Safe Implementations of Supervisory Commands

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Abstract

This paper compares two different types of control strategies used to safely implement supervisory commands of hybrid dynamical systems. Both approaches considered in this paper switch between members of a family of control agents to ensure that constraints on the plant state are not violated at any time. The first approach is motivated by a hybrid system architecture outlined in [6] and uses a Fliess functional series of the plant's output to form a system of linear inequalities characterizing safe control inputs. Control signals are determined by solving a sequence of linear programs. The second approach is a model reference control approach to hybrid systems introduced in [7] and uses a known safe dynamical reference model to characterize the desired plant behaviour. The controller is determined by representing the resulting error dynamics as a linear parameter varying system and applying linear robust control techniques to enforce a bounded amplitude performance level. The fundamental results underlying each of the methods are derived; both approaches are compared with regard to their complexity, performance, and sensitivity to modeling uncertainty. A numerical example is included for illustration.

1 Introduction

This paper considers the high level supervision of continuous time dynamical control systems evolving over a state set which is dense in $\Re^n$. It is assumed that a supervisory command is characterized by a set of guard conditions and a goal condition. These guard and goal conditions are inequality conditions on the plant's state. A control system is used to implement the supervisory command. This controller is said to be "safe" when the controlled plant's state trajectory triggers the goal condition in finite time without triggering any of the guard conditions. This paper compares two different types of controllers used to safely implement supervisory commands.

Both approaches considered in this paper switch between members of a family of control agents to ensure the guard conditions are not triggered. The first approach is motivated by a hybrid system architecture outlined in [6]. This approach uses a Fliess functional series of the plant's output to form a system of linear inequalities characterizing safe control inputs. In this method, control signals

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can be determined by solving a sequence of linear programs (LP). The second approach is a model reference control approach to hybrid systems introduced in [7]. In this approach, the controlled plant follows a reference model which is known to be safe. The error dynamics of this system are represented as a linear parameter varying (LPV) system whose controllers enforce a bounded amplitude performance level. This paper formally derives the fundamental results behind both of these methods and compares both approaches with regard to their complexity, performance, and sensitivity to modeling uncertainty.

A formal definition of "safe" controllers is given in section 2. The remainder of the paper discusses the two methods for characterizing safe controllers which were outlined above. The first method will be referred to as the LP-method since it solves a sequence of linear programs to determine safe control signals. The LP-method is discussed in section 3. The fundamental result in section 3 is a set of inequality constraints characterizing locally safe piecewise constant control signals. The second method is referred to as the MRC-method since it uses a model reference control (MRC) approach to formulate the controller synthesis problem. The MRC method is discussed in section 4. The fundamental results in this section are sufficient conditions characterizing controllers ensuring bounded-amplitude performance for the switched control system. Section 5 compares both methods and draws some general conclusions about their relative strengths and weaknesses.

2 Safe Supervisory Controllers

Hybrid dynamical systems arise when the time and/or the state space have mixed continuous and discrete natures. Such systems frequently arise when computers are used to control continuous state systems. In recent years, specific attention has been focused on hybrid systems in which a discrete-event system is used to supervise the behaviour of plants whose state spaces are dense in $\Re^n$. In this class of hybrid control systems, commands are issued by a discrete-event system to direct the behaviour of the plant. These commands are high-level directives to the plant which require that the supervised plant satisfy logical conditions on the plant's state. The simplest set of conditions are inequality conditions on the plant's state.

Assume that the plant's dynamics are generated by the differential equation

$$\dot{x} = f(x,u) \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control input, and $f : \Re^n \times \Re^m \to \Re^n$ is a Lipschitz continuous mapping. A *supervisory directive* to this system is characterized by a set of functionals, $h_j : \Re^n \to \Re$ for $j = 0, \ldots, N$, that separate the state space. The functionals, h_j, are said to *separate* the state space if and only if for all $x, y \in \Re^n$ such that $h_j(x) > 0$ and $h_j(y) < 0$, there exists $0 < \lambda < 1$ such that $h_j(\lambda x + (1-\lambda)y) = 0$. The functional, h_0, is said to be the *goal trigger* and the other functionals, h_j for $j = 1, \ldots, N$, are called the *guard triggers*. Consider a state feedback controller,

$$u = k(x) \tag{2}$$

Such a controller is said to be *safe* if and only if there exist finite times T_1 and T_2 ($T_1 < T_2$) such that

- $h_j(x(t)) < 0$ for all $t_0 \leq t < T_2$ ($j = 1, \ldots, N$),
- $h_0(x(t)) < 0$ for all $t_0 \leq t < T_1$,

- and $h_0(x(t)) > 0$ for all $T_1 < t < T_2$.

Essentially, these conditions state that the goal condition is triggered in finite time without any of the guard triggers being violated. Assume that we have a monotone increasing function $r(t)$ such that $r(0) = h_0(x(0))$ and $r(T_1) = 0$. We can use this "reference" function to rewrite the preceding list of conditions as a set of inequality constraints such that the guard triggers ($j = 1, \ldots, N$) satisfy $h_j(x(t)) < 0$ and the goal trigger satisfies $h_0(x(t)) - r(t) > 0$ for all $t \in [0, T_2]$.

3 LP-Method

The LP-method is motivated by a hybrid system architecture outlined in [6]. This method characterizes safe control signals as a set of linear inequality constraints. The LP-method assumes that the plant's differential equation has the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i(t) \tag{3}$$

where $f_i : \Re^n \to \Re^n$ are analytic functions forming a nonsingular distribution of vector fields in $\Re^n$. It is also assumed that the set of trigger functions $\{h_j\}_{j=1}^N$ is analytic.

Assume that the trigger functions, $h_j(x(t))$, are known at time, t. Under appropriate conditions, it is possible to represent the trigger functions at time $t+\delta$ as a Fliess functional series. To formally state these results, some notational conventions need to be introduced. Let $f : \Re^n \to \Re^n$ be a vector of analytic functions, $f' = \begin{bmatrix} f_1 & f_2 & \cdots f_n \end{bmatrix}$ where $f_i : \Re^n \to \Re^n$ ($i = 1, \ldots, n$). The Lie derivative of an analytic function $h : \Re^n \to \Re$ with respect to vector field f is

$$L_f h(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x) \tag{4}$$

Let $i \in \{1, \ldots, m\}$ be an *index* and let $i_1, \ldots, i_k$ be a sequence of indices of length k called a *multi-index*. The set of all multi-indices will be denoted as I^*. Associated with the multi-index $i_1, \ldots, i_k$ is the iterated integral,

$$E_{i_k, \ldots, i_1}(t) = \int_0^t d\xi_{i_k} \ldots \xi_{i_1} \tag{5}$$

where for $i = 1, \ldots, m$,

$$\begin{aligned} \xi_i(t) &= \int_0^t u_i(\tau) d\tau & (6) \\ \int_0^t d\xi_{i_k} \ldots d\xi_{i_1} &= \int_0^t d\xi_{i_k}(\tau) \int_0^\tau d\xi_{i_{k-1}} \ldots d\xi_{i_1} & (7) \end{aligned}$$

The following theorem which is proven in [5] will be used in our following development.

Proposition 1 *[5] Consider the system given by equations 3. If there exist $K > 0$ and $M > 0$ such that*

$$\left| L_{f_{i_1}} \ldots L_{f_{i_k}} h_j(x(t)) \right| \leq K k! M^k \tag{8}$$

for all k, j, and all multi-indices in I^*, then there exists a real $\Delta > 0$ such that for all $\delta \in [0, \Delta]$ and piecewise continuous control functions $u_i(t)$ defined over $[t, t+\Delta]$ subject to the constraint

$$\max_{\delta \in [0,\Delta]} |u_i(t+\delta)| < 1 \tag{9}$$

then the series

$$h_j(x(t)) + \sum_{k=1}^{\infty} \sum_{I^*} L_{f_{i_1}} \ldots L_{f_{i_k}} h_j(x(t)) \int_0^{\delta} d\xi_{i_k} \ldots d\xi_{i_1} \tag{10}$$

is uniformly and absolutely convergent to $h_j(x(t+\delta))$.

If we can find a control signal, u, so that the safety conditions are satisfied over $[t, t+\delta]$, for all t then we say the control is locally safe. The Fliess series is a formal series over the control symbols, u_i. It provides a means of expressing the values of the trigger functions, h_j, over a finite interval, $[t, t+\delta]$. It therefore makes sense to use the Fliess series in characterizing control inputs, u_i, ensuring local safety of the control system. The following proposition provides just such a characterization.

Proposition 2 *Consider the system given by equation 3 and let $r(t)$ be a known reference trigger such that $\dot{r}(t) = R > 0$ and $r(0) = h_0(x(0))$. Assume that proposition 1 holds and that $x(0)$ is safe. If there exist $\gamma > 0$, $\gamma_1 > 0$, and $\Delta > 0$ such that the constant vector $u^* \in \Re^m$ satisfies*

$$-\gamma < h_j(x(0)) + \sum_{i=1}^{m} [L_{f_i} h_j(x(0))] \, u_i^* \Delta \quad (j = 1, \ldots, N) \tag{11}$$

and

$$|u_i^*| \leq 1 \quad i = 1, \ldots, m \tag{12}$$

and

$$R - \gamma_1 < \sum_{i=1}^{m} [L_{f_i} h_0(x(0))] \, u_i^* \tag{13}$$

then the constant control $u(t) = u^*$ generates a safe state trajectory in $[0, \Delta)$.

Proof: Assuming that proposition 1 holds,then there exists $K > 0$ and $M > 0$ such that the growth constraint 8 is satisfied. Given inequality 12, we know that the Fliess series is uniformly convergent in an interval, $[0, \Delta]$, and that for any $\delta \in [0, \Delta]$, we can expand $h_j(x(\delta))$ as

$$h_j(x(\delta)) = h_j(x(0)) + \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k} L_{f_{i_1}} \cdots L_{f_{i_k}} h_j(t) E_{i_1, \ldots, i_k}(\delta) \tag{14}$$

Assuming a constant u^* over this interval, we see that

$$\begin{aligned} h_j(x(\delta)) &= h_j(x(0)) + \sum_{i=1}^{m} L_{f_{i_1}} \cdots L_{f_{i_k}} h_j(x(0)) u_{i_1}^* \cdots u_{i_k}^* \frac{\delta^k}{k!} \\ &= h_j(x(0)) + \sum_{i=1}^{m} L_{f_i} h_j(x(0)) u_i^* \delta + o_j(\delta) \end{aligned} \tag{15}$$

The tail term is

$$o_j(\delta) = \sum_{k=2}^{\infty} \sum_{i_1,\ldots,i_k} L_{f_{i_1}} \cdots L_{f_{i_k}} h_j(x(0)) u_{i_1}^* \cdots u_{i_k}^* \frac{\delta^k}{k!} \tag{16}$$

The magnitude of the tail is bounded as

$$|o_j(\delta)| \leq K(Mm\delta)^2 \left(\frac{1}{1 - Mm\delta} \right) \tag{17}$$

for $\delta < 1/Mm$.

We now take $\Delta = \rho/Mm$ where $\rho < 1$, then

$$|o_j(\delta)| \leq K \frac{\rho^2}{1-\rho} = \gamma \tag{18}$$

We take the righthand side of this inequality to be the γ of our theorem and immediately conclude that inequality 15 can be written as

$$h_j(\delta) \leq h_j(x(0)) + \sum_{i=1}^{m} L_{f_i} h_j(x(0)) u_i^* \Delta + \gamma \tag{19}$$

For $j = 1, \ldots, N$, this implies that the state is safe at time Δ. It is also safe at time zero. Since our bound is linear this must also hold for all δ between 0 and Δ. So for all time in $[0, \Delta)$, the desired inequality constraints ensure the guard triggers are not violated.

We now turn to the terminating trigger, $h_0(x(t))$. In this case, we require that $h_0(x(\delta)) > r(\delta)$ for all $\delta \in [0, \Delta]$. By assumption, $h_0(x(0)) \geq r(0)$ and we know by that $r(\delta) = r(0) + R\delta$. To ensure our other constraint is satisfied, we require

$$r(0) + R\delta < h_0(x(0)) + \sum_{i=1}^{m} L_{f_i} h_0(x(0)) u_i^* \delta + K(Mm\delta)^2 \left(\frac{1}{1 - Mm\delta} \right) \tag{20}$$

Assuming that $r(0) = h_0(x(0))$, we see that the condition reduces to

$$R < \sum_{i=1}^{m} L_{f_i} h_0(x(0)) u_i^* + KMm \frac{\rho}{1-\rho} \tag{21}$$

We treat this last quantity as γ_1, and our result follows. **QED**

Proposition 2 characterizes the *set* of locally safe control signals. In practice, a specific control signal will need to be chosen from this set. This selection is made with respect to an assumed cost functional, $J(u)$. The "optimal" locally safe control is determined by finding the control signal that minimizes this given cost subject to the local safety conditions represented by the inequality constraints in proposition 2. A particularly simple choice for the cost is a linear function of u. If we restrict $0 < u_i < 1$ for all $i = 1, \ldots, m$, then our cost functional becomes

$$J(u) = w'u = \sum_{i=1}^{m} w_i u_i \tag{22}$$

where w is an m-vector of positive weights. The control signal minimizing this cost is obtained by solving the following linear programming problem

$$\begin{array}{ll} \text{minimize:} & w'u \\ \text{with respect to:} & u \\ \text{subject to:} & A(t)u < b \\ & 0 < u_i < 1 \end{array} \tag{23}$$

where

$$A(t) = \begin{bmatrix} L_{f_1}h_0 & L_{f_2}h_0 & \cdots & L_{f_m}h_0 \\ L_{f_1}h_1 & L_{f_2}h_1 & \cdots & L_{f_m}h_1 \\ \cdot & \cdot & \cdots & \cdot \\ L_{f_1}h_p & L_{f_2}h_p & \cdots & L_{f_m}h_p \end{bmatrix} \tag{24}$$

and

$$b = \begin{bmatrix} R - \gamma_1 \\ -\gamma \\ \cdot \\ -\gamma \end{bmatrix} \tag{25}$$

Note that the constraint matrix $A(t)$ is a function of time.

The preceding discussion solved an LP problem to find a constant control, u^*, for a time $t \in [0, T)$ which was locally safe. A safe control trajectory, $u^*(t)$, for all $t \in [0, T)$, can be determined by solving a sequence of linear programs at the time instants $t_0 + n\Delta$, where n is the set of positive integers and Δ is given by the growth constants in proposition 2. The constraint matrices $A(t)$ are obtained from our knowledge of the distribution, $\{f_0, f_1, \ldots, f_n\}$ as well as the current state vector. This essentially means that an LP problem must be solved at the sampling instant $t_0 + n\Delta$ to determine the piecewise constant control u^* that is used over the interval $[t_0 + n\Delta, t_0 + (n+1)\Delta]$. Provided these constraint matrices do not change too quickly, we can expect such an on-line approach to control-signal generation to ensure the safety of the commanded transition.

Example: A simple example of this approach is used to illustrate the approach. The following example has been modified from [4] to yield a plant which is affine in the control. The modified plant equations are

$$\dot{x}_1 = -x_1 + u_1 \tag{26}$$

$$\dot{x}_2 = -x_2 + (1 + x_1^2)u_2 \tag{27}$$

We can rewrite this as a linear combination of vector fields,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 1 + x_1^2 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ -(1 + x_2)^2 \end{bmatrix} u_4 \tag{28}$$

The control objective is to move the plant from an initial state near the operating point $(0, 0)$ to a point near $(2.5, 2)$.

The guard triggers are

$$h_1(x) = x_2 - 1.25x_1 + .5 \tag{29}$$
$$h_2(x) = -x_2 + 1.25x_1 + .5 \tag{30}$$

with a goal trigger,

$$h_0(x,t) = x_1(0) - Rt \tag{31}$$

where R is the desired rate at which we want to achieve the desired goal set. These regions are illustrated in figure 1.

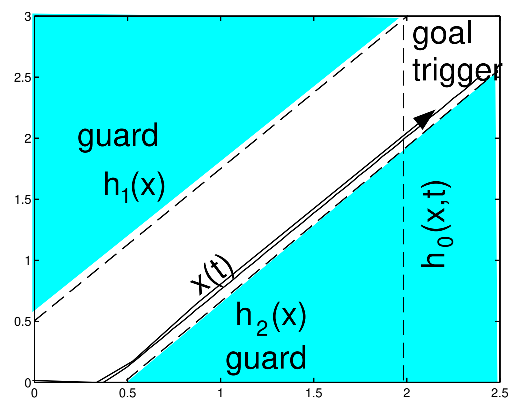

Figure 1: Guard and goal triggers for example.

A simple Matlab script was written to simulate this system. Figure 1 illustrates the state trajectory that was generated by this approach. In this case, the LP-problems determining safe controls were computed at a rate $\Delta = .1$. The weighting vector w was chosen to be a vector of ones. As can be seen, the selected controls basically select one control strategy that drives the system in the direction of the h_2 guard trigger. Once within a distance γ of that guard trigger, the control strategy changes to a chattering policy which drives the system state along the boundary until the terminal condition is satisfied. The chattering nature of the control policy is seen in figure 2.

This example illustrates some fundamental characteristics of the LP-approach to safe controller generation. In the first place, this is an on-line procedure which requires the solution of an LP problem at each sampling instant. The computation of the control requires significant information about the underlying vector fields generating the system's dynamics. Finally, this approach tends to produce a chattering control strategy, as shown in figure 2.

4 MRC-Method

A model reference control (MRC) approach for implementing safe controllers was introduced in [7]. In this approach, the plant is forced to follow a reference trajectory, $x_m(t)$, which is known to be safe with a worst case tracking error of γ. Provided there exists a time T such that $h_0(x_m(T)) > \gamma$ and

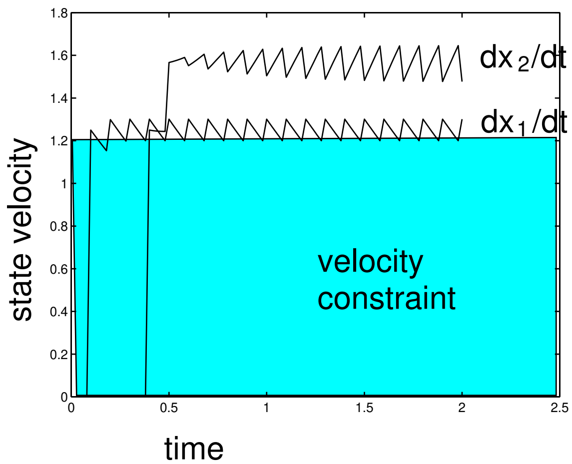

Figure 2: Chattering control policy

for all $0 < t < T$ and $j = 1, \ldots, N$ that $h_j(x_m(t)) > -\gamma$, the plant trajectory, $x_p(t)$,is guaranteed to be safe.

In this framework, synthesis of safe switched controllers is accomplished by examining the error between the plant and reference trajectories. Suppose that the plant state dynamics are generated by

$$\dot{x}_p = f_p(x_p, u) \tag{32}$$

and let the reference trajectory be generated by

$$\dot{x}_m = f_m(x_m) \tag{33}$$

Defining the state error signal, $x = x_m - x_p$, yields the differential equation

$$\dot{x} = f(x_m, x, u) = f_m(x_m) - f_p(x_p, u) \tag{34}$$

The control input is generated by a controller $u = k(x_m, x)$ which is dependent on the reference model state and the reference error.

One control strategy is to choose a collection of setpoints along the reference trajectory, $x_m(t)$, and design linear control agents at each of the setpoints using the plant model obtained from linearizing about the corresponding setpoint. This is the basic idea behind the switched linear control agent approach introduced in [7]. Note that each of the control agents designed using this approach are designed for local performance near each of the setpoints; performance of the switched system will be difficult to guarantee, in general. Thus, the linear control agents should demonstrate robustness to the nonlinearities in the system lost in the linearization. One way of incorporating this robustness requirement into the design is to use linear parameter varying (LPV) plant models at each of the setpoints.

An LPV model of the error dynamics may be obtained by rewriting the dynamics of equation 34 as

$$\begin{aligned} \dot{x} &= A(\theta)x + B_u(\theta)u + B_w(\theta)w && (35) \\ z &= Cx + Du && (36) \end{aligned}$$

where $w = 1$ is introduced as a fictitious disturbance. The s-dimensional parameter vector, θ, is a function $S(x_m, x, u)$. The vector θ is assumed to vary continuously over a compact subset $\tilde{\Theta} \subset \Re^s$; this assumption is denoted $\theta \in \mathcal{F}_{\tilde{\Theta}}$. For each of the local plant models, θ is assumed to vary continuously over a compact subset $\Theta \subseteq \tilde{\Theta}$ for a time interval $[\tau_s, \tau_f]$; this assumption is denoted $\theta \in \mathcal{F}_\Theta[\tau_s, \tau_f]$. This notation distinguishes a *parameter variation* over Θ from a point in Θ which will be denoted $\theta \in \Theta$. The vector z will be called the *objective* signal and is chosen (via C and D) to reflect not only the the trigger constraints, but also control energy constraints. The entire LPV system will be denoted as $\Sigma(\tilde{\Theta}, A, B, C, D)$ where $B' = [B'_u \; B'_w]$.

Let $\mathcal{T}_i = [t_i, t_{i+1})$ denote the time interval over which the ith setpoint controller issued. Note that if each individual setpoint controller satisies the performance requirement,

$$\sup_{t \in \mathcal{T}_i} \|z(t)\| < \gamma \tag{37}$$

then local safety of the control directive will be preserved. Local setpoint controllers are therefore obtained by solving what is called a finite horizon $\mathcal{L}_1$, or *bounded-amplitude*, optimal control problem for LPV systems.

There are, unfortunately, relatively few results for the solution of $\mathcal{L}_1$ optimal control problems. In [3], it was shown that optimal solutions to this problem are irrational or infinite dimensional, even for rational and finite-dimensional plants. For deterministic linear time-invariant systems [8] an approach to $\mathcal{L}_1$ optimal control synthesized a sub-optimal controller minimizing an upper bound on the bounded-amplitude gain by solving a set of linear matrix inequalities. To use this prior work in our synthesis problems, however, existing synthesis methods must be extended to LPV systems. The following theorem provides a characterization of systems whose $\mathcal{L}_1$ gains are bounded.

Note: The remainder of the paper will use the following notation: the infinite-horizon ∞-norm of a signal $x(t)$ is defined as $\|x(t)\|_\infty := \sup_t \|x(t)\|$ where $\|\cdot\|$ is the Euclidean norm. $\mathcal{L}^n_\infty$ is the space of n-dimensional vector signals with finite ∞-norm; $\mathcal{BL}^n_\infty$ is the space of n-dimensional vector signals with ∞-norm bounded by 1. For a constants $\tau < T$, finite-horizon ∞-norm of a signal $x(t)$ defined on the interval $[\tau, T]$ is $\|x(t)\|_{\infty,[\tau,T]} := \sup_{t\in[\tau,T]} \|x(t)\|$. $\mathcal{L}^n_\infty[\tau, T]$ and $\mathcal{BL}^n_\infty[\tau, T]$ are defined in an analogous manner. Recall that $\theta \in \mathcal{F}_\Theta[\tau, T]$ is an s-dimensional signal $\theta(t)$ which takes values on a compact subset $\Theta \subset \Re^s$ for $t \in [\tau, T]$. This implies that $\theta \in \mathcal{L}^s_\infty[\tau, T]$

Proposition 3

Given constants $r > 0$, $\gamma > 0$ and $T > 0$ and the LPV system $\Sigma(\tilde{\Theta}, A, B, C, D)$ with $u = 0$. Let Θ be a compact subset of $\tilde{\Theta}$ and suppose there exists $\alpha > 0$ and $\beta \geq 0$ and a positive definite matrix $P \in \Re^{n\times n}$ satisfying

$$P \geq \frac{r}{\gamma^2} C'C \tag{38}$$

and

$$A'(\theta)P + PA(\theta) + \left(2\beta + \frac{\alpha}{r}\right) P + \frac{1}{\alpha} PB_w(\theta)B_w(\theta)'P \leq 0 \tag{39}$$

for all $\theta \in \Theta$. If $\theta \in \mathcal{F}_\Theta[0, T]$ and $w \in \mathcal{BL}^{n_w}_\infty[0, T]$, then

- *if* $x'(0)Px(0) \leq r$ *then* $x'(t)Px(t) \leq r$ *and* $z'(t)z(t) \leq \gamma^2$ *for all* $t \in [0,T]$

- *if* $\beta > 0$ *and* $x'(0)Px(0) = r_0 > r$, *then* $x'(t)Px(t) \leq r$ *for all* $t \in [t_d, T]$ *where*

$$t_d := -\frac{1}{2\beta}\log\left(\frac{r}{r_0}\right) \tag{40}$$

(assuming $t_d \leq T$*).*

Proof: Let $r > 0$, $\gamma > 0$ and $T > 0$ and assume there are constants $\alpha > 0$ and $\beta \geq 0$ and a positive definite matrix P so that the conditions of the theorem are satisfied. For any $\theta \in \Theta$,

$$\frac{1}{\alpha}PB(\theta)B'(\theta)P \geq 0 \tag{41}$$

If equation 39 holds for all $\theta \in \Theta$, then

$$A'(\theta)P + PA(\theta) + \left(\frac{\alpha}{r} + 2\beta\right)P \leq -\frac{1}{\alpha}PB(\theta)B'(\theta)P \leq 0 \tag{42}$$

Using Schur complements, this inequality is true if and only if

$$\begin{bmatrix} A'(\theta)P + P(\theta) + \rho P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \leq 0 \tag{43}$$

where $\rho = 2\beta + \alpha/r$. This inequality is true if and only if

$$\begin{bmatrix} \xi \\ \nu \end{bmatrix}' \begin{bmatrix} A'(\theta)P + PA(\theta) + \rho P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \begin{bmatrix} \xi \\ \nu \end{bmatrix} \leq 0 \tag{44}$$

for all $\xi \in \Re^n$ and $\nu \in \Re^{n_w}$. Expanding, it is apparent that

$$\xi'\left[A'(\theta)P + PA(\theta) + 2\beta P\right]\xi + \nu'B'(\theta)P\xi + \xi'PB(\theta)\nu - \frac{\alpha}{r}\left[\xi'P\xi - r\right] + \alpha\left[1 - \nu'\nu\right] \leq 0 \tag{45}$$

This last equation implies that

$$\xi'\left[A'(\theta)P + PA(\theta)\right]\xi + \nu'B'(\theta)P\xi + \xi'PB(\theta)\nu \leq -2\beta\xi'P\xi \leq 0 \tag{46}$$

for all ξ and ν such that $\xi'P\xi \geq r$ and $\nu'\nu \leq 1$.

Now consider a function, $V : \Re^n \rightarrow \Re$, such that $V(\xi) = \xi'P\xi$. Along trajectories of the LPV system with $u = 0$, the time derivative of $V(x(t))$ is

$$\frac{dV}{dt}(x(t)) = x'(t)\left[A'(\theta(t))P + PA(\theta(t))\right]x(t) + w'(t)B'(\theta(t))Px(t) + x'(t)PB(\theta(t))w(t) \tag{47}$$

and from equation 46, it is immediately evident that

$$\frac{dV}{dt}(x(t)) \leq -2\beta V(x(t)) \leq 0 \tag{48}$$

for any $x(t)$ and $w(t)$ such that $x'(t)Px(t) \geq r$ and $w'(t)w(t) \leq 1$ with $t \in [0,T]$.

Assume, for some $w \in \mathcal{BL}_{\infty}^{n_w}[0,T]$, that there is a trajectory with initial state $x(0)$ satisfying $V(x(0)) = x'(0)Px(0) \leq r$ and $V(x(T)) > r$. Since $V(x(t))$ is differentiable in t, the mean value theorem may be used to imply the existence of a time $\tau \in [0,T]$ such that $V(x(\tau)) \geq r$ and $\dot{V}(x(\tau)) > 0$. This is a contradiction of equation 48, so one must conclude that $x'(t)Px(t) \leq r$, hence $z'(t)z(t) \leq \gamma^2$, for all $t \in [0,T]$.

If $V(x(0)) > r$, then the differential inequality implies that

$$V(x(t) \leq V(x(0)) - \int_0^t 2\beta V(x(\tau))d\tau \tag{49}$$

and the Bellman-Gronwall inequality may be used to conclude that

$$V(x(t)) \leq V(x(0))e^{-2\beta t} \tag{50}$$

Now suppose that $V(x(0)) = r_0 > r$, $\beta > 0$ and let t_d be the *dwell time* given in equation 40. If $t_d \leq T$, then

$$V(x(t)) \leq r_0 e^{-2\beta t_d} = r \tag{51}$$

for all $t \in [t_d, T]$. •

Proposition 3 characterizes a class of uncontrolled ($u = 0$) LPV systems where $\|z\|_{\infty,[0,T]} \leq \gamma$ and where the parameter variation is confined to the set Θ. The next result helps characterize a class of controlled LPV systems using linear state feedback, $u = Kx$.

Proposition 4

Given $\gamma > 0$ and an LPV system $\Sigma(\tilde{\Theta}, A, B, C, D)$ with state space realization

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \left[\begin{array}{c|cc} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C & 0 & D \end{array}\right] \begin{bmatrix} x \\ u \\ w \end{bmatrix} \tag{52}$$

Let $\Theta \subseteq \tilde{\Theta}$ be a compact subset and consider a state feedback control law $u = Kx$ where $K \in \Re^{n_u \times n}$. Define $\tilde{A}(\theta) = A(\theta) + B_u(\theta)K$. Then, there exist constants $\alpha_1 \geq \alpha_2 > 0$, a positive definite matrix $P \in \Re^{n \times n}$ and a controller K satisfying

$$P \geq \frac{1}{\gamma^2}(C + DK)'(C + DK) \tag{53}$$

and

$$\tilde{A}(\theta)P + P\tilde{A}(\theta) + \alpha_1 P + \frac{1}{\alpha_2} P B_w(\theta) B_w(\theta)' P \leq 0 \tag{54}$$

for all $\theta \in \Theta$ if and only if there exists a positive definite matrix $Q \in \Re^{n \times n}$ and a matrix $V \in \Re^{n_u \times n}$ such that for all $\theta \in \Theta$

$$\begin{bmatrix} Q & QC' + V'D' \\ CQ + DV & \gamma^2 I \end{bmatrix} \geq 0 \tag{55}$$

and

$$QA'(\theta) + A(\theta)Q + \alpha_1 Q + \frac{1}{\alpha_2} B_w(\theta) B_1'(\theta) + B_u(\theta) V + V' B_u'(\theta) \leq 0 \tag{56}$$

Proof: Assume that there exists a positive definite matrix Q and a matrix V such that

$$\begin{bmatrix} Q & QC' + V'D' \\ CQ + DV & \gamma^2 I \end{bmatrix} \geq 0 \tag{57}$$

Using Schur complements, this holds if and only if

$$Q - \frac{1}{\gamma^2}(QC' + V'D')(CQ + DV) \geq 0 \tag{58}$$

If we let $P = Q^{-1}$ and $K = VQ^{-1}$, then this holds if and only if

$$P \geq \frac{1}{\gamma^2}(C + DK)'(C + DK) \tag{59}$$

which establishes the first condition in the proposition.

Now assume that there also exist constants $\alpha_1 \geq \alpha_2 > 0$ such that

$$\tilde{A}(\theta)P + P\tilde{A}(\theta) + \alpha_1 P + \frac{1}{\alpha_2} P B_w(\theta) B_w(\theta)' P \leq 0$$

for all $\theta \in \Theta$. Substituting $P = Q^{-1}$ and $K = VQ^{-1}$ as above,

$$[A(\theta) + B_u(\theta)K]' P + P[A(\theta) + B_u(\theta)K] + \alpha_1 P + \frac{1}{\alpha_2} P B_w(\theta) B_w'(\theta) P \tag{60}$$

$$= \quad Q^{-1}\left[QA'(\theta) + A(\theta)Q + \alpha_1 Q + \frac{1}{\alpha_2} B_u(\theta) B_u'(\theta) + B_w(\theta)V + V' B_w'(\theta)\right] Q^{-1} \tag{61}$$

Since $Q^{-1} > 0$, the conclusion of the theorem immediately follows. •

The following remarks summarize the importance of propositions 3 and 4.

- Under the assumptions of proposition 3 and 4, it should be apparent that if the inequalities 55 and 56 hold, then under control $u = Kx$, the objective function $z = (C + DK)x$ will have a finite horizon sup-norm less than γ provided that the parameter variation is bounded according to $\theta \in \mathcal{F}_\Theta[0, T]$.
- From the proof of proposition 4 it should be apparent that the matrices Q and V satisfying inequalities 55 and 56 parameterize a set of locally safe controllers. In particular, for any such Q and V, the controller is $K = VQ^{-1}$.

- The importance of inequalities 55 and 56 is that these can be used to form matrix inequalities which are *linear* in Q and V. These inequalities need only be satisfied *pointwise* over Θ without regard to parameter variation rate, as long as the parameter variation is bounded according to $\theta \in \mathcal{F}_\Theta[0,T]$.

- Note that the θ dependence of inequality 56 limits its usefulness: verifying the condition for all $\theta \in \Theta$ may be unreasonable or infeasible. In certain cases, however, the computational burden can be signficantly reduced. For instance, if $A(\theta)$, $B_u(\theta)$ and $B_w(\theta)$ can be written as linear fractional transformations in θ, and if the parameter set Θ is a polytope, then it is possible to express inequality 56 as a matrix inequality which is independent of θ and linear in the variables Q and V. Derivation of such LMIs is a straightforward application of the results in [2]; a detailed proof is beyond the scope of this paper but can be found in [1].

The results in proposition 3 are extremely important in determining whether or not a given set of linear setpoint controllers will safely execute a supervisory directive. Let $\mathcal{T}_i$ be the time interval when the ith setpoint controller is used. This controller is characterized by the matrices P_i, the radius r_i, and constants, α_i and β_i. The results in this proposition state that the controlled system will be locally safe if the error satisfies $x'(t_i)P_i x(t_i) \leq r_i^2$. To ensure that the plant behaviour is safe under the next $(i+1\text{st})$ setpoint controller, one must ensure that $x'(t_{i+1})P_{i+1}x(t_{i+1}) \leq r_{i+1}^2$. The problem here is that the second condition is not guaranteed if the switch occurs too quickly. This is where the second part of proposition 3 has something to add. Specifically, if the state at time t_i starts outside of the invariant set for the $i+1$st setpoint controller, then there is a minimum time called the dwell time, after which the state is guaranteed of being within the required distance. In particular, let $r_i P_{i+1} \leq r_{i+1} P_i$ and assume that P_i and P_{i+1} both satisfy the conditions for setpoint controllers in proposition 3. It is readily apparent that if $t_{i+1} - t_i \geq t_d$, where

$$t_d = -\frac{1}{2\beta_i} \log \frac{r_{i+1}}{r_i}, \tag{62}$$

then

$$\|z\|_{\infty,[t_i,t_{i+2}]} \leq \gamma \tag{63}$$

The satisfaction of the inequality constraints, of course, also requires that $\theta(t)$ lie in Θ_1 for $t_i \leq t < t_{i+1}$ (i.e. $\theta \in \mathcal{F}_{\Theta_1}[t_i, t_{i+1}]$) and in Θ_2 for $t_{i+1} \leq t \leq t_{i+2}$. Satisfaction of this parameter variation condition is non-trivial to verify.

The preceding discussion has outlined how the conditions determined in proposition 3 can be used to ensure safe behaviour between the switch of two different setpoint controllers. These conditions are summarized in the following proposition.

Proposition 5 (LPV Switching Lemma)

Given LPV systems $\Sigma(\tilde{\Theta}, A_1, B_1, C_1, D_1)$, and $\Sigma(\tilde{\Theta}, A_2, B_2, C_2, D_2)$ with associated controllers K_1 and K_2, let the ith controller $(i = 1,2)$ be characterized by matrix P_i, and constants, r_i, α_i, and β_i so that the conditions of propositions 3 and 4 are satisfied for compact parameter sets $\Theta_i \subseteq \tilde{\Theta}$. Assume that controller K_1 is used over time interval $t \in [t_0, t_s)$ and that controller K_2 is used for *time interval $t \in [t_s, T]$ for any $T > t_s$. If $r_1 P_2 \leq r_2 P_1$ and the switch time t_s satisfies*

$$t_s - t_0 \geq -\frac{1}{2\beta_1} \log \frac{r_2}{r_1}, \tag{64}$$

and if $\theta \in \{\mathcal{F}_{\Theta_1}[t_0, t_s], \mathcal{F}_{\Theta_2}[t_s, T]\}$ then $\|z\|_{\infty,[0,T]} \leq \gamma$.

The LPV switching lemma suggests a means of testing to see whether or not a given collection of linear setpoint controllers will generate a safe trajectory. Essentially, this involves verifying the dwell-time condition for all possible switching times and verifying the conditions on the parameter variation. The required dwell-times may be computed from the synthesis LMIs and the coupling condition $r_1 P_2 \leq r_2 P_1$. Switching times and parameter variation bounds are more difficult to verify, but preliminary results indicate that a *nominal parameter trajectory*, $S(x_m(t), 0, 0)$, may be used to estimate these quantities off-line. These estimates may then be compared to the dwell-time results as a sufficient condition for safeness.

Example: As an illustration of some of the important aspects of the MRC-approach, the methods described above were applied to the process control example described in section 3. The reference model

$$\begin{array}{rcl} \dot{x}_{m1} & = & x_{m1} - 1.63 - x_{m1}/(1 + 0.5 \sin 10(x_{m1} - 1.63)) \\ \dot{x}_{m2} & = & 1 \end{array} \tag{65}$$

is specified to move the plant from an initial state near the operating point $(x_{p1}, x_{p2}) = (2.5, 2)$ to a point near $(x_{p1}, x_{p2}) = (1, 3)$ in 1 second. The performance weights for the objective function were chosen as $C = I$ and $D = 0.1I$; the desired bound on the objective function was $\gamma = 0.5$.

The LPV error system is derived as

$$\begin{array}{rcl} \dot{x} & = & A(\theta)x + B_u(\theta)u + B_w(\theta)w \\ z & = & Cx + Du \end{array} \tag{66}$$

with

$$A(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_w(\theta) = \begin{bmatrix} \theta_1 - 1.63 \\ \theta_2 + 1 \end{bmatrix} \quad \text{and} \quad B_u(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -\theta_3 - 1 \end{bmatrix}, \tag{67}$$

and parameter mapping

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = S(x_m, x, u) = \begin{bmatrix} 2x_{m1} - x_{m1}/(1 + 0.5 \sin 10(x_{m1} - 1.63)) \\ x_{m2} + (2x_{m1}x_1 - x_1^2)u_2 \\ x_{m1}^2 \end{bmatrix}. \tag{68}$$

Linear state feedback control agents, $u = Kx$, were designed by choosing setpoints, $\theta^{nom} = S(x_m^{nom}, 0, 0)$ and solving the appropriate synthesis LMIs for parameter sets

$$\Theta := \left\{ \theta \,\middle|\, \sup_{i=1,\ldots,s} |\theta_i - \theta_i^{nom}| \leq \vartheta \right\}$$

for a design parameter $\vartheta > 0$. Switching control was achieved by switching a new feedback controller into the loop whenever the parameter variation evolved onto the boundary of the current agent's parameter set. The new control agent was chosen to minimize a distance measure in the parameter space.

A Matlab program was written to solve the appropriate synthesis LMIs, as indicated above, and simulate the closed loop system. Simulations were performed for various values of ϑ, resulting

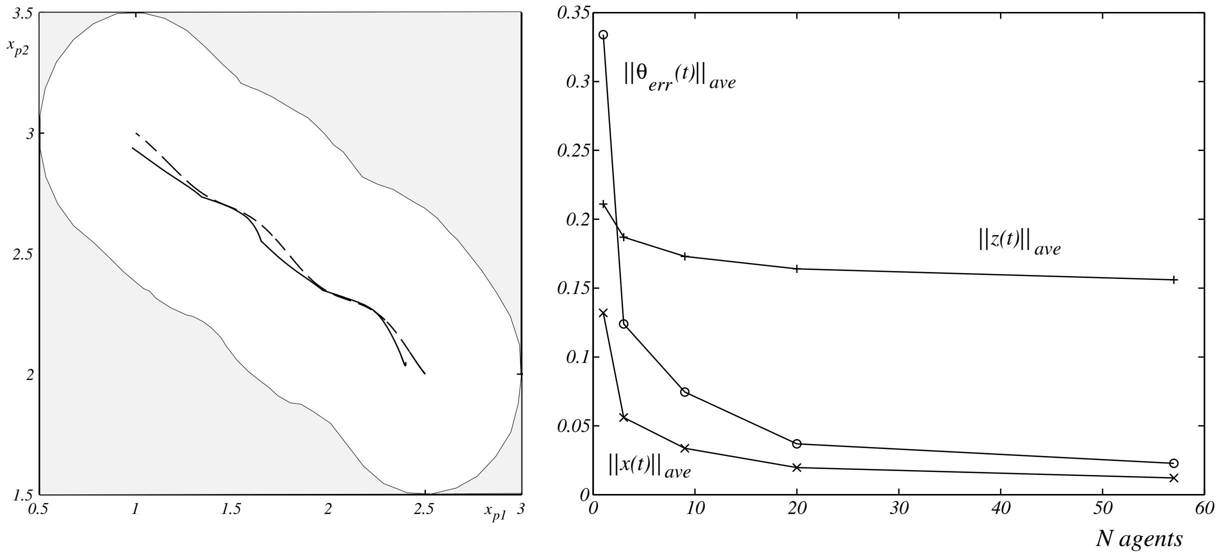

Figure 3: Simulation Results: *(a)* reference trajectory (dashed) and controlled plant state (solid) with forbidden regions (shaded). *(b)* Average performance versus number of agents

in experiments requiring varying numbers of models. The resulting trajectory for nine models is depicted in figure 3a. Figure 3a also depicts the reference trajectory and forbidden (shaded) regions of the state space. Note that the resulting trajectory is safe and nonchattering. Similar results were observed for different numbers of models. Figure 3b depicts the resulting trend observed for increasing numbers of models. Note the monotonic improvement in performance with increasing numbers of models. The average $\|\theta_{err}\|$ represents a mismatch between the reference model dynamics and the multiple agent controlled system; the result in the figure indicates that an increase in the number of agents (via a reduction in ϑ) results in improved dynamical model matching. The other performance curves are self-explanatory.

As with the previous example of the LP-approach, this example depicts some of the fundamental characteristics of the MRC-approach. The approach is an off-line procedure which requires the solution of LMI problems. In the present form, the computation of the control requires explicit knowledge of the plant dynamics and direct measurement of the plant state. However, because the approach is based primarily on Lyapunov and structured uncertainty methods for robust control design, the approach should be extendable to uncertain systems. The computational burden is large, but it is off-line and the payoff is a nonchattering control which satisfies amplitude constraints.

5 Conclusions

This paper has compared two methods for safe implementation of supervisory commands in hybrid dynamical control systems, called the LP method and the MRC method. Both methods appear to be able to guarantee the bounded amplitude performance requirements dictated by the hybrid design problem, assuming knowledge of the plant dynamics. The LP method produces a chattering control policy versus the nonchattering control policy generated by the MRC method. Both methods require that the plant dynamics do not vary too rapidly.

While both methods require explicit knowledge of the plant dynamics, the MRC method appears to be more amenable to incorporation of modelling uncertainty and disturbances into the design, yielding robust control policies. Furthermore, the designs may be accomplished using the same tools as for the nominal case since the design tools are linear robust design techniques. The LP method may offer such advantages, but they are not apparent.

In the area of numerical complexity, the LP method requires the solution of simple linear programming problems which, of course, can be solved quickly and efficiently. This advantage is offset, somewhat, by the fact that the linear programs must be solved on-line and often. On the other hand, the MRC method requires the solution of a series of larger convex optimization problems. However, while this requires a more computationally intensive effort, the procedure is performed off-line and must only be performed once.

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