\beta_j \geq 1$ and $Z_j \in \mathbb{S}_+^n$ for $j = 1, 2, \ldots, N$.

We next present a more realistic game-theoretic interpretation in terms of the lowest upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for all $j \in \mathcal{N}$ (that prescribe the N-tuple uncertainty set $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \ldots, u_{\hat{\rho}_N}) \in \prod_{j\in\mathcal{N}} [-\hat{\rho}_j, \hat{\rho}_j]$) together with the existence of stabilizing state-feedback gains that provide a sufficient condition for obtaining a set of feedback Nash equilibria.

In light of above discussion and Theorem 1 (as well as Lemma 1), we have the following theorem which provides a sufficient condition for the existence of feedback Nash equilibrium.

Theorem 2 *Let $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Assume that $\alpha_j > 0$, $\beta_j \geq 1$ and $Z_j \in \mathbb{S}_+^n$ for $j = 1, 2, \ldots, N$. Then, there exit $X_j \in \mathbb{S}_+^n$, $U_j \in \mathrm{GL}_n(\mathbb{R})$, $j = 1, 2, \ldots, N$ and an N-tuple $(\, L_1^*,\, L_2^*,\, \ldots,\, L_N^* \,) \in \prod_{j\in\mathcal{N}} \mathbb{R}^{r_j \times n}$ such that*

$$\begin{bmatrix} 0_{n\times n} & E\langle X_j, \widetilde{X}_j\rangle \\ \langle X_j, \widetilde{X}_j\rangle E^T & 0_{(N+1)n\times(N+1)n} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} [A,B]_{U_j,\widetilde{L}^*_{\neg j}} \\ -\langle U_j, \widetilde{W}\rangle \end{bmatrix}\begin{bmatrix} E^T & \epsilon_j I_{(N+1)n\times(N+1)n} \end{bmatrix}\right) \prec 0, \tag{14}$$

where, for some $L_j \in \mathbb{R}^{r_j\times n}$, $j = 1, 2, \ldots, N$,

$$[A,B]_{U_j,\widetilde{L}^*_{\neg j}} = \left[\, AU_j \;\, B_1L_1^* \cdots B_{j-1}L_{j-1}^* \;\, B_jL_j \;\, B_{j+1}L_{j+1}^* \cdots B_NL_N^* \,\right],$$

and

$$\langle U_j, \widetilde{W}\rangle = \mathrm{block\,diag}\{U_j, W_1, \ldots, W_{j-1}, W_j, W_{j+1}, \ldots, W_N\}.$$

Furthermore, there exist $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ and $\hat{\rho}_j(x_0, K^*)$ for $j = 1, 2, \ldots, N$ such that

$$\sup_{K_j\in\mathcal{K}_j} \tilde{J}_j(K_j, K^*_{\neg j}) \rightsquigarrow \hat{\rho}_j(x_0, K^*), \tag{15}$$

for which all perturbed systems in (10) *are robustly stable for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ with $K_j^* \in \arg\sup_{K_j\in\mathcal{K}_j} \tilde{J}_j(K_j, K^*_{\neg j})$ for all $j = 1, 2, \ldots, N$.*[2]

Proof Suppose all the perturbed systems in (10) satisfy the following dissipativity inequalities

$$V_j(x(0)) + \int_0^t w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t))dt \geq V_j(x(t)), \tag{16}$$

for all $t \geq 0$ with non-negative quadratic storage functions $V_j(x(t)) = x(t)^T Y_j x(t)$ and $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ that satisfy $V_j(0) = 0$ for $j = 1, 2, \ldots, N$.

[2] Notice here that we write $\sup_{K_j\in\mathcal{K}_j} \tilde{J}_j(K_j, K^*_{\neg j}) \rightsquigarrow \hat{\rho}_j(x_0, K^*)$ to refer to following expression

$$\sup_{K_j\,\in\,\mathcal{K}_j} \left\{ \inf\left\{ J_j^{u_{\hat{\rho}_j}}(K_j, K^*_{\neg j}) \;\middle|\; u_{\hat{\rho}_j} \in \left[-\hat{\rho}_j(x_0, K_j, K^*_{\neg j}), \hat{\rho}_j(x_0, K_j, K^*_{\neg j})\right] \;\&\; K^*_{\neg j} \in \hat{\mathcal{K}}_{\neg j} \right\}\right\}$$

(cf. Equation (8)).

Moreover, the upper bounds $\hat{\rho}_j(x_0, K_j, K^*_{\neg j}) \in \mathbb{R}_+$ continuously depend (*in a weak sense*) on x_0 and K_j for all $j \in \mathcal{N}$. This further guarantees the existence of such upper bounds for which the dissipativity conditions in (16) will hold for all instances of perturbation in the system.

Thus, the trajectories of each perturbed closed-loop system (i.e., for $j = 1, 2, \dots, N$)

$$\dot{x}(t) = \left((A + u_{\rho_j} A_j^{\delta}) + \sum_{i \in \mathcal{N}} B_i K_i^* \right) x(t),$$

satisfy

$$\begin{aligned} \frac{d}{dt}\left(x^T(t) Y_j x(t)\right) &= x^T(t) \mathrm{He}\left(\left((A + u_{\rho_j} A_j^{\delta}) + B_j K_j^* + \sum_{i \in \mathcal{N}_{\neg j}} B_i K_i^* \right)^T Y_j \right) x(t), \\ &\le -\alpha_j x^T(t) Z_j x(t), \\ &\le -\alpha_j x^T(t) Y_j x(t), \end{aligned} \tag{17}$$

for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ in the system.

Then, the rest is to follow the same lines as that of Lemma 1. In fact, replacing the following

$$[A, B]_{U, \widetilde{L}} \to [A, B]_{U_j, \widetilde{L}^*_{\neg j}}, \quad \langle U, \widetilde{W} \rangle \to \langle U_j, \widetilde{W} \rangle \quad \text{and} \quad \langle X, \widetilde{X} \rangle \to \langle X_j, \widetilde{X}_j \rangle,$$

in Lemma 1 immediately gives the condition in (14) of Theorem 2. Note that K_j and K_i^* are given by $K_j = L_j W_j^{-1}$, $j \in \mathcal{N}$ and $K_i^* = L_i^* W_i^{-1}$, $i \in \mathcal{N}_{\neg j}$, respectively. Moreover, the N-tuple $(Y_1, Y_2, \cdots, Y_N) \in \prod_{j \in \mathcal{N}} \mathcal{I}_{[\beta_j, Z_j]}$ is a collection of dissipativity certificates corresponding to a set of supply rates (11) for all instances of perturbation in (10).

We now make a claim about the *supremum* of the worst-case objective function $\tilde{J}_j(K_j, K_{\neg j})$ for any $j \in \mathcal{N}$ as the corresponding value of the lowest upper uncertainty bound $\hat{\rho}_j(x_0, K^*)$ (cf. Equation (8)). Note that $\tilde{J}_j(K_j, K_{\neg j})$ is continuous and finite for any $(K_j, K^*_{\neg j}) \in \mathcal{K}_j \times \mathcal{K}_{\neg j}$; and moreover, it is semi-convex on $\mathcal{K}_j$ for $K^*_{\neg j} \in \mathcal{K}_{\neg j}$ for all $j \in \mathcal{N}$.

Then, we have

$$\sup_{K_j \in \mathcal{K}_j} \tilde{J}_j(K_j, K^*_{\neg j}) \rightsquigarrow \hat{\rho}_j(x_0, K^*),$$

with $K_j^* \in \underset{K_j \in \mathcal{K}_j}{\arg\sup} \tilde{J}_j(K_j, K^*_{\neg j})$ for all $j = 1, 2, \dots, N$. □

Remark 3 Here we remark that the existence of a solution for the state trajectories is well-defined and it is always upper semicontinuous in x_0 (see [10]).

We next state the following equivalent statements that characterize the set of feedback Nash equilibria:

(i). $\exists K^* \in \mathcal{K},\ \forall x_0,\ \forall u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j],\ \forall K \in \mathcal{K},\ \forall j \in \mathcal{N}$ such that

$$\tilde{J}_j(K_j, K^*_{\neg j}) \le \tilde{J}_j(K^*). \tag{18}$$

(ii). The extended LMIs condition in (14) and the dissipativity inequalities of (16) with a set of supply rates $\mathcal{W}$ in (11) describes completely the set of robust stabilizing state-feedback gains.

The equivalence between (i) and (ii) leads to characterization of feedback Nash equilibria over an infinite-time horizon in terms of a set of stabilizing solutions of the extended LMIs.

Furthermore, the exact characterization of the feedback Nash equilibria is given by the following two theorems.

Theorem 3 *Let $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Suppose $X_j \in \mathbb{S}_+^n$, $U_j \in \mathrm{GL}_n(\mathbb{R})$, $L_j^* \in \mathbb{R}^{r_j \times n}$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$ satisfy the condition in* (14) *of Theorem 2. Then, there exists an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$ feedback Nash equilibrium with respect to the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for $j = 1, 2 \ldots, N$ of* (15).

Proof The first part of this theorem is already provided in Theorem 2, i.e., from the standard argument of the stabilizability of the pair $(A, [\,B_1\ B_2\ \cdots\ B_N\,])$, we can always find an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$ and for all $K_j = L_j^* W_j^{-1} \in \mathbb{R}^{r_j \times n}$ and $j = 1, 2, \ldots, N$ such that (14) holds. Applying (15) of Theorem 2 together with the dissipativity certificates $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ of (13) and a set of supply rate functions $\mathcal{W}$ of (11). Then, for a fixed $(x_0, K^*) \in \mathbb{R}^n \times \mathcal{K}$, we will obtain an upper bound $\hat{\rho}_j \in \mathbb{R}_+$ for all instances of perturbation in (10) and so that

$$\tilde{J}_j(K_j, K_{\neg j}^*) \leq \tilde{J}_j(K^*),$$

for all $K_j \in \mathcal{K}_j$ and for all $j \in \mathcal{N}$. Hence, we immediately see that the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$ satisfies the feedback Nash equilibrium.[3] □

Note that the class of admissible strategies for all agents are generated through a set of individual objective functions that are induced from dissipativity inequalities of (16) with a set of supply rates (11).

[3] Here we remark that a strong version of fixed-point theorem is required to establish the existence of feedback Nash equilibria for the game, which is defined on compact topological spaces with continuous objective functions (e.g., see [12]). To this end, if we introduce the following continuous map $\Phi_{[x_0, u_{\hat{\rho}}]} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ defined by

$$\Phi_{[x_0, u_{\hat{\rho}}]}(K, \bar{K}) = \sum_{j \in \mathcal{N}} \left(\bar{J}_j(x_0, u_{\hat{\rho}_j}, K) - \bar{J}_j(x_0, u_{\hat{\rho}_j}, \bar{K}_j, K_{\neg j}) \right),$$

where $K = \left(K_1, K_2, \ldots, K_N\right) \in \mathcal{K}$, $\bar{K} = \left(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_N\right) \in \mathcal{K}$ and $u_{\hat{\rho}} \triangleq (u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \ldots, u_{\hat{\rho}_N}) \in \prod_{j \in \mathcal{N}} [-\hat{\rho}_j, \hat{\rho}_j]$. Note that for such a map whose fixed-point is an equilibrium is called a Nash map for the game (i.e., if the N-tuple $\left(K_1^*, K_2^*, \ldots, K_N^*\right)$ is a feedback Nash equilibrium), then we have $\bar{J}_j(x_0, u_{\hat{\rho}_j}, K_j, K_{\neg j}^*) \leq \bar{J}_j(x_0, u_{\hat{\rho}_j}, K^*)$ for all $j \in \mathcal{N}$ and $K_j \in \mathbb{R}^{r_j \times n}$. This shows that the map Φ satisfies $\Phi_{[x_0, u_{\hat{\rho}}]}(K^*, K) \geq 0$ for any arbitrary $K = \left(K_1, K_2, \ldots, K_N\right) \in \mathcal{K}$. Therefore, the feedback Nash equilibrium K^* is an equilibrium point, i.e., a fixed point, for the map $\Phi_{[x_0, u_{\hat{\rho}}]}(.,.)$.

Theorem 4 *Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$ is a feedback Nash equilibrium with respect to the values of the objective functions of* (15)*. Let also $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Then, there exists a solution set $X_j \in \mathbb{S}_+^n$, $U_j \in \mathrm{GL}_n(\mathbb{R})$ and $L_j^* \in \mathbb{R}^{r_j \times n}$ for $j = 1, 2, \ldots, N$ that satisfies the condition in* (14) *of Theorem 2.*

Proof Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$ is a feedback Nash equilibrium such that

$$\tilde{J}_j(K_j, K_{\neg j}^*) \leq \tilde{J}_j(K^*),$$

where the value for the continuous objective function $\tilde{J}_j : \mathcal{K}_j \times \mathcal{K}_{\neg j} \to \mathbb{R}_+$ is claimed as

$$\sup_{K_j \in \mathcal{K}_j} \tilde{J}_j(K_j, K_{\neg j}^*) \rightsquigarrow \hat{\rho}_j(x_0, K^*),$$

with $K_j^* \in \arg\sup_{K_j \in \mathcal{K}_j} \tilde{J}_j(K_j, K_{\neg j}^*)$ for all $j \in \mathcal{N}$.

Then, we can always find a solution set that satisfies the condition in (14) of Theorem 2 for which the closed-loop systems in (10) are robustly stable for all instances of perturbations $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j \in \mathcal{N}} [-\hat{\rho}_j, \hat{\rho}_j]$. □

Finally, the feedback Nash equilibrium has a strong time consistency property. This fact corresponds to the class of admissible strategies for all agents that are generated through a set of individual objective functions where the latter are induced from dissipativity inequalities of (16) with a set of supply rates (11). Note that the equivalence between (i) and (ii) (i.e., Theorem 3: (ii) $\Rightarrow$ (i) and Theorem 4: (i) $\Rightarrow$ (ii)) leads exactly to characterization of the feedback Nash equilibrium via a set of robust stabilizing state-feedback solutions of the extended linear matrix inequalities.

5 Concluding remarks

In this paper, we have looked at the problem of robust state-feedback stabilization for a multi-channel system using a game-theoretic framework. Specifically, we presented a sufficient condition for the existence of a robust feedback Nash equilibrium where each agent aims to optimize different type of objective function and when agents may unaware of all the aspects or the structure of the game. Moreover, we characterized the robust feedback Nash equilibrium solutions for such a game using a set of extended linear matrix inequalities and set-valued mappings – where the latter is employed for the game with an incomplete information.

Acknowledgements This work was supported in part by the National Science Foundation under Grants No. CNS-1035655; GKB acknowledges support from the College of Engineering, University of Notre Dame.

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