Passivity based Supervisory Control

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On a notion of passivity for discrete systems

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Abstract— **Cyber Physical Systems (CPS) often have continuous components interconnected with systems described by discrete state space. Examples of such discrete components may be supervisory controllers or components implemented in software. To design large scale CPS in a compositional manner it would be useful to define a notion of passivity for these discrete components as well. As a first step towards this goal we assign notions of passivity and passivity indices for a finite state model that is abstracted from an infinite state continuous system. We also characterize the degradation of passivity under this abstraction and analyze the stability properties of such passive discrete systems.**

I. Introduction

There is a resurgence of interest in the classical concepts of passivity and dissipativity for the design of large scale cyber physical systems owing to the property of compositionality that these concepts offer. The reader is referred to texts such as [1], [2] and articles such as [3], [4] for a survey of the research landscape in this direction. Somewhat surprisingly, however, there is limited understanding of passivity for discrete (state) systems, such as those obtained when systems are implemented in software or from higher order controllers such as supervisory controllers or trajectory planners.

Interaction between continuous plants and discrete controllers is an important feature of modern day embedded systems and cyber physical systems. The problem is not trivial since it is known that several conventional discretization methods (where discretization is of the time variable and the state is deemed continuous) and quantization procedures (quantization of input and output) degrade passivity, and more generally, dissipativity properties of a continuous-time system [5]. Hence careful design of sampling time, input and output signals, or quantization methods is needed to ensure passivity [6], [7], [8]. The problem is even more difficult when the state is discretized (as in a finite state model) or when a discrete supervisor is interconnected with a continuous plant.

A relevant line of research is passivity of hybrid systems where both continuous and discrete dynamics are considered in the same framework [9], [10]. Recently, new passivity conditions have also been proposed for switched systems [11], linear complimentarity systems [12] and piecewise affine systems [13]. Although the hybrid systems framework is well established, systematic compositional methods to analyze the interconnection between discrete controllers and continuous plants are not yet available.

Instead, we follow an alternate approach based on abstracting finite state models from infinite state models (continuous-time systems) which may further interact with other finite state systems. There is now significant work available on abstracting finite state models from continuous systems while maintaining equivalence in a certain sense between the two [14], [15] , [16], [17]. The common approach between most of these works is to design discrete controllers for the finite state discrete abstractions of a plant in order to satisfy given discrete specifications. The abstractions are such that the same discrete controller works for the continuous system. This requirement imposes certain requirements on the abstraction methodology in order to maintain a certain equivalence between the continuous systems and the finite state systems. In works such as [17], [18], notions of input output stability and Lyapunov stability have been proposed for systems in the discrete domain; however, they do not allow a natural description of system's passive behavior. Passivity property for continuous plants is described using a notion of inner product over a real space, thus motivating abstractions which allow a notion of inner product to defined for the input-output vectors of the finite state abstraction. To this end, we follow the approach of abstracting a finite state model using the concepts of simulations and bisimulations. In particular, we follow the notions of approximate simulation and approximate bisimulation [19], and in particular their extensions for continuous nonlinear systems that are incrementally forward complete [20].

Specifically, we note that [20] incorporates notions of a metric and a vector space, as well as provides us conditions on sampling time and quantization parameters that guarantee approximately similar models. The notions of approximate similarity are based on bounding the distance between the output trajectories of the continuous-time system and its abstraction. However, in order to preserve properties like passivity, which are described using both inputs and outputs, we need to extend the notions of approximate simulation further. In Section II we introduce new notions of approximate input output simulation and approximate input output alternating simulation. Similar notions of input output simulations were also proposed by [23] and [24]. In Section

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III, we show how to obtain such finite abstractions based on certain modifications to the methods proposed by [20]. Then we use these approximate relations to quantify the degradation of passivity (in terms of passivity indices) under abstraction. In Section IV, we generalize notions of passivity and dissipativity for a class of finite state transition systems. Then we analyze such finite abstractions to find conditions under which they attain stable behavior in a practical sense. A companion paper [29] considers the interaction of these finite state models with higher order supervisory controllers to ensure that the entire interconnection is passive. We would also like to mention the paper [30] that considers the related but complementary problem of discretizing a controller designed in the continuous space so that passivity indices of the closed loop system are maintained despite discretization. Note that while [30] requires a bisimulation of the controller, we require a simulation of the plant for our purpose.

II. Preliminaries

A. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\iota_A : A \hookrightarrow B$ or simply by ι the natural inclusion map taking any $a \in A$ to $\iota(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+_0$ denote the set of natural, real, positive, and nonnegative real numbers, respectively. The transpose of a general matrix M is denoted by M^T. A matrix P is symmetric if $P^T = P$. A symmetric matrix P is positive (negative) definite if $x^T P x > 0$ ($x^T P x < 0$) for all non-zero x and we denote this by $P > 0$ ($P < 0$). A symmetric matrix P is positive (negative) semi-definite if $x^T P x \geq 0$ ($x^T P x \leq 0$) for all x and we denote this by $P \geq 0$ ($P \leq 0$). The inner product of signals $u(t)$, $y(t)$ is denoted by $\langle u, y\rangle$ defined as $\langle u, y\rangle = \int_0^t u^T(\tau) y(\tau) \mathrm{d}\tau$. Given a vector $x \in \mathbb{R}^n$, x_i is the i-th element of x and we denote infinity norm and euclidean norms of x by $\|x\|$ and $\|x\|_2$. Given a measurable function $f : \mathbb{R}^+ : 0 \leftarrow \mathbb{R}^n$ the (essential) supremum (sup norm) of f is denoted by $\|\|\|_\infty$. If $A \subseteq \mathbb{R}^n$ and $\eta \in \mathbb{R}^+$, $[A]_\eta$ denotes the subset $[A]_\eta \subseteq A$ defined by:

$$[A]_\eta = \left\{ z \in A \mid z_i = k_i \eta \ \text{ for some } k_i \in \mathbb{A} \text{ and } i = 1, 2, \ldots, n \right\}.$$

The set $[A]_\eta$ will be used as an approximation of the set A with precision η. If we define $\mathbf{B}_\varepsilon(x) = \{y \in \mathbb{R}^n | \, \|x - y\| \leq \varepsilon\}$. For set $A \subseteq \mathbb{R}^n$ of the form $A = \bigcup_{j=1}^M A_j$ for some $M \in \mathbb{N}$, where $A_j = \Pi_{i=1}^n [c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$ and positive constant $\eta \leq \hat{\eta}$, where $\hat{\eta} = \min_{j=1,\ldots,M} \eta_{A_j}$ and $\eta_{A_j} = \min\{|d_1^j - c_1^j|, \cdots, |d_n^j - c_n^j|\}$. Note that $[A]_\eta \neq \varnothing$ for any $\eta \leq \hat{\eta}$. Geometrically, for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$ the collection of sets $\{\mathbf{B}_\lambda(p)\}_{p \in [A]_\eta}$ is a covering of A, i.e. $A \subseteq \bigcup_{p \in [A]_\eta} \mathbf{B}_\lambda(p)$. A continuous function $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ belongs to class $\mathscr{K}_\infty$ if $\gamma \in \mathscr{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\gamma : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ belongs to class $\mathscr{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathscr{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. A relation $R \subseteq A \times B$ is defined by a map of the form $R : A \to 2^B$ where $b \in R(a)$ if and only if $(a, b) \in R$. For a set $S \in A$ the set $R(S)$ is defined as $R(S) = \{b \in B : \exists \ a \in S, (a, b) \in R\}$. We also denote by $\mathbf{d} : X \times X \to \mathbb{R}^+_0$ a metric in the space X.

B. Incremental forwardness and stability

In this paper we restrict ourselves to control systems of the form $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathscr{U}, f)$ where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input space;
- $\mathscr{U} : \mathbb{R} \to \mathsf{U}$ is a subset of the set of all locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a Lipschitz continuous map.

If $\xi :]a, b[\longrightarrow \mathbb{R}^n$ is a trajectory of Σ (or equivalently a solution of the differential equation $\dot{x} = f(x, u)$), then we will use $\xi(\tau, x, \nu)$ to denote a unique point reached at time τ under the input ν from an initial condition x. System Σ is said to be forward-complete if such a solution is defined for all $t \in]0, \infty[$. In this paper we use an incremental version of this property, defined as:

*Definition 1 (**Incremental forward-completeness**):* A control system Σ is δ-FC if there exist continuous functions $\beta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and $\gamma : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that for every $s \in \mathbb{R}^+$, the functions $\beta(\cdot, s)$ and $\gamma(\cdot, s)$ belong to class $\mathscr{K}_\infty$, and for any $x, x' \in \mathbb{R}^n$, any $\tau \in \mathbb{R}^+$, and any $\nu, \nu' \in \mathscr{U}$, where $\nu, \nu' : [0, \tau) \to \mathsf{U}$, the following condition is satisfied for all $t \in [0, \tau]$:

$$\|\xi(t, x, \nu) - \xi(t, x', \nu')\| \leq \beta(\|x - x'\|, t) + \gamma(\|\nu - \nu'\|_\infty, t). \tag{1}$$

*Definition 2 (**Asymptotic Stability**):* The origin of Σ with $\dot{x} = f(x, 0)$ is asymptotically stable if and only if there exists a $\beta(\cdot, \cdot) \in \mathscr{KL}$ such that when $\|x(0)\| \leq \delta$ we have

$$\|\xi(t, x(0), 0)\| \leq \beta(\|x(0)\|, t) \quad \forall t \geq 0. \tag{2}$$

C. Dissipativity

Consider the system Σ and an output function $y = h(x, u) \in \mathbb{R}^p$. Further, assume that $f(0, 0) = 0$ and $h(0, 0) = 0$. Σ is dissipative w.r.t. $y = h(x, u)$ if there exists a $\mathscr{C}^1$ storage function $V(x) : \mathbb{R}^n \to \mathbb{R}^+_0$ and a supply rate $\omega : \mathsf{U} \times \mathbb{R}^p \to \mathbb{R}^+_0$ such that $V(0) = 0$ and the following inequality is satisfied:

$$V(x(t)) - V(x(t_0)) \leq \int_{t_0}^t \omega(u, y) dt \tag{3}$$

for any $t \geq t_0$ and $u \in \mathscr{U}$. A special case of dissipativity is (ρ, ν)-***input output strict passivity (IOSP)*** when $\omega(u, y) = u^T y - \nu u^T u - \rho y^T y$ with $\rho, \nu \geq 0$. In this definition, parameters ν and ρ are known as passivity indices. The next Lemma presents an important implication of passivity on stability properties of Σ.

Lemma 1: [25]*(**Zero-input asymptotic stability**)* The origin of Σ with $\dot{x} = f(x, 0)$ is asymptotically stable if Σ is (ρ, ν) - IOSP with $\rho > 0$ and $\nu \geq 0$ and $\rho y^T y > 0$. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

One of our main goals in this paper is to understand the implications of finite state abstraction on such stability properties of a passive system.

D. Transition systems and system relations

Definition 3: [21] A system T is a quintuple $T = (X, U, \longrightarrow, Y, H)$ consisting of:

- A set of states X;
- A set of inputs U;
- A transition relation $\longrightarrow \subseteq X \times U \times X$;
- An output set Y;
- An output function $H : X \to Y$.

T is said to be metric, if the output set Y is equipped with a metric; $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$. To define notions of stability for transition systems we assume that the finite sets X, U, and Y are equipped with the metric given by $\mathbf{d}(x,y) = \|x-y\|$ where x, y are elements of X, U, or Y. If for any state $x \in X$ and $u \in U$ there exists at most one state $x' \in X$ such that $x \xrightarrow{u} x'$. If the system is nondeterministic, then for a transition $x \xrightarrow{u} x'$ the state x' may not unique, x' is also known as the u-successor of x. In such a case x' belongs to a set of all possible u-successors given by $\mathbf{Post}_u(x)$ and we will use $U(x)$ to denote the set of inputs $u \in U$ for which $\mathbf{Post}_u(x)$ is nonempty. We will further use $\mathbf{Post}_u(Q)$ to denote the set $\bigcup_{x \in Q} \mathbf{Post}_u(x)$.

E. System relations

Now we present certain system relations, that are essential in obtaining faithful finite abstractions for control systems. Approximate simulation relations used in [19] and [22] are primarily based on bounding the distance between the outputs or states of the continuous-time and its abstraction. However, in order to account for behaviors like passivity (which are defined using both inputs and outputs), we introduce new notions of approximate input output simulation and approximate input output alternating simulation.

*Definition 4 (**Approximate input output simulation**):* Let $T_1 := (X_1, U_1, \xrightarrow[1]{}, Y_1, H_1)$, $T_2 := (X_2, U_2, \xrightarrow[2]{}, Y_2, H_2)$ be metric transition systems with the same sets of inputs $U = U_1 = U_2$ and outputs $Y = Y_1 = Y_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon_u, \varepsilon_y \in \mathbb{R}_0^+$ be given precision requirements then a relation $R \subseteq X_1 \times X_2$ is said to be an $(\varepsilon_u, \varepsilon_y)$ - approximate input output simulation (IOS) relation between T_1 and T_2 if the following two conditions are satisfied:

(i) for every $(x_1, x_2) \in R$ we have $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon_y$;

(ii) for every $(x_1, x_2) \in R$ and for every $u_1 \in U_1$ there exists $u_2 \in U_2$ such that $\mathbf{d}(u_1, u_2) \leq \varepsilon_u$ and $x_1 \xrightarrow[1]{u_1} x_1'$ in T_1 implies the existence of $x_2 \xrightarrow[2]{u_2} x_2'$ in T_2 such that $(x_1', x_2') \in R$.

*Definition 5: (**Approximate input output alternating simulation**)*: Let $T_1 := (X_1, U_1, \xrightarrow[1]{}, Y_1, H_1)$, $T_2 := (X_2, U_2, \xrightarrow[2]{}, Y_2, H_2)$ be metric transition systems with the same sets of inputs $U = U_1 = U_2$ and outputs $Y = Y_1 = Y_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon_u, \varepsilon_y \in \mathbb{R}_0^+$ be a given precision requirements, a relation $R \subseteq X_1 \times X_2$ is said to be an $(\varepsilon_u, \varepsilon_y)$ - approximate input output alternating simulation (IOAS) relation from T_1 to T_2 if condition (i) of Definition 4 and the following condition are satisfied:

(iii) for every $(x_1, x_2) \in R$ and for every $u_1 \in U_1(x_1)$ there exists $u_2 \in U_2(x_2)$ such that $\mathbf{d}(u_1, u_2) \leq \varepsilon_u$ and for every $x_2' \in \mathbf{Post}_{u_2}(x_2)$ there exists $x_1' \in \mathbf{Post}_{u_1}(x_1)$ satisfying $(x_2', x_1') \in R$.

The two notions of alternating approximate simulation and approximate simulation coincide in the special case of deterministic systems. If T_1 is $(\varepsilon_u, \varepsilon_y)$- approximately input output simulated (or approximately input output alternatingly simulated) by T_2, then we denote this fact by $T_1 \preceq_{IOS}^{(\varepsilon_u, \varepsilon_y)} T_2$ ($T_1 \preceq_{IOAS}^{(\varepsilon_u, \varepsilon_y)} T_2$).

F. Finitely abstracted transition systems

In this paper, we make a modification to the procedure described in [20], to obtain finite state transition systems. Method described in [20] is based on selection of appropriate sampling time and quantization parameters which guarantee approximate simulation and alternating simulation relations between the original system and its abstraction. Initially we consider a discrete time sub-transition system $T_\tau(\Sigma)$ corresponding to $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathscr{U}, f)$ with a sampling time period $\tau \in \mathbb{R}^+$. We further assume that control inputs are piecewise-constant over the sampling time period τ, the class of inputs considered are:

$$\mathscr{U}_\tau := \{u \in \mathscr{U} \mid u(t) = u(0),\ t \in [0, \tau]\}.$$

For $T_\tau(\Sigma)$ we use identity map as the output function, however, for stability and passivity analysis, we use an alternate output corresponding to $y = h(x, u)$.

Definition 6: [21] Let Σ be a control system and $T(\Sigma)$ its associated transition system. For any $\tau > 0$, the sub transition system $T_\tau(\Sigma) := (X_\tau, U_\tau, \xrightarrow[\tau]{u_\tau}, Y_\tau, H_\tau)$ is defined by:

- $X_\tau = \mathbb{R}^n$;
- $U_\tau = \mathscr{U}_\tau$;
- $x_\tau \xrightarrow[\tau]{u_\tau} x_\tau'$, if there exists a trajectory $\xi : [0, \tau] \longrightarrow \xi(\tau, x_\tau, u_\tau) = x_\tau'$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = 1_{\mathbb{R}^n}$.

Now we restrict the input set and the state set to a hyperrectanlges $\mathsf{U} \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$ such that $\{0\} \in \mathsf{U}$ and $\{0\} \in X$. Then we choose input and state quantization factors such that $\mu \leq \hat{\mu}$ and $\eta \leq \hat{\eta}$ (see Section II.A on how to calculate $\hat{\mu}$, $\hat{\eta}$).

Definition 7: For any δ-FC control system Σ and parameters $\tau > 0$, $\eta > 0$, $\mu > 0$ and a design parameters $\theta_1, \theta_2 \in \mathbb{R}^+$, a countable transition system can be defined as:

$$T_{\tau,\mu,\eta}(\Sigma) := (X_\mathbf{q}, U_\mathbf{q}, \xrightarrow[\tau]{u_\mathbf{q}}, Y_\mathbf{q}, H_\mathbf{q}) \tag{4}$$

where:

- $X_\mathbf{q} = [X]_\eta$;
- $U_\mathbf{q} = [\mathsf{U}]_\mu$;

- $x_{\mathbf{q}} \xrightarrow[\tau]{u_{\mathbf{q}}} x'_{\mathbf{q}}$, if $\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$;
- $Y_{\mathbf{q}} = [X]_\eta$
- $H_{\mathbf{q}} = \imath : X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$

where β and γ are functions from Definition 1.
Finite transition system $T_{\tau,\mu,\eta}(\Sigma)$ is different from the transition systems abstracted in [20], because of an extra design parameter θ_2. This extra design parameter is necessary to obtain finite abstractions which are approximately input output (alternatingly) similar to the original system. In the next section we provide sufficient conditions to guarantee the existence of such approximate abstractions. We also quantify the degradation of passivity condition for a continuous-time IOSP system under such abstractions. We illustrate this degradation through a new dissipation inequality satisfied by the finitely abstracted transition system. Based on this new inequality, we define a new notion of practical passivity for transition systems. Further, we show that this notion of practical passivity guarantees a notion of zero-input practical asymptotic stability transition system akin to continuous-time passive systems.

III. Degradation of passivity

Initially we provide sufficient conditions for $T_{\tau,\eta,\mu}(\Sigma)$ to be $(\varepsilon_u, \varepsilon_y)$ - approximately input output (alternatingly) similar $T_\tau(\Sigma)$.

Proposition 1: Consider a control system Σ and any desired precision parameters $\varepsilon_y > 0$, $\varepsilon_u > 0$. If Σ is δ-FC then for any $\tau > 0$, $\theta_1 > 0$, $\theta_2 > 0$, $\eta > 0$ and $\mu > 0$ satisfying the following inequality:

$$\beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta \leq \varepsilon_y, \tag{5}$$

such that $\eta \leq \varepsilon_y \leq \theta_1$ and $\mu \leq \varepsilon_u \leq \theta_2$, we have:

$$T_{\tau,\eta,\mu}(\Sigma) \preceq_{IOAS}^{(\varepsilon_u,\varepsilon_y)} T_\tau(\Sigma) \preceq_{IOS}^{(\varepsilon_u,\varepsilon_y)} T_{\tau,\eta,\mu}(\Sigma). \tag{6}$$

Proof: Initially we show that $T_\tau(\Sigma) \preceq_{IOS}^{(\varepsilon_u,\varepsilon_y)} T_{\tau,\eta,\mu}(\Sigma)$. Consider any $x_\tau \in X_\tau$ and $u_\tau \in U_\tau$, then there exists $x_{\mathbf{q}} \in X_{\mathbf{q}} = [X]_\eta$ and $u_{\mathbf{q}} \in U_{\mathbf{q}} = [\mathsf{U}]_\mu$ such that

$$\|x_\tau - x_{\mathbf{q}}\| \leq \eta \leq \varepsilon_y \tag{7}$$

and

$$\|u_\tau - u_{\mathbf{q}}\| \leq \mu \leq \varepsilon_u \tag{8}$$

This is possible because of the nature of quantization which allows $X_\tau \subseteq \bigcup_{p\in[X]_\eta} \mathbf{B}_\lambda(p)$ and $U_\tau \subseteq \bigcup_{p\in[\mathsf{U}]_\mu} \mathbf{B}_\lambda(p)$. From the definitions of output functions $H_\tau = 1_{\mathbb{R}^n}$ and $H_{\mathbf{q}} = \imath : X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$, we have $\|H_\tau(x_\tau) - H_{\mathbf{q}}(x_{\mathbf{q}})\| = \|x_\tau - x_{\mathbf{q}}\| \leq \varepsilon_u$, hence condition (i) of Definition 4 is satisfied.

Now if we consider the transition $x_\tau \xrightarrow[\tau]{u_\tau} x'_\tau$ in the transition system $T_\tau(\Sigma)$, then the distance between x'_τ and $\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})$ can estimated based on the δ - FC property of Σ and inequalities (7) and (8) i.e.,

$$\|x'_\tau - \xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})\| \leq \beta(\varepsilon_y, \tau) + \gamma(\varepsilon_u, \tau) \tag{9}$$

Since $X_\tau \subseteq \bigcup_{p\in[X]_\eta} \mathbf{B}_\lambda(p)$, there exists $x'_{\mathbf{q}} \in X_{\mathbf{q}}$ such that

$$\|x'_\tau - x'_{\mathbf{q}}\| \leq \eta \tag{10}$$

From the triangular inequality we have

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_\tau\| + \|x'_\tau - x'_{\mathbf{q}}\|$$

From inequalities (9) and (10) we have

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\varepsilon_y, \tau) + \gamma(\varepsilon_u, \tau) + \eta$$

Finally we use $\eta \leq \varepsilon_y \leq \theta_1$ and $\mu \leq \varepsilon_u \leq \theta_2$ to show that

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$$

which, by the definition of $T_{\tau,\mu,\eta}(\Sigma)$ implies the existence of $x_{\mathbf{q}} \xrightarrow{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$. Therefore, from inequality (10) and since $\eta \leq \varepsilon_y$ we conclude that $(x'_\tau, x'_{\mathbf{q}}) \in R$ and condition (ii) in Definition 4 holds.

Now we show that $T_{\tau,\eta,\mu}(\Sigma) \preceq_{IOAS}^{(\varepsilon_u,\varepsilon_y)} T_\tau(\Sigma)$. For $R \subseteq X_\tau \times X_{\mathbf{q}}$ we consider an $x_\tau = x_{\mathbf{q}} \in X_{\mathbf{q}}$. This is possible because $X_{\mathbf{q}} \subseteq X_\tau$ and it satisfies condition (i) of Definition 4 (i.e., $\|x_\tau - x_{\mathbf{q}}\| = 0 < \varepsilon_y$). Now we choose an input $u_\tau = u_{\mathbf{q}} \in U_{\mathbf{q}}$ (satisfying $\|u_\tau - u_{\mathbf{q}}\| = 0 < \varepsilon_u$) and consider the unique transition $x_\tau \xrightarrow[\tau]{u_\tau} x'_\tau = \xi(\tau, x_\tau, u_\tau) \in \mathbf{Post}_{u_\tau}(x_\tau)$. The distance between x'_τ and $\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})$ can be bounded using the δ - FC properties of Σ, i.e.,

$$\|x'_\tau - \xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})\| \leq \beta(0, \tau) + \gamma(0, \tau) \tag{11}$$

Since $X_\tau \subseteq \bigcup_{p\in[X]_\eta} \mathbf{B}_\lambda(p)$, we can always find $x'_{\mathbf{q}} \in X_{\mathbf{q}}$ such that

$$\|x'_{\mathbf{q}} - x'_\tau\| \leq \eta \tag{12}$$

From the triangular inequality and inequalities (11) and (12) we have

$$\begin{aligned} \|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| &\leq \|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_\tau\| + \|x'_\tau - x'_{\mathbf{q}}\| \\ &\leq \beta(0, \tau) + \gamma(0, \tau) + \eta \end{aligned}$$

Finally we use $0 < \theta_1$ and $0 < \theta_2$ to show that

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$$

which, by the definition of $T_{\tau,\mu,\eta}(\Sigma)$ implies the existence of $x_{\mathbf{q}} \xrightarrow{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$. Therefore, from inequality (12) and since $\eta \leq \varepsilon_y$ we conclude that $(x'_\tau, x'_{\mathbf{q}}) \in R$ and condition (iii) in Definition 5 holds.

■

Now we analyze degradation of passivity of Σ under approximate input output similarity. For this purpose, we use an assumption from [27]. If the control system Σ is passive w.r.t. the passive output function $y = h(x, u)$, this assumption bounds the rate at which y can change w.r.t. time.

Assumption 1: [27] Assume that the operator from $u(t)$ to $\dot{y}(t)$ has the finite L_2 gain, γ, that is

$$\int_0^\tau \|\dot{y}(t)\|_2^2 \mathrm{d}t \le \gamma^2 \int_0^\tau \|u(t)\|_2^2 \mathrm{d}t$$

for any $\tau \ge 0$ and admissible $u(t)$.

Theorem 1: Suppose that the original continuous-time system Σ is δ - FC and (ν, ρ) - IOSP w.r.t. the passive output function $y = h(x,u)$ and a storage function V with a Lipschitz constant K. We also assume that such that Assumption 1 is satisfied. Let $T_\tau(\Sigma)$ be the transition system corresponding to Σ with a sampling time τ. If the state and input quantization parameters η and μ are chosen such that $T_{\tau,\mu,\eta}(\Sigma)$ is $(\varepsilon_u, \varepsilon_y)$ - approximately input output similar (or alternatingly similar) to $T_\tau(\Sigma)$, then $T_{\tau,\mu,\eta}(\Sigma)$ satisfies

$$\begin{aligned}\frac{1}{\tau}\left(V(x'_{\mathbf{q}}) - V(x_{\mathbf{q}})\right) \quad &\le \quad (u_{\mathbf{q}}^T, y_{\mathbf{q}}) - \rho_F(y_{\mathbf{q}}^T y_{\mathbf{q}}) \\ &\quad -\nu_F(u_{\mathbf{q}}^T u_{\mathbf{q}}) + \frac{K\varepsilon_y}{\tau} \qquad (13)\end{aligned}$$

for all transitions of the form $x_{\mathbf{q}} \xrightarrow[\tau]{u_{\mathbf{q}}} x'_{\mathbf{q}}$ with $y_{\mathbf{q}} = h(x_{\mathbf{q}}, u_{\mathbf{q}})$ and

$$\nu_F = \nu - \gamma\tau - \frac{\rho\gamma\tau}{1+\tau} \text{ and } \rho_F = \frac{\rho}{(1+\tau)\tau}. \qquad (14)$$

PROOF: Since Σ is (ρ, ν) - IOSP w.r.t. the passive output $y(t) = h(x(t), u(t))$ we have

$$\begin{aligned} V(x(\tau+t_0)) - V(x(t_0)) \quad &\le \quad \int_{t_0}^{\tau+t_0} (u^T y - \nu u^T u - \rho y^T y)\mathrm{d}t \\ &= \quad \langle u, y\rangle_\tau - \nu\langle u, u\rangle_\tau - \rho\langle y, y\rangle_\tau \end{aligned}$$

for any $t_0, \tau \ge 0$ and $u \in \mathscr{U}$. The passivity inequality can be interpreted as

$$\begin{aligned}\langle u, y\rangle_\tau - \nu\langle u, u\rangle_\tau - \rho\langle y, y\rangle_\tau + V(x(t_0)) - V(x(\tau+t_0)) \ge 0 \\ \forall x(t_0) \in \mathbb{R}^n,\ u \in \mathscr{U} \text{ and } \tau \ge 0\end{aligned}$$

Without loss of any generality we let $t_0 = 0$. For the sub transition system $T_\tau(\Sigma)$, if we consider any transition $x_\tau \xrightarrow[\tau]{u_\tau} x'_\tau = \xi(\tau, x_\tau, u_\tau)$, where u_τ is a piecewise constant input and $y(t) = h(x(t), u_\tau)$ is the passive output function with $x(0) = x_\tau$, then we have

$$\begin{aligned}\langle u_\tau, h(x(t), u_\tau)\rangle_\tau - \rho\langle h(x(t), u_\tau), h(x(t), u_\tau)\rangle_\tau - \nu\langle u_\tau, u_\tau\rangle_\tau \\ +V(x_\tau) - V(\xi(\tau, x_\tau, u_\tau)) \ge 0 \\ \forall\, u_\tau \in U_\tau, x_\tau \in X_\tau \text{ and } 0 \le t \le \tau. \qquad (15)\end{aligned}$$

The finite transition system $T_{\tau,\mu,\eta}(\Sigma) := (X_{\mathbf{q}}, U_{\mathbf{q}}, \xrightarrow[\tau]{u_{\mathbf{q}}}, Y_{\mathbf{q}}, H_{\mathbf{q}})$ is $(\varepsilon_u, \varepsilon_y)$ - approximately input output similar to $T_\tau(\Sigma)$. Hence we can always find a transition $x_{\mathbf{q}} \xrightarrow[\tau]{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$ such that $\|x_\tau - x_{\mathbf{q}}\| \le \varepsilon_y$, $\|u_\tau - u_{\mathbf{q}}\| \le \varepsilon_u$ and $\|x'_\tau - x'_{\mathbf{q}}\| \le \varepsilon_y$. Since inequality (15) is valid for all $u_\tau \in U_\tau$ and $x_\tau \in X_\tau$, it will be valid if we substitute $u_\tau = u_{\mathbf{q}}$ and $x_\tau = x_{\mathbf{q}}$ and it is always possible to find such u_τ and x_τ, because $U_{\mathbf{q}} \subseteq U_\tau$ and $X_{\mathbf{q}} \subseteq X_\tau$. Thus, we have

$$\begin{aligned}\langle u_{\mathbf{q}}, h(x(t), u_{\mathbf{q}})\rangle_\tau - \rho\langle h(x(t), u_{\mathbf{q}}), h(x(t), u_{\mathbf{q}})\rangle_\tau - \nu\langle u_{\mathbf{q}}, u_{\mathbf{q}}\rangle_\tau \\ +V(x_{\mathbf{q}}) - V(\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})) \ge 0 \\ \forall\, 0 \le t \le \tau. \ (16)\end{aligned}$$

For clarity of exposition we define $\mathbf{y}(t) = h(x(t), u_{\mathbf{q}})$ hence $\mathbf{y}(0) = h(x_{\mathbf{q}}, u_{\mathbf{q}})$. Now inequality (16) can be written as

$$\begin{aligned}\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau - \rho\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau - \nu\langle u_{\mathbf{q}}, u_{\mathbf{q}}\rangle_\tau \\ +V(x_{\mathbf{q}}) - V(\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})) \ge 0 \quad \forall\, 0 \le t \le \tau. \qquad (17)\end{aligned}$$

Form inequality (17) we will obtain the corresponding inequality for the transition system $T_{\tau,\mu,\eta}(\Sigma)$ in terms of its inputs, states and outputs. To do that we bound each term from inequality (17).

Bounds for $\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau$: Here we compare $\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau$ and $\tau(u_{\mathbf{q}}^T, \mathbf{y}(0))$ and their difference can be bounded as

$$\begin{aligned}|\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau - \tau(u_{\mathbf{q}}^T, \mathbf{y}(0))| \quad &= \quad |u_{\mathbf{q}}^T \int_0^\tau (\mathbf{y}(t) - \mathbf{y}(0))\,\mathrm{d}t| \\ &= \quad |u_{\mathbf{q}}^T \int_0^\tau \int_0^t \dot{\mathbf{y}}(s)\mathrm{d}s\mathrm{d}t| \\ &\le \quad \|u_{\mathbf{q}}\|_2 \int_0^\tau \int_0^t \|\dot{\mathbf{y}}(s)\|_2 \mathrm{d}s\mathrm{d}t\end{aligned}$$

Further, we can bound the integral w.r.t. s as

$$\int_0^t \|\dot{\mathbf{y}}(s)\|_2 \mathrm{d}s \le \int_0^\tau \|\dot{\mathbf{y}}(s)\|_2 \mathrm{d}s \le \sqrt{\tau}\sqrt{\int_0^\tau \|\dot{\mathbf{y}}(s)\|_2^2 \mathrm{d}s}$$

hence

$$|\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau - \tau(u_{\mathbf{q}}^T, \mathbf{y}(0))| \le \tau\sqrt{\tau}\|u_{\mathbf{q}}\|_2 \sqrt{\int_0^\tau \|\dot{\mathbf{y}}(s)\|_2^2 \mathrm{d}s}$$

Since Assumption 1 is valid for all admissible inputs, it is also valid for inputs $u_{\mathbf{q}}$ and the corresponding passive output $\mathbf{y}(t)$, hence we have

$$\begin{aligned}|\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau - \tau(u_{\mathbf{q}}^T, \mathbf{y}(0))| \le \tau\sqrt{\tau}\|u_{\mathbf{q}}\|_2 \sqrt{\int_0^\tau \|u_{\mathbf{q}}\|_2^2 \mathrm{d}s} \\ = \tau^2 \gamma(u_{\mathbf{q}}^T u_{\mathbf{q}})\end{aligned}$$

and

$$\langle u_{\mathbf{q}}, \mathbf{y}(t)\rangle_\tau \le \tau^2\gamma(u_{\mathbf{q}}^T u_{\mathbf{q}}) + \tau(u_{\mathbf{q}}^T, \mathbf{y}(0)) \qquad (18)$$

Bounds for $-\rho\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau$: Here we compare $\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau$ and $\mathbf{y}(0)^T\mathbf{y}(0)$ and their difference can be bounded as

$$\begin{aligned} &|\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau - \mathbf{y}(0)^T\mathbf{y}(0)| \qquad (19) \\ \le\ & \left|\int_0^\tau (\mathbf{y}(t)^T\mathbf{y}(t))\mathrm{d}t - \tau(\mathbf{y}(0)^T\mathbf{y}(0))\right| \\ \le\ & \left|\int_0^\tau \left(\int_0^s \frac{\mathrm{d}}{\mathrm{d}s}(\|\mathbf{y}(s)\|_2^2)\mathrm{d}s\right)\mathrm{d}t\right| \\ \le\ & \left|\tau\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s}(\|\mathbf{y}(s)\|_2^2)\mathrm{d}s\right| \qquad (20)\end{aligned}$$

we can further bound this integral in s as

$$\begin{aligned}\left|\tau\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s}(\|\mathbf{y}(s)\|_2^2)\mathrm{d}s\right| \quad &= \quad \left|2\tau\int_0^\tau \mathbf{y}(s)^T\dot{\mathbf{y}}(s)\mathrm{d}s\right| \\ &\le \quad 2\tau\int_0^\tau \|\mathbf{y}(s)\|_2\|\dot{\mathbf{y}}(s)\|_2\mathrm{d}s \\ &\le \quad \tau\int_0^\tau (\|\mathbf{y}(s)\|_2^2 + \|\dot{\mathbf{y}}(s)\|_2^2)\mathrm{d}s \\ &\le \quad \tau\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau + \tau^2\gamma(u_{\mathbf{q}}^T u_{\mathbf{q}})\end{aligned}$$

Hence inequality (19) results in

$$|\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau - \mathbf{y}(0)^T\mathbf{y}(0)| \leq \tau\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau + \tau^2\gamma(u_{\mathbf{q}}^T u_{\mathbf{q}})$$

thus $-\tau\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau - \tau^2\gamma(u_{\mathbf{q}}^T u_{\mathbf{q}}) + \mathbf{y}(0)^T\mathbf{y}(0) \leq \langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau$ and

$$\frac{\rho\tau^2\gamma}{1+\tau}(u_{\mathbf{q}}^T u_{\mathbf{q}}) - \frac{\rho}{1+\tau}\mathbf{y}(0)^T\mathbf{y}(0) \geq -\rho\langle \mathbf{y}(t), \mathbf{y}(t)\rangle_\tau \tag{21}$$

Bounds for $-\nu\langle u_{\mathbf{q}}, u_{\mathbf{q}}\rangle_\tau$: Since $\langle u_{\mathbf{q}}, u_{\mathbf{q}}\rangle_\tau = \tau(u_{\mathbf{q}}^T u_{\mathbf{q}})$ we have

$$-\nu\langle u_{\mathbf{q}}, u_{\mathbf{q}}\rangle_\tau = -\nu\tau(u_{\mathbf{q}}^T u_{\mathbf{q}}) \tag{22}$$

Bounds for $-V(\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}))$: Now we consider a transition $x_{\mathbf{q}} \xrightarrow[\tau]{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$ and by Definition of $T_{\tau,\mu,\eta}(\Sigma)$ we have $\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$. For Lipschitz continuous storage functions

$$V(x'_{\mathbf{q}}) \leq V(\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})) + K(\|x'_{\mathbf{q}} - \xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})\|) \tag{23}$$

Since $T_\tau(\Sigma)$ is $(\varepsilon_y, \varepsilon_u)$ - approximately input output (alternatingly) similar to $T_{\tau,\mu,\eta}(\Sigma)$, we have $\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \varepsilon_y$. Hence inequality (23) results in

$$-V(x'_{\mathbf{q}}) + K\varepsilon_y \quad \geq \quad -V(\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})) \tag{24}$$

From (18), (21), (22) and (24) we obtain

$$\begin{aligned}\tau(u_{\mathbf{q}}^T, \mathbf{y}(0)) - \rho_F\tau\mathbf{y}(0)^T\mathbf{y}(0) - \nu_F\tau(u_{\mathbf{q}}^T u_{\mathbf{q}}) \\ + V(x_{\mathbf{q}}) - V(x'_{\mathbf{q}}) + K\varepsilon_y \geq 0.\end{aligned}$$

Thus $T_{\tau,\mu,\eta}(\Sigma)$ satisfies (13) with $\mathbf{y}(0) = h(x_{\mathbf{q}}, u_{\mathbf{q}})$ and ρ_F, ν_F given by (14).

■

In the next section, the terms ρ_F and ν_F will be interpreted as the passivity indices for $T_{\tau,\mu,\eta}(\Sigma)$. Apart from degradation of passivity indices from (ρ, ν) to (ρ_F, ν_F), the presence of $\frac{K\varepsilon_y}{\tau} > 0$ on the right hand side of (13) indicates further deterioration of passivity under abstraction.

IV. Passivity of transition systems

Finitely abstracted transition system (4) is a quantized version of the sampled-data system $T_\tau(\Sigma)$. The finite state transition system (4) can be thought of as a discrete time system with a finite state run

$$x_{\mathbf{q}_0} \xrightarrow[\tau]{u_{\mathbf{q}0}} x_{\mathbf{q}_1} \xrightarrow[\tau]{u_{\mathbf{q}1}} x_{\mathbf{q}_2} \xrightarrow[\tau]{u_{\mathbf{q}2}} \cdots \xrightarrow{u_{\mathbf{q}(n-2)}} x_{\mathbf{q}_{n-1}} \xrightarrow[\tau]{u_{\mathbf{q}(n-1)}} x_{\mathbf{q}_n}$$

where $x_{\mathbf{q}_0} \in X_{\mathbf{q}}$ is the initial state and $x_{\mathbf{q}i} \xrightarrow[\tau]{u_{\mathbf{q}i}} x_{\mathbf{q}(i+1)}$ for all $0 \leq i \leq n$. The subscript i corresponds to the sampling time instants $t = 0, 1\tau, 2\tau, \ldots, n\tau$ and $x_{\mathbf{q}i}$ corresponds to the state of (4) at the time instant $i\tau$. In some cases, a finite state run can be extended to an infinite state run with $i \in \mathbb{N}$. Based on inequality (13) and discrete time nature of these transition systems we define notions of dissipativity and passivity for the transition system (4).

*Definition 8 (**Practical dissipativity**):* Let $\mathscr{C}^1$ function $V : X_{\mathbf{q}} \to \mathbb{R}_0^+$ be a storage function with $V(0) = 0$ and let $\omega : U_{\mathbf{q}} \times X_{\mathbf{q}} \to \mathbb{R}$ be a supply rate, then the transition system (4) is practically dissipative with respect to ω if

$$\frac{1}{\tau}\left(V(x_{\mathbf{q}(i+1)}) - V(x_{\mathbf{q}i})\right) \leq \omega(u_{\mathbf{q}i}, x_{\mathbf{q}i}) + \delta \quad \forall i \in \mathbb{N} \text{ and } \delta > 0 \tag{25}$$

for all the transitions $x_{\mathbf{q}i} \xrightarrow[\tau]{u_{\mathbf{q}i}} x_{\mathbf{q}(i+1)}$.

In the above definition $V(x_{\mathbf{q}(i+1)}) - V(x_{\mathbf{q}i})$ represents the increase in stored energy during the transition $x_{\mathbf{q}_i} \xrightarrow[\tau]{u_{\mathbf{q}i}} x_{\mathbf{q}(i+1)}$ and $\omega(u_{\mathbf{q}i}, x_{\mathbf{q}i})$ is the energy supplied before the transition. Presence of $\delta > 0$ on the right hand side of (25) indicates indicates the energy generated due to the error introduced by the abstraction process (see Theorem 1 and [5]). A special case of dissipativity can be obtained when the supply rate

$$\begin{aligned}\omega(u_{\mathbf{q}i}, x_{\mathbf{q}i}) = (u_{\mathbf{q}i}^T h(x_{\mathbf{q}i}, u_{\mathbf{q}i})) - \rho_F(h^T(x_{\mathbf{q}i}, u_{\mathbf{q}i})h(x_{\mathbf{q}i}, u_{\mathbf{q}i})) \\ -\nu_F(u_{\mathbf{q}i}^T u_{\mathbf{q}i})\end{aligned} \tag{26}$$

where $\rho_F \geq 0$ and $\nu_F \geq 0$ are passivity indices for the finite transition system. For a supply rate given by (26) we define (4) to be (ρ_F, ν_F, δ)-***practically input output strictly passive*** w.r.t. an output function $h(x_{\mathbf{q}i}, u_{\mathbf{q}i})$. The function $\mathbf{y}_i = h(x_{\mathbf{q}i}, u_{\mathbf{q}i})$ is an output function for the finite transition system at a time instant $i\tau$ and it is analogous to $y = h(x, u)$ for the continuous-time system (see Subsection II-C). Note that this output function is different from the output function $H_{\mathbf{q}} = \imath : X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$. The output function $\mathbf{y}_i = h(x_{\mathbf{q}i}, u_{\mathbf{q}i})$ will be used only to study passivity and the stability behavior of the transition system. In order to avoid confusion between the output functions $H_{\mathbf{q}}$ and $\mathbf{y}_i = h(x_{\mathbf{q}i}, u_{\mathbf{q}i})$, we will refer to $\mathbf{y}_i = h(x_{\mathbf{q}i}, u_{\mathbf{q}i})$ as the ***passive output function***, i.e., the output with respect to which the system is passive. The passive output function for a transition system can be obtained using $H_{\mathbf{q}} = \imath : X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$, whenever $H_{\mathbf{q}} = 1_{X_{\mathbf{q}}}$, i.e., $\mathbf{y} = h(H_{\mathbf{q}}(x_{\mathbf{q}}), u_{\mathbf{q}}) = h(x_{\mathbf{q}}, u_{\mathbf{q}})$.

Two important properties of continuous-time IOSP systems are zero-input asymptotic stability (see Lemma 1) and compositionalilty. Compositionalilty refers to the fact that negative feedback interconnection of two passive systems is also passive. In a companion paper [29], we discuss the compositionality of practically IOSP transition systems. In this paper we show that practically IOSP transition systems are also zero-input asymptotically stable, however, in a practical sense. For this purpose we define notions of practical asymptotic stability and we also derive Lyapunov like sufficient conditions that guarantee practical asymptotic stability. These Lyapunov like conditions allow us to recover practical asymptotic stability from practically IOSP transition systems.

*Definition 9 (**Practical asymptotic stability**):* The transition system (4) is practically asymptotically stable for zero input $u_{\mathbf{q}} \equiv 0$, if for any strictly positive real numbers $\Delta > \delta > 0$ there exists a class $\mathscr{KL}$ function β such that for all initial states with $\|x_{\mathbf{q}0}\| \leq \Delta$, all the transitions $x_{\mathbf{q}i} \xrightarrow[\tau]{0} x_{\mathbf{q}(i+1)}$

(for all $i \in \mathbb{N}$) satisfy

$$\|x_{\mathbf{q}i}\| \le \beta(\|x_{\mathbf{q}0}\|, i) + \delta, \quad i \in \mathbb{N}. \tag{27}$$

This notion of stability for a transition system can be described in terms of Lyapunov-like functions which provide sufficient conditions for a transition system to be practically asymptotically stable.

*Theorem 2 (**Practical stability Lyapunov function**):* A $\mathscr{C}^1$ function $V : X_{\mathbf{q}} \to \mathbb{R}_0^+$ is called a practical stability Lyapunov function for the finite state transition system (4) for zero input $u_{\mathbf{q}i} \equiv 0$, if there exists class $\mathscr{K}_\infty$ functions $\overline{\alpha}$, $\underline{\alpha}$, α such that for any strictly positive real numbers Δ, δ and $x_{\mathbf{q}} \in X_{\mathbf{q}}$ such that $\|x_{\mathbf{q}}\| \le \Delta$, the following holds

$$\underline{\alpha}(\|x_{\mathbf{q}}\|) \le V(x_{\mathbf{q}}) \le \overline{\alpha}(\|x_{\mathbf{q}}\|), \tag{28}$$

$$V(x_{\mathbf{q}(i+1)}) - V(x_{\mathbf{q}i}) \le -\alpha(\|x_{\mathbf{q}i}\|) + \delta \tag{29}$$

$$\alpha \circ \overline{\alpha}^{-1} \circ \underline{\alpha}(\Delta) > \delta. \tag{30}$$

for all the transitions $x_{\mathbf{q}i} \xrightarrow[\tau]{0} x_{\mathbf{q}(i+1)}$ (for all $i \in \mathbb{N}$).

PROOF: From $V(x_{\mathbf{q}}) \le \overline{\alpha}(\|x_{\mathbf{q}}\|)$, $\forall x_{\mathbf{q}} \in X_{\mathbf{q}}$, thus we have $\overline{\alpha}^{-1}(V(x_{\mathbf{q}})) \le \|x_{\mathbf{q}}\|$, and

$$-\alpha(\|x_{\mathbf{q}i}\|) \le -\alpha \circ \overline{\alpha}^{-1}(V(x_{\mathbf{q}i})) = -\alpha_3(V(x_{\mathbf{q}i}))$$

where $\alpha_3 = \alpha \circ \overline{\alpha}^{-1} \in \mathscr{K}_\infty$. Then (29) can be written as

$$V(x_{\mathbf{q}(i+1)}) - V(x_{\mathbf{q}i}) \le -\alpha_3(V(x_{\mathbf{q}i})) + \delta. \tag{31}$$

Define $D = \{\zeta : V(\zeta) \le b\}$ where $b = \alpha_3^{-1}(\delta)$. We show that if there is some $x_{\mathbf{q}0} \in D$ then $x_{\mathbf{q}i} \in D$ for all $i \in \mathbb{N}$, i.e., D is an invariant set. Consider $x_{\mathbf{q}0} \in D$ (i.e. $V(x_{\mathbf{q}0}) \le b$) and inequality (31) results in

$$V(x_{\mathbf{q}1}) \quad \le \quad V(x_{\mathbf{q}0}) - \alpha_3(V(x_{\mathbf{q}0})) + \delta \tag{32}$$

Without loss of generality, we can assume that $\mathrm{Id} - \alpha_3 \in \mathscr{K}$, where Id is the identity function (see [26]). Hence we can write (32) as $V(x_{\mathbf{q}1}) \le (\mathrm{Id} - \alpha_3)(V(x_{\mathbf{q}0})) + \delta$. Since $\mathrm{Id} - \alpha_3 \in \mathscr{K}$, we can write $V(x_{\mathbf{q}1}) \le (\mathrm{Id} - \alpha_3)(b) + \delta = b - (\delta) + \delta = b$. Using induction we can show that $V(x_{\mathbf{q}(0+i)}) \le b$ for all $i \in \mathbb{N}$ thus $\|x_{\mathbf{q}i}\| \le \underline{\alpha}^{-1}\alpha_3^{-1}(\delta)$.

Now consider $\|x_{\mathbf{q}0}\| \in \left(\underline{\alpha}^{-1}\alpha_3^{-1}(\delta), \Delta\right)$ and let $j_0 = \min\{i \in \mathbb{N} : x_{\mathbf{q}i} \in D\}$. For $i < j_0$, we have $\alpha_3(V(x_{\mathbf{q}i})) \ge \delta$, then we can always find $c \in (0,1)$ such that $c\alpha_3(V(x_{\mathbf{q}i})) = \delta$ and hence

$$V(x_{\mathbf{q}(i+1)}) - V(x_{\mathbf{q}i}) \le -(1-c)\alpha_3(V(x_{\mathbf{q}i})). \tag{33}$$

From comparison principle for discrete-time systems [28, Lemma 4.3], we have

$$V(x_{\mathbf{q}i}) \le \beta(V(x_{\mathbf{q}0}), i) \tag{34}$$

where $\beta \in \mathscr{KL}$. Thus $\|x_{\mathbf{q}i}\| \le \underline{\alpha}^{-1}(V(x_{\mathbf{q}i})) \le \underline{\alpha}^{-1}(\beta(V(x_{\mathbf{q}0}), i)) \le \underline{\alpha}^{-1}(\beta(\overline{\alpha}(\|x_{\mathbf{q}0}\|), i)) = \tilde{\beta}(\|x_{\mathbf{q}0}\|, i)$ where $\tilde{\beta} \in \mathscr{KL}$.

Hence we can write $\|x_{\mathbf{q}i}\| \le \max\{\tilde{\beta}(\|x_{\mathbf{q}0}\|, i), \underline{\alpha}^{-1}\alpha_3^{-1}(\delta)\} \le \tilde{\beta}(\|x_{\mathbf{q}0}\|, i) + \underline{\alpha}^{-1}\alpha_3^{-1}(\delta)$ $\forall i \in \mathbb{N}$.

■

Based on the definitions of practical asymptotic stability and practical stability Lyapunov functions, we can now show that practically IOSP are zero-input practically asymptotically stable.

Corollary 1: The transition system (4) is practically asymptotically stable for zero input ($u_{\mathbf{q}i} \equiv 0$) if there exists class $\mathscr{K}_\infty$ functions $\overline{\alpha}$, $\underline{\alpha}$, θ such that for any strictly positive real numbers Δ, δ and for all $x_{\mathbf{q}} \in X_{\mathbf{q}}$ such that $\|x_{\mathbf{q}}\| \le \Delta$, the following holds

$$\underline{\alpha}(\|x_{\mathbf{q}}\|) \le V(x_{\mathbf{q}}) \le \overline{\alpha}(\|x_{\mathbf{q}}\|), \tag{35}$$

$$\theta(\|x_{\mathbf{q}}\|) \ge \rho_F \tau h^T(x_{\mathbf{q}}, 0) h(x_{\mathbf{q}}, 0), \tag{36}$$

$$\theta(\overline{\alpha}^{-1} \circ \underline{\alpha}(\Delta)) > \delta \tag{37}$$

and (4) is (ρ_F, ν_F, δ)-practically IOSP with $\rho_F > 0$ and $\nu_F \ge 0$.

PROOF: The proof follows directly from Theorem 2.

V. NUMERICAL EXAMPLE

An LTI system $\Sigma : \dot{x} = -x + u$ is $(0.25, 0.5)$ - IOSP with an output function $y = x + u$ and a storage function $V(x) = \frac{1}{2}x^T(0.5154)x = 0.2577x^2$. Now we construct a approximately input output similar symbolic model for Σ. It is readily seen that Σ is incrementally forward complete, thus we can apply Corollary 1. We work on the subset $D = [-0.2, 0.2]$ of state space and subset $U = [-0.1, 0.1]$ of the input space. To construct the symbolic model of precision $\varepsilon_y = 1$, we construct a symbolic model $T_{\tau,\eta,\mu}(\Sigma)$ by choosing $\theta_1 = 1$, $\eta = 0.1$, $\theta_2 = \varepsilon_u = \mu = 0.1$ and $\tau = 0.2$ so that assumptions of Corollary 1 are satisfied. Since $\mu = 0.1$ and $\tau = 0.2$, the control inputs are piecewise constant of duration τ such that

$$\{-\mu, 0, \mu\} = \{u_{-1}, u_0, u_1\} = \{-0.1, 0, 0.1\} \in U.$$

And the states of the symbolic system are described by

$$\{-2\eta, -\eta, 0, \eta, 2\eta\} = \{-0.2, -0.1, 0, 0.1, 0.2\} \in D.$$

The transitions between states upon the action of a control input can be calculated using the differential equation describing Σ. The symbolic system in figure 1 represents $T_{\tau,\eta,\mu}(\Sigma)$ and it can be observed $T_{\tau,\eta,\mu}(\Sigma)$ is nondeterministic, i.e., $\mathbf{Post}_u(x)$ may not be a single element. For example, if we consider the state -2η, the $\mathbf{Post}_{u_0}(-2\eta) = \{-2\eta, -\eta\}$, i.e., under the input u_0, the next possible state may be -2η or $-\eta$. In Fig. 1, multiple inputs on the arrows represent all the possible inputs that can cause that transition.

Now we discuss the effect of symbolic abstraction on the passivity properties of Σ. It can be verified that the output $y = x + u$ satisfies Assumption 1 for $\gamma = 1$. Hence Theorem 4 states that $T_{\tau,\eta,\mu}(\Sigma)$ is $\left(\rho_F, \nu_F, \frac{K\varepsilon_y}{\tau}\right)$-practically IOSP where

$$\nu_F = \nu - \gamma\tau - \frac{\rho\gamma\tau}{1+\tau} = 0.2583, \quad \rho_F = \frac{\rho}{(1+\tau)\tau} = 1.0417.$$

and K is the Lipschitz constant for $V(x)$. For the state space D as the Lipschitz constant $K = 0.0773$, hence $T_{\tau,\eta,\mu}(\Sigma)$ is

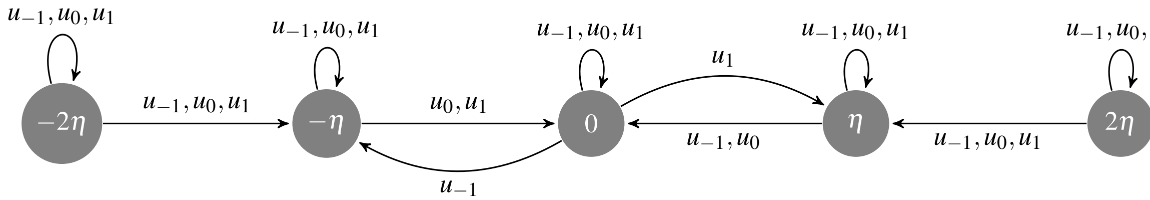

Fig. 1: Symbolic model for Σ.

(0.2, 0.24, 0.3865) - practically IOSP.

For the symbolic transition system, Theorem 4 can be alternatively verified by checking if

$$V(\mathbf{q})+\tau(\ell^T o)-\rho_F\tau(\mathbf{o}^T\mathbf{o})-\nu_F\tau(\ell^T\ell)+\frac{K\varepsilon_y}{\tau}-V(\mathbf{p})\geq 0$$

is satisfied for all transitions $\mathbf{q}\xrightarrow[\tau]{\ell}\mathbf{p}$ where $\mathbf{p}\in\mathbf{Post}_\ell(\mathbf{q})$, $\mathbf{o}=C\mathbf{q}+D\ell$ and $V(\mathbf{q})=\frac{1}{2}\mathbf{q}^TP\mathbf{q}$, i.e.,

$$\mathbf{q}^TF\mathbf{q}+\ell^TG\mathbf{q}+\mathbf{q}^TG^T\ell+\ell^TH\ell+\frac{K\varepsilon_y}{\tau}-\frac{1}{2}\mathbf{p}^TP\mathbf{p}\geq 0$$

where

$$\begin{aligned}F &= \frac{1}{2}P-\rho_F\tau C^TC, \quad G=\frac{\tau}{2}C-\rho_F\tau D^TC,\\ H &= \frac{\tau}{2}(D+D^T)-\rho_F\tau D^TD-\nu_F\tau I.\end{aligned}$$

For the symbolic system, we assume that there are M quantized inputs denoted by $\{\ell_1,\ell_2,\ldots,\ell_M\}$ and there are N quantized states denoted by $\{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_N\}$. All the transitions in the symbolic system can be represented by $\mathbf{q}_i\xrightarrow[\tau]{\ell_j}\mathbf{p}_i^j$ for $i=1,\ldots N$ and $j=1,\ldots,M$, where $\mathbf{p}_i^j$ represents the next state after time τ with an initial state $\mathbf{q}_i$, under the action input ℓ_j. Hence, passivity verification would entail verification of the inequality

$$\begin{aligned}\mathbf{q}_i^TF\mathbf{q}_i+\ell_j^TG\mathbf{q}_i+\mathbf{q}_i^TG^T\ell_j+\ell_j^TH\ell_j+\frac{K\varepsilon_y}{\tau}\\ -\frac{1}{2}(\mathbf{p}_i^j)^TP(\mathbf{p}_i^j)\geq 0\end{aligned} \tag{38}$$

for $i=1,\ldots N$ and $j=1,\ldots,M$. In order to verify the above inequality for all transitions in a systematic fashion, we consider each state $\mathbf{q}_i\in D$ and evaluate the next states corresponding to all possible inputs ℓ_j, $j=1,\ldots,M$. We continue this procedure for all states $\mathbf{q}_i$, $i=1,\ldots,N$. If we let $\bar{\mathbf{q}}=\left[\mathbf{q}_1,\cdots,\mathbf{q}_N\right]^T$ and $\bar{\ell}=\left[\ell_1,\cdots,\ell_M\right]^T$ then the vectors $\bar{\mathbf{p}}^1=\left[\mathbf{p}_1^1,\cdots,\mathbf{p}_N^1\right]^T,\ldots,\bar{\mathbf{p}}^M=\left[\mathbf{p}_1^M,\cdots,\mathbf{p}_N^M\right]^T$ can be calculated together as $\bar{\bar{\mathbf{p}}}=\left[\bar{\mathbf{p}}^1,\cdots,\bar{\mathbf{p}}^M\right]^T$

$$=\left[(I_M\otimes e^{A\tau})(I_M\otimes\bar{\mathbf{q}})+\left(\left(\int_0^\tau e^{A(\tau-\alpha)}B\mathrm{d}\alpha\right)\otimes I_N\right)(\bar{\ell}\otimes I_N)\right]_\eta$$

for a system described by $\dot{x}=Ax+Bu$. Verification of (38) for $i=1,\ldots N$ and $j=1,\ldots,M$. would require us to verify positivity of MN scalars. All these MN scalars will be arranged along the diagonal of an $MN\times MN$ matrix, and this diagonal matrix would be checked for its positive definiteness. This approach allows us to represent all the inequalities together in a compact fashion. This compact representation will be achieved using the Kronecker product as given by

$$\begin{aligned}PASSIVE &= I_M\otimes((I_N\otimes\bar{\mathbf{q}}^T)(I_N\otimes F)(I_N\otimes\bar{\mathbf{q}}))\\ &+ ((I_N\otimes\bar{\ell}^T)(I_N\otimes G)(I_N\otimes\bar{\mathbf{q}}))\otimes I_M\\ &+ ((I_N\otimes\bar{\mathbf{q}}^T)(I_N\otimes G^T)(I_N\otimes\bar{\ell}))\otimes I_M\\ &+ ((I_M\otimes\bar{\ell}^T)(I_M\otimes H)(I_M\otimes\bar{\ell}))\otimes I_N\\ &+ I_{MN}\otimes\frac{K\varepsilon_y}{\tau}-\bar{\bar{\mathbf{p}}}^T(I_{MN}\otimes P)\bar{\bar{\mathbf{p}}}\geq 0\end{aligned} \tag{39}$$

For the nondeterministic cases where $\bar{\bar{\mathbf{p}}}$ is not unique, we verify (39) for all possible values of $\bar{\bar{\mathbf{p}}}$. Performing this test for our numerical example, we obtain the diagonal elements of the PASSIVE matrix for two possible values of $\bar{\bar{\mathbf{p}}}$ and it can verified that all the diagonal elements are positive,

$$\begin{aligned}\mathrm{diag}(PASSIVE_1)=\big[&0.3732, 0.3817, 0.3833, 0.3886, 0.3896,\\ &0.3782, 0.3844, 0.3865, 0.3870, 0.3859,\\ &0.3819, 0.3860, 0.3859, 0.3817, 0.3810\big]\\ \mathrm{diag}(PASSIVE_2)=\big[&0.3279, 0.3623, 0.3835, 0.3839, 0.3635,\\ &0.3526, 0.3787, 0.3865, 0.3761, 0.3448,\\ &0.3712, 0.3865, 0.3809, 0.3623, 0.3202\big]\end{aligned}$$

hence confirming practical passivity of the symbolic model.

VI. Conclusion

In this paper we introduce approximate input output (alternating) simulation relations which allow us to describe passivity and dissipativity for finite state abstractions of continuous-time systems. These relations further allow us to quantify degradation of passivity under abstraction in terms of passivity indices. We also analyze the implication of such relations on zero-input stability of a passive system. Future work will focus other properties of passive systems like input-output stability. In a companion paper [29] we consider the interaction between a continuous-time passive system with a finite state transition system.

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On passivity of a continuous plant interconnected with a discrete supervisory controller

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***Abstract*— Consider a continuous plant interconnected with a discrete supervisory controller. In what sense can the interconnection be termed passive and stable? This question is useful for application of passivity based control techniques in cyberphysical systems where controllers may be implemented in software. Using a notion of passivity for a finite state model abstracted from an infinite state continuous system that was proposed in [13], hence we consider the interaction of such finite state models of a plant with discrete controllers. The chief result of this paper is to find conditions that ensure that the entire interconnection is passive and hence stable.**

I. Introduction

Close interaction between dynamic systems and the computational elements in a cyberphysical system makes it desirable that these computational elements are designed while exploiting the properties (for e.g. passivity, dissipativity, symmetry, invariance, etc.) of a dynamic system. In this paper, we are primarily interested in the interaction between computational elements and passive dynamic systems. Passivity is an important property used to design stable control systems that also offers compositionality [1]. Hence, there is much interest in using passivity as design tool for cyberphysical systems [2]. For a full theory of passivity in cyberphysical systems we need a better understanding of the interaction between passive dynamic systems with computational elements that are discrete state systems. However, this needs a well defined notion of passivity for discrete state systems. One broad approach to solve this problem has been through the hybrid systems framework. Hybrid system consider both continuous and discrete dynamics are considered in the same framework and several definitions of passivity have proposed for certain special classes of hybrid systems [3], [4], [5], [6], [7]. Although hybrid systems framework is well established, systematic compositional methods to analyze the interconnection between discrete controllers and continuous plants are not yet available. We follow an alternate approach based on abstracting finite state models from infinite state models (continuous-time systems) which further interact with computational elements modeled using finite state controllers (see figure 1). This approach provides us with a unified framework to analyze both discrete and continuous components of a cyberphysical system. In [13] we showed that if the finite state transition systems are abstracted using the concepts of approximate input output simulation relationships then properties like passivity and dissipativity can be defined for the finite state abstractions. Further, we quantified the degradation of passivity under such abstractions and showed that finite abstractions of a certain class of continuous-time passive systems are zero input asymptotically stable in a practical sense.

In this paper we present results on passivity for systems obtained by composing of such practically passive finite state abstractions. Specifically, we analyze feedback composition of computational elements with finite abstractions of continuous-time passive systems. We adopt a modified notion of approximate feedback composition of transition systems proposed by [10], which requires two transition systems to satisfy certain approximate simulation relations for feedback composition. We show that once two transition systems are approximately feedback composable, then practical passivity of one of those transition systems implies practical passivity of entire composition, although the passivity indices may be different. We also show that these results can be used to develop practically passivating discrete controllers for the interconnection of a continuous-time systems and the discrete controller. However, actual design of such controllers in out of the scope of this paper. A major assumption in our work is that both continuous and discrete components of the cyberphyiscal system receive the same quantized inputs and is the subject of our future work.

This paper is organized as follows. In section II of this paper we introduce our notation and some preliminary definitions. In section III, we present some initial results for feedback composition of finite transition systems. We further interpret these results for the case when a discrete supervisor is connected to a continuous-time passive system. We show that if one of the components in the interconnection is passive then their approximate feedback composition is also passive in a practical sense. We would also like to mention the paper [14] that considers the related but complementary problem of discretizing a controller designed in the continuous space so that passivity indices of the closed loop system are maintained despite discretization. Note that while [14] requires a bisimulation of the controller, we require a simulation of the plant for our purpose.

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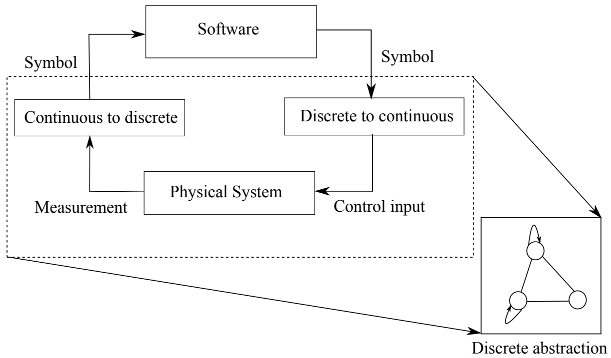

Fig. 1: Finite state approximation of a continuous-time plant interacting with a finite state controllers (software).

II. Preliminaries

A. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\iota_A : A \hookrightarrow B$ or simply by ι the natural inclusion map taking any $a \in A$ to $\iota(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, integer, real, positive, and nonnegative real numbers, respectively. The inner product of signals $u(t)$, $y(t)$ is denoted by $\langle u, y\rangle$ defined as $\langle u, y\rangle = \int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau$. Given a vector $x \in \mathbb{R}^n$, x_i is the i-th element of x and we denote infinity norm and euclidean norms of x by $\|x\|$ and $\|x\|_2$. Given a measurable function $f : \mathbb{R}^+ : 0 \leftarrow \mathbb{R}^n$ the (essential) supremum (sup norm) of f is denoted by $\|\|_\infty$. If $A \subseteq \mathbb{R}^n$ and $\eta \in \mathbb{R}^+$, $[A]_\eta$ denotes the subset $[A]_\eta \subseteq A$ defined by: $[A]_\eta = \left\{z \in A \mid z_i = k_i\eta \;\text{ for some } k_i \in \mathbb{A} \text{ and } i = 1, 2, \ldots, n\right\}$. The set $[A]_\eta$ will be used as an approximation of the set A with precision η. If we define $\mathbf{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. For set $A \subseteq \mathbb{R}^n$ of the form $A = \bigcup_{j=1}^M A_j$ for some $M \in \mathbb{N}$, where $A_j = \Pi_{i=1}^n [c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$ and positive constant $\eta \leq \hat{\eta}$, where $\hat{\eta} = \min_{j=1,\ldots,M} \eta_{A_j}$ and $\eta_{A_j} = \min\{|d_1^j - c_1^j|, \cdots, |d_n^j - c_n^j|\}$. Note that $[A]_\eta \neq \varnothing$ for any $\eta \leq \hat{\eta}$. Geometrically, for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$ the collection of sets $\{\mathbf{B}_\lambda(p)\}_{p \in [A]_\eta}$ is a covering of A, i.e. $A \subseteq \bigcup_{p \in [A]_\eta} \mathbf{B}_\lambda(p)$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ belongs to class $\mathscr{K}_\infty$ if $\gamma \in \mathscr{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathscr{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathscr{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. A relation $R \subseteq A \times B$ is defined by a map of the form $R : A \to 2^B$ where $b \in R(a)$ if and only if $(a, b) \in R$. For a set $S \in A$ the set $R(S)$ is defined as $R(S) = \{b \in B : \exists\; a \in S, (a, b) \in R\}$. Also, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$. We also denote by $\mathbf{d} : X \times X \to \mathbb{R}_0^+$ a metric in the space X and by $\pi_X : X \times U \to X$ the projection sending $(x, u) \in X \times U$ to $x \in X$.

B. Incremental forwardness and stability

In this work we restrict ourselves to control systems of the form

$$\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathscr{U}, f) \tag{1}$$

where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input space;
- $\mathscr{U} : \mathbb{R} \to \mathsf{U}$ is a subset of the set of all locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a Lipschitz continuous map.

If $\xi :]a, b[\longrightarrow \mathbb{R}^n$ is a trajectory of Σ (or equivalently a solution of the differential equation $\dot{x} = f(x, u)$), then we will use $\xi(\tau, x, v)$ to denote a unique point reached at time τ under the input v from an initial condition x. System Σ is said to be forward-complete if such a solution is defined for all $t \in]0, \infty[$. In this paper we use an incremental version of this property, defined as:

*Definition 1 (**Incremental forward-completeness**):* A control system Σ is δ-FC if there exist continuous functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for every $s \in \mathbb{R}^+$, the functions $\beta(\cdot, s)$ and $\gamma(\cdot, s)$ belong to class $\mathscr{K}_\infty$, and for any $x, x' \in \mathbb{R}^n$, any $\tau \in \mathbb{R}^+$, and any $v, v' \in \mathscr{U}$, where $v, v' : [0, \tau) \to \mathsf{U}$, the following condition is satisfied for all $t \in [0, \tau]$:

$$\|\xi(t, x, v) - \xi(t, x', v')\| \leq \beta(\|x - x'\|, t) + \gamma(\|v - v'\|_\infty, t). \tag{2}$$

C. Transition systems and system relations

Definition 2: [10] A system $T_\mathbf{q}$ is a quintuple $T_\mathbf{q} = (Q, L, \xrightarrow[\tau]{\ell}, O, H)$ consisting of:

- a finite set of states $\mathbf{q} \in Q$;
- a finite set of inputs $\ell \in L$;
- a transition relation $\longrightarrow \subseteq Q \times L \times \{\tau\} \times Q$;
- an output set O;
- an output function $H : Q \to O$.

$T_\mathbf{q}$ can be thought of as a discrete time system with a sample time τ and a finite state run

$$\mathbf{q}_0 \xrightarrow[\tau]{\ell_0} \mathbf{q}_1 \xrightarrow[\tau]{\ell_1} \mathbf{q}_2 \xrightarrow[\tau]{\ell_2} \cdots \xrightarrow{\ell_{n-2}} \mathbf{q}_{n-1} \xrightarrow[\tau]{\ell_{n-1}} \mathbf{q}_n$$

where $\mathbf{q}_0 \in Q$ is the initial state and $\mathbf{q}_i \xrightarrow[\tau]{\ell_i} \mathbf{q}_{i+1}$ for all $0 \leq i \leq n$. The subscript i corresponds to the sampling time instants $t = 0, 1, 2, \ldots, n\tau$ and $\mathbf{q}_i$ corresponds to the state of $T_\mathbf{q}$ at the time instant $i\tau$. In some cases, a finite state run can be extended to an infinite state run with $i \in \mathbb{N}$. To define notions of stability for transition systems we assume that the finite sets Q, L, and O are equipped with the metric given by $\mathbf{d}(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$ where $\mathbf{p}, \mathbf{q}$ are elements of Q, L or O. $T_\mathbf{q}$ is deterministic, if for any state $\mathbf{q} \in Q$ and $\ell \in L$ there exists at most one state $\mathbf{q}' \in Q$ such that $\mathbf{q} \xrightarrow{\ell} \mathbf{q}'$; if the system is nondeterministic, then for a transition $\mathbf{q} \xrightarrow{\ell} \mathbf{q}'$ the state $\mathbf{q}'$ may not unique, $\mathbf{q}'$ is also known as the $\boldsymbol{\ell}$-successor of $\mathbf{q}$. In such a case $\mathbf{q}'$ belongs to a set of all possible $\boldsymbol{\ell}$-successors given by $\mathbf{Post}_{\boldsymbol{\ell}}(\mathbf{q})$ and we will use $L(\mathbf{q})$ to denote the set of inputs $\boldsymbol{\ell} \in L$ for which $\mathbf{Post}_{\boldsymbol{\ell}}(\mathbf{q})$ is nonempty. We will further use $\mathbf{Post}_{\boldsymbol{\ell}}(Q)$ to denote the set $\bigcup_{\mathbf{q}\in Q} \mathbf{Post}_{\boldsymbol{\ell}}(\mathbf{q})$. Now we present certain system relations, that are used in this paper.

Definition 3: [11](***ε*-Approximate Simulation and Alternating Simulation**) Let $T_1 := (Q_1, L_1, \xrightarrow[1]{}, O_1, H_1)$, $T_2 := (Q_2, L_2, \xrightarrow[2]{}, O_2, H_2)$ be metric transition systems with the same sets of inputs $L = L_1 = L_2$ and outputs $O = O_1 = O_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon \in \mathbb{R}_0^+$ be given precision requirements then a relation $R \subseteq Q_1 \times Q_2$ is said to be an

(a) an ε-approximate Simulation (ε-S) relation between T_1 and T_2 if the following two conditions are satisfied:
   (i) for every $(\mathbf{q}_1, \mathbf{q}_2) \in R$ we have $\mathbf{d}(H_1(\mathbf{q}_1), H_2(\mathbf{q}_2)) \leq \varepsilon$;
   (ii) for every $(\mathbf{q}_1, \mathbf{q}_2) \in R$ we have that $\mathbf{q}_1 \xrightarrow[1]{\boldsymbol{\ell}_1} \mathbf{q}_1'$ in T_1 implies the existence of $\mathbf{q}_2 \xrightarrow[2]{\boldsymbol{\ell}_2} \mathbf{q}_2'$ in T_2 satisfying $(\mathbf{q}_1', \mathbf{q}_2') \in R$.

(b) an ε-approximate Alternating Simulation (ε-AS) relation from T_1 to T_2 if conditions (i), (ii) and the following condition are satisfied:
   (iii) for every $(\mathbf{q}_1; \mathbf{q}_2) \in R$ and for every $\boldsymbol{\ell}_1 \in L_1(\mathbf{q}_1)$ there exists $\boldsymbol{\ell}_2 \in L_2(\mathbf{q}_2)$ such that for every $\mathbf{q}_2' \in \mathbf{Post}_{\boldsymbol{\ell}_2}(\mathbf{q}_2)$ there exists $\mathbf{q}_1' \in \mathbf{Post}_{\boldsymbol{\ell}_1}(\mathbf{q}_1)$ satisfying $(\mathbf{q}_2', \mathbf{q}_1') \in R$.

If T_1 is ε- approximately simulated (or alternatingly simulated) by T_2, then we denote this fact by $T_1 \preceq_S^{\varepsilon} T_2$ ($T_1 \preceq_{AS}^{\varepsilon} T_2$). Approximate simulation relations used in [11], are primarily based on bounding the distance between the outputs or states of the continuous-time and its approximation. However, in order to account for behaviors like passivity (which are defined using both inputs and outputs) we introduced notions of approximate input output simulation and approximate input output alternating simulation. These notions allow us to bound the distances between outputs as well as inputs. If ε_u and ε_y are the given precision requirements for inputs and outputs respectively and if T_1 is $(\varepsilon_u, \varepsilon_y)$- approximately input output simulated (or approximately input output alternatingly simulated) by T_2, then we denote this fact by $T_1 \preceq_{IOS}^{(\varepsilon_u,\varepsilon_y)} T_2$ $\left(T_1 \preceq_{IOAS}^{(\varepsilon_u,\varepsilon_y)} T_2\right)$.

Definition 4 (***Approximate input output simulation***)*:* Let $T_1 := (X_1, U_1, \xrightarrow[1]{}, Y_1, H_1)$, $T_2 := (X_2, U_2, \xrightarrow[2]{}, Y_2, H_2)$ be metric transition systems with the same sets of inputs $U = U_1 = U_2$ and outputs $Y = Y_1 = Y_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon_u, \varepsilon_y \in \mathbb{R}_0^+$ be given precision requirements then a relation $R \subseteq X_1 \times X_2$ is said to be an $(\varepsilon_u, \varepsilon_y)$ - approximate input output simulation (IOS) relation between T_1 and T_2 if the following two conditions are satisfied:

(i) for every $(x_1, x_2) \in R$ we have $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon_y$;
(ii) for every $(x_1, x_2) \in R$ and for every $u_1 \in U_1$ there exists $u_2 \in U_2$ such that $\mathbf{d}(u_1, u_2) \leq \varepsilon_u$ and $x_1 \xrightarrow[1]{u_1} x_1'$ in T_1 implies the existence of $x_2 \xrightarrow[2]{u_2} x_2'$ in T_2 such that $(x_1', x_2') \in R$.

Definition 5: (***Approximate input output alternating simulation***)*:* Let $T_1 := (X_1, U_1, \xrightarrow[1]{}, Y_1, H_1)$, $T_2 := (X_2, U_2, \xrightarrow[2]{}, Y_2, H_2)$ be metric transition systems with the same sets of inputs $U = U_1 = U_2$ and outputs $Y = Y_1 = Y_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon_u, \varepsilon_y \in \mathbb{R}_0^+$ be a given precision requirements, a relation $R \subseteq X_1 \times X_2$ is said to be an $(\varepsilon_u, \varepsilon_y)$ - approximate input output alternating simulation (IOAS) relation from T_1 to T_2 if condition (i) of Definition 4 and the following condition are satisfied:

(iii) for every $(x_1, x_2) \in R$ and for every $u_1 \in U_1(x_1)$ there exists $u_2 \in U_2(x_2)$ such that $\mathbf{d}(u_1, u_2) \leq \varepsilon_u$ and for every $x_2' \in \mathbf{Post}_{u_2}(x_2)$ there exists $x_1' \in \mathbf{Post}_{u_1}(x_1)$ satisfying $(x_2', x_1') \in R$.

The two notions of alternating approximate simulation and approximate simulation coincide in the special case of deterministic systems. If T_1 is $(\varepsilon_u, \varepsilon_y)$- approximately input output simulated (or approximately input output alternatingly simulated) by T_2, then we denote this fact by $T_1 \preceq_{IOS}^{(\varepsilon_u,\varepsilon_y)} T_2$ ($T_1 \preceq_{IOAS}^{(\varepsilon_u,\varepsilon_y)} T_2$). In this paper, we use a special case of approximate input-output simulation described in [13]. This special case is obtained when we set $\varepsilon_u = 0$ and such a case arises when both transition systems T_1 and T_2 are connected to the same input signal.

Definition 6: (**$(0, \varepsilon_y)$*-Approximate input output simulation***) Let $T_1 := (Q_1, L_1, \xrightarrow[1]{}, O_1, H_1)$, $T_2 := (Q_2, L_2, \xrightarrow[2]{}, O_2, H_2)$ be metric transition systems with the same sets of inputs $L = L_1 = L_2$ and outputs $O = O_1 = O_2$ and equipped with the metric $\mathbf{d}$. Let $\varepsilon_y \in \mathbb{R}_0^+$ be given precision requirement for the outputs, then a relation $R \subseteq Q_1 \times Q_2$ is said to be

(a) an $(0, \varepsilon_y)$ - approximate input output simulation (IOS) relation between T_1 and T_2 if the following two conditions are satisfied:
   (i) for every $(\mathbf{q}_1, \mathbf{q}_2) \in R$ we have $\mathbf{d}(H_1(\mathbf{q}_1), H_2(\mathbf{q}_2)) \leq \varepsilon_y$;
   (ii) for all $\boldsymbol{\ell} \in L$ and for every $(\mathbf{q}_1, \mathbf{q}_2) \in R$ we have that $\mathbf{q}_1 \xrightarrow[1]{\boldsymbol{\ell}} \mathbf{q}_1'$ in T_1 implies the existence of $\mathbf{q}_2 \xrightarrow[2]{\boldsymbol{\ell}} \mathbf{q}_2'$ in T_2 satisfying $(\mathbf{q}_1', \mathbf{q}_2') \in R$.

(b) an $(0, \varepsilon_y)$ - approximate input output alternating simulation (IOAS) relation between T_1 and T_2 if conditions (i), (ii) and the following condition is satisfied:
   (iii) for every $(\mathbf{q}_1; \mathbf{q}_2) \in R$, $L_1(\mathbf{q}_1) = L_2(\mathbf{q}_2) = L$ and for every $\boldsymbol{\ell} \in L$ and for every $\mathbf{q}_2' \in \mathbf{Post}_{\boldsymbol{\ell}}(\mathbf{q}_2)$ there exists $\mathbf{q}_1' \in \mathbf{Post}_{\boldsymbol{\ell}}(\mathbf{q}_1)$ satisfying $(\mathbf{q}_2', \mathbf{q}_1') \in R$.

Proposition 1: Consider two transitions systems T_1 and T_2 such that both transition systems receive the same input signal. Then $(0, \varepsilon_y)$ - approximate input output (alternating) simulation and ε-approximate (alternating) simulation (from definition 3) are equivalent.

PROOF: The proof follows from Definition 3 by setting $\boldsymbol{\ell}_1 = \boldsymbol{\ell}_2$.

■

D. Finite abstractions

In order to a obtain finite abstraction for $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathscr{U}, f)$, we begin by considering a discrete time sub-transition system $T_\tau(\Sigma)$ with a sampling time period $\tau \in \mathbb{R}^+$. We further assume that control inputs are piecewise-constant over the sampling time period τ, the class of inputs considered are:

$$\mathscr{U}_\tau := \{u \in \mathscr{U} \mid u(t) = u(0),\ t \in [0, \tau]\}.$$

For $T_\tau(\Sigma)$ we use identity map as the output function, however, for stability and passivity analysis, we use an alternate output corresponding to $y = h(x, u)$.

Definition 7: [10] Let Σ be a control system and $T(\Sigma)$ its associated transition system. For any $\tau > 0$, the sub transition system $T_\tau(\Sigma) := (X_\tau, U_\tau, \xrightarrow[\tau]{u_\tau}, Y_\tau, H_\tau)$ is defined by:

- $X_\tau = \mathbb{R}^n$;
- $U_\tau = \mathscr{U}_\tau$;
- $x_\tau \xrightarrow[\tau]{u_\tau} x'_\tau$, if there exists a trajectory $\xi : [0, \tau] \longrightarrow \xi(\tau, x_\tau, u_\tau) = x'_\tau$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = 1_{\mathbb{R}^n}$.

Now we restrict the input set to a hyper-rectanlge $\mathsf{U} \subseteq \mathbb{R}^m$ such that $\{0\} \in \mathsf{U}$. Then we choose the input quantization factor such that $\mu \leq \hat{\mu}$ (see *A. Notation* on how to calculate $\hat{\mu}$). Now we consider the transition system $T_\tau(\Sigma)$ with a quantized input space to obtain $T_{\tau,\mu}(\Sigma)$ defined as:

$$T_{\tau,\mu}(\Sigma) := (X_\tau, U_{\mathbf{q}}, \xrightarrow{u_{\mathbf{q}}}, Y_\tau, H_\tau)$$

where

- $X_\tau = \mathbb{R}^n$;
- $U_{\mathbf{q}} = [\mathsf{U}]_\mu$;
- $x_\tau \xrightarrow[\tau]{u_{\mathbf{q}}} x'_\tau$, if there exists a trajectory $\xi : [0, \tau] \longrightarrow \xi(\tau, x_\tau, u_{\mathbf{q}}) = x'_\tau$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = 1_{\mathbb{R}^n}$.

In this final stage, we restrict the state set to a hyper-rectanlge $X \subseteq \mathbb{R}^n$ such that $\{0\} \in X$. Then we choose the state quantization factor such that $\eta \leq \hat{\eta}$ (see *A. Notation* on how to calculate $\hat{\eta}$). Then for any δ-FC control system Σ and parameters $\tau > 0$, $\eta > 0$, $\mu > 0$ and a design parameters $\theta_1, \theta_2 \in \mathbb{R}^+$, a countable transition system [9] can be defined as:

$$T_{\tau,\mu,\eta}(\Sigma) := (X_{\mathbf{q}}, U_{\mathbf{q}}, \xrightarrow[\tau]{u_{\mathbf{q}}}, Y_{\mathbf{q}}, H_{\mathbf{q}}) \tag{3}$$

where:

- $X_{\mathbf{q}} = [X]_\eta$;
- $U_{\mathbf{q}} = [\mathsf{U}]_\mu$;
- $x_{\mathbf{q}} \xrightarrow[\tau]{u_{\mathbf{q}}} x'_{\mathbf{q}}$, if $\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$;
- $Y_{\mathbf{q}} = [X]_\eta$
- $H_{\mathbf{q}} = \imath : X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$

where β and γ are functions from Definition 1. The transition system $T_{\tau,\mu,\eta}(\Sigma)$ is finite and countable [9]. A slight modification of Theorem 4.1 from [9] can be used to obtain finite countable abstractions $T_{\tau,\eta,\mu}(\Sigma)$ which are $(0, \varepsilon_y)$ - approximately input output alternatingly similar to $T_{\tau,\mu}(\Sigma)$ and hence we can state the following result.

Proposition 2: [13] Consider a control system Σ and any desired precision $\varepsilon_u = 0$, $\varepsilon_y > 0$. If Σ is δ-FC then for any $\tau > 0$, $\theta > 0$, $\eta > 0$ and $\mu > 0$ satisfying the following inequality:

$$\beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta \leq \varepsilon_y, \tag{4}$$

such that $\eta \leq \varepsilon_y \leq \theta_1$ and $\mu \leq \theta_2$, we have:

$$T_{\tau,\eta,\mu}(\Sigma) \preceq^{(0,\varepsilon_y)}_{IOAS} T_{\tau,\mu}(\Sigma) \preceq^{(0,\varepsilon_y)}_{IOS} T_{\tau,\eta,\mu}(\Sigma). \tag{5}$$

PROOF: To show $T_{\tau,\mu}(\Sigma) \preceq^{(0,\varepsilon_y)}_{IOS} T_{\tau,\eta,\mu}(\Sigma)$ we consider any $x_\tau \in X_\tau$ and any $u_{\mathbf{q}} \in U_{\mathbf{q}} = [\mathsf{U}]_\mu$, then there exists $x_{\mathbf{q}} \in X_{\mathbf{q}} = [X]_\eta$ such that

$$\|x_\tau - x_{\mathbf{q}}\| \leq \eta \leq \varepsilon_y \tag{6}$$

hence condition (i) of Definition 6 is satisfied.

Now if we consider the transition $x_\tau \xrightarrow[\tau]{u_{\mathbf{q}}} x'_\tau$ in the transition system $T_{\tau,\mu}(\Sigma)$, then the distance between x'_τ and $\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})$ can estimated based on the δ - FC property of Σ and inequality (6) i.e.,

$$\|x'_\tau - \xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})\| \leq \beta(\varepsilon_y, \tau) + \gamma(0, \tau) \tag{7}$$

Since $X_\tau \subseteq \bigcup_{p \in [X]_\eta} \mathbf{B}_\lambda(p)$, there exists $x'_{\mathbf{q}} \in X_{\mathbf{q}}$ such that

$$\|x'_\tau - x'_{\mathbf{q}}\| \leq \eta \tag{8}$$

From the triangular inequality we have

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_\tau\| + \|x'_\tau - x'_{\mathbf{q}}\|$$

From inequalities (7) and (8) we have

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\varepsilon_y, \tau) + \gamma(0, \tau) + \eta$$

Finally we use $\eta \leq \varepsilon_y \leq \theta_1$ and $0 < \theta_2$ to show that

$$\|\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}}) - x'_{\mathbf{q}}\| \leq \beta(\theta_1, \tau) + \gamma(\theta_2, \tau) + \eta$$

which, by the definition of $T_{\tau,\mu,\eta}(\Sigma)$ implies the existence of $x_{\mathbf{q}} \xrightarrow{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$. Therefore, from inequality (8) and since $\eta \leq \varepsilon_y$ we conclude that $(x'_\tau, x'_{\mathbf{q}}) \in R$ and condition (ii) in Definition 6 holds.

Now we show that $T_{\tau,\eta,\mu}(\Sigma) \preceq^{(0,\varepsilon_y)}_{IOAS} T_{\tau,\mu}(\Sigma)$. For $R \subseteq X_\tau \times X_{\mathbf{q}}$ we consider an $x_\tau = x_{\mathbf{q}} \in X_{\mathbf{q}}$. This is possible because $X_{\mathbf{q}} \subseteq X_\tau$ and it satisfies condition (i) of Definition 4 (i.e., $\|x_\tau - x_{\mathbf{q}}\| = 0 < \varepsilon_y$). Now we choose an input $u_{\mathbf{q}} \in U_{\mathbf{q}}$ and consider the unique transition $x_\tau \xrightarrow[\tau]{u_{\mathbf{q}}} x'_\tau = \xi(\tau, x_\tau, u_{\mathbf{q}}) \in \mathbf{Post}_{u_{\mathbf{q}}}(x_\tau)$. The distance between x'_τ and $\xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})$ can be bounded using the δ - FC properties of Σ, i.e.,

$$\|x'_\tau - \xi(\tau, x_{\mathbf{q}}, u_{\mathbf{q}})\| \leq \beta(0, \tau) + \gamma(0, \tau) \tag{9}$$

Since $X_\tau \subseteq \bigcup_{p \in [X]_\eta} \mathbf{B}_\lambda(p)$, we can always find $x'_{\mathbf{q}} \in X_{\mathbf{q}}$ such that

$$\|x'_{\mathbf{q}} - x'_\tau\| \leq \eta \tag{10}$$

From the triangular inequality and inequalities (9) and (10) we have

$$\begin{aligned}\|\xi(\tau,x_{\mathbf{q}},u_{\mathbf{q}})-x'_{\mathbf{q}}\| &\leq \|\xi(\tau,x_{\mathbf{q}},u_{\mathbf{q}})-x'_{\tau}\|+\|x'_{\tau}-x'_{\mathbf{q}}\| \\ &\leq \beta(0,\tau)+\gamma(0,\tau)+\eta\end{aligned}$$

Finally we use $0<\theta_1$ and $0<\theta_2$ to show that

$$\|\xi(\tau,x_{\mathbf{q}},u_{\mathbf{q}})-x'_{\mathbf{q}}\| \leq \beta(\theta_1,\tau)+\gamma(\theta_2,\tau)+\eta$$

which, by the definition of $T_{\tau,\mu,\eta}(\Sigma)$ implies the existence of $x_{\mathbf{q}} \xrightarrow{u_{\mathbf{q}}} x'_{\mathbf{q}}$ in $T_{\tau,\mu,\eta}(\Sigma)$. Therefore, from inequality (10) and since $\eta \leq \varepsilon_y$ we conclude that $(x'_{\tau},x'_{\mathbf{q}}) \in R$ and condition (iii) in Definition 6 holds.

■

Finitely abstracted transition system (3) is a quantized version of the sampled-data system $T_\tau(\Sigma)$. The finite state transition system (3) can be thought of as a discrete time system with a finite state run

$$x_{\mathbf{q}_0} \xrightarrow[\tau]{u_{\mathbf{q}0}} x_{\mathbf{q}_1} \xrightarrow[\tau]{u_{\mathbf{q}1}} x_{\mathbf{q}_2} \xrightarrow[\tau]{u_{\mathbf{q}2}} \cdots \xrightarrow{u_{\mathbf{q}(n-2)}} x_{\mathbf{q}_{n-1}} \xrightarrow[\tau]{u_{\mathbf{q}(n-1)}} x_{\mathbf{q}_n}$$

where $x_{\mathbf{q}_0} \in X_{\mathbf{q}}$ is the initial state and $x_{\mathbf{q}i} \xrightarrow[\tau]{u_{\mathbf{q}i}} x_{\mathbf{q}(i+1)}$ for all $0 \leq i \leq n$. The subscript i corresponds to the sampling time instants $t=0,1\tau,2\tau,\ldots,n\tau$ and $x_{\mathbf{q}i}$ corresponds to the state of (3) at the time instant $i\tau$.

E. Dissipativity and passivity

Consider the system Σ and an output function $y=h(x,u) \in \mathbb{R}^p$. Further, assume that $f(0,0)=0$ and $h(0,0)=0$. Σ is dissipative w.r.t. $y=h(x,u)$ if there exists a $\mathscr{C}^1$ storage function $V(x):\mathbb{R}^n \to \mathbb{R}_0^+$ and a supply rate $\omega:\mathsf{U}\times\mathbb{R}^p \to \mathbb{R}_0^+$ such that $V(0)=0$ and the following inequality is satisfied:

$$V(x(t_2))-V(x(t_1)) \leq \int_{t_1}^{t_2} \omega(u,y,x)dt \qquad (11)$$

for any $t_2 \geq t_1$ and $u \in \mathscr{U}$. A special case of dissipativity is (ρ,ν) - ***input output strict passivity (IOSP)*** when $\omega(u,y,x)=u^Ty-\nu u^Tu-\rho y^Ty$ with $\rho,\nu \in \mathbb{R}_0^+$. In this definition, parameters ν and ρ are known as passivity indices. Corresponding to continuous-time notions of dissipativity and passivity, we introduced the notions of practical dissipativity and practical passivity for transition systems in [13].

*Definition 8 (**Practical dissipativity**):* Let $\mathscr{C}^1$ function $V: X_{\mathbf{q}} \to \mathbb{R}_0^+$ be a storage function with $V(0)=0$ and let $\omega: U_{\mathbf{q}}\times X_{\mathbf{q}} \to \mathbb{R}$ be a supply rate, then the transition system (3) is practically dissipative with respect to ω if

$$\frac{1}{\tau}\left(V(x_{\mathbf{q}(i+1)})-V(x_{\mathbf{q}i})\right) \leq \omega(u_{\mathbf{q}i},x_{\mathbf{q}i})+\delta \quad \forall i \in \mathbb{N} \text{ and } \delta>0 \qquad (12)$$

for all the transitions $x_{\mathbf{q}i} \xrightarrow[\tau]{u_{\mathbf{q}i}} x_{\mathbf{q}(i+1)}$.

The transition system (3) is (ρ_F,ν_F,δ)-***practically IOSP*** when the supply rate is $\omega(u_{\mathbf{q}i},x_{\mathbf{q}i})=(u_{\mathbf{q}i}^T h(x_{\mathbf{q}i},u_{\mathbf{q}i}))-\rho_F(h^T(x_{\mathbf{q}i},u_{\mathbf{q}i})h(x_{\mathbf{q}i},u_{\mathbf{q}i}))-\nu_F(u_{\mathbf{q}i}^T u_{\mathbf{q}i})$. The function $\mathbf{y}_i=h(x_{\mathbf{q}i},u_{\mathbf{q}i})$ is an output function for the finite transition system at a time instant $i\tau$ and it is analogous to $y=h(x,u)$ for the continuous-time system. Note that this output function is different from the output function $H_{\mathbf{q}}=\imath: X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$. The output function $\mathbf{y}_i=h(x_{\mathbf{q}i},u_{\mathbf{q}i})$ will be used only to study passivity and the stability behavior of the transition system. In order to avoid confusion between the output functions $H_{\mathbf{q}}$ and $\mathbf{y}_i=h(x_{\mathbf{q}i},u_{\mathbf{q}i})$, we will refer to $\mathbf{y}_i=h(x_{\mathbf{q}i},u_{\mathbf{q}i})$ as the ***passive output function***, i.e., the output with respect to which the system is passive. The passive output function for a transition system can be obtained using $H_{\mathbf{q}}=\imath: X_{\mathbf{q}} \hookrightarrow Y_{\mathbf{q}}$, whenever $H_{\mathbf{q}}=1_{X_{\mathbf{q}}}$, i.e., $\mathbf{y}=h(H_{\mathbf{q}}(x_{\mathbf{q}}),u_{\mathbf{q}})=h(x_{\mathbf{q}},u_{\mathbf{q}})$. An important consequence of practical IOSP is practical asymptotic stability.

Corollary 1: The transition system (3) is practically asymptotically stable for zero input ($u_{\mathbf{q}i}\equiv 0$) if there exists class $\mathscr{K}_\infty$ functions $\overline{\alpha}$, $\underline{\alpha}$, θ such that for any strictly positive real numbers Δ,δ and for all $x_{\mathbf{q}} \in X_{\mathbf{q}}$ such that $\|x_{\mathbf{q}}\| \leq \Delta$, the following holds

$$\underline{\alpha}(\|x_{\mathbf{q}}\|) \leq V(x_{\mathbf{q}}) \leq \overline{\alpha}(\|x_{\mathbf{q}}\|), \qquad (13)$$

$$\theta(\|x_{\mathbf{q}}\|) \geq \rho_F \tau h^T(x_{\mathbf{q}},0)h(x_{\mathbf{q}},0), \qquad (14)$$

$$\theta(\overline{\alpha}^{-1}\circ\underline{\alpha}(\Delta)) > \delta \qquad (15)$$

and (3) is (ρ_F,ν_F,δ)-practically IOSP with $\rho_F>0$ and $\nu_F \geq 0$.

Now we present a result which quantifies the degradation of passivity under finite state approximations which are approximately input output similar to continuous-time systems. This result was presented in [13] and it is based on an assumption from [12].

Theorem 1: [13] Suppose that the original continuous-time system Σ is δ - FC and (ν,ρ) - IOSP w.r.t. the passive output function $y=h(x,u)$ and a storage function V with a Lipschitz constant K. We also assume that the operator from $u(t)$ to $\dot{y}(t)$ has the finite L_2 gain, γ, that is

$$\int_0^\tau \|\dot{y}(t)\|_2^2 \mathrm{d}t \leq \gamma^2 \int_0^\tau \|u(t)\|_2^2 \mathrm{d}t$$

for any $\tau \geq 0$ and admissible $u(t)$. Let $T_\tau(\Sigma)$ be the transition system corresponding to Σ with a sampling time τ. If the state and input quantization parameters η and μ are chosen such that $T_{\tau,\mu,\eta}(\Sigma)$ is $(\varepsilon_u,\varepsilon_y)$ - approximately input output similar (or alternatingly similar) to $T_\tau(\Sigma)$, then $T_{\tau,\mu,\eta}(\Sigma)$ is $\left(\rho_F,\nu_F,\frac{K\varepsilon_y}{\tau}\right)$-practically IOSP w.r.t. to the passive output function $y_{\mathbf{q}}=h(x_{\mathbf{q}},u_{\mathbf{q}})$ and

$$\nu_F=\nu-\gamma\tau-\frac{\rho\gamma\tau}{1+\tau} \quad \rho_F=\frac{\rho}{(1+\tau)\tau}.$$

In the next section we consider the problem of feedback composition of a discrete controller $T_{\mathbf{q}}$ and the finite state abstraction of continuous-time IOSP system Σ given by $T_{\tau,\mu,\eta}(\Sigma)$. Discrete controllers $T_{\mathbf{q}}$ are designed for continuous plants to satisfy certain discrete and/or continuous specifications, for example, discrete supervisory controllers are used for mode selection, trajectory planning etc. In the section, our main goal is analyze the extra conditions imposed on $T_{\mathbf{q}}$ and on the nature of feedback composition such that the interconnection of $T_{\mathbf{q}}$ and $T_{\tau,\mu,\eta}(\Sigma)$ is also practically IOSP and hence practically asymptotically stable.

III. Composition of transition systems

We begin this section by presenting a modified notion of approximate feedback composition of transition systems from [10]. In Sub-section III-B we consider the approximate feedback composition of two transition systems where one of them is practically IOSP. We show that once two transition systems are approximately feedback composable, then practical passivity of one of those transition systems implies practical passivity of entire composition, although with different passivity indices. Thus, guaranteeing practical stability for the composed transition system. In Sub-section III-C we show that these results have the potential to develop passivating discrete controllers for continuous-time systems.

A. Feedback composition

It was shown in [10] that feedback composition of two transition systems is possible for state feedback if there exists an approximate alternating simulation relation between the two systems.

Definition 9: [10] A system T_2 is said to be ε-approximate feedback composable with a system T_1 if there exists an ε-approximate alternating simulation relation R from T_2 to T_1. According to Proposition 1, existence of ε_y-approximate alternating simulation relation between two transition systems T_1 and T_2 is equivalent to the existence of $(0,\varepsilon_y)$-approximate input output alternating simulation between T_1 and T_2 . This equivalence is possible when both T_1 and T_2 receive the same input signal. Thus we define approximate feedback composition between T_1 and T_2 with the same input signal.

*Definition 10 (**Approximate feedback composition**):* Let $T_1 := (Q_1, L_1, \xrightarrow[\tau]{}, O_1, H_1)$, $T_2 := (Q_2, L_2, \xrightarrow[\tau]{}, O_2, H_2)$ be two transition systems with a common time period τ and common input and output sets equipped with euclidean norm as the metric. Let R be a $(0,\varepsilon_y)$ - approximate input output alternating simulation relation from T_2 to T_1. The feedback composition of T_2 and T_1 with interconnection relation $\mathscr{F}$, denoted by $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1$, is the transition system $(Q_{12}, Q_{120}, L_{12}, \xrightarrow[\tau]{}, O_{12}, H_{12})$ consisting of

- $Q_{12} = \pi_Q(\mathscr{F}) = R \Rightarrow \mathbf{d}(H_1(\mathbf{q}_1), H_2(\mathbf{q}_2)) \leq \varepsilon_y$; or equivalently

$$Q_{12} = \{(\mathbf{q}_1, \mathbf{q}_2) \in (Q_1 \times Q_2) \,|\, \mathbf{d}(H_1(\mathbf{q}_1), H_2(\mathbf{q}_2)) \leq \varepsilon_y\}$$

- $Q_{120} = Q_{12} \cap (Q_{10} \times Q_{20})$;
- $L_{12} = L_1 = L_2$;
- $(\mathbf{q}_1, \mathbf{q}_2) \xrightarrow[\tau]{(\ell,\ell)} (\mathbf{p}_1, \mathbf{p}_2)$ if the following three conditions hold:
  1) $\mathbf{q}_1 \xrightarrow[\tau]{\ell} \mathbf{p}_1$ in T_1;
  2) $\mathbf{q}_2 \xrightarrow[\tau]{\ell} \mathbf{p}_2$ in T_2;
  3) $(\mathbf{q}_1, \mathbf{q}_2, \ell, \ell) \in \mathscr{F}$;
- $O_{12} = O_1 = O_2$;
- $H_{12}(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2}(H_1(\mathbf{q}_1) + H_2(\mathbf{q}_2))$.

This symmetrical choice of output allows $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1$ to be commutative, however, we can also choose an output for the composition as $H_{12}(\mathbf{q}_1, \mathbf{q}_2) = H_1(\mathbf{q}_1)$ or $H_{12}(\mathbf{q}_1, \mathbf{q}_2) = H_2(\mathbf{q}_2)$.

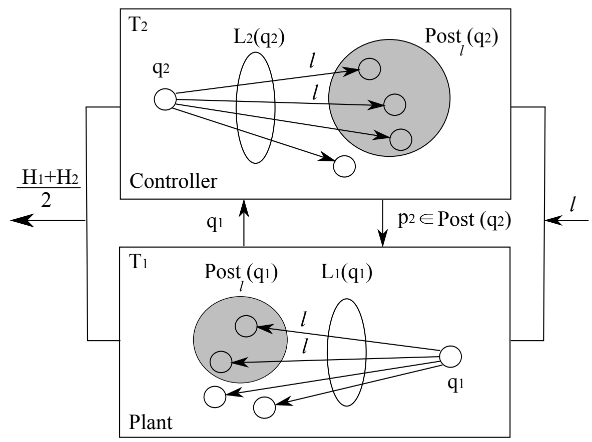

Fig. 2: A schematic of the state feedback interconnection with an alternating simulation relation.

See figure 2 for a schematic of the state feedback interconnection between a transition system representing the continuous time plant and a finite state controller. Also the feedback nature of this composition can be observed through the alternating simulation relation between the plant and controller. In the figure 2, the current plant state $q_1 \in Q_1$ will be communicated to the controller and the controller makes a transition to a state $q_2 \in Q_2$ such that $(q_1; q_2) \in R$. Now if we consider the common input ℓ, then for every $\ell \in L_1(q_1) = L_2(q_2)$ and the next state of the controller $p_2 \in \mathbf{Post}_\ell(q_2)$ will be communicated to the plant. This leads to a transition in the plant state to $p_1 \in \mathbf{Post}_\ell(q_1)$ such that $(p_2; p_1) \in R$. These transitions happen simultaneously after every time period τ.

B. Practical passivity of the feedback composition

Before analyzing passivity of the feedback composition, we present some preliminary results. Initially we consider the consequences of $(0,\varepsilon_y)$ - approximate input output similarity between two transition systems when one of them is a practically IOSP transition system.

Lemma 1: Let $T_1 := (Q_1, L_1, \xrightarrow[\tau]{}, O_1, H_1)$, $T_2 := (Q_2, L_2, \xrightarrow[\tau]{}, O_2, H_2)$ be two transition systems with a common time period τ and common input and output sets equipped with euclidean norm as the metric. Assume that $H_1 = 1_{Q_1}$ and $H_2 = 1_{Q_2}$ and let T_2 be $(0,\varepsilon_y)$ - approximately input output similar to T_1. If T_1 is (ρ_1, ν_1, β_1)-practically IOSP w.r.t. $h(\mathbf{q}_1, \ell_1)$ where $\mathbf{q}_1 \in Q_1$ and $\ell_1 \in L_1$. Then T_2 is (ρ_2, ν_2, β_2)-practically IOSP w.r.t. $h(\mathbf{q}_2, \ell_2)$ where $\mathbf{q}_2 \in Q_2$,

$\ell_1 = \ell_2 = \ell \in L = L_1 = L_2$ and

$$\begin{aligned} \rho_2 &= \rho_1(1-\alpha_2) \\ \nu_2 &= \left(\nu_1 - \frac{\alpha_1}{2}\right) \\ \beta_2 &= \frac{1}{2\alpha_1} M^2 \varepsilon_y^2 + \rho_1 \left(\frac{1}{\alpha_2}+1\right) M^2 \varepsilon_y^2 + \beta_1 + \frac{2K\varepsilon_y}{\tau} \end{aligned} \quad (16)$$

where K and M are Lipschitz constants of the storage function V and the passive output function $h(\mathbf{q},\ell)$, i.e., for any $\mathbf{p},\mathbf{q} \in Q_1 \cup Q_2$ and an arbitrary $\ell \in L$ we have $|V(\mathbf{p}) - V(\mathbf{q})| \leq K\|\mathbf{p}-\mathbf{q}\|$ and $\|h(\mathbf{p},\ell) - h(\mathbf{q},\ell)\|_2 \leq M\|\mathbf{p}-\mathbf{q}\|$ and α_1 and $\alpha_2 \in \mathbb{R}^+$ are such that

$$\nu_1 - \frac{\alpha_1}{2} \geq 0 \text{ and } 1-\alpha_2 \geq 0.$$

PROOF: Consider $(\mathbf{q}_1,\mathbf{q}_2) \in R$ and an input $\ell \in L = L_1(\mathbf{q}_1) = L_2(\mathbf{q}_2)$, then for every $\mathbf{p}_2 \in \mathbf{Post}_\ell(\mathbf{q}_2)$, we have $\mathbf{p}_1 \in \mathbf{Post}_\ell(\mathbf{q}_1)$ such that $\|\mathbf{p}_1 - \mathbf{p}_2\| \leq \varepsilon_y$. Since T_1 is (β_1,ρ_1,ν_1)-IOSP w.r.t. to the passive output function $h(\mathbf{q},\ell)$, for any transition $\mathbf{q}_1 \xrightarrow[\tau]{\ell} \mathbf{p}_1$ in T_1 we have

$$\begin{aligned} V(\mathbf{p}_1) - V(\mathbf{q}_1) \leq\ & (\ell^T h(\mathbf{q}_1,\ell))\tau - \rho_1(h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell))\tau \\ & -\nu_1(\ell^T\ell)\tau + \beta_1\tau \end{aligned} \quad (17)$$

Also from the Lipschitz continuity of the storage function and the passive output function, we have

$$V(\mathbf{p}_2) - V(\mathbf{p}_1) \leq K\|\mathbf{p}_1 - \mathbf{p}_2\| = K\varepsilon_y \quad (18)$$

and

$$\|h(\mathbf{q}_1,\ell) - h(\mathbf{q}_2,\ell)\|_2 \leq M\varepsilon_y.$$

From inequalities (17) and (18) we have

$$\begin{aligned} V(\mathbf{p}_2) &\leq V(\mathbf{p}_1) + K\varepsilon_y \\ &\leq V(\mathbf{q}_1) + (\ell^T h(\mathbf{q}_1,\ell))\tau - \rho_1(h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell))\tau \\ &\quad -\nu_1(\ell^T\ell)\tau + \beta_1\tau + K\varepsilon_y \\ &\leq V(\mathbf{q}_2) + (\ell^T h(\mathbf{q}_1,\ell))\tau - \rho_1(h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell))\tau \\ &\quad -\nu_1(\ell^T\ell)\tau + \beta_1\tau + 2K\varepsilon_y \end{aligned} \quad (19)$$

Let $\Delta h = h(\mathbf{q}_1,\ell) - h(\mathbf{q}_2,\ell)$ then $\|\Delta h\|_2 \leq M\varepsilon_y$. Now we obtain bounds for different terms in the inequality (19).

Bounds on $\tau(\ell^T h(\mathbf{q}_1,\ell))$: Here we compare the terms $\ell^T h(\mathbf{q}_1,\ell)$ and $\ell^T h(\mathbf{q}_2,\ell)$ using

$$|\ell^T h(\mathbf{q}_1,\ell) - \ell^T h(\mathbf{q}_2,\ell)| = |\ell^T \Delta h|.$$

For any $\alpha_1 \in \mathbb{R}^+$, we have

$$|\ell^T\Delta h| \leq \frac{\alpha_1}{2}\ell^T\ell + \frac{1}{2\alpha_1}\Delta h^T\Delta h \leq \frac{\alpha_1}{2}\ell^T\ell + \frac{1}{2\alpha_1}M^2\varepsilon_y^2$$

hence

$$(\ell^T h(\mathbf{q}_1,\ell))\tau \leq (\ell^T h(\mathbf{q}_2,\ell))\tau + \frac{\alpha_1}{2}(\ell^T\ell)\tau + \frac{1}{2\alpha_1}M^2\varepsilon_y^2\tau. \quad (20)$$

Bounds on $h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell)$: Here we compare the terms $h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell)$ and $h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell)$ using

$$\begin{aligned} & |h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell) - h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell)| \\ =\ & |(h(\mathbf{q}_2,\ell)+\Delta h)^T(h(\mathbf{q}_2,\ell)+\Delta h) - h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell)| \\ =\ & |2h^T(\mathbf{q}_2,\ell)\Delta h + \Delta h^T\Delta h| \\ \leq\ & 2|h^T(\mathbf{q}_2,\ell)\Delta h| + \Delta h^T\Delta h \end{aligned} \quad (21)$$

For any $\alpha_2, \in \mathbb{R}^+$, we have

$$2|h^T(\mathbf{q}_2,\ell)\Delta h| \leq \alpha_2 h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell) + \frac{1}{\alpha_2}\Delta h^T\Delta h \quad (22)$$

From inequalities (21) and (22) we have

$$\begin{aligned} -\rho_1(h^T(\mathbf{q}_1,\ell)h(\mathbf{q}_1,\ell))\tau \leq & -\rho_1(1-\alpha_2)(h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell))\tau \\ & +\rho_1\left(\frac{1}{\alpha_2}+1\right)M^2\varepsilon_y^2\tau \end{aligned} \quad (23)$$

Finally bounds from (19), (20) can be used for inequality (23) to obtain

$$\begin{aligned} V(\mathbf{p}_2) \leq\ & V(\mathbf{q}_2) + (\ell^T h(\mathbf{q}_2,\ell))\tau - \rho_2(h^T(\mathbf{q}_2,\ell)h(\mathbf{q}_2,\ell))\tau \\ & -\nu_2(\ell^T\ell)\tau + \beta_2\tau \end{aligned}$$

■

Based on the definition of approximate feedback composition presented in this section, the following results were derived in [10]. Even though the results in [10] were derived for approximate (alternating) simulation relationships they also hold true for approximate input output (alternating) simulation relationships.

Proposition 3: Let T_1 and T_2 be metric systems with $O_1 = O_2$ and $L_1 = L_2$ normed vector spaces with the same norm-induced metric, and let $\mathscr{F}$ be an interconnection relation between T_1 and T_2 with a common input and satisfying

$$(\mathbf{q}_1,\mathbf{q}_2) \in \pi_Q(\mathscr{F}) \Rightarrow \mathbf{d}(H_1(\mathbf{q}_1),H_2(\mathbf{q}_2)) \leq \varepsilon_y.$$

If we define the output of the composition as $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = \frac{1}{2}(H_1(\mathbf{q}_1)+H_2(\mathbf{q}_2))$ then the following holds:

1) $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1 \preceq_{IOS}^{(0,\varepsilon_y/2)} T_2$,
2) $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1 \preceq_{IOS}^{(0,\varepsilon_y/2)} T_1$

if $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = H_1(\mathbf{q}_1)$, then

$$T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1 \preceq_{IOS}^{(0,\varepsilon_y)} T_2$$

and if $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = H_2(\mathbf{q}_2)$, then

$$T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1 \preceq_{IOS}^{(0,\varepsilon_y)} T_1.$$

PROOF: The proof is direct consequence of Proposition 1 and Proposition 11.8 of [10]. For completeness sake we provide the following proof. We prove that $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1 \preceq_{IOS}^{(0,\varepsilon_y/2)} T_2$ for the case when $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = \frac{1}{2}(H_1(\mathbf{q}_1)+H_2(\mathbf{q}_2))$ and other results follow directly. The desired $(0,\varepsilon_y)$ - approximate input output similarity relation from $T_2 \times_{\mathscr{F}}^{\varepsilon_y} T_1$ to T_2 can be written as

$$R_{\varepsilon_y} = \{((\mathbf{q}_1,\mathbf{q}_2),\mathbf{q}_2) \in (Q_{12}\times Q_2)\,|\,\mathbf{d}(H_{12}(\mathbf{q}_1,\mathbf{q}_2),H_2(\mathbf{q}_2)) \leq \varepsilon_y/2\}$$

It can be observed that for any $((\mathbf{q}_1,\mathbf{q}_2),\mathbf{q}_2) \in (Q_{12} \times Q_2)$ we have

$$\begin{aligned}\mathbf{d}(H_{12}(\mathbf{q}_1,\mathbf{q}_2),H_2(\mathbf{q}_2))) &= \left\|\frac{1}{2}(H_1(\mathbf{q}_1)+H_2(\mathbf{q}_2))-H_2(\mathbf{q}_2)\right\| \\ &= \left\|\frac{1}{2}(H_1(\mathbf{q}_1)-H_2(\mathbf{q}_2))\right\| \leq \varepsilon_y/2\end{aligned}$$

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Based on Proposition 3 and Lemma 1 we obtain the following corollary. This result states that once two transition systems are approximately feedback composable, then practical passivity of one of those transition systems implies practical passivity of the composed transition system.

Theorem 2: Let $T_1 := (Q_1, L_1, \xrightarrow[\tau]{}, O_1, H_1)$, $T_2 := (Q_2, L_2, \xrightarrow[\tau]{}, O_2, H_2)$ be two transition systems with a common time period τ and common input and output sets equipped with euclidean norm as the metric. Assume that $H_1 = 1_{Q_1}$ and $H_2 = 1_{Q_2}$ and let T_2 be $(0,\varepsilon_y)$ - approximately input output alternatingly similar to T_1. If T_1 is (ρ_1,ν_1,β_1) - practically IOSP w.r.t. a function $h(\mathbf{q}_1,\ell_1)$ where $\mathbf{q}_1 \in Q_1$ and $\ell_1 \in L_1$. Then $T_2 \times^{\varepsilon_y}_{\mathscr{F}} T_1$ is $(\rho_{12},\nu_{12},\beta_{12})$-practically IOSP w.r.t. $h\left(\frac{1}{2}(\mathbf{q}_1+\mathbf{q}_2),\ell\right)$ where $(\mathbf{q}_1,\mathbf{q}_2) \in \pi_Q(\mathscr{F})$, $\ell \in L_{12} = L_1 = L_2$ and

$$\begin{aligned}\rho_{12} &= \rho_1(1-\alpha_2) \\ \nu_{12} &= \left(\nu_1 - \frac{\alpha_1}{2}\right) \\ \beta_{12} &= \frac{1}{2\alpha_1}M^2(\varepsilon_y/2)^2 + \rho_1\left(\frac{1}{\alpha_2}+1\right)M^2(\varepsilon_y/2)^2 \\ &\quad +\beta_1 + \frac{K\varepsilon_y}{2\tau}.\end{aligned} \tag{24}$$

Also, $T_2 \times^{\varepsilon_y}_{\mathscr{F}} T_1$ is $(\rho_{12},\nu_{12},\beta'_{12})$-practically IOSP w.r.t. $h(\mathbf{q}_2,\ell)$ where $\mathbf{q}_2 \in Q_2$ such that $(\mathbf{q}_1,\mathbf{q}_2) \in \pi_Q(\mathscr{F})$, $\ell \in L_{12}$ and

$$\begin{aligned}\beta'_{12} &= \frac{1}{2\alpha_1}M^2(\varepsilon_y)^2 + \rho_1\left(\frac{1}{\alpha_2}+1\right)M^2(\varepsilon_y)^2 \\ &\quad +\beta_1 + \frac{K\varepsilon_y}{\tau}\end{aligned} \tag{25}$$

where K and M are Lipschitz constants of the storage function V and the passive output function $h(q,\ell)$ i.e., for any $\mathbf{p},\mathbf{q} \in Q_1 \cup Q_2$ and an arbitrary $\ell \in L$ we have $|V(\mathbf{p}) - V(\mathbf{q})| \leq K\|\mathbf{p}-\mathbf{q}\|$ and $\|h(\mathbf{p},\ell)-h(\mathbf{q},\ell)\|_2 \leq M\|\mathbf{p}-\mathbf{q}\|$ and α_1 and $\alpha_2 \in \mathbb{R}^+$ such that $\nu_1 - \frac{\alpha_1}{2} \geq 0$ and $1-\alpha_2 \geq 0$.

PROOF: Output of T_1 is $H_1(\mathbf{q}_1)$ and we consider two possible outputs of $T_{12} = T_2 \times^{\varepsilon_y}_{\mathscr{F}} T_1$. From the definition of approximate feedback composition and Proposition 3, possible relations between T_{12} and T_1 are given by

Case 1: $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = \frac{1}{2}(H_1(\mathbf{q}_1)+H_2(\mathbf{q}_2))$
$= \frac{1}{2}(\mathbf{q}_1+\mathbf{q}_2) \Rightarrow T_{12} \preceq^{(0,\varepsilon_y/2)}_{IOS} T_1$

Case 2 : $H_{12}(\mathbf{q}_1,\mathbf{q}_2) = H_2(\mathbf{q}_2) = \mathbf{q}_2 \Rightarrow T_{12} \preceq^{(0,\varepsilon_y)}_{IOS} T_1$.

Now we use Lemma 1 for both cases mentioned above.

(i) **Case 1:** T_{12} is $(\rho_{12},\nu_{12},\beta_{12})$-practically IOSP w.r.t.
$$h(H_{12}(\mathbf{q}_1,\mathbf{q}_2),\ell) = h\left(\frac{1}{2}(\mathbf{q}_1+\mathbf{q}_2),\ell\right).$$
(ii) **Case 2 :** T_{12} is $(\rho_{12},\nu_{12},\beta'_{12})$ - practically IOSP w.r.t.
$$h(H_{12}(\mathbf{q}_1,\mathbf{q}_2),\ell) = h(\mathbf{q}_2,\ell).$$

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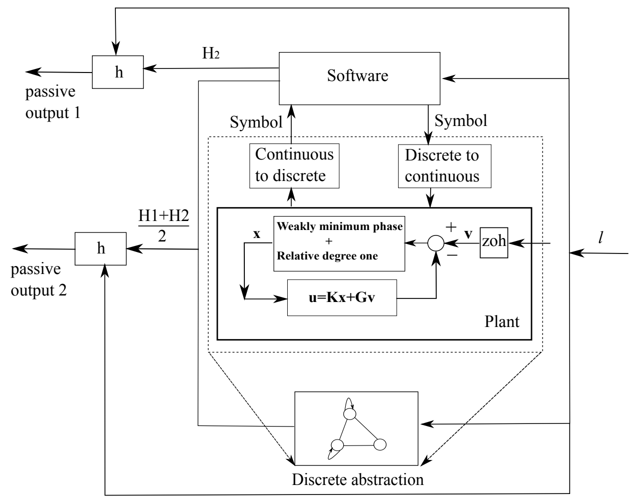

Fig. 3: A schematic of the state feedback interconnection between a controller (software) and a passive continuous-time system (or a feedback passive system with appropriate passivating feedback).

Theorem 2 can be applied to design discrete supervisory controllers for passive plants, while preserving passive nature for the interconnection. Figure 3 shows a schematic for implementing Theorem 2, where a common quantized input is provided for the discrete controller (software) and the continuous-time passive system (or a feedback passive system with appropriate passivating feedback). The continuous-time system should be preceded by a sample and hold element to convert the common quantized input symbol into a piecewise constant input. Under this framework, the interconnected system is practically passive w.r.t. passive outputs

(i) $h(H_2(\mathbf{q}_2),\ell) = h(\mathbf{q}_2,\ell)$, where $\mathbf{q}_2$ is a discrete state of the controller and
(ii) $h\left(\frac{(H_1(\mathbf{q}_1)+H_2(\mathbf{q}_2))}{2},\ell\right) = h\left(\frac{\mathbf{q}_1+\mathbf{q}_2}{2},\ell\right)$ where $\mathbf{q}_1$ is the discrete plant state and $\mathbf{q}_2$ is the discrete controller state.

Once the interconnection is practically passive, we can guarantee practical asymptotic stability of the interconnection, if the conditions of Theorem 1 are satisfied.

C. Practical passivation

Theorem 2 can be used to design a discrete controller for a passive system which guaranteeing practical passivity of the interconnected system. However, one might be interested in designing a discrete controller to practically passivate a continuous-time system. For this purpose we use the general methodology proposed in [10] to design discrete controllers to satisfy a discrete specification provided in the form a transition system T_{spec}. In order to practically passivate an interconnection of a continuous-time plant and the discrete controller we choose the discrete specification to be practically IOSP transition system $T_{passive}$. For this purpose we present another preliminary result from [10] for approximate alternating simulations which is also valid for approximate input and output alternating simulations.

Proposition 4: Let T_1, T_2 and T_3 be be metric systems with the same input and output sets. If we assume that all three transition systems receive the same input, then the following statements hold:

(i) for any $\varepsilon_1 \leq \varepsilon_2$, $T_1 \preceq_{IOS}^{(0,\varepsilon_1)} T_2$ implies $T_1 \preceq_{IOS}^{(0,\varepsilon_2)} T_2$;

(ii) if $T_1 \preceq_{IOS}^{(0,\varepsilon_{12})} T_2$ and $T_2 \preceq_{IOS}^{(0,\varepsilon_{23})} T_3$ then $T_1 \preceq_{IOS}^{(0,\varepsilon_{12}+\varepsilon_{23})} T_3$.

PROOF: The proof is direct consequence of Proposition 1 and Proposition 11.10 of [10]. For completeness sake we provide proofs.

(i) Let R_{ε_1} denote the relation $T_1 \preceq_{IOS}^{(0,\varepsilon_1)} T_2$, then we have $(\mathbf{q}_1,\mathbf{q}_2) \in R_{\varepsilon_1}$ if and only if $\mathbf{d}(H_1(\mathbf{q}_1),H_2(\mathbf{q}_2)) \leq \varepsilon_1$. Which further implies that $\mathbf{d}(H_1(\mathbf{q}_1),H_2(\mathbf{q}_2)) \leq \varepsilon_1 \leq \varepsilon_2$, hence $(\mathbf{q}_1,\mathbf{q}_2) \in R_{\varepsilon_2}$.

(ii) Now consider $(\mathbf{q}_1,\mathbf{q}_2) \in R_{\varepsilon_{12}}$ and $(\mathbf{q}_2,\mathbf{q}_3) \in R_{\varepsilon_{23}}$, then $\mathbf{d}(H_1(\mathbf{q}_1),H_3(\mathbf{q}_3)) \leq \mathbf{d}(H_1(\mathbf{q}_1),H_2(\mathbf{q}_2)) + \mathbf{d}(H_2(\mathbf{q}_2),H_3(\mathbf{q}_3)) \leq \varepsilon_{12} + \varepsilon_{23}$, hence $(\mathbf{q}_1,\mathbf{q}_3) \in R_{\varepsilon_{12}+\varepsilon_{23}}$.

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Now we present a result that can be used to design a discrete controller to practically passivate an interconnection of a continuous-time plant and the discrete controller. For this purpose we consider controller specifications for the interconnection in the form of a (ρ,ν,δ)-practically IOSP transition system $T_{passive}$.

Proposition 5: Let Σ be a δ - FC control system and let $T_{\tau,\mu,\eta}(\Sigma)$ be $(0,\varepsilon_y)$ - approximately input output similar to $T_{\tau,\mu}(\Sigma)$. If there exists a controller T_{cont} satisfying

$$T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma) \preceq_{IOS}^{(0,0)} T_{passive}$$

then the controller $T'_{cont} = T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma)$ satisfies

$$T'_{cont} \times_{\mathscr{F}}^{\varepsilon} T_{\tau,\mu}(\Sigma) \preceq_{IOS}^{(0,\varepsilon)} T_{passive}.$$

PROOF: Let us consider the controller $T'_{cont} = T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma)$. From Theorem 3 we have

$$T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma) \preceq_{IOS}^{(0,0)} T_{\tau,\mu,\eta}(\Sigma).$$

However, it is given that $T_{\tau,\eta,\mu}(\Sigma) \preceq_{IOS}^{(0,\varepsilon_y)} T_{\tau,\mu}(\Sigma)$. Hence from Lemma 4 we have

$$T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma) \preceq_{IOS}^{(0,0+\varepsilon)} T_{\tau,\mu}(\Sigma). \tag{26}$$

Now we consider the ε - approximate feedback composition of the controller T'_{cont} and the plant with quantized inputs given by $T_{\tau,\mu}(\Sigma)$. This feedback composition is possible because of the $(0,\varepsilon)$ - approximate input output similarity from T'_{cont} to $T_{\tau,\mu}(\Sigma)$ (from (26)). Then using Theorem 3 again we have

$$T'_{cont} \times_{\mathscr{F}}^{\varepsilon} T_{\tau,\mu}(\Sigma) \preceq_{IOS}^{\varepsilon} T'_{cont} = T_{cont} \times_{\mathscr{F}}^{0} T_{\tau,\mu,\eta}(\Sigma) \preceq_{IOS}^{(0,0)} T_{spec}.$$

Hence $T'_{cont} \times_{\mathscr{F}}^{\varepsilon} T_{\tau,\mu}(\Sigma) \preceq_{IOS}^{(0,\varepsilon)} T_{spec}$.

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This results implies that we can design a controller to satisfy a specification $T_{passive}$ with an output error bounded by ε. Designing T_{cont} is beyond the scope of this paper. Interested readers may refer to [10]. However, under the assumption that T_{cont} is available, we can always practically passivate an interconnection of $T_{\tau,\mu}(\Sigma)$ and a discrete controller using the same framework as shown in figure 3. Practical passivation of $T_{\tau,\mu}(\Sigma)$ instead of $T_\tau(\Sigma)$ implies that a common quantized input is available for both the continuous-time plant and the discrete controller.

IV. Conclusion

In this paper we consider approximate feedback composition of discrete controllers and finite state abstractions of continuous-time systems. We show that if one of the components in the interconnection is practically passive then their approximate feedback composition is also practically passive. We also provide preliminary guidelines to practically passivate a continuous-time system with quantized inputs. Future work will focus on the exact methodologies to design discrete supervisory controllers to passivate continuous-time systems.

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