Multi-graded Macaulay dual spaces

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Abstract

We describe an algorithm for computing Macaulay dual spaces for multi-graded ideals. For homogeneous ideals, the natural grading is inherited by the Macaulay dual space which has been leveraged to develop algorithms to compute the Macaulay dual space in each homogeneous degree. Our main theoretical result extends this idea to multi-graded Macaulay dual spaces inherited from multi-graded ideals. This natural duality allows ideal operations to be translated from homogeneous ideals to their corresponding operations on the multi-graded Macaulay dual spaces. In particular, we describe a linear operator with a right inverse for computing quotients by a multi-graded polynomial. By using a total ordering on the homogeneous components of the Macaulay dual space, we also describe how to recursively construct a basis for each component. Several examples are included to demonstrate this new approach.

Keywords Macaulay dual spaces; Hilbert functions; multi-grading; symbolic-numeric computing

1 Introduction

For a polynomial system $F \subseteq \mathbb{C}[x_1, \ldots, x_N]$, many algebraic properties of the ideal $I = \langle F \rangle$ generated by F can, for example, be deduced from a Gröbner basis of I, such as its Hilbert function. In many instances, one often knows a generating set F for an ideal I, but computing a Gröbner basis of I could be computationally infeasible. From a generating set F, another approach to compute information about the corresponding ideal is to use Macaulay dual spaces which Macaulay formulated as inverse systems in [18] and have been utilized in a variety of scenarios such as [1, 2, 6, 7, 11, 13, 15, 17, 19, 22, 24–26]. One particular application of interest here is to compute Hilbert functions of ideals up to a given degree which are graded by a finitely generated abelian group M, called multi-graded ideals.

Since multi-graded ideals are a generalization of homogeneous ideals, multi-graded Macaulay dual spaces are a generalization of homogeneous Macaulay dual spaces. Moreover, multi-graded ideals naturally arise when considering multi-projective varieties, and more generally, subvarieties of a smooth toric variety. One key theoretical result is Thm. 3.2 which states that the Macaulay dual space of a multi-graded ideal inherits the multi-grading from the ideal. This is applied in Section 4 to ideal operations with another key theoretical result being Thm. 4.8 for computing ideal quotients

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using Macaulay dual spaces. For $\mathbb{Z}^k$-gradings, an algorithm is described for computing each graded piece of the dual space sequentially up to a given degree. This is in contrast to other known Gröbner basis techniques over semigroup algebras [3,8].

The rest of the paper is organized as follows. Section 2 summarizes necessary background regarding multi-graded ideals and Macaulay dual spaces. Section 3 describes multi-graded Macaulay dual spaces which are used in ideal operations in Section 4. Section 5 provides an algorithm for computing multi-graded Macaulay dual spaces and summarizes a proof-of-concept implementation which is used in the examples presented in Section 6. A short conclusion is provided in Section 7.

2 Background

The following summarizes necessary background information on multi-graded ideals and Macaulay dual spaces.

2.1 Multi-graded ideals

The first step in defining a multi-graded ideal is to have a multi-grading on a polynomial ring.

Definition 2.1 *Let $R = \mathbb{C}[x_1, \ldots, x_N]$ and M be a finitely generated abelian group. Then, R is* M-graded *if there is a direct sum decomposition of the form*

$$R = \bigoplus_{m \in M} R_m$$

where $R_{m_1} \cdot R_{m_2} \subseteq R_{m_1+m_2}$ for all $m_1, m_2 \in M$. Moreover, if $m \in M$ and $f \in R_m$, then f is M-homogeneous of degree m, *denoted* $\deg(f) = m$. *Finally, if $I \subseteq R$ is an ideal, then I is* M-graded *if I is generated by M-homogeneous polynomials.*

If $\alpha \in (\mathbb{Z}_{\geqslant 0})^N$, then consider

$$|\alpha| = \sum_{i=1}^{N} \alpha_i, \quad \alpha! = \prod_{i=1}^{N} \alpha_i!, \quad \text{and} \quad x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}. \tag{1}$$

Hence, the standard grading of R is a $\mathbb{Z}$-grading with

$$R_m = \operatorname{span}_{\mathbb{C}} \left\{ x^\alpha \ : \ |\alpha| = m, \alpha \in (\mathbb{Z}_{\geqslant 0})^N \right\}.$$

The following is an example with a different grading.

Example 2.2 *Consider $R = \mathbb{C}[x_1, x_2]$ and $M = \mathbb{Z}$ such that* $\deg(x_1) = 2$ *and* $\deg(x_2) = 1$. *Then, for $m \in \mathbb{Z}$,*

$$R_m = \operatorname{span}_{\mathbb{C}}\{x_1^a x_2^{m-2a} \ : \ m \geqslant 2a, a \in \mathbb{Z}_{\geqslant 0}\}. \tag{2}$$

Hence, $f = x_1 - 3x_2^2 \in R_2$, i.e., f is an M-homogeneous polynomial with $\deg(f) = 2$.

Such a construction used in Ex. 2.2 can be naturally generalized to define an M-grading on R, namely select $m_1, \ldots, m_N \in M$ and assign $\deg(x_i) = m_i$. Thus, for any $\alpha \in (\mathbb{Z}_{\geqslant 0})^N$,

$$\deg(x^\alpha) = \sum_{i=1}^{N} \alpha_i m_i.$$

In particular, for $m \in M$, one has

$$R_m = \operatorname{span}_{\mathbb{C}} \left\{ x^\alpha \ : \ \deg(x^\alpha) = m, \alpha \in (\mathbb{Z}_{\geqslant 0})^N \right\}.$$

Remark 2.3 If $M = \mathbb{Z}$ and $\deg(x_1) = \cdots = \deg(x_N) = 1$, then the $\mathbb{Z}$-grading on R is the standard grading.

Example 2.4 *Let $M = \mathbb{Z}^2$, $R = \mathbb{C}[x_1, x_2, x_3, x_4]$, and $r \in \mathbb{Z}_{>0}$. Set*

$$\begin{array}{ll} \deg(x_1) = (1, -r), & \deg(x_2) = (0, 1), \\ \deg(x_3) = (1, 0), & \deg(x_4) = (0, 1). \end{array}$$

This is the Cox ring of the r^{th} Hirzebruch surface $\mathcal{H}_r$ [5, § 5.2]. Just as there is a correspondence between homogeneous ideals and projective varieties, there is a correspondence between M-graded ideals of R and subvarieties of $\mathcal{H}_r$.

For an M-graded ideal I, the multi-graded Hilbert function simply records information about the corresponding dimensions of homogeneous components of R/I [20, §8.2]. In order to have finite dimensions, we will only consider M-gradings in the remainder of this article such that $\dim_{\mathbb{C}}(R_m) < \infty$ for all $m \in M$. This is equivalent to $R_0 = \mathbb{C}$, that is, every polynomial of degree 0 is constant.

Definition 2.5 *If R is M-graded and $I \subseteq R$ is an M-graded ideal, then the* multi-graded Hilbert function of I *is the function $H_I : M \to \mathbb{Z}$ defined by*

$$H_I(m) = \dim_{\mathbb{C}}(R_m) - \dim_{\mathbb{C}}(R_m \cap I).$$

Example 2.6 *Following the setup from Ex. 2.2 with $I = \langle x_1 - 3x_2^2 \rangle$, one can easily verify that*

$$H_I(m) = \begin{cases} 0 & \text{if } m < 0, \\ 1 & \text{if } m \geqslant 0. \end{cases}$$

Example 2.7 *An illustration of a grading which will not be considered is $R = \mathbb{C}[x_1, x_2]$ and $M = \mathbb{Z}$ with $\deg(x_1) = 1$ and $\deg(x_2) = -1$. Thus, for example, $\deg(x_1^k x_2^k) = 0$ for any $k \in \mathbb{Z}_{\geqslant 0}$ so that $\dim_{\mathbb{C}}(R_0) = \infty$.*

2.2 Macaulay dual spaces

Macaulay dual spaces are a modern form of inverse systems studied by Macaulay [18]. Let $R = \mathbb{C}[x_1, \ldots, x_N]$, $\alpha \in (\mathbb{Z}_{\geqslant 0})^N$, and $y \in \mathbb{C}^N$. Following (1), consider the operator $\partial_\alpha[y] : R \to \mathbb{C}$ defined by

$$\partial_\alpha[y](g) = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \bigg|_{x=y}.$$

When the context is clear, we will write ∂_α instead of $\partial_\alpha[y]$.

Example 2.8 *For $R = \mathbb{C}[x_1, x_2]$, $\alpha = (3, 2)$, and $y = (1, 2)$, we have*

$$\partial_\alpha[y](x_1^4 x_2^3 + 3x_1^3 x_2^2 - 2x_1^2 + 3x_2 - 1) = \frac{144 x_1 x_2 + 36}{3!2!} \bigg|_{x=(1,2)} = 27.$$

In particular, 27 is the coefficient of $(x_1 - 1)^3 (x_2 - 2)^2$ in a Taylor series expansion of $x_1^4 x_2^3 + 3x_1^3 x_2^2 - 2x_1^2 + 3x_2 - 1$ centered at $y = (1, 2)$.

The Macaulay dual space is a $\mathbb{C}$-vector space contained inside of

$$D_y = \mathrm{span}_{\mathbb{C}} \left\{ \partial_\alpha[y] \ : \ \alpha \in (\mathbb{Z}_{\geqslant 0})^N \right\}.$$

Definition 2.9 *Let $I \subseteq R$ be an ideal and $y \in \mathbb{C}^N$. The* Macaulay dual space of I at y *is the $\mathbb{C}$-vector space*

$$D_y(I) = \{\partial \in D_y \ : \ \partial(g) = 0 \text{ for all } g \in I\}.$$

If the dimension of $D_y(I)$ is finite, then $\dim_{\mathbb{C}} D_y(I)$ is the multiplicity of y with respect to I. If the dimension of $D_y(I)$ is infinite, then y is a nonisolated solution in $\mathbb{C}^N$ of the variety corresponding to I. This fact was exploited in [1] to develop a numerical local dimension test.

Example 2.10 *Let $R = \mathbb{C}[x_1, x_2]$ and $I = \langle 29/16x_1^3 - 2x_1x_2, x_2 - x_1^2 \rangle$ arising from the Griewank-Osborne system [10]. It is well-known that $y = (0,0)$ has multiplicity 3 with respect to I and one can easily verify that*

$$D_0(I) = \mathrm{span}_{\mathbb{C}} \left\{ \partial_{(0,0)}, \partial_{(1,0)}, \partial_{(0,1)} + \partial_{(2,0)} \right\} \tag{3}$$

is a 3-dimensional vector space.

From [24], for $i = 1, \ldots, N$, there are linear anti-differentiation operators $\Phi_i : D_y \to D_y$ which are defined via

$$\Phi_i(\partial_\alpha) = \begin{cases} \partial_{\alpha - e_i} & \text{if } \alpha_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where e_i is the i^{th} standard basis vector. From the Leibniz rule, one can easily verify that, for any $f \in R$ and $\partial \in D_y$,

$$\Phi_i(\partial)(f) = \partial((x_i - y_i)f) \tag{5}$$

The following, from [24, 26], uses these linear operators to compute $D_y(I)$ via the so-called *closedness subspace condition* which has been exploited to improve the efficiency of computing dual spaces [6, 13].

Proposition 2.11 *Let $I = \langle f_1, \ldots, f_t \rangle \subseteq \mathbb{C}[x_1, \ldots, x_N]$, $y \in \mathbb{C}^N$, and $\partial \in D_y$. Then, $\partial \in D_y(I)$ if and only if $\partial(f_i) = 0$ for all $1 \leqslant i \leqslant t$ and $\Phi_j(\partial) \in D_y(I)$ for all $1 \leqslant j \leqslant N$.*

One key aspect of this closedness condition is that any basis for the ideal I can be utilized.

Example 2.12 *Continuing with Ex. 2.10 where $f_1 = 29/16x_1^3 - 2x_1x_2$ and $f_2 = x_2 - x_1^2$, consider $\delta = \partial_{(0,1)} + \partial_{(2,0)}$. Clearly, $\delta(f_1) = 0$ since the monomials x_1^2 and x_2 do not appear in f_1. Next, it is easy to verify that $\delta(f_2) = 1 - 1 = 0$. Finally, $\Phi_1(\delta) = \partial_{(1,0)}$ and $\Phi_2(\delta) = \partial_{(0,0)}$. Hence, given that $\partial_{(0,0)}, \partial_{(1,0)} \in D_0(I)$, Prop. 2.11 allows one to conclude that $\delta \in D_0(I)$.*

3 Multi-Graded Macaulay Dual Spaces

For a multi-graded ideal $I \subseteq R = \mathbb{C}[x_1, \ldots, x_N]$, one can consider investigating the Macaulay dual space at $y = 0 \in \mathbb{C}^N$ to determine properties about I. The following shows that the multi-graded structure of I extends to $D_0(I)$.

Suppose that R is M-graded where the M-grading is induced by assigning $\deg(x_i) = m_i \in M$ such that $\dim_{\mathbb{C}}(R_0) = 1$. In particular, after selecting a basis of M, say $\beta = \{\beta_1, \ldots, \beta_k\}$, one can express each m_i in terms of β. Let A be the $k \times N$ matrix whose i^{th} column corresponds with m_i in terms of β. Hence, for any $m \in R$, a basis of monomials for R_m is

$$\{x^\alpha \ : \ A \cdot \alpha = m, \alpha \in \mathbb{Z}^N_{\geqslant 0}\}.$$

In particular, $\dim_{\mathbb{C}}(R_0) = 1$ is equivalent to null $A \cap \mathbb{Z}^N_{\geqslant 0} = \{0\}$.

Example 3.1 *With the setup from Ex. 2.4, using a standard basis $\beta = \{e_1, e_2\}$ for $M = \mathbb{Z}^2$, one has*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -r & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 0 \\ r & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the columns of N span null A. *The first and third rows of N clearly show* null $A \cap \mathbb{Z}^4_{\geqslant 0} = \{0\}$.

One can extend the M-grading to D_0, namely, for each $m \in M$,

$$D_0^m = \operatorname{span}_{\mathbb{C}} \left\{\partial_\alpha[0] \ : \ A \cdot \alpha = m, \alpha \in \mathbb{Z}^N_{\geqslant 0}\right\}.$$

Hence, there is a direct sum decomposition of the form

$$D_0 = \bigoplus_{m \in M} D_0^m. \tag{6}$$

The following is the key theoretical result that $D_0(I)$ inherits the multi-grading from I.

Theorem 3.2 *Suppose that $I \subseteq R$ is an M-graded ideal. Then, the Macaulay dual space $D_0(I)$ is also M-graded, that is,*

$$D_0(I) = \bigoplus_{m \in M} D_0^m(I)$$

where $D_0^m(I) = D_0^m \cap D_0(I)$.

Proof. Suppose that $\partial \in D_0(I)$. Thus, from (6), one can write

$$\partial = \sum_{m \in M} \partial_m$$

where each $\partial_m \in D_0^m$. The result follows by showing $\partial_m \in D_0(I)$ for all $m \in M$. To that end, let $g \in I$. Since I is M-graded, one has

$$g = \sum_{m \in M} g_m$$

where each $g_m \in R_m \cap I$. We claim that, for any $m \in M$,

$$\partial_m(g) = \partial_m(g_m) = \partial(g_m) = 0.$$

The first equality follows from ∂_m being a linear operator such that $\partial_m(p) = 0$ for any $p \in R_q$ for $q \neq m$. The second equality follows from linearity along with $\delta(g_m) = 0$ for any $\delta \in D_q$ for $q \neq m$. The last equality follows from $g_m \in I$ and $\partial \in D_0(I)$. The result now follows since both $g \in I$ and $m \in M$ were arbitrary. $\square$

Example 3.3 *Continuing with the setup from Ex. 2.10, one can view I as M-graded by taking $M = \mathbb{Z}$ such that* $\deg(x_i) = i$*, i.e.,*

$$A = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Hence, one can interpret (3) *as*

$$D_0(I) = D_0^0(I) \oplus D_0^1(I) \oplus D_0^2(I)$$

where

$$\begin{gathered} D_0^0(I) = \operatorname{span}_{\mathbb{C}}\{\partial_{(0,0)}\}, D_0^1(I) = \operatorname{span}_{\mathbb{C}}\{\partial_{(1,0)}\}, \text{ and} \\ D_0^2(I) = \operatorname{span}_{\mathbb{C}}\{\partial_{(0,1)} + \partial_{(2,0)}\}. \end{gathered} \tag{7}$$

Adapting, for example, the proof of [11, Thm. 3.2], the multi-graded Hilbert function is simply the dimension of the corresponding Macaulay dual space.

Proposition 3.4 *Suppose that R is M-graded and $I \subseteq R$ is an M-graded ideal. Then, for all $m \in M$,*

$$H_I(m) = \dim_{\mathbb{C}}(D_0^m(I)).$$

Example 3.5 *From Ex. 3.3, one has $H_I(0) = H_I(1) = H_I(2) = 1$ and otherwise equal to* 0*.*

With the M-grading on D_0, one can view the anti-differentiation operators Φ_i in (4) as operators from D_0^m to $D_0^{m-A\cdot e_i}$ and refine the closedness subspace condition in Prop. 2.11 to the multi-graded case.

Corollary 3.6 *Suppose that $I = \langle f_1, \ldots, f_t \rangle \subseteq R$ is an M-graded ideal where each f_i is M-homogeneous. For each $m \in M$, let*

$$C_0^m(I) = \left\{ \partial \in D_0^m \ : \ \Phi_i(\partial) \in D_0^{m-Ae_i}(I) \text{ for } i = 1, \ldots, N \right\}.$$

be the closedness subspace of degree m. Then,

$$D_0^m(I) = \{ \partial \in C_0^m(I) \ : \ \partial(f_i) = 0 \text{ for all } i \text{ such that } \deg(f_i) = m \}.$$

The equation-by-equation approach described in [13] for computing closedness subspaces can easily be adapted to this multi-graded situation. Moreover, to compute $C_0^m(I)$, one must have already computed $D_0^{m-Ae_i}(I)$ for each $i = 1, \ldots, N$. There is a natural question about which order one has to compute these spaces. To answer this, we make the following definition.

Definition 3.7 *For an M-grading, the* weight semigroup of M *is*

$$\omega = \{ m \in M \ : \ R_m \neq 0 \}. \tag{8}$$

The partial ordering induced by ω*, denoted $\preceq_\omega$ is defined by*

$$m_1 \preceq_\omega m_2 \quad \Longleftrightarrow \quad m_2 - m_1 \in \omega.$$

Note that ω is indeed a semigroup and, by our assumptions on the M-grading of R, the positive hull of ω, denoted $\omega_{\mathbb{R}}$, in $M \otimes \mathbb{R}$ is a pointed polyhedral cone, called the *weight cone* of M.

Proposition 3.8 *$\preceq_\omega$ is a partial ordering on ω.*

Proof. Reflexivity follows since $m - m = 0 \in \omega$ so that $m \preceq_\omega m$. For anti-symmetry, suppose $m_1 \preceq_\omega m_2$ and $m_2 \preceq_\omega m_1$. Hence, both $m_1 - m_2$ and $-(m_1 - m_2)$ are in ω. Our assumptions on the M-grading imply that $a, -a \in \omega$ if and only if $a = 0$. Hence, $m_1 = m_2$. For transitivity, suppose $m_1 \preceq_\omega m_2$ and $m_2 \preceq_\omega m_3$. Then, since $m_2 - m_1, m_3 - m_2 \in \omega$ and since ω is a semigroup,

$$m_3 - m_1 = (m_3 - m_2) + (m_2 - m_1) \in \omega.$$

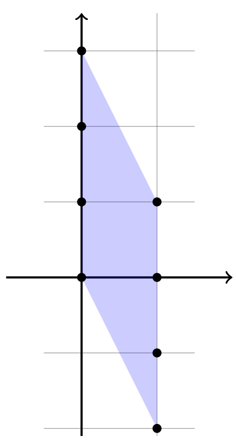
Hence, $m_1 \preceq_\omega m_3$. $\square$

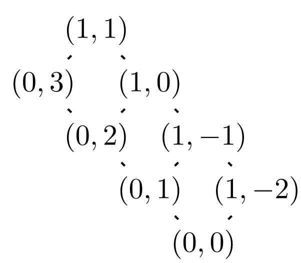
Let $\leqslant_\omega$ be any linear extension of $\preceq_\omega$. Thus, $C_0^m(I)$ can be computed from knowing $D_0^s(I)$ for all $s <_\omega m$ as illustrated next.

Example 3.9 *Consider Ex. 2.4 with $r = 2$ so that $\deg(x_i)$ is the i^{th} column of*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}.$$

Consider $I = \langle f \rangle$ where $f = x_3 - x_1 x_2^2$ is M-homogeneous with $\deg(f) = (1,0)$. Suppose that one aims to build up to compute $C_0^{(1,1)}(I) = D_0^{(1,1)}(I)$ via Cor. 3.6. The first step is to order all the points $v \in \omega$ such that $v \preceq_\omega (1,1)$. There are 8 such points corresponding to the lattice points in $\omega_\mathbb{R} \cap ((1,1) - \omega)_\mathbb{R}$, i.e., the lattice points in the quadrilateral with vertices $(0,0), (0,3), (1,1)$, and $(1,-2)$. The following illustrates the lattice points and the Hasse diagram of the interval $[(0,0),(1,1)]$.





There are 8 linear extensions of the partial order $\preceq_\omega$ and we just pick one, say

$$(0,0) <_\omega (1,-2) <_\omega (0,1) <_\omega (1,-1) <_\omega (0,2)$$

$$<_\omega (0,3) <_\omega (1,0) <_\omega (1,1).$$

By Cor. 3.6, for every $\alpha <_\omega (1,0)$, we know $D_0^\alpha(I) = D_0^\alpha$ since I has no generators of these degrees and the closedness subspace condition is trivial in this range. Thus, one just needs to compute $D_0^{(1,0)}(I)$ and then lift to $C_0^{(1,1)}(I) = D_0^{(1,1)}(I)$.

For $(1,0)$, we have that

$$C_0^{(1,0)}(I) = \operatorname{span}_\mathbb{C}\left\{\partial_{(1,2,0,0)}, \partial_{(1,1,0,1)}, \partial_{(1,0,0,2)}, \partial_{(0,0,1,0)}\right\}.$$

Imposing the vanishing condition for f yields

$$D_0^{(1,0)}(I) = \mathrm{span}_{\mathbb{C}} \left\{ \partial_{(0,0,1,0)} + \partial_{(1,2,0,0)}, \partial_{(1,1,0,1)}, \partial_{(1,0,0,2)} \right\}.$$

For $(1,1)$, there are four linear maps that need to be considered to compute $C_0^{(1,1)}(I) = D_0^{(1,1)}(I)$. The maps $\Phi_1 : D_0^{(1,1)} \to D_0^{(3,0)}$ and $\Phi_3 : D_0^{(1,1)} \to D_0^{(0,1)}$ can safely be ignored since the corresponding Macaulay dual spaces of degrees $(3,0)$ and $(0,1)$ are spanned by all mononomials of their respective degrees and thus do not add any restrictions to $C_0^{(1,1)}(I)$. Now, the maps $\Phi_2, \Phi_4 : D_0^{(1,1)} \to D_0^{(1,0)}$ do need to be considered as $D_0^{(1,0)}(I)$ has a non-trivial relation. In particular, $C_0^{(1,1)}(I) = \Phi_2^{-1}(C_0^{(1,0)}(I)) \cap \Phi_4^{-1}(C_0^{(1,0)}(I))$, namely

$$\begin{aligned} C_0^{(1,1)}(I) &= D_0^{(1,1)}(I) \\ &= \mathrm{span}_{\mathbb{C}} \left\{ \begin{array}{l} \partial_{(1,3,0,0)} + \partial_{(0,1,1,0)}, \partial_{(1,1,0,2)} \\ \partial_{(1,2,0,1)} + \partial_{(0,0,1,1)}, \partial_{(1,0,0,3)} \end{array} \right\}. \end{aligned}$$

Hence, $H_I(1,1) = 4$.

4 Ideal Operations

For an M-graded ideal $I \subseteq R = \mathbb{C}[x_1, \ldots, x_N]$, there is an expected duality between I_m and $D_0^m(I)$ for every $m \in M$. This allows for ideal operations to be translated to operations of multi-graded Macaulay dual spaces as summarized below.

4.1 Ideal membership test

The following summarizes testing membership using a multi-graded Macaulay dual space.

Corollary 4.1 *If R is M-graded, $I \subseteq R$ is an M-graded ideal, and $g \in R_m$, then $g \in I$ if and only if $\partial(g) = 0$ for all $\partial \in D_0^m(I)$.*

Proof. This follows immediately from the definition of $D_0^m(I)$ and Thm. 3.2. □

Note that the key to this membership test is the multi-grading provided by Thm. 3.2. Since this was not included in the statement of [17, Thm. 4.6], a counter example for that statement was provided in [14, §. 4], which is considered next in the multi-graded context.

Example 4.2 *For $R = \mathbb{C}[x_1, x_2]$, consider the $M = \mathbb{Z}$-grading with $\deg(x_i) = i$. The ideal $J = \langle x_2 - x_1^2, x_2^2 \rangle$ is M-graded and $g = x_2 \in R_2$. It is easy to verify that*

$$D_0(J) = D_0^0(J) \oplus D_0^1(J) \oplus D_0^2(J) \oplus D_0^3(J)$$

where

$$\begin{array}{c} D_0^0(J) = \mathrm{span}_{\mathbb{C}}\{\partial_{(0,0)}\}, D_0^1(J) = \mathrm{span}_{\mathbb{C}}\{\partial_{(1,0)}\}, \\ D_0^2(J) = \mathrm{span}_{\mathbb{C}}\{\partial_{(0,1)} + \partial_{(2,0)}\}, D_0^3(J) = \mathrm{span}_{\mathbb{C}}\{\partial_{(1,1)} + \partial_{(3,0)}\}. \end{array} \tag{9}$$

In particular, using $D_0^2(J)$, since $\left(\partial_{(0,1)} + \partial_{(2,0)}\right)(g) = 1 \neq 0$, one concludes $g \notin J$.

4.2 Inclusion, sum, and intersection

The following considers additional ideal operations.

Corollary 4.3 *Suppose that R is M-graded and $I, J \subseteq R$ are M-graded ideals.*

1. *$I \subset J$ if and only if $D_0^m(I) \supset D_0^m(J)$ for all $m \in M$.*
2. *$D_0^m(I + J) = D_0^m(I) \cap D_0^m(J)$ for every $m \in M$.*
3. *$D_0^m(I \cap J) = D_0^m(I) + D_0^m(J)$ for every $m \in M$.*

Proof. The first statement immediately follows from Cor. 3.6.

Since $I, J \subseteq I + J$, we know by the first statement that

$$D_0^m(I) \cap D_0^m(J) \supseteq D_0^m(I + J)$$

for all $m \in M$. On the other hand, if $\partial \in D_0^m(I) \cap D_0^m(J)$ and $f + g \in I + J$, then $\partial(f + g) = \partial(f) + \partial(g) = 0$. Hence, $\partial \in D_0^m(I + J)$ showing the second statement.

Since $I \cap J \subseteq I, J$, the first statement implies

$$D_0^m(I \cap J) \supseteq D_0^m(I) + D_0^m(J)$$

for all $m \in M$. One way to see equality is by verifying that they have the same dimension, namely, for all $m \in M$,

$$\begin{aligned}
\dim_{\mathbb{C}} D_0^m(I \cap J) &= H_{I \cap J}(m) \\
&= H_I(m) + H_J(m) - H_{I+J}(m) \\
&= \dim_{\mathbb{C}}(D_0^m(I)) + \dim_{\mathbb{C}}(D_0^m(J)) \\
&\qquad - \dim_{\mathbb{C}}(D_0^m(I) \cap D_0^m(J)) \\
&= \dim_{\mathbb{C}}(D_0^m(I) + D_0^m(J)).
\end{aligned}$$

□

Although the first statement in Cor. 4.3 regarding ideal containment suggests that one needs to test all $m \in M$, coupling with Cor. 3.6 provides that one only needs to test the values of $m \in M$ for which there is a generator of either I or J.

Example 4.4 *Since ideals I from Ex. 2.10 and J from Ex. 4.2 have the same grading, one can observe from* (7) *and* (9) *that $J \subsetneq I$. In fact, $D_0^k(I) = D_0^k(J)$ for $k = 0, 1, 2$ and $D_0^3(I) = \{0\} \subsetneq D_0^3(J)$.*

4.3 Ideal quotient

For M-graded ideals $I, J \subseteq R$, the *quotient of I by J* is the ideal

$$I : J = \{f \in R : \ f \cdot J \subseteq I\}.$$

In particular, if $g \in R$, then

$$I : \langle g \rangle = I : g = \{f \in R \ : \ f \cdot g \in I\}.$$

Hence, if $J = \langle g_1, \ldots, g_t \rangle$, then

$$I : J = \bigcap_{i=1}^{t} I : g_i$$

so that, for every $m \in M$, Cor. 4.3 yields

$$D_0^m(I : J) = D_0^m\left(\bigcap_{i=1}^{t} I : g_i\right) = \sum_{i=1}^{t} D_0^m(I : g_i). \tag{10}$$

Thus, one needs to only consider quotients by principal ideals.

The maps Φ_i from (4) arise from quotients by variables.

Proposition 4.5 *For every $m \in M$ and $i = 1, \ldots, N$,*

$$\Phi_i(D_0^m(I)) = \Phi_i(D_0^m(I \cap \langle x_i \rangle)) = D_0^{m-Ae_i}(I : x_i).$$

Proof. Let $\partial \in D_0^m(I)$ and $g \in I : x_i$. Since $x_i g \in I$, (5) yields

$$\Phi_i(\partial)(g) = \partial(x_i g) = 0.$$

Hence, $\Phi_i(D_0^m(I)) \subseteq D_0^{m-Ae_i}(I : x_i)$.

Let $\delta \in D_0^{m-Ae_i}(I : x_i)$ and $f \in I \cap \langle x_i \rangle$. Define $h = f/x_i \in I : x_i$. Consider the linear map $\Psi_i : D_0^{m-Ae_i} \to D_0^m$ with $\Psi_i(\partial_\alpha) = \partial_{\alpha+e_i}$. Clearly, $\Phi_i \circ \Psi_i$ is the identity map. Hence,

$$\Psi_i(\delta)(f) = \Psi_i(\delta)(x_i h) = \Phi_i(\Psi_i(\delta)(h)) = \delta(h) = 0$$

so that $D_0^{m-Aei}(I : xi) \subseteq \Phi_i(D_0^m(I \cap \langle x_i \rangle))$.

Finally, suppose $\delta = \Phi_i(\partial) \in \Phi_i(D_0^m(I \cap \langle x_i \rangle))$ and $f \in I$. Then,

$$\delta(f) = \Phi_i(\partial)(f) = \partial(x_i f) = 0$$

so that $\Phi_i(D_0^m(I \cap \langle x_i \rangle)) \subseteq \Phi_i(D_0^m(I))$. $\square$

Example 4.6 *Continuing with the setup from Ex. 2.10,* (3) *provides*

$$\begin{aligned} D_0(I : x_1) &= \Phi_1(D_0(I)) = \operatorname{span}_{\mathbb{C}}\left\{0, \partial_{(0,0)}, \partial_{(1,0)}\right\}, \\ D_0(I : x_2) &= \Phi_2(D_0(I)) = \operatorname{span}_{\mathbb{C}}\left\{0, 0, \partial_{(0,0)}\right\}. \end{aligned}$$

Hence, the multiplicity of 0 *with respect to $I : x_1$ and $I : x_2$ is* 2 *and* 1*, respectively.*

One can generalize from quotients by a variable x_i using Φ_i from [24] via (5) to quotients by a M-homogeneous polynomial g by defining the linear operator $\Phi_g : D_0 \to D_0$ where

$$\Phi_g(\partial)(f) = \partial(gf).$$

The Leibniz rule provides

$$\Phi_g(\partial_\alpha) = \sum_{\substack{\gamma \in \mathbb{Z}_{\geqslant 0}^N \\ A \cdot \gamma = \deg g}} \partial_\gamma(g) \partial_{\alpha-\gamma}$$

which has degree $A \cdot (\alpha - \gamma) = A \cdot \alpha - \deg g$. The following shows that Φ_g has a right-sided inverse by providing an explicit construction.

Theorem 4.7 *For $g \in R_m$, there is a linear function Ψ_g such that $\Phi_g \circ \Psi_g$ is the identity map.*

Proof. Let $\prec$ be a lexicographic ordering on $(\mathbb{Z}_{\geqslant 0})^N$ and write

$$g = \sum_{\alpha} g_\alpha x^\alpha.$$

Define $\alpha_0 = \min_\prec\{\alpha \ : \ g_\alpha \neq 0\}$. For any $\beta, \gamma \in (\mathbb{Z}_{\geqslant 0})^N$, define

$$G(\beta,\gamma) = \begin{cases} g_{\gamma-\beta} & \text{if } \gamma \preceq \beta \\ 0 & \text{otherwise.} \end{cases}.$$

Thus, define $\Psi_g(\partial_\beta) = \sum_\alpha c_\alpha(\beta)\partial_\alpha$ where

$$c_\alpha(\beta) = \begin{cases} \frac{1}{g_{\alpha_0}}\left(\delta(\alpha-\alpha_0,\beta) - \sum_{\gamma \succ \alpha} G(\alpha-\alpha_0,\gamma)c_\gamma(\beta)\right) & \text{if } \alpha_0 \preceq \alpha \\ 0 & \text{otherwise} \end{cases}$$

with $\delta(\gamma,\beta)$ being Kronecker's delta. Consider the following

$$\begin{aligned} \Phi_g(\Psi_g(\partial_\beta)) &= \sum_\alpha c_\alpha(\beta)\Phi_g(\partial_\alpha) \\ &= \sum_\alpha c_\alpha(\beta) \sum_{\substack{\gamma \\ A\cdot\gamma = \deg g}} \partial_\gamma(g)\partial_{\alpha-\gamma} \\ &= \sum_\alpha c_\alpha(\beta) \sum_{\substack{\gamma \\ A\cdot\gamma = \deg g}} G(\gamma,\alpha)\partial_{\alpha-\gamma} \\ &= \sum_{\substack{\gamma \\ A\cdot\gamma = \deg g}} \left(\sum_\alpha G(\gamma,\alpha)c_\alpha(\beta)\right)\partial_\gamma. \end{aligned}$$

All that remains is to show $\sum_\alpha G(\gamma,\alpha)c_\alpha(\beta) = \delta(\gamma,\beta)$. To that end, we break up the sum as follows:

$$\begin{aligned} \sum_\alpha G(\gamma,\alpha)c_\alpha(\beta) = &\sum_{\alpha \prec \gamma+\alpha_0} G(\gamma,\alpha)c_\alpha(\beta) \\ &+ G(\gamma,\gamma+\alpha_0)c_{\gamma+\alpha_0}(\beta) + \sum_{\alpha \succ \gamma+\alpha_0} G(\gamma,\alpha)c_\alpha(\beta). \end{aligned}$$

Suppose $\alpha \prec \gamma + \alpha_0$. If $\alpha \not\succcurlyeq \alpha_0$, then $c_\alpha(\beta) = 0$. Otherwise, $G(\gamma,\alpha) = 0$ by construction of α_0. Thus,

$$\sum_{\alpha \prec \gamma+\alpha_0} G(\gamma,\alpha)c_\alpha(\beta) = 0.$$

The definition of $c_{\gamma+\alpha_0}(\beta)$ finishes the claim due to the following.

$$G(\gamma,\gamma+\alpha_0)c_{\gamma+\alpha_0}(\beta) = \delta(\gamma,\beta) - \sum_{\alpha \succ \gamma+\alpha_0} G(\gamma,\alpha)c_\alpha(\beta).$$

□

Although the following could be proved using ideal operations, it also follows from the right inverse.

Theorem 4.8 *Let $I, J \subseteq R$ be M-graded ideals, $g \in R$ be an M-homogeneous polynomial, and $m \in M$. Suppose that $J = \langle g_1, \ldots, g_t \rangle$ where each g_i is M-homogeneous.*

1. $\Phi_g(D_0^{m+\deg g}(I)) = \Phi_g(D_0^{m+\deg g}(I \cap \langle g \rangle)) = D_0^m(I : g)$.
2. $\sum_{i=1}^t \Phi_{g_i}(D_0^{m+\deg g_i}(I)) = \sum_{i=1}^t D_0^m(I : g_i) = D_0^m(I : J)$.

Proof. The first follows a similar approach as the proof of Prop. 4.5 using Thm. 4.7. The second follows from the first and (10). □

Example 4.9 *Consider computing $J : I$ where I is from Ex. 2.10 and J is from Ex. 4.2. Let f_1 and f_2 as in Ex. 2.12 be generators for I with the Macaulay dual space for J provided in* (9). *Since* $\deg f_1 = 3$*, one only needs to compute*

$$\Phi_{f_1}\left(\partial_{(1,1)} + \partial_{(3,0)}\right) = \Phi_{f_1}\left(\partial_{(1,1)}\right) + \Phi_{f_1}\left(\partial_{(3,0)}\right) = -3/16\ \partial_{(0,0)}$$

to see that $D_0(J : f_1) = \operatorname{span}_{\mathbb{C}}\{\partial_{(0,0)}\}$*. Now, since* $\deg f_2 = 2$*, we start with*

$$\Phi_{f_2}\left(\partial_{(0,1)} + \partial_{(2,0)}\right) = \partial_{(0,0)} - \partial_{(0,0)} = 0$$

so that $D_0^0(J : f_2) = \{0\}$*. For completeness, one can verify that*

$$\Phi_{f_2}\left(\partial_{(1,1)} + \partial_{(3,0)}\right) = \partial_{(1,0)} - \partial_{(1,0)} = 0.$$

Hence, $D_0(J : f_2) = \{0\}$ *which was expected since* $f_2 \in J$ *yields* $J : f_2 = \langle 1 \rangle$*. Therefore,*

$$D_0(J : I) = \operatorname{span}_{\mathbb{C}}\{\partial_{(0,0)}\}$$

which corresponds with $J : I = \langle x_1, x_2 \rangle$*.*

In general, one can repeatedly compute ideal quotients, say

$$I : J, \quad (I : J) : J, \quad ((I : J) : J) : J, \ldots$$

denoted $I : J$, $I : J^2$, $I : J^3$, $\ldots$, respectively. This sequence stabilizes after finitely many terms and is equal to the *saturation of I with respect to J*, namely

$$I : J^\infty = \{f \in R \ : \ f \cdot J^n \subseteq I \text{ for some } n \geqslant 1\}.$$

In particular, $I : J^p = I : J^{p+1}$ if and only if $I : J^p = I : J^\infty$. Saturation is useful, for example, to compute information regarding a non-homogeneous ideal by homogenizing and saturating away the component at infinity.

5 Algorithm and Software

The results from Sec. 3 and 4 lead to algorithms for computing multi-graded Macaulay dual spaces as summarized below. Our proof-of-concept implementation using `Macaulay2` [9] is available at `https://doi.org/10.7274/j098z894548`.

In an effort to simplify our procedures and implementation, we assume that the multi-grading is a $\mathbb{Z}^k$-grading. Moreover, we assume, for every $m \in \mathbb{Z}^k$, R_m, the m-graded component of $R =$

$\mathbb{C}[x_1, \ldots, x_N]$, is a finite dimensional complex vector space. Additionally, we will only consider gradings that arise from a matrix $A \in \mathbb{Z}^{k \times N}$ where $\deg(x_i)$ is the i^{th} column of A. Since it is useful for us to have a half-space description of the weight semigroup ω as defined in (8), we assume there is a matrix $B \in \mathbb{Z}^{p \times k}$ where the rows are the normal vectors of the half-spaces whose intersection is the weight cone $\omega_{\mathbb{R}}$, that is,

$$\omega_{\mathbb{R}} = \{y \in \mathbb{R}^k \ : \ By \geqslant 0\}$$

and $\omega = \omega_{\mathbb{R}} \cap \mathbb{Z}^k$, i.e., ω is *saturated*. Given $m \in \mathbb{Z}^k$, this enables one to quickly ascertain whether or not m is contained in ω or not.

As stated in Sec. 3, in order to compute $D_0^m(I)$ for some $m \in \mathbb{Z}^k$, we first fix a total ordering on the elements on the set

$$\omega_m = \{s \in \omega \ : \ s \preceq_\omega m\}.$$

Since ω is a saturated semigroup, the set ω_m can be realized as the lattice points in a polyhedron, e.g., see Ex. 3.9. Hence, a lattice point $s \in \omega$ is less than m in the partial order if and only if $B(m-s) \geqslant 0$ and $Bs \geqslant 0$. Therefore,

$$\omega_m = \{s \in \mathbb{Z}^k \ : \ Bm \geqslant Bs \geqslant 0\}.$$

Our first procedure below details how to find a linear extension of the partial order $\prec_\omega$ on ω_m. Note that the most expensive part of this procedure is in computing the lattice points in ω_m. Our implementation used the `Polyhedra` package [4].

Procedure SortLatticePoints(A, B, m)

Input The matrix $A \in \mathbb{Z}^{k \times N}$ where the i^{th} column is $\deg(x_i)$. The matrix $B \in \mathbb{Z}^{p \times k}$ where $\omega_{\mathbb{R}} = \{y \ : \ By \geqslant 0\}$. A lattice point $m \in \omega$.

Output A total ordering of the lattice points in ω less than m in the partial ordering.

Begin

1. Let **Unsorted** $:= \{s \in \omega \ : \ s \preceq_\omega m\} \backslash \{0\}$ be the non-zero lattice points in ω less than m in the partial order. This list is the set of non-zero integral solutions, s, to the system of inequalities $Bm \geqslant Bs \geqslant 0$. Let **Sorted** $:= \{0\}$
2. For every $s \in$ **Unsorted**, check for every $i = 1, \ldots, N$ if
   (a) $s - Ae_i \in$ **Sorted** or
   (b) $B(s - Ae_i) \ngeqslant 0$, so $s - Ae_i \notin \omega$.
3. If one of (2a) or (2b) is true for every i, then add s to **Sorted** and delete it from **Unsorted**.
4. Repeat steps 2 and 3 until **Unsorted** is empty.

Return Sorted

The following show the correctness of this procedure.

Lemma 5.1 *The set ω_m is finite.*

Proof. Note that ω_m is the set of lattice points in the polyhedron $\omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$. Our assumption that $R_0 = \mathbb{C}$ implies that $\omega_{\mathbb{R}}$ is a pointed polyhedral cone, i.e., $\mathrm{null}(B) = \omega \cap (-\omega) = \{0\}$. In order to show that ω_m is finite, it is enough to show that $\omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$ is bounded. We show this via contradiction.

Suppose $\omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$ is unbounded. Then, there must exist $s \in \omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$ and a $v \in \mathbb{R}^k \backslash \{0\}$ so that for every $\lambda \geqslant 0$, $s + \lambda v \in \omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$. Since $s + \lambda v \in \omega_{\mathbb{R}}$, we have

$$B(s + \lambda v) \geqslant 0,$$

and since $s + \lambda v \in m - \omega_{\mathbb{R}}$, we have

$$B(m - s - \lambda v) \geqslant 0$$

for all $\lambda \geqslant 0$. Solving each inequality for $\lambda B v$ yields the following

$$B(m - s) \geqslant \lambda B v \geqslant -Bs$$

for every $\lambda \geqslant 0$. Since s, m, v, and B are all fixed, the only way this holds true is if $Bv = 0$. This contradicts $\omega_{\mathbb{R}}$ is pointed yielding that $\omega_{\mathbb{R}} \cap (m - \omega_{\mathbb{R}})$ is bounded. $\square$

Theorem 5.2 *The procedure SortLatticePoints terminates and the output is a linear extension of the partial order* $\preceq_\omega$.

Proof. Since, at each step in the procedure, at least one element is sorted. Finiteness from Lemma 5.1 yields this procedure must terminate in finitely many steps. For the second claim, suppose we are at the step in the procedure where we are about to add s to **Sorted**. The elements $t \in \omega$ which are covered by s are all of the form $s - Ae_j$ for some $j \in \{1, \ldots, N\}$. Therefore, by induction, when s is sorted, all elements less than s in the partial order have already been sorted and that no elements greater than s have been sorted yielding a linear extension. $\square$

Our second procedure below computes $D_0^m(I)$ by utilizing the closedness subspace condition. The correctness of this procedure is the content of Cor. 3.6 and illustrated in Ex. 3.9.

Procedure DualSpace(m, I, A, B)

Input The matrix $A \in \mathbb{Z}^{k \times N}$ where the i^{th} column is $\deg(x_i)$. A lattice point $m \in \omega$. A $\mathbb{Z}^k$-graded ideal $I = \langle f_1, \ldots, f_\ell \rangle$ with $\deg(f_i) = d_i$. The matrix $B \in \mathbb{Z}^{p \times k}$ with $\omega_{\mathbb{R}} = \{y \ : \ By \geqslant 0\}$.

Output A basis for $D_0^m(I)$.

Begin

1. Sort the lattice points in ω less than or equal to m, say $\{s_1, \ldots, s_r\} := \text{SortLatticePoints}(A, B, m)$ where $s_1 = 0$ and $s_r = m$.
2. Set $C_0^0(I) := \text{span}_{\mathbb{C}}\{\partial_{(0,\ldots,0)}\}$ and $D_0^0(I) := C_0^0(I)$.
3. For i from 2 to r do
   (a) Compute a basis of $C_0^{s_i}(I) := \bigcap_{j=1}^{N} \Phi_j^{-1}(D_0^{s_i - Ae_j}(I))$.
   (b) Impose the linear conditions that $\partial(f_j) = 0$ on the basis for $C_0^{s_i}(I)$ for $j = 1, \ldots, t$ to compute a basis for

$$D_0^{s_i}(I) := \{\partial \in C_0^{s_i}(I) \ : \ \partial(f_j) = 0 \text{ for } 1 \leqslant j \leqslant t\}$$

Return a basis for $D_0^{s_r}(I) = D_0^m(I)$

One approach to performing the computations in the procedure DualSpace is to utilize closedness subspaces [13,26] computations. Another approach is to use integration [21] (see also [15,16]).

6 Examples

The following three examples were computed using our `Macaulay2` implementation described in Sec. 5. Since our implementation is a proof-of-concept, it is not yet competitive with highly researched and optimized Gröbner basis methods. However, as mentioned in the Introduction, one advantage of a dual space approach is that one can start computing dual spaces immediately up to a given degree which would be particularly useful for problems in which computing a Gröbner basis is computationally more difficult than the following examples. See Sec. 7 for comments regarding future research directions including improved efficiency and incorporating parallel linear algebra routines.

6.1 Hirzebruch surface

Examples 2.4, 3.1, and 3.9 consider aspects of the Hirzebruch surface. The following considers $\mathcal{H}_2$, which is a smooth projective toric surface. The Cox ring is a polynomial ring $R = \mathbb{C}[x_1, x_2, x_3, x_4]$ which is graded by the Picard group, namely $\mathbb{Z}^2$. The degree of each x_i is given by the equivalence class of e_i in the cokernel of the transpose of

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{pmatrix}.$$

After choosing a basis, $\deg(x_i) \in \mathbb{Z}^2$ is given by the i^{th} column of

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}$$

which corresponds with the $r = 2$ case in Ex. 3.1.

Consider $f = x_1^2 x_2^6 + x_1^2 x_2^3 x_4^3 - x_3^2 x_4^2$ which is irreducible with $\deg f = (2, 2)$. Let $I = \langle f \rangle$. By the toric ideal-variety correspondence [5, Prop. 5.2.4], I cuts out an irreducible curve $C \subset \mathcal{H}_2$. The values from $(0, 0)$ to $(4, 4)$ of the multi-graded Hilbert function $H_I(i, j)$ are given in the table below. A dash is put in position (i, j) if $(i, j) \succ (4, 4)$ or $H_I(i, j) = 0$.

j \ i	0	1	2	3	4
12	13	-	-	-	-
11	12	-	-	-	-
10	11	24	-	-	-
9	10	22	-	-	-
8	9	20	26	-	-
7	8	18	24	-	-
6	7	16	22	28	-
5	6	14	20	26	-
4	5	12	18	24	30
3	4	10	16	22	28
2	3	8	14	20	26
1	2	6	12	18	24
0	1	4	9	15	21
-1	-	2	6	12	18
-2	-	1	4	9	15
-3	-	-	2	6	12
-4	-	-	1	4	9
-5	-	-	-	2	6
-6	-	-	-	1	4
-7	-	-	-	-	2
-8	-	-	-	-	1

Since the class of $(1, 1)$ in $\mathrm{Pic}(\mathcal{H}_2)$ is very ample, we can embed C in $\mathbb{P}^5$ via this divisor. By looking at the values $H_I(a, a)$ for $a \geqslant 2$, we see that the Hilbert polynomial of $C \subseteq \mathbb{P}^5$ is $8a - 2$ from which we conclude that C has degree 8 and arithmetic genus 3.

6.2 Parameter geography

In [23], the authors study the following parameterized system $\Phi_1(u,v;\sigma) = \Phi_2(u,v;\sigma) = 0$ where $\theta_1,\ldots,\theta_8$ are taking to be generic and $\zeta = 1$:

$$\begin{gathered}\Phi_1(u,v;\sigma) = \\ \theta_1 v^2 + \zeta uv + \theta_2\zeta^2 u^2 + (\theta_1\theta_3 - \theta_1\theta_3\sigma - \theta_1 + \theta_7\zeta)uv^2 \\ +(\theta_4\zeta - \theta_4\zeta\sigma - \zeta + \theta_2\theta_8\zeta^2)u^2 v \\ +(\theta_2\theta_5\zeta^2 - \theta_2\theta_5\zeta^2\sigma - \theta_2\zeta^2)u^3 + \theta_1\theta_6 v^3 - (\theta_1\theta_3 + \theta_7\zeta)u^2v^2 \\ -(\theta_4\zeta + \theta_2\theta_8\zeta^2)u^3 v - \theta_2\theta_5\zeta^2 u^4 - \theta_1\theta_6 uv^3 \\ \Phi_2(u,v;\sigma) = \\ \theta_1 v^2 + \zeta uv + \theta_2\zeta^2 u^2 + (\theta_1\theta_6 - \theta_1\theta_6 - \theta_1)v^3 \\ +(\theta_7\zeta - \theta_7\zeta\sigma - \zeta + \theta_1\theta_3)uv^2 \\ +(\theta_2\theta_8\zeta^2 - \theta_2\theta_8\zeta^2\sigma - \theta_2\zeta^2 + \theta_4\zeta)u^2 v + \theta_2\theta_5\zeta^2 u^3 \\ -(\theta_1\theta_3 + \theta_7\zeta)uv^3 - (\theta_4\zeta + \theta_2\theta_8\zeta^2)u^2v^2 - \theta_2\theta_5\zeta^2 u^3 v - \theta_1\theta_6 v^4\end{gathered}$$

Consider homogenizing by adding τ and w to consider the polynomial ring $\mathbb{C}[\sigma,\tau,u,v,w]$ where $\deg\sigma = \deg\tau = (1,0)$ and $\deg u = \deg v = \deg w = (0,1)$. This yields a $\mathbb{Z}^2$-graded ideal with 2 generators and we view the zero locus of this system as a reducible curve in $\mathbb{P}^1 \times \mathbb{P}^2$. After slicing this system with a generic linear form of degree $(0,1)$, the following table lists multi-graded Hilbert function up to $(10,10)$ computed via Macaulay dual spaces.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10	11
1	2	4	6	8	8	8	8	8	8	8	8
2	3	6	9	12	11	10	9	8	8	8	8
3	4	8	12	16	14	12	10	8	8	8	8
4	5	10	15	20	17	14	11	8	8	8	8
5	6	12	18	24	20	16	12	8	8	8	8
6	7	14	21	28	23	18	13	8	8	8	8
7	8	16	24	32	26	20	14	8	8	8	8
8	9	18	27	36	29	22	15	8	8	8	8
9	10	20	30	40	32	24	16	8	8	8	8
10	11	22	33	44	35	26	17	8	8	8	8

Using Macaulay dual spaces, we saturated away the components lying along coordinate axes resulting in the following multi-graded Hilbert function.

$i \backslash j$	0	1	2	3	4	5	6
0	1	2	3	4	5	6	7
1	2	4	6	7	7	7	7
2	3	6	8	7	7	7	7
3	4	8	9	7	7	7	7
4	5	10	10	7	7	7	7
5	6	12	11	7	7	7	7
6	7	14	12	7	7	7	7

Although this table provides information when viewed as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^2$, the Hilbert function of the system viewed in $\mathbb{P}^4$ via a Segre product is exactly $H_I(i,i)$. Hence, since the values along the main diagonal stabilize at 7, there are 7 non-zero complex solutions to this system for a generic choice of σ matching the results in [23].

6.3 Chemical reaction network

Finally, consider a chemical reaction network known as the one-site phosphorylation cycle [12]. The *steady-state degree* of this chemical reaction network is the number of complex solutions to the following system for generic parameters c_A and k_{ij}.

$$\begin{aligned}
f_1 &= x_E + x_{X_1} - c_E - c_{X_1} \\
f_2 &= x_F + x_{Y_1} - c_F - c_{Y_1} \\
f_3 &= x_{S_0} + x_{S_1} - x_E - x_F - c_{S_0} - c_{S_1} + c_E + c_F \\
f_4 &= -k_{01} x_{S_0} x_E + k_{10} x_{X_1} + k_{45} x_{Y_1} \\
f_5 &= -k_{34} x_{S_1} x_F + k_{12} x_{X_1} + k_{43} x_{Y_1} \\
f_6 &= k_{01} x_{S_0} x_E - (k_{10} + k_{12}) x_{X_1} \\
f_7 &= k_{34} x_{S_1} x_F - (k_{43} + k_{45}) x_{Y_1}
\end{aligned}$$

One way to compute the steady-state degree is to homogenize with respect to a new variable t, saturate away the hyperplane at infinity, and compute the degree of the resulting projective variety. Letting I be the ideal generated by the homogenization with respect to t of $f_1, \ldots, f_7$, one obtains the following using Macaulay dual spaces.

k	0	1	2	3	4	5	6	7	8	9	10
$H_I(k)$	1	4	7	8	8	8	8	8	8	8	8
$H_{I:t}(k)$	1	3	4	4	4	4	4	4	4	4	-
$H_{I:t^2}(k)$	1	3	3	3	3	3	3	3	3	-	-
$H_{I:t^3}(k)$	1	3	3	3	3	3	3	3	-	-	-

Since $H_{I:t^2} = H_{I:t^3}$, we can conclude that $H_{I:t^2} = H_{I:t^\infty}$. Hence, this computation shows the steady-state degree is 3 in agreement with the results found in [12].

7 Conclusion

Building on a key theoretical contribution in Thm. 3.2 which shows that the Macaulay dual space of a multi-graded ideal is multi-graded, algorithms are presented for performing computations related to such dual spaces including using Thm. 4.8 which describes how to compute ideal quotients using dual spaces. Using a proof-of-concept implementation in `Macaulay2` [9], ideal computations were performed using multi-graded dual spaces on several different examples.

Some future research directions include incorporating more efficient and parallel numerical linear algebra routines into the implementation to improve the performance, consider examples where obtainig a Gröbner basis is computationally more challenging, consider errors and stability when performing numerical linear algebra routines with dual spaces, and investigate the complexity of performing computations using dual spaces.

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