Notes on Lie Groups, Lie algebras, and the Exponentiation Map

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1. Preliminaries.

In these notes, we concern ourselves with special objects called *matrix Lie groups* and their corresponding *Lie algebras*, connected via the exponentiation map. Our main goal is to establish that the Lie algebra corresponding to a matrix Lie group is a real vector space. Thus, in order to further study a Lie group, one might more easily study its corresponding Lie algebra, since vector spaces are well-studied.

In these notes, we refer to the set of $m \times n$ matrices over the field F by $M_{m\times n}(F)$. We also use the notation $\mathbb{R}$ for field of real numbers and $\mathbb{C}$ for the field of complex numbers.

We expect that the reader is familiar with the exponential of a matrix. Namely, if A is an $n \times n$ matrix, the matrix

$$e^A := \sum_{i=0}^{\infty} \frac{A^i}{i!} \tag{1}$$

is the exponential of A, often called "e to the A". It can be shown that the exponential function $\exp : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ defined by $\exp(A) = e^A$ is well-defined with domain $M_{n\times n}(\mathbb{C})$. In other words, the series in (1) converges for each $A \in M_{n\times n}(\mathbb{C})$.

2. Matrix Lie Groups.

Before we can answer the question of what a matrix Lie group is, we must first define a group.

Definition 1. *A* ***group*** *is a set G together with a map $* : G \times G \to G$ $[*(g_1, g_2) \mapsto g_1 * g_1]$ such that*

1. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3), \quad g_1, g_2, g_3 \in G;$
2. There exists $e \in G$ such that $e * g = g * e = g$ for each $g \in G$;
3. For each $g \in G$ there exists $h \in G$ such that $h * g = g * h = e$.

Example 2. We give several examples of matrix groups. In each case, the product is given by matrix multiplication. Recall that matrix multiplication is associative.

1. $GL(n,\mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) : A^{-1} \text{ exists}\}$. The general linear group (over $\mathbb{R}$). Note that

   (a) $GL(n,\mathbb{R})$ is closed with respect to matrix multiplication, since $(AB)^{-1} = B^{-1}A^{-1}$.

   (b) The $n \times n$ identity matrix is an element of $GL(n,\mathbb{R})$.

   (c) For each A, A^{-1} is an element of $GL(n,\mathbb{R})$ satisfying property 3.

2. $SL(n,\mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 1\}$. The special linear group (over $\mathbb{R}$).

   (a) $SL(n,\mathbb{R})$ is closed with respect to matrix multiplication, since if $\det A = 1$ and $\det B = 1$, then $\det AB = 1$.

   (b) The $n \times n$ identity is an element of $SL(n,\mathbb{R})$.

   (c) For each A, A^{-1} exists and $\det A^{-1} = 1$ as well.

3. $U(n) := \{A \in M_{n\times n}(\mathbb{C}) : AA^* = I\}$. The unitary group.

4. $SU(n) := \{A \in M_{n\times n}(\mathbb{C}) : AA^* = 1, \det A = 1\}$. The special unitary group.

5. H. The Heisenberg group. The set of real matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

   Note that H is closed under multiplication and that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

All of these examples are actually more precisely called matrix *Lie* groups. We give the precise definition of a matrix Lie group below. But first, we give a remark on Sophus Lie.

Remark 3. Sophus Lie (1842 - 1899) was a Norwegian mathematician. His major achievement was discovering that such transformation groups as mentioned above could be better understood by "linearizing" them and studying the corresponding linear spaces, called Lie algebras.

Definition 4. *Let A_n be a sequence of complex matrices. We say that A_n* ***converges to a matrix*** *A if each entry of A_n converges to the corresponding entry of A.*

Definition 5. *A* ***matrix Lie group*** *is any subgroup H of $GL(n,\mathbb{C})$ with the property that if A_n is a sequence of matrices in H and A_n converges to a matrix A, then either A belongs to H or A is not invertible.*

Essentially, what the definition of a matrix Lie group requires is that H be a closed subset of $GL(n, \mathbb{C})$. Geometrically, one can think of matrix Lie groups as manifolds embedded in a space. Although we do not discuss at length on this notion, it does help motivate the following discussion.

3. Connectedness.

Definition 6. *A matrix Lie Group G is **connected** if for each $E, F \in G$, there is a continuous path $A : [a, b] \to G$, such that $A(t) \in G$ for each t, $A(a) = E$, and $A(b) = F$.*

It turns out that most of the groups from Example 2 are connected. We show now the case for $U(n)$. In our proof, we will construct a path from any unitary matrix to the identity matrix. It follows that one can construct a path between any two unitary matrices, by first traveling to the identity, then out to the other unitary matrix.

Proposition 7. *The group $U(n)$ is connected for each $n \geq 1$.*

Proof. Let U be unitary. Recall that if U is unitary, then $U^*U = I = UU^*$, so U is normal. By the spectral theorem for normal operators, there exists an orthonormal system of eigenvectors for U. Recall that each e-value of U satisfies $|\lambda| = 1$, so we may write

$$U = U_1 \begin{bmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_n} \end{bmatrix} U_1^* \qquad (*)$$

where U_1 is unitary and $\theta_i \in \mathbb{R}$. Conversely, one can show that a matrix of the form $(*)$ is unitary by noting that $UU^* = I$. For each $t \in [0, 1]$ define

$$A(t) = U_1 \begin{bmatrix} e^{i(1-t)\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i(1-t)\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i(1-t)\theta_n} \end{bmatrix} U_1^*.$$

Then $A(0) = U$ and $A(1) = I$. Furthermore, $A(t)$ is unitary for every t. So each unitary matrix can be connected to the identity. So $U(n)$ is connected. □

We have constructed the following table displaying whether each group from Example 2 is connected.

Group	Connected?	Components
$GL(n, \mathbb{C})$	Yes	1
$GL(n, \mathbb{R})$	No	2
$SL(n, \mathbb{C})$	Yes	1
$SL(n, \mathbb{R})$	Yes	1
$U(n)$	Yes	1
H	Yes	1

Note 8. The group $GL(n, \mathbb{R})$ is not connected for the following reason. Suppose that $\det E > 0$ and $\det F < 0$. Let $A : [0, 1] \to M_{n\times n}(\mathbb{R})$ be a continuous path connecting E and F. Compose $\det \circ A$, a continuous function. By the intermediate value theorem, there is a $t \in (0, 1)$ such that $(\det \circ A)(t) = 0$, i.e., the path A leaves $GL(n, \mathbb{R})$.

4. Lie Algebra of a Matrix Lie Group.

As was mentioned earlier, it is often easier to study the so-called "Lie algebra" of a matrix Lie group than the group itself. This is because the algebra has some nice properties which we begin to discuss now.

Definition 9. *Let G be a matrix Lie group. The* ***Lie algebra of*** *G, denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} belongs to G for each real number t.*

Example 10. We now discuss the corresponding Lie algebras of those groups in Example 2.

1. $gl(n, \mathbb{R}) := \{X : e^{tX} \in GL(n, \mathbb{R})\ \forall\, t \in \mathbb{R}\}$.

   Claim 0.1. $gl(n, \mathbb{R}) = M_{n\times n}(\mathbb{R})$.

   Proof. Let $X \in M_{n\times n}(\mathbb{R})$. Then e^{tX} is invertible and real for each real t, i.e., $e^{tX} \in GL(n, \mathbb{R})$.
   Suppose X is a matrix such that e^{tX} is real for each t. Then $X = \frac{d}{dt}|_{t=0} e^{tX}$ is real, i.e., $X \in M_{n\times n}(\mathbb{R})$. □

2. $sl(n, \mathbb{R}) := \{X \in M_{n\times n}(\mathbb{R}) : e^{tX} \in SL(n, \mathbb{R})\ \forall\, t \in R\}$.

   Claim 0.2. $sl(n, \mathbb{R}) = \{X \in M_{n\times n}(\mathbb{R}) : tr(X) = 0\}$.

   Proof. We require the theorem $\det e^X = e^{tr(X)}$ (which we do not prove in these notes). Let X satisfy $tr(X) = 0$. Then $tr(tX) = 0$ for each t. So $\det e^{tX} = 1$ for each t as required.
   Conversely, suppose that $\det e^{tX} = 1$ for each t. Then $e^{t\cdot tr(X)} = 1$ for each t. So $t \cdot tr(X) = (2\pi i)k$, $k \in \mathbb{Z}$ for each t. So $tr(X) = 0$. □

3. $u(n) := \{X \in M_{n\times n}(\mathbb{C}) : e^{tX} \in U(n)\, \forall\, t \in \mathbb{R}\}$.

   Claim 0.3. $u(n) = \{X \in M_{n\times n} : X^* = -X\}$ (the set of skew-symmetric matrices)

   Proof. Recall that if $e^{tX} \in U(n)$, then $(e^{tX})^* = (e^{tX})^{-1} = e^{-tX}$. Note that
   $$(e^{tX})^* = e^{t^* X^*} = e^{tX^*}.$$
   So $X^* = -X$. Conversely, we can see that such X satisfy $e^{tX} \in U(n)$ for each t. □

4. $\mathfrak{h} := \{X \in M_{n\times n}(R) : e^{tX} \in H\, \forall\, t \in \mathbb{R}\}$ (The Heisenberg algebra). The Heisenberg algebra $\mathfrak{h}$ consists of all matrices of the form
$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

We want to discuss some properties of a Lie algebra. But first, we require some knowledge about the matrix exponential and the matrix logarithm.

5. The Exponential and the Logarithm.

We first make two observations about the matrix exponential function. Suppose we have $n\times n$ matrices A and X where A is invertible. Then $(AXA^{-1})^m = AX^mA^{-1}$. So
$$e^{AXA^{-1}} = Ae^XA^{-1} \tag{2}$$
by looking at the series in (1) term by term.

Now suppose X is an $n \times n$ matrix and t is a real number. Then
$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X. \tag{3}$$
This follows by differentiating the series in (1) term-by-term and recognizing that X commutes with powers of itself.

We now define the matrix logarithm. Although we do not concern ourselves too much with the construction of this function, it can be shown that such a function is well-defined and satisfies the following properties. (In the theorem below, let $\|\cdot\|$ denote the operator norm on $n \times n$ matrices.)

Theorem 11. *The function* $\log : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ *defined by*

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$$

is well-defined and continuous on the set of all $n\times n$ *matrices satisfying* $\|A-I\| < 1$. *And if* A *is real, then* $\log A$ *is real.*

Furthermore, for each A *with* $\|A-I\| < 1$,

$$e^{\log A} = A.$$

For each X *satisfying* $\|X\| < \log 2$, *we have*

$$\|e^X - I\| < 1 \qquad \text{and} \qquad \log e^X = X.$$

Hence, for matrices of a certain "distance" from the identity, the logarithm is the inverse of the exponential. Note that log is defined as we would expect: an extension of the Taylor expansion for log to matrices.

Lemma 12. *There exists a constant* c *such that for all* $n \times n$ *matrices* A *with* $\|A\| < 1/2$, *we have*

$$\|\log(I+A) - A\| \leq c\|A\|^2.$$

Proof. We note that

$$\log(I+A) - A = \sum_{m=2}^{\infty} (-1)^m \frac{A^m}{m} = A^2 \sum_{m=2}^{\infty} (-1)^m \frac{A^{m-2}}{m}.$$

A property of the operator norm is that $\|XY\| \leq \|X\|\|Y\|$. A further property is that $\|X+Y\| \leq \|X\| + \|Y\|$. So

$$\begin{aligned} \|\log(I+A) - A\| &\leq \|A\|^2 \| \sum_{m=2}^{\infty} (-1)^m \frac{A^{m-2}}{m} \| \\ &\leq \|A\|^2 \sum_{m=2}^{\infty} \frac{\|A\|^{m-2}}{m} \\ &\leq \|A\|^2 \sum_{m=2}^{\infty} \frac{(1/2)^{m-2}}{m}. \end{aligned}$$

The series on the right converges to some $c \in \mathbb{R}$ by the ratio test. So we obtain

$$\|\log(I+A) - A\| \leq c\|A\|^2$$

as required. ∎

Theorem 13. *Let* X *and* Y *belong to* $M_{n\times n}(\mathbb{C})$. *Then*

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{1}{m}X} e^{\frac{1}{m}Y} \right)^m.$$

Proof. Expanding the first few terms of the power series multiplication for the exponential, we obtain

$$e^{\frac{1}{m}X}e^{\frac{1}{m}Y} = I + \frac{1}{m}X + \frac{1}{m}Y + C_m$$

where C_m is a series such that $\|C_m\| \leq k/m^2$ for some $k \in R$. By choosing m large enough, we can make $e^{\frac{1}{m}X}e^{\frac{1}{m}Y}$ close enough to I, thus making $e^{\frac{1}{m}X}e^{\frac{1}{m}Y}$ in the domain of the logarithm. Therefore, for sufficiently large m, applying Lemma 12 gives

$$\begin{aligned} \log\left(e^{\frac{1}{m}X}e^{\frac{1}{m}Y}\right) &= \log\left(I + \frac{1}{m}X + \frac{1}{m}Y + C_m\right) && (4) \\ &= \frac{1}{m}X + \frac{1}{m}Y + C_m + E_m, && (5) \end{aligned}$$

where E_m is an error term which satisfies

$$\|E_m\| \leq c\|\frac{1}{m}X + \frac{1}{m}Y + C_m\|^2 \leq c\|C_m\|^2 \leq \frac{ck}{m^2},$$

for each $m \geq 2$ and for some $c \in \mathbb{R}$. Applying exp to both sides of (4) gives

$$e^{\frac{1}{m}X}e^{\frac{1}{m}Y} = \exp\left(\frac{1}{m}X + \frac{1}{m}Y + C_m + E_m\right).$$

So

$$\left(e^{\frac{1}{m}X}e^{\frac{1}{m}Y}\right)^m = \exp\left(X + Y + mC_m + mE_m\right). \qquad (6)$$

Since both C_m and E_m are on the order of $1/m^2$ and since the exponential is a continuous function, we take the limit of both sides of (6) to obtain

$$\lim_{m\to\infty}\left(e^{\frac{1}{m}X}e^{\frac{1}{m}Y}\right)^m = \exp(X+Y),$$

as required. □

6. Properties of a Lie Algebra

Using our work in the previous section, we now obtain a nice result concerning Lie algebras, namely, that they are real vector spaces. First we require a small proposition.

Proposition 14. *Let G be a matrix Lie group and $\mathfrak{g}$ its corresponding Lie algebra. Let X be an element of $\mathfrak{g}$ and let A be an element of G. Then AXA^{-1} belongs to $\mathfrak{g}$.*

Proof. By (2) of section 5, we see that

$$e^{t(AXA^{-1})} = Ae^{tX}A^{-1}.$$

Since A, A^{-1} and e^{tX} belong to G for each t, we see that AXA^{-1} belongs to $\mathfrak{g}$ by definition. □

Theorem 15. *Let G be a matrix Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $X, Y \in \mathfrak{g}$. Then*

1. $sX \in \mathfrak{g}$ for each $s \in \mathbb{R}$.
2. $X + Y \in \mathfrak{g}$.

Proof. For (1), if $X \in \mathfrak{g}$, then $e^{tX} \in G$ for each t. So $e^{tsX} \in G$ for each s since $ts \in \mathbb{R}$ for each $t, s \in \mathbb{R}$. Therefore, sX belongs to $\mathfrak{g}$ by definition.

For (2), let X and Y belong to $\mathfrak{g}$. Theorem 13 gives

$$e^{t(X+Y)} = \lim_{m\to\infty} \left(\exp\left(\frac{t}{m}X\right) \exp\left(\frac{t}{m}Y\right)\right)^m. \tag{7}$$

Since X and Y belong to $\mathfrak{g}$, $\exp\left(\frac{t}{m}X\right)$ and $\exp\left(\frac{t}{m}Y\right)$ belong to G. Moreover, since G is a group, their product belongs to G, as does any power of their product. To make sure that the limit is in G, we need to make sure that the limit is invertible. Since the left hand side of (7) is invertible with inverse $e^{-t(X+Y)}$, the limit is also invertible. Therefore, $e^{t(X+Y)}$ belongs to G. By definition, $X + Y$ belongs to $\mathfrak{g}$. □

Thus, we have established that the Lie algebra is a real vector space. Because of this, we immediately get an interesting property about the Lie algebra: that it is closed under the so-called bracket. The **bracket** of two elements X and Y in a matrix Lie algebra $\mathfrak{g}$, denoted $[X, Y]$ is given by $[X, Y] = XY - YX$. In a general Lie algebra, the bracket need not be defined in this way. But in the case of matrices, this bracket definition is commonly used.

Corollary 16. *Let G be a matrix Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $X, Y \in \mathfrak{g}$. Then $[X, Y] = XY - YX$ belongs to $\mathfrak{g}$.*

Proof. Recall the property of the exponential given by (3) in section 5: $\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X$. By the Leibniz rule of differentiation, we have

$$\begin{aligned} \frac{d}{dt}e^{tX}Ye^{-tX} &= e^{tX}Y\frac{d}{dt}\left(e^{-tX}\right) + \frac{d}{dt}\left(e^{tX}Y\right)e^{-tx} \\ &= -e^{tX}YXe^{-tX} + e^{tX}XYe^{-tX}. \end{aligned}$$

Evaluation at $t = 0$ gives

$$\frac{d}{dt}|_{t=0}e^{tX}Ye^{-tX} = -YX + XY = XY - YX.$$

A property of vector spaces is that the derivative of any smooth curve lying in $\mathfrak{g}$ must also be in $\mathfrak{g}$. Since by Proposition 14, $e^{tX}Ye^{-tX}$ belongs to $\mathfrak{g}$ for each t, we see that $XY - YX$ belongs to $\mathfrak{g}$. □

References

Brian C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics, 2003.

Karin Erdmann & Mark J. Wildon, *Introduction to Lie Algebras.* Springer, 2006.