Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme

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Abstract

In numerical algebraic geometry, a witness point set W is a key object for performing numerical computations on a projective scheme X of pure dimension $d > 0$ defined over $\mathbb{C}$. If X is arithmetically Cohen-Macaulay, W can also be used to obtain information about X, such as the initial degree of the ideal generated by X and its Castelnuovo-Mumford regularity. Due to this relationship, we develop a new numerical algebraic geometric test for deciding if X is arithmetically Cohen-Macaulay using points which lie (approximately) on a general curve section C of X. For any curve, we also compute other information such as the arithmetic genus and index of regularity. Several examples are presented showing the effectiveness of this method, even when the ideal of X is unknown.

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1 Introduction

Let $\mathbb{P}^n$ denote the projective space of dimension n over $\mathbb{C}$ and let $X \subset \mathbb{P}^n$ be a pure-dimensional projective scheme of dimension $d > 0$. A fundamental goal in computational algebraic geometry is to compute information about X, especially when the ideal of X is not known. If X is arithmetically Cohen-Macaulay (aCM), information about X, such as the initial degree of the ideal generated by X, Castelnuovo-Mumford regularity, Hilbert function, Hilbert polynomial, and Hilbert series, can be recovered from general hyperplane (and hypersurface) sections of X. Therefore, a procedure for deciding the arithmetically Cohen-Macaulayness of a scheme is a key problem in computational algebraic geometry. For any curve (a pure-dimensional projective scheme of dimension 1), invariants such as the arithmetic genus, Castelnuovo-Mumford regularity, and index of regularity also provide important information.

Since defining equations for X may not be known, e.g., X may be a pure-dimensional component of some other scheme Y or the image of an algebraic set under an algebraic map, we propose a test for deciding if X is aCM given the following. First, one needs the ability to sample

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points lying (approximately) on each irreducible component of a general curve section of X. If each such component is generically reduced, then this is all that is required. For each generically nonreduced component C of the general curve section of X, one needs to be able to compute the local multiplicity structure, via a Macaulay dual space, with respect to the intersection of C and a general hyperplane. For example, suppose that f is a system of homogeneous polynomials and, by abuse of notation, define $f^{-1}(0)$ to be the scheme defined by f. These computations can be performed when X is a pure-dimensional subset of the scheme $f^{-1}(0)$. That is, X may be a proper subset of $f^{-1}(0)$ so that equations defining X exactly are not known, e.g., see Section 5.5.

One key fact of schemes of dimension at least 2 is that arithmetically Cohen-Macaulayness is preserved under slicing by a general hyperplane (or hypersurface). In particular, a pure-dimensional scheme X of positive dimension is aCM if and only if a general curve section of X is aCM. A numerical test is provided in [17] that determines if a curve C is aCM. This test relies on computing Hilbert functions of zero-dimensional schemes defined by intersecting C with general hypersurfaces of various degrees. Due to the increasingly higher degree zero-dimensional schemes under consideration, this test becomes impractical for curves of even moderate degree.

The main result of this paper is an effective version of a test for arithmetically Cohen-Macaulayness (Corollary 3.3) which immediately yields an algorithm given the ability to compute Hilbert functions up to a specified degree. Section 2.5 considers this Hilbert function computation building on [13] (see also [23]). The upper bound on our test is sharp, as demonstrated by the example in Section 5.3, which is also used to compare our new approach with that of [17]. We also describe how to compute other invariants for any curve.

Two important topics related to symbolic computations in algebraic geometry are minimal free resolutions and complexity of Gröbner basis computations. The arithmetically Cohen-Macaulayness of a scheme is related to the length of a minimal free resolution via the relationship between projective dimension and depth in the Auslander-Buchsbaum formula (see [8] for a general overview). Over fields of characteristic zero, e.g., $\mathbb{Q}$ and $\mathbb{C}$, the Castelnuovo-Mumford regularity of the ideal I defining a scheme X is equal to the maximum degree of the elements in a Gröbner basis when working with generic coordinates in the reverse lexicographic ordering (see [6] for more information). Thus, the Castelnuovo-Mumford regularity provides a measure of complexity for performing symbolic computations on I.

The arithmetic genus and geometric genus are two invariants of a curve C of particular interest in computational algebraic geometry. These genera must be equal if C is smooth. A numerical algebraic geometric procedure for computing the geometric genus is presented in [5] which was extended in [19] to curves which arise as the image of an algebraic set under a polynomial map. The geometric genus of a general four-bar coupler curve was verified to be one in [5] with the arithmetic genus of such a curve computed in Section 5.1.

Even though it is not directly related to deciding the arithmetically Cohen-Macaulayness of a projective scheme, we note that a symbolic-numeric approach for computing Hilbert functions and Hilbert polynomials in local rings is described in [23]. This approach is based on computing the Macaulay dual space of an ideal at a point that (approximately) lies in the solution set of the ideal. There are no assumptions related to the point, e.g., multiple components could pass through the point including embedded components. The practicality of this approach, especially for high dimensional components, is limited by the stopping criterion which requires that the Macaulay dual space is computed in degree up to twice the maximum degree of a "g-corner."

The rest of this article is organized as follows. Section 2 provides the necessary background information to describe the numerical algebraic geometry methods used throughout. Section 3 develops an algorithm for deciding the arithmetically Cohen-Macaulayness of a curve with Section 4 considering the general case. Several examples are presented in Section 5. In particular, Section 5.2 considers $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, which is related to the so-called Salmon problem [1]. Section 5.4 studies a problem arising from theoretical physics related to the nature of the vacuum space in the Minimal Supersymmetric Standard Model.

2 Background

2.1 Arithmetically Cohen-Macaulay

A positive dimensional projective scheme X with ideal sheaf $\mathcal{I}_X$ is said to be *arithmetically Cohen-Macaulay* (aCM) if

$$H^i_*(\mathcal{I}_X) = 0 \quad \text{for} \quad 1 \leq i \leq \dim X \tag{1}$$

where $H^i_*(\mathcal{I}_X)$ denotes the i^{th} cohomology module of $\mathcal{I}_X$. Equivalently, a projective scheme X is aCM if and only if its coordinate ring has Krull dimension equal to its depth [24] (in this case, the coordinate ring is called a Cohen-Macaulay ring). All zero-dimensional schemes are aCM and a consequence of the above definition is that all aCM schemes must be pure-dimensional.

Example 2.1 Consider the curves in $\mathbb{P}^3$:

$$C = \{(s^3, s^2t, st^2, t^3) \mid (s,t) \in \mathbb{P}^1\} \quad \text{and} \quad Q = \{(s^4, s^3t, st^3, t^4) \mid (s,t) \in \mathbb{P}^1\}$$

with corresponding ideals

$$I(C) = \langle xz - y^2, yw - z^2, xw - yz \rangle \quad \text{and} \quad I(Q) = \langle xw - yz, x^2z - y^3, xz^2 - y^2w, z^3 - yw^2 \rangle.$$

The curve C is the twisted cubic curve while Q is a smooth rational quartic curve. The twisted cubic curve C is well-known to be aCM while [10, Ex. 1.7] shows that Q is not. We verify this statement using the definition by comparing the Krull dimension and depth. Clearly, both coordinates rings have Krull dimension 2. Computations using the `Depth` package of `Macaulay2` [12] find that the depth of C is 2 and the depth of Q is 1.

The cohomology characterization presented in (1) imposes conditions on the Hilbert function, which is defined next. For a curve, Corollary 3.3 presents an effective test of arithmetically Cohen-Macaulayness that can be performed using numerical algebraic geometric computations.

2.2 Hilbert functions, genus, and regularity

Let $X \subset \mathbb{P}^n$ be a projective scheme with corresponding homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$. Let $\mathbb{C}[x_0, \ldots, x_n]_t$ denote the vector space of homogeneous polynomials of degree t, which has dimension $\binom{n+t}{t}$, and $I_t = I \cap \mathbb{C}[x_0, \ldots, x_n]_t$. The *Hilbert function* of X is defined as

$$HF_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \binom{n+t}{t} - \dim I_t & \text{otherwise.} \end{cases} \tag{2}$$

The *initial degree* of X is the smallest t such that $\dim I_t > 0$. If $X = \mathbb{P}^n$, that is, $I = \langle 0 \rangle$, then the initial degree is defined as $-\infty$. If $X = \emptyset$, that is, $I = \langle 1 \rangle$, then the initial degree is 0. For all other schemes $X \subset \mathbb{P}^n$, the initial degree is a positive integer.

Since $HF_X(t) = 0$ for $t < 0$, we will express HF_X via the list $HF_X(0), HF_X(1), HF_X(2), \ldots$. The generating function of HF_X is called the *Hilbert series* of X, namely

$$HS_X(t) = \sum_{j=0}^{\infty} HF_X(j) \cdot t^j.$$

One key operation on Hilbert functions is taking differences, e.g., the first difference of HF_X is

$$\Delta HF_X(t) = HF_X(t) - HF_X(t-1) \quad \text{for all } t \in \mathbb{Z}.$$

By (2), we know $\Delta HF_X(t) = 0$ for $t < 0$ and $\Delta HF_X(0) = 1$. One can also iterate this process. For example, the k^{th} difference of HF_X is

$$\Delta^k HF_X(t) = \underbrace{\Delta \circ \cdots \circ \Delta}_{k \text{ times}} HF_X(t).$$

The Hilbert function of X becomes polynomial in t for $t \gg 0$. That is, there exists a polynomial HP_X, called the *Hilbert polynomial* of X, such that $HF_X(t) = HP_X(t)$ for all $t \gg 0$. The Hilbert polynomial has rational coefficients with highest degree term $\frac{\deg X}{(\dim X)!} \cdot t^{\dim X}$. When X is a curve, the Hilbert polynomial of X has the form

$$HP_X(t) = \deg X \cdot t + (1 - g_X) \tag{3}$$

where g_X is the *arithmetic genus* of X.

Example 2.2 Consider the quartic curve $Q \subset \mathbb{P}^3$ from Ex. 2.1. From the generators of $I(Q)$, it is easy to compute, e.g., via `Macaulay2` [12], the following:

$$HF_Q = 1, 4, 9, 13, 17, 21, 25, \ldots, \quad HS_Q(t) = (1 + 2t + 2t^2 - t^3)/(1-t)^2, \quad \text{and} \quad HP_Q(t) = 4t + 1.$$

From HF_Q, the initial degree of Q is 2. From HP_Q and (3), the arithmetic genus of Q is $g_Q = 0$.

We will discuss two types of regularity for X. The *index of regularity* of X is the smallest integer ρ_X such that $HF_X(t) = HP_X(t)$ for all $t \geq \rho_X$. Let $\mathcal{I}_X$ be the sheafification of the ideal I corresponding to X. The *Castelnuovo-Mumford regularity* of X is

$$\operatorname{reg} X = \min\{m \mid H^i(\mathcal{I}_X(m-i)) = 0 \quad \text{for all } i > 0\}.$$

If X is aCM, ρ_X, $\operatorname{reg} X$, and $\dim X$ are related as follows.

Proposition 2.3 *Suppose that $X \subset \mathbb{P}^n$ is an aCM scheme.*

1. $\operatorname{reg} X = \rho_X + \dim X + 1$.
2. *Let $\mathcal{L} \subset \mathbb{P}^n$ be a general linear space with* $\operatorname{codim} \mathcal{L} \leq \dim X$ *and $Z = X \cap \mathcal{L}$. Then,* $\operatorname{reg} Z = \operatorname{reg} X$ *and* $\rho_Z = \rho_X + \operatorname{codim} \mathcal{L}$.

Proof. See, for example, [7, Remark 2.5a] for Item 1. Item 2 follows immediately by combining [24, pg. 30] and Item 1. $\square$

For aCM schemes, this proposition shows that the index of regularity increases under intersection with a general hyperplane. Thus, the index of regularity can be negative so that the Hilbert polynomial has roots at negative integers. Section 5.2 presents an example of this.

The following will be used in Section 3 for computing $\operatorname{reg} C$ where $C \subset \mathbb{P}^n$ is a curve, that is, a union of irreducible one-dimensional projective schemes.

Proposition 2.4 *Let $C \subset \mathbb{P}^n$ be a curve and $\mathcal{H} \subset \mathbb{P}^n$ be a general hyperplane. If $W = C \cap \mathcal{H}$,*

$$\operatorname{reg} C = \max\{\rho_C + 1, \rho_W + 1\} = \min\{t \geq \rho_W + 1 \mid \Delta HF_C(t) = HF_W(\rho_W)\}. \tag{4}$$

Proof. By [7, Lemma 2.6], $\operatorname{reg} C = \max\{\rho_C + 1, \operatorname{reg} W\}$. Since $\dim W = 0$, W is aCM yielding $\operatorname{reg} W = \rho_W + 1$ by Item 1 of Prop. 2.3 and $HF_W(\rho_W) = \deg C$. The last equality thus follows from [7, § 3]. $\square$

For example, using the notation of Prop. 2.4, if C is also aCM, then $\rho_W = \rho_C + 1$ so that

$$\operatorname{reg} W = \operatorname{reg} C = \rho_W + 1 = \rho_C + 2. \tag{5}$$

Example 2.5 From HF_Q and HP_Q presented in Ex. 2.2, we have $\rho_Q = 2$. If $W = Q \cap \mathcal{H}$ for a general hyperplane $\mathcal{H} \subset \mathbb{P}^3$, one can use [13] to compute $HF_W = 1, 3, 4, 4, \ldots$ and $\rho_W = 2$. Hence, (4) yields $\operatorname{reg} Q = 3$ and, since (5) does not hold, this again shows Q is not aCM.

2.3 Witness sets

Both Prop. 2.3 and 2.4 consider general linear sections of schemes. This is amenable to numerical algebraic geometry where the fundamental data structure of an irreducible algebraic set, a witness set, is based on linear sections of complimentary dimension.

Let f be a system of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ and $V \subset \mathbb{P}^n$ be an irreducible algebraic set of dimension d which is an irreducible component of $\mathcal{V}(f) = \{x \in \mathbb{P}^n \mid f(x) = 0\}$. Then, a *witness set* for V is a triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{P}^n$ is a general linear space of codimension d and $W = V \cap \mathcal{L} \subset \mathbb{P}^n$, a *witness point set* consisting of $\deg V$ points. We refer the reader to [25] for more details about witness sets.

Example 2.6 Let $C \subset \mathbb{P}^3$ be the twisted cubic curve, as described in Ex. 2.1, and

$$f(x, y, z, w) = \begin{bmatrix} xz - y^2 \\ yw - x^2 \end{bmatrix}.$$

Clearly, $\mathcal{V}(f) = C \cup \mathcal{V}(x, y)$. So, a witness set for C is $\{f, \mathcal{L}, W\}$ where $\mathcal{L} = \mathcal{V}(x+y-z+2w)$ and

$$W = \{(8, -4, 2, -1), (2, 1 + \sqrt{-3}, -1 + \sqrt{-3}, -2), (2, 1 - \sqrt{-3}, -1 - \sqrt{-3}, -2)\} \subset \mathbb{P}^3.$$

We note that $\mathcal{L}$ is defined via a linear polynomial with integer coefficients only for illustration.

One key operation in numerical algebraic geometry, called *sampling*, is the ability to use a witness set $\{f, \mathcal{L}, W\}$ for V to produce a collection of arbitrarily close numerical approximations of arbitrarily many smooth points on V. In particular, suppose that $x \in W = V \cap \mathcal{L}$ and let $\mathcal{L}^* \subset \mathbb{P}^n$ be a linear space of codimension d. Consider the path $z(t) : [0, 1] \to V$ defined by $z(1) = x$ and $z(t) \in V \cap (t \cdot \mathcal{L} + (1 - t) \cdot \mathcal{L}^*)$. Except on a Zariski closed proper subset of choices for $\mathcal{L}^*$, $z(t)$ is a smooth point of V for all $t \in [0, 1]$, i.e., $z(0)$ is also a smooth point of V. We note the smooth points of V are (path) connected since V is irreducible.

Rather than consider V as an irreducible component of $\mathcal{V}(f) \subset \mathbb{P}^n$, one can consider V as a subscheme of the scheme defined by f which, by abuse of notation, we will denote as $f^{-1}(0)$. For simplicity, we also say that $\{f, \mathcal{L}, W\}$ is a witness set for the scheme V. The system f can be used to obtain the local scheme structure at generic points of V. If the multiplicity of V with respect to f is greater than 1, we can use isosingular deflation [20] to compute another system f_V of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ such that, as a set, $V \subset \mathcal{V}(f_V)$ and has multiplicity 1 with respect to f_V. That is, one uses f_V to sample points on V but uses f to compute the Macaulay dual space, which is described in the following subsection.

We close this subsection by considering one other "type" of witness set, namely a *pseudowitness set* [18, 19]. For simplicity, suppose that $f : \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial system and and $\pi : \mathbb{C}^n \to \mathbb{C}^k$ is a projection onto the first k coordinates. Let $V \subset \mathcal{V}(f)$ be an irreducible component of dimension d and thus $Y = \overline{\pi(V)} \subset \mathbb{C}^k$ is an irreducible algebraic set. The scheme of interest is the projectivization of Y, denoted $X = \mathbb{P}(Y) \subset \mathbb{P}^k$, which can be sampled via Y. Thus, a *pseudowitness set* for Y is the quadruple $\{f, \pi, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{C}^n$ is the linear space of codimension d constructed below and $W = V \cap \mathcal{L}$. If $\ell = \dim Y \leq d$, let $L : \mathbb{C}^k \to \mathbb{C}^\ell$ and $L' : \mathbb{C}^n \to \mathbb{C}^{d-\ell}$ be systems consisting of general linear polynomials. Then, $\mathcal{L} = \mathcal{V}(L(x_1, \ldots, x_k), L'(x_1, \ldots, x_n)) \subset \mathbb{C}^n$. With this setup, $|\pi(W)| = \deg Y$.

2.4 Macaulay dual spaces

One way to computationally understand the local structure of a scheme is via Macaulay dual spaces. Since this will be exploited in Section 2.5, the following provides a brief overview with expanded details presented in the books [22, 26].

For $\alpha = (\alpha_0, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{Z}_{\geq 0}$, define

$$|\alpha| = \alpha_0 + \cdots + \alpha_n, \quad \alpha! = \alpha_0! \cdots \alpha_n!, \quad \text{and} \quad x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}.$$

For a fixed $x^* \in \mathbb{C}^{n+1}$, let $\partial_\alpha : \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}$ be the operator defined by

$$f \mapsto \frac{1}{\alpha!} \left. \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right|_{x=x^*} \quad \text{so that} \quad \partial_\alpha \left(\sum_{\beta \in (\mathbb{Z}_{\geq 0})^{n+1}} c_\beta (x - x^*)^\beta \right) = c_\alpha.$$

For convenience, we often write ∂_α as ∂_{x^α} which simply computes the coefficient of $(x - x^*)^\alpha$ in a Taylor series expansion centered at x^*. In particular, for $\alpha = \mathbf{0}$, since $x^\alpha = 1$, we will write ∂_α as ∂_1 which simply yields the constant term of the Taylor series expansion centered at x^*. For $j \geq 0$, consider the linear space of all such differential operators spanned by elements of the form ∂_α with $|\alpha| \leq j$, namely

$$D_{x^*}^j = \left\{ \sum_{|\alpha| \leq j} c_\alpha \partial_\alpha \;\middle|\; c_\alpha \in \mathbb{C} \right\}.$$

Then, for an ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$, the j^{th} *Macaulay dual space of I at x^** is

$$D_{x^*}^j[I] = \left\{ \partial \in D_{x^*}^j \;\middle|\; \partial(g) = 0 \text{ for all } g \in I \right\}.$$

Numerical approaches for computing $D_{x^*}^j[I]$ using closedness subspaces are presented in [15, 27].

Example 2.7 For $I = \langle x^2 - z^2, xz - z^2, xy - z^2 \rangle \subset \mathbb{C}[x, y, z]$ and $x^* = \mathbf{0} \in \mathbb{C}^3$, one has

$D_{\mathbf{0}}^0[I] = \text{span}\{\partial_1\}$,
$D_{\mathbf{0}}^1[I] = \text{span}\left(D_{\mathbf{0}}^0[I] \cup \{\partial_x, \partial_y, \partial_z\}\right)$,
$D_{\mathbf{0}}^2[I] = \text{span}\left(D_{\mathbf{0}}^1[I] \cup \{\partial_{x^2} + \partial_{xy} + \partial_{xz} + \partial_{z^2}, \partial_{y^2}, \partial_{yz}\}\right)$,
$D_{\mathbf{0}}^3[I] = \text{span}\left(D_{\mathbf{0}}^2[I] \cup \{\partial_{x^3} + \partial_{x^2y} + \partial_{x^2z} + \partial_{xy^2} + \partial_{xyz} + \partial_{xz^2} + \partial_{yz^2} + \partial_{z^3}, \partial_{y^3}, \partial_{y^2z}\}\right)$.

The particular case of interest is when $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal such that $\widehat{x} \in \mathbb{P}^n$ is an isolated point of $\mathcal{V}(I) = \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in I\}$. By treating $\widehat{x} \in \mathbb{P}^n$ as a line through the origin in $\mathbb{C}^{n+1}$, a random affine hyperplane $\mathcal{A} \subset \mathbb{C}^{n+1}$ will intersect this line in one point, say $x^* \in \mathbb{C}^{n+1}$. If p is a linear polynomial such that $\mathcal{A} = \mathcal{V}(p)$, consider the ideal $J = I + \langle p \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$. Since x^* is an isolated point in $\mathcal{V}(J) \subset \mathbb{C}^{n+1}$,

$$D_{x^*}[J] = \bigcup_{j=0}^{\infty} D_{x^*}^j[J]$$

is a vector space of dimension equal to the multiplicity of x^* with respect to J. In particular, there must exist $j^* \geq 0$ such that $D_{x^*}^{j^*}[J] = D_{x^*}^j[J]$ for all $j \geq j^*$. The vector space $D_{x^*}[J] = D_{x^*}^{j^*}[J]$ is called the *Macaulay dual space of J at x^** and a *Macaulay dual basis* is any basis of this finite-dimensional vector space.

Example 2.8 Reconsider $I \subset \mathbb{C}[x, y, z]$ from Ex. 2.7 with $\widehat{x} = (0, 1, 0) \in \mathbb{P}^2$. By taking $\mathcal{A} = \mathcal{V}(p)$ where $p(x, y, z) = 2x + 3y - 4z - 6$, we have

$$J = \langle x^2 - z^2, xz - z^2, xy - z^2, 2x + 3y - 4z - 6 \rangle \quad \text{and} \quad x^* = (0, 2, 0) \in \mathbb{C}^3.$$

One can easily compute

$$\begin{aligned} &D_{x^*}^0[J] = \operatorname{span}\{\partial_1\}, \\ &D_{x^*}^1[J] = \operatorname{span}\left(D_{x^*}^0[J] \cup \{4\partial_y + 3\partial_z\}\right), \\ &D_{x^*}^2[J] = D_{x^*}^1[J] \end{aligned}$$

so that $D_{x^*}[J] = D_{x^*}^1[J]$. Thus, x^* and $\widehat{x}$ have multiplicity 2 with respect to J and I, respectively.

2.5 Interpolation

The key computational step in our approach for deciding the arithmetically Cohen-Macaulayness of a curve is the computation of the Hilbert function of the curve and its general hyperplane section in particular degrees. For zero-dimensional schemes, such as the general hyperplane section of a curve, the Hilbert function and index of regularity can be computed using the approach of [13] after fixing a general affine patch as in Section 2.4. For curves (and higher dimensional schemes), the Hilbert function in a particular degree can be computed by considering a sufficiently large zero-dimensional subscheme (see also [23]). This computation is summarized in Algorithm 1 which uses the ability to sample points on each irreducible component of a curve and compute Macaulay dual spaces when the multiplicity is greater than one.

For a finite set of points W, $HF_W(t)$ is simply the rank of the matrix whose rows are the t^{th} degree Veronese embedding of the points in W. If a point has scheme structure defined via a Macaulay dual space, the corresponding matrix constructed via [13] has a row for each element of a Macaulay dual basis for each point.

Theorem 2.9 *Subject to genericity, Algorithm 1 is an algorithm that computes $HF_C(t)$.*

Proof. Using the notation of Algorithm 1, we clearly have $0 \leq h_1 \leq \binom{n+t}{t}$. Since either h_1 must increase by at least one or remain the same during each loop, Algorithm 1 terminates in at most $\binom{n+t}{t}$ loops. Since $I(C) \subset I(S)$, we always have $h_1 = HF_S(t) \leq HF_C(t) \leq \binom{n+t}{t}$. Thus, if $h_1 = \binom{n+t}{t}$, then $HF_C(t) = \binom{n+t}{t}$.

Algorithm 1 Numerical computation of $HF_C(t)$

Input: A collection of irreducible curves $C_1, \ldots, C_r \subset \mathbb{P}^n$ where each C_i is represented via a witness set $\{f_i, \mathcal{L}_i, W_i\}$ and an integer $t \geq 1$.

Output: $HF_C(t)$ where $C := C_1 \cup \cdots \cup C_r$.

Initialize $h_0 := -1$, $h_1 := 0$, and $S := \emptyset$.
while $h_0 \neq h_1$ and $h_1 < \binom{n+t}{t}$ **do**
  Set $h_0 := h_1$.
  for $i = 1, \ldots, r$ **do**
    Use sampling (possibly facilitated using isosingular deflation [20]) to compute a point $\widehat{c} \in \mathbb{P}^n$ contained in $C_i \cap \mathcal{H}$ where $\mathcal{H} \subset \mathbb{P}^n$ is a random hyperplane. Let ℓ be a linear form so that $\mathcal{H} = \mathcal{V}(\ell)$.
    Pick a random affine patch of $\mathbb{P}^n$ via a random affine linear polynomial p. Compute the point in $\mathbb{C}^{n+1}$ corresponding to $\widehat{c}$ in $\mathcal{V}(p)$. Denote it by $c^* \in \mathbb{C}^{n+1}$.
    Compute a basis B_{c^*} for $D_{c^*}[\langle f_i, \ell, p\rangle]$ and append $\{c^*, B_{c^*}\}$ to S.
  Compute $h_1 := HF_S(t) = \binom{n+t}{t} - \dim I(S)_t$ via [13].
return $HF_C(t) := h_1$.

Suppose that $r = 1$, that is, C is irreducible and assume that we have reached where $h_0 = h_1$. If $f \in I(C)_t$, then clearly $f \in I(S)_t$. If $f \in I(S)_t$, then, since $\partial_1 \in D_c[\langle f_i, \ell, p\rangle]$, standard interpolation theory provides that $f \in \sqrt{I(C)}$. Since $I(C)$ is a saturated homogeneous primary ideal, if $f \notin I(C)$, then the multiplicity of C must be smaller with respect to $I(C) + \langle f\rangle$ than with respect to $I(C)$. However, this is impossible since $h_0 = h_1$ ensures that the multiplicity has not decreased, i.e., $I(S)_t = I(C)_t$.

When $r > 1$, since the algorithm uses points on every irreducible component of C, the algorithm is simply performing the intersection $I(C_1)_t \cap \cdots \cap I(C_r)_t = I(C)_t$ via [13]. This follows immediately from linear algebra since the null space of a matrix of the form $\begin{bmatrix} A \\ B \end{bmatrix}$ is the intersection of the null space of A and the null space of B. $\square$

Example 2.10 Reconsider $I = \langle x^2 - z^2, xz - z^2, xy - z^2\rangle$ from Ex. 2.7 now as an ideal in $\mathbb{C}[x, y, z, w]$. Thus, I defines a curve $C \subset \mathbb{P}^3$ having two irreducible components C_1 and C_2 with

$$I(C_1) = \langle x - z, y - z\rangle, \quad I(C_2) = \langle x, z^2\rangle, \text{ and } I = I(C) = I(C_1) \cap I(C_2).$$

To use Algorithm 1, we will only assume we are given the polynomial system

$$F(x, y, z, w) = \begin{bmatrix} x^2 - z^2 \\ xz - z^2 \\ xy - z^2 \end{bmatrix}$$

which will be the polynomial system in a witness set one can compute for C_1 and C_2. From the witness set, it is easy verify that C_j has multiplicity j with respect to F (and I). Thus, isosingular deflation [20] is used to construct a polynomial system for sampling C_2.

Applying to C_1: Since C_1 has multiplicity one, applying Algorithm 1 to C_1 is simply computing $HF_{C_1}(t)$ by performing standard interpolation at arbitrarily many points in C_1. In this case, each $c^* \in \mathbb{C}^4$ is of the form (a, a, a, b) for some $a, b \in \mathbb{C}$. For example, to compute $HF_{C_1}(2)$, each loop adds a row of the form

$$\begin{bmatrix} a^2 & a^2 & a^2 & ab & a^2 & a^2 & ab & a^2 & ab & b^2 \end{bmatrix}. \tag{6}$$

The resulting values of h_1 computed by Algorithm 1 are $1, 2, 3, 3$ yielding $HF_{C_1}(2) = 3$.

Applying to C_2: Since C_2 has multiplicity two, each new point under consideration in Algorithm 1 applied to C_2 imposes two conditions. In each loop, suppose that $\ell = \ell_1 x + \ell_2 y + \ell_3 z + \ell_4 w$ and $p = p_1 x + p_2 y + p_3 z + p_4 w - 1$ for random $\ell_i, p_i \in \mathbb{C}$. Then, $c^* \in \mathbb{C}^4$ is precisely

$$c^* = \left(0, \frac{\ell_4}{\ell_4 p_2 - \ell_2 p_4}, 0, \frac{\ell_2}{\ell_2 p_4 - \ell_4 p_2}\right)$$

and a basis B_{c^*} for $D_{c^*}[\langle F, \ell, p\rangle]$ consists of the following two elements:

$$\partial_1 \quad \text{and} \quad \delta := (\ell_3 p_4 - \ell_4 p_3)\partial_y + (\ell_4 p_2 - \ell_2 p_4)\partial_z + (\ell_2 p_3 - \ell_3 p_2)\partial_w.$$

For example, to compute $HF_{C_2}(2)$, each loop adds the two rows

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\ell_4^2}{(\ell_2 p_4 - \ell_4 p_2)^2} & 0 & \frac{-\ell_2 \ell_4}{(\ell_2 p_4 - \ell_4 p_2)^2} & 0 & 0 & \frac{\ell_2^2}{(\ell_2 p_4 - \ell_4 p_2)^2} \\ 0 & 0 & 0 & 0 & \frac{2\ell_4(\ell_4 p_3 - \ell_3 p_4))}{\ell_2 p_4 - \ell_4 p_2} & \ell_4 & \frac{\ell_2 \ell_3 p_4 - 2\ell_2 \ell_4 p_3 + \ell_3 \ell_4 p_2}{\ell_2 p_4 - \ell_4 p_2} & 0 & -\ell_2 & \frac{2\ell_2(\ell_2 p_3 - \ell_3 p_2))}{\ell_2 p_4 - \ell_4 p_2} \end{bmatrix}. \tag{7}$$

The resulting values of h_1 computed by Algorithm 1 are $2, 4, 5, 5$ yielding $HF_{C_2}(2) = 5$.

Applying to C: At each loop, three conditions are added: one from C_1 and two from C_2. For example, to compute $HF_C(2)$, each loop adds the three rows in (6) and (7). The resulting values of h_1 computed by Algorithm 1 are $3, 6, 7, 7$ yielding $HF_C(2) = 7$.

Since the previous example is actually arithmetically Cohen-Macaulay, we now consider a similar example which is not.

Example 2.11 Consider the ideal $I = \langle xy - xz, x^2 - xw, yz^2 - z^3, xz^2 - z^2 w\rangle \subset \mathbb{C}[x, y, z, w]$ which defines a curve $C' \subset \mathbb{P}^3$ having two irreducible components C_1' and C_2' with

$$I(C_1') = \langle x - w, y - z\rangle, \quad I(C_2') = \langle x, z^2\rangle, \text{ and } I = I(C') = I(C_1') \cap I(C_2').$$

Applying to C_1': Just as with C_1 in Ex. 2.10, C_1' has multiplicity one. Each $c^* \in \mathbb{C}^4$ computed by Algorithm 1 is of the form (a, b, b, a) for some $a, b \in \mathbb{C}$. For example, to compute $HF_{C_1'}(2)$, each loop adds a row of the form

$$\begin{bmatrix} a^2 & ab & ab & a^2 & b^2 & b^2 & ab & b^2 & ab & a^2 \end{bmatrix}. \tag{8}$$

The resulting values of h_1 computed by Algorithm 1 are $1, 2, 3, 3$ yielding $HF_{C_1'}(2) = 3$.

Applying to C_2': This is the same as described above in Ex. 2.10 since C_2' is the same as C_2.

Applying to C': As with Ex. 2.10, each loop adds three conditions: one from C_1' and two from C_2'. For example, to compute $HF_{C'}(2)$, each loop adds the three rows in (8) and (7). The resulting values of h_1 computed by Algorithm 1 are $3, 6, 8, 8$ yielding $HF_{C'}(2) = 8$.

3 Computations for a curve

The following considers curves with Section 4 exploring higher-dimensional cases.

3.1 Computing invariants

Let $C \subset \mathbb{P}^n$ be a curve, that is, C is a union of one-dimensional irreducible schemes. The defining equations for C may be unknown, but we assume that we have either a witness set or a pseudo-witness set for each irreducible component of C, thereby providing the ability to sample points from each irreducible component of C. We also need the ability to compute $HF_C(t)$ for specified values of t, which, for example using Algorithm 1 only additionally requires the ability to compute the corresponding Macaulay dual spaces for components of multiplicity greater than one.

For a curve $C \subset \mathbb{P}^n$, six invariants of interest are the Castelnuovo-Mumford regularity, index of regularity, arithmetic genus, geometric genus, Hilbert polynomial, and Hilbert series. The geometric genus can be computed using [5] from a witness set for C. The following uses the ability to compute $HF_C(t)$ via Algorithm 1 given HF_W and ρ_W, both of which can be computed via [13], to compute the other five invariants.

Castelnuovo-Mumford regularity The Castelnuovo-Mumford regularity $\operatorname{reg} C$ is derived from (4) by using Algorithm 1 to compute enough terms of HF_C.

Hilbert polynomial, arithmetic genus, and index of regularity If $\operatorname{reg} C > \rho_W + 1$, then (4) also yields $\rho_C = \operatorname{reg} C - 1$. Thus, $HF_C(\rho_C) = HP_C(\rho_C)$ so that (3) yields

$$g_C = \deg C \cdot \rho_C - HF_C(\rho_C) + 1 = HP_W(\rho_W) \cdot \rho_C - HF_C(\rho_C) + 1 \tag{9}$$
$$HP_C(t) = \deg C \cdot t + (1 - g_C) = HP_W(\rho_W) \cdot t + (HF_C(\rho_C) - HP_W(\rho_W) \cdot \rho_C). \tag{10}$$

If $\operatorname{reg} C \leq \rho_W + 1$, then (4) yields $\rho_C \leq \rho_W$. Since $HF_C(\rho_W) = HP_C(\rho_W)$, (3) yields

$$g_C = \deg C \cdot \rho_W - HF_C(\rho_W) + 1 = HP_W(\rho_W) \cdot \rho_W - HF_C(\rho_W) + 1 \tag{11}$$
$$HP_C(t) = \deg C \cdot t + (1 - g_C) = HP_W(\rho_W) \cdot t + (HF_C(\rho_W) - HP_W(\rho_W) \cdot \rho_W). \tag{12}$$

In this case, $\rho_C = \min\{-1 \leq t \leq \rho_W \mid HF_C(t) = HP_C(t)\}$.

Hilbert series By adapting [24, p. 28] to this situation, we have

$$HS_C(t) = \frac{\sum_{j=0}^{\rho_C+1} \Delta^2 HF_C(j) \cdot t^j}{(1-t)^2}. \tag{13}$$

Example 3.1 Consider the degree 8 curve in $\mathbb{P}^3$ derived from [10, Ex. 1.7]:

$$C = \{(s^8, s^7 t, s t^7, t^8) \mid (s,t) \in \mathbb{P}^1\}.$$

It is easy to verify that the corresponding ideal is

$$I(C) = \langle xw - yz, x^6 z - y^7, x^5 z^2 - y^6 w, x^4 z^3 - y^5 w^2, x^3 z^4 - y^4 w^3, x^2 z^5 - y^3 w^4, x z^6 - y^2 w^5, z^7 - y w^6 \rangle.$$

Let $\mathcal{H}$ be a general hyperplane and $W = C \cap \mathcal{H}$. Using [13], we find that

$$HF_W = 1, 3, 5, 7, 8, 8 \quad \text{and} \quad \rho_W = 4.$$

Using Algorithm 1, we find that

$$HF_C = 1, 4, 9, 16, 25, 36, 49, 57, \ \Delta HF_C = 1, 3, 5, 7, 9, 11, 13, 8, \ \Delta^2 HF_C = 1, 2, 2, 2, 2, 2, 2, -5.$$

Hence, $\operatorname{reg} C = 7$ and $\rho_C = \operatorname{reg} C - 1 = 6$. Additionally, (9), (10), and (13) yield

$$g_C = 8 \cdot 6 - 49 + 1 = 0, \ HP_C(t) = 8t + 1, \ HS_C(t) = \frac{1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 - 5t^7}{(1-t)^2}.$$

The geometric genus of a curve is the arithmetic genus of the desingularization of the curve. Since the curve in Ex. 3.1 is smooth, its geometric genus is equal to its arithmetic genus, namely 0. Section 5.1 compares these genera on a nonsmooth curve.

3.2 Testing arithmetically Cohen-Macaulayness of a curve

The following tests the arithmetically Cohen-Macaulayness of a curve.

Theorem 3.2 *Let $C \subset \mathbb{P}^n$ be a curve, $\mathcal{H} \subset \mathbb{P}^n$ be a general hyperplane, and $W = C \cap \mathcal{H}$. Then, C is aCM if and only if $\Delta HF_C(t) = HF_W(t)$ for all $t \geq 0$.*

Proof. Since C is a curve, C is aCM if and only if $H^1_*(\mathcal{I}_C) = 0$. By [24, Prop. 1.3.4], this is equivalent to $J = I(C) + \langle \ell \rangle$ being a saturated ideal in $R := \mathbb{C}[x_0, \ldots, x_n]$ where $\mathcal{H} = \mathcal{V}(\ell)$. That is, C is aCM if and only if $J = I(W)$. Since $J \subset I(W)$, this is equivalent to $HF_{R/J}(t) = HF_W(t)$ for all $t \geq 0$. The result now follows since $HF_{R/J}(t) = \Delta HF_C(t)$ because C is a curve. $\square$

The following is a so-called *effective version* of Theorem 3.2.

Corollary 3.3 *Let $C \subset \mathbb{P}^n$ be a curve, $\mathcal{H} \subset \mathbb{P}^n$ be a general hyperplane, and $W = C \cap \mathcal{H}$. Then, C is aCM if and only if $\Delta HF_C(t) = HF_W(t)$ for all $1 \leq t \leq \rho_W + 1$.*

Proof. Clearly, $\Delta HF_C(0) = HF_W(0) = 1$ and $HF_W(\rho_W) = HF_W(\rho_W + t)$ for all $t \geq 0$. If $\Delta HF_C(\rho_W + 1) = HF_W(\rho_W + 1) = HF_W(\rho_W)$, then $\rho_C + 1 \leq \rho_W + 1$ so that $\rho_C \leq \rho_W$. Hence, $HP_C(\rho_W + t) = HF_C(\rho_W + t)$ for all $t \geq 0$ which yields

$$\Delta HF_C(\rho_W + t) = HF_W(\rho_W + t) = HF_W(\rho_W) \text{ for all } t \geq 1.$$

In particular, we have shown that $\Delta HF_C(t) = HF_W(t)$ for $1 \leq t \leq \rho_W + 1$ is equivalent to $\Delta HF_C(t) = HF_W(t)$ for all $t \geq 0$. Therefore, the statement holds by Theorem 3.2. $\square$

Section 5.3 provides an example that is not aCM such that $\Delta HF_C(t) = HF_W(t)$ for all $1 \leq t \leq \rho_W$. Hence, the effective upper bound $\rho_W + 1$ provided in Corollary 3.3 is sharp.

Corollary 3.3 immediately yields an algorithm for determining the arithmetically Cohen-Macaulayness of a curve C. As discussed in Section 2.5, [13] can be used to compute both HF_W and ρ_W, where W is a general hyperplane section of C, upon fixing a general affine patch. Additionally, Algorithm 1 can be used to compute $HF_C(1), \ldots, HF_C(\rho_W + 1)$ with $HF_C(0) = 1$. Thus, C is aCM if and only if $HF_C(t) - HF_C(t-1) = HF_W(t)$ for $t = 1, \ldots, \rho_W + 1$.

Example 3.4 Recall the curves C and Q in $\mathbb{P}^3$ introduced in Ex. 2.1. Let $\mathcal{H}$ be a general hyperplane, $W_C = C \cap \mathcal{H}$, and $W_Q = Q \cap \mathcal{H}$. For the twisted cubic curve C, we compute

$$HF_{W_C} = 1, 3, 3, \quad \rho_{W_C} = 1, \quad HF_C = 1, 4, 7, \quad \Delta HF_C = 1, 3, 3.$$

Since $\Delta HF_C(t) = HF_{W_C}(t)$ for $1 \leq t \leq \rho_{W_C} + 1 = 2$, Corollary 3.3 shows C is aCM.

Similarly, for the quartic curve Q, we compute

$$HF_{W_Q} = 1, 3, 4, 4, \quad \rho_{W_Q} = 2, \quad HF_Q = 1, 4, 9, 13, \quad \Delta HF_Q = 1, 3, 5, 4.$$

Since $\Delta HF_Q(2) = 5 \neq 4 = HF_{W_Q}(2)$, Corollary 3.3 shows Q is not aCM.

4 Higher-dimensional cases

4.1 Testing arithmetically Cohen-Macaulayness

The key to testing the arithmetically Cohen-Macaulayness of a scheme of dimension at least 2 is to test the arithmetically Cohen-Macaulayness of a general curve section.

Theorem 4.1 *Let $X \subset \mathbb{P}^n$ be a pure-dimensional scheme of dimension $d > 1$ and $\mathcal{L} \subset \mathbb{P}^n$ be a general linear space of codimension $d-1$. Then, X is aCM if and only if the curve $X \cap \mathcal{L}$ is aCM.*

Proof. If X is aCM, then [24, Thm. 1.3.3] yields that $X \cap \mathcal{L}$ is also aCM. Conversely, if the projective curve $X \cap \mathcal{L}$ is aCM, then [21, Prop. 2.1] provides that X must also be aCM. $\square$

The combination of Theorem 4.1 and Corollary 3.3 yields a test for deciding the arithmetically Cohen-Macaulayness of a pure-dimensional scheme of dimension at least 2 by determining the arithmetically Cohen-Macaulayness of a general curve section. Additional information about this general curve section can be computed via Section 3.1, such as its arithmetic genus.

Example 4.2 Let $X \subset \mathbb{P}^4$ be the degree 4 surface defined by the ideal

$$I = \langle x_0x_1 - x_2^2, x_0x_3 - x_4^2 \rangle \subset \mathbb{C}[x_0, x_1, x_2, x_3, x_4].$$

Let $\mathcal{L}$ and $\mathcal{H}$ be general hyperplanes with $C = X \cap \mathcal{L}$ and $W = C \cap \mathcal{H}$. Since

$$HF_W = 1, 3, 4, 4, \quad \rho_W = 2, \quad HF_C = 1, 4, 8, 12, \quad \Delta HF_C = 1, 3, 4, 4,$$

Corollary 3.3 yields that C is aCM so that X is aCM by Theorem 4.1.

If $X \subset \mathbb{P}^n$ is aCM of dimension $d > 1$ and $W \subset \mathbb{P}^n$ is a general linear section of complimentary dimension, the index of regularity of X and Castelnuovo-Mumford regularity of X can be computed directly from the index of regularity of W via Prop. 2.3. The remainder of this section describes how to compute the Hilbert function, Hilbert series, and Hilbert polynomial of X given the Hilbert function and index of regularity of W.

Hilbert function Using [24, Cor. 1.3.8(d)] applied d times, we have

$$\Delta^d HF_X(t) = HF_W(t) \text{ for all } t \geq 0.$$

In particular, unrolling this formula provides

$$HF_X(t) = \sum_{j_1=0}^{t} \sum_{j_2=0}^{j_1} \cdots \sum_{j_d=0}^{j_{d-1}} HF_W(j_d). \tag{14}$$

Hilbert series By adapting [24, p. 28] to this situation, we have

$$HS_X(t) = \frac{\sum_{j=0}^{\rho_W} \Delta^{d+1} HF_X(j) \cdot t^j}{(1-t)^{d+1}} = \frac{\sum_{j=0}^{\rho_W} \Delta HF_W(j) \cdot t^j}{(1-t)^{d+1}}. \tag{15}$$

Hilbert polynomial Since $HP_X(t)$ is a polynomial of degree d with rational coefficients and $HP_X(\rho_X+j) = HF_X(\rho_X+j)$ for all $j \geq 0$, standard polynomial interpolation computes HP_X.

Example 4.3 Let $X \subset \mathbb{P}^4$ be the surface from Ex. 4.2, which is aCM. From Prop. 2.3 and (5),

$$\rho_C = 1, \quad \rho_X = 0, \quad \operatorname{reg} X = 3.$$

Following (14) and (15) with data from Ex. 4.2, we have

$$HF_X(t) = 1, 5, 13, 25, 41, 61, \ldots \quad \text{and} \quad HF_S(t) = \frac{1+2t+t^2}{(1-t)^3}.$$

Since $\rho_X = 0$, one can easily verify that $HP_X(t) = 2t^2+2t+1$ with $HP_X(t) = HF_X(t)$ for $t \geq 0$.

4.2 Minimal generators

Let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal. For each $j \geq 0$, there exists $d_j(I) \geq 0$ such that every minimal generating set consisting of homogeneous polynomials for I consists of exactly $d_j(I)$ polynomials of degree j. For a scheme $X \subset \mathbb{P}^n$, $d_j(X)$ is defined as $d_j(I)$ where I is the corresponding homogeneous ideal. In fact, $d_j(X) = 0$ for $j > \operatorname{reg} X$.

When intersecting with linear spaces, say $Z = X \cap \mathcal{L}$, one can consider $Z \subset X \subset \mathbb{P}^n$ or $Z \subset \mathcal{L}$ where $\mathcal{L}$ may be viewed as a projective space. In the following, when we write $Z \subset \mathcal{L}$, we mean that $d_j(Z)$ is taken with respect to $\mathcal{L}$.

For arithmetically Cohen-Macaulay X, the following provides an approach to compute $d_j(X)$.

Proposition 4.4 *Let $X \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay scheme of dimension $d > 0$, $\mathcal{L} \subset \mathbb{P}^n$ be a general linear space of codimension $0 < \ell \leq d$, and $Z = X \cap \mathcal{L} \subset \mathcal{L}$. Then, $d_j(X) = d_j(Z)$ for all j. In particular, the initial degree of X is the initial degree of Z.*

Proof. By treating Z as a subscheme of $\mathcal{L}$, the result now follows from [24, Thm. 1.3.6]. □

Example 4.5 Let $X \subset \mathbb{P}^4$ be the aCM surface introduced in Ex. 4.2. By looking at the generating set of the ideal, one sees $d_2(X) = 2$ with $d_j(X) = 0$ for all other j. Thus, by Prop. 4.4, we have $d_2(W) = 2$ with $d_j(W) = 0$ for all other j. This can be verified directly by performing computations on W as follows. Clearly, $d_0(W) = 0$ and, since $\operatorname{reg} W = 3$, $d_j(W) = 0$ for $j \geq 4$. Since $HF_W(1) = \binom{2+1}{1}$ and $HF_W(2) = \binom{2+2}{2} - 2$, we know $d_1(W) = 0$ and $d_2(W) = 2$. Using linear algebra, it is easy to verify this two dimensional space of quadratic polynomials generates a six dimensional space of cubic polynomials. Since $HF_W(3) = \binom{2+3}{3} - 6$, $d_3(W) = 0$.

5 Examples

Code for easily reproducing the following examples using `Bertini` [2, 3] and `MATLAB` is available at `www.nd.edu/~jhauenst/aCM`.

5.1 The coupler curve of a planar four-bar linkage

Since the curve in Ex. 3.1 was smooth, the arithmetic genus and geometric genus are equal. Here, we investigate a nonsmooth curve arising in kinematics. In particular, the coupler curve of a planar four-bar linkage describes the motion allowed by a mechanism consisting of four hinged bars arranged as a quadrilateral in the plane. The arrangement of the mechanism is described by ten parameters $(p, \overline{p}, q, \overline{q}, s, \overline{s}, t, \overline{t}, r, R) \in \mathbb{C}^{10}$. If

$$\begin{array}{lll} a_1 = s(\overline{z} - \overline{p}), & \overline{a}_1 = \overline{s}(z - p), & \alpha_1 = (z - p)(\overline{z} - \overline{p}) + s\overline{s} - r, \\ a_2 = t(\overline{z} - \overline{q}), & \overline{a}_2 = \overline{t}(z - q), & \alpha_2 = (z - q)(\overline{z} - \overline{q}) + t\overline{t} - R, \end{array}$$

the coupler curve is the set of points $(z, \overline{z}) \in \mathbb{C}^2$ satisfying

$$\begin{vmatrix} \overline{a}_1 & \alpha_1 \\ \overline{a}_2 & \alpha_2 \end{vmatrix} \cdot \begin{vmatrix} a_1 & \alpha_1 \\ a_2 & \alpha_2 \end{vmatrix} + \begin{vmatrix} a_1 & \overline{a}_1 \\ a_2 & \overline{a}_2 \end{vmatrix}^2 = 0. \tag{16}$$

By fixing random values for the parameters and homogenizing (16), we will treat a general coupler curve C as a projective scheme on $\mathbb{P}^2$. The degree of C is 6, and the numerical algebraic geometry approach of [5] verified that the geometric genus is 1.

Let $W = C \cap \mathcal{H}$ where $\mathcal{H} \subset \mathbb{P}^2$ is a random hyperplane. Then,

$$HF_W = 1, 2, 3, 4, 5, 6, 6, \quad HF_C = 1, 3, 6, 10, 15, 21, 27, \quad \Delta HF_C = 1, 2, 3, 4, 5, 6, 6$$

shows that C is aCM by Corollary 3.3. In particular, (11) yields the arithmetic genus is $g_C = 10$.

5.2 A secant variety example

Consider the fourth secant variety of the Segre product for $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$, namely

$$X = \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3) \subset \mathbb{P}^{35}.$$

In [4], numerical computations showed that X was set-theoretically defined by 10 polynomials of degree 6 and 20 polynomials of degree 9. This result was shown without the use of a computer in [9]. Here, we show that X is aCM and use this to show that $I(X)$ is minimally generated by 10 polynomials of degree 6 and 20 polynomials of degree 9.

Rather than start with known polynomials vanishing on X, we derive our results from a parameterization of X. In particular, consider the map $\pi : \mathbb{C}^{3\times4} \times \mathbb{C}^{3\times4} \times \mathbb{C}^{4\times4} \to \mathbb{C}^{36}$ defined by

$$(S, T, U) \mapsto \sum_{\ell=1}^{4} S_{i\ell} T_{j\ell} U_{k\ell} \text{ for } 1 \leq i, j \leq 3 \text{ and } 1 \leq k \leq 4.$$

For $Y = \overline{\pi(\mathbb{C}^{3\times4} \times \mathbb{C}^{3\times4} \times \mathbb{C}^{4\times4})} \subset \mathbb{C}^{36}$, X is the projectivization of Y, namely $X = \mathbb{P}(Y) \subset \mathbb{P}^{35}$. Using π, it is easy to verify that X is non-defective with $\dim X = 31$. After selecting a random linear space $\mathcal{L} \subset \mathbb{P}^{35}$ of codimension 30 and random hyperplane $\mathcal{H} \subset \mathbb{P}^{35}$, consider the curve $C = X \cap \mathcal{L}$ and witness point set $W = C \cap \mathcal{H}$. We used `Bertini` to compute W and a pseudowitness set [18] for $C = X \cap \mathcal{L}$. This computation, in particular, verified that $\deg X = 345$ as reported in [4]. Algorithm 1 and [13] produced

$$\begin{aligned} HF_W &= 1, 5, 15, 35, 70, 126, 200, 280, 345, 345 \\ HF_C &= 1, 6, 21, 56, 126, 252, 452, 732, 1077, 1422 \\ \Delta HF_C &= 1, 5, 15, 35, 70, 126, 200, 280, 345, 345 \end{aligned}$$

which, by Corollary 3.3 and Theorem 4.1, shows that both C and X are aCM. Since $\rho_W = 8$, we know $\operatorname{reg} X = \operatorname{reg} C = \operatorname{reg} W = 9$, $\rho_C = 7$, and $\rho_X = -23$. In particular, (11) yields $g_C = 1684$

and the strategy outlined in Section 4 provides

$$HF_X = 1, 36, 666, 8436, 82251, 658008, 4496378, 26977968, 145001853, 708846128, \ldots$$
$$HS_X(t) = (1 + 4t + 10t^2 + 20t^3 + 35t^4 + 56t^5 + 74t^6 + 80t^7 + 65t^8)/(1 - t)^{32}$$
$$HP_X(t) = 345/31! \cdot t^{31} + \cdots + 299405047890287/72201776446800 \cdot t + 1.$$

In fact, since $\rho_X = -23$, $HP_X(j) = 0$ for $-23 \leq j \leq -1$ so that $HP_X(t)$ can be written as

$$HP_X(t) = \frac{G(t)}{31!} \prod_{j=1}^{23} (t + j) \quad \text{where}$$
$$G(t) = 345 \cdot t^8 + 13032 \cdot t^7 + 484578 \cdot t^6 + 11904840 \cdot t^5 + 218110185 \cdot t^4 + 2831500368 \cdot t^3 + 24772341372 \cdot t^2 + 131202341280 \cdot t + 318073392000.$$

We now turn to describing a minimal generating set for $I(X)$ using Prop. 4.4. Since $\operatorname{reg} X = 9$, we know that $I(X)$ is minimally generated by polynomials of degree at most 9, that is, $d_j(W) = d_j(X) = 0$ for $j \geq 10$. Moreover, $d_j(W) = d_j(X) = 0$ for $0 \leq j \leq 5$ since $HF_W(t) = \binom{4+t}{t}$ for $0 \leq t \leq 5$. Also, $HF_W(6) = \binom{4+6}{6} - 10$ yields that $d_6(W) = d_6(X) = 10$ with the initial degree of X being 6. Using linear algebra, we verified that this 10 dimensional space of sextic polynomials vanishing on W generates a 50 dimensional space of septic polynomials, a 150 dimensional space of octic polynomials, and a 350 dimensional space of nonic polynomials. Since $HF_W(7) = \binom{4+7}{7} - 50$, $HF_W(8) = \binom{4+8}{8} - 150$, and $HF_W(9) = \binom{4+9}{9} - 370$, we know $d_W(7) = d_X(7) = d_W(8) = d_X(8) = 0$ and $d_W(9) = d_X(9) = 20$. Therefore, $I(X)$ is minimally generated by 10 sextic polynomials and 20 nonic polynomials.

5.3 A non-aCM example

Consider the map $\pi : \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \to \mathbb{C}^{16}$ defined by

$$(S, T, U, V) \mapsto S_{k\ell} T_{ij} + U_{ik} V_{j\ell} \text{ for } i, j, k, \ell = 1, 2.$$

Let $Y = \overline{\pi(\mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2})} \subset \mathbb{C}^{16}$. Our object of interest is the projectivization of Y, which we will denote by $X = \mathbb{P}(Y) \subset \mathbb{P}^{15}$. Using π, it is easy to compute that $\dim X = 13$. After selecting a random linear space $\mathcal{L} \subset \mathbb{P}^{15}$ of codimension 12 and random hyperplane $\mathcal{H} \subset \mathbb{P}^{15}$, consider the curve $C = X \cap \mathcal{L}$ and witness point set $W = C \cap \mathcal{H}$. We used `Bertini` to compute W and a pseudowitness set [18] for C which yields that $\deg X = 28$. Algorithm 1 and [13] produced

$$HF_W = 1, 3, 6, 10, 15, 21, 28, 28$$
$$HF_C = 1, 4, 10, 20, 35, 56, 84, 120$$
$$\Delta HF_C = 1, 3, 6, 10, 15, 21, 28, 36.$$

In particular, $\rho_W = 6$ with $\Delta HF_C(7) = 36 \neq 28 = HF_W(7)$ yielding that C is not aCM. Therefore, by Theorem 4.1, X is not aCM.

For this non-aCM example, the terms of HF_C computed while testing C for arithmetically Cohen-Macaulayness are not enough to determine ρ_C. Since $\operatorname{reg} C > \rho_W + 1$, we use (4) to compute $\operatorname{reg} C$ with $\rho_C = \operatorname{reg} C - 1$. The additional terms of HF_C needed are

$$HF_C = 1, 4, 10, 20, 35, 56, 84, 120, 165, 196, 224, \quad \Delta HF_C = 1, 3, 6, 10, 15, 21, 28, 36, 45, 31, 28$$

showing that $\operatorname{reg} C = 10$ with $\rho_C = 9$. Using (9), the arithmetic genus of C is $g_C = 57$ with

$$HP_C(t) = 28t - 56$$
$$HS_C(t) = (1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 - 14t^9 - 3t^{10})/(1 - t)^2.$$

For comparison, consider [17, Alg. 2.4] for numerically testing the arithmetically Cohen-Macaulayness of C. This test requires an *a priori* bound on $\operatorname{reg} C$. One could use (4) to compute $\operatorname{reg} C$ exactly. However, this computation provides enough data needed to use Corollary 3.3 to decide the arithmetically Cohen-Macaulayness of C. Alternatively, one could bound $\operatorname{reg} C$, for example, by using [14] to conclude that $\operatorname{reg} C \leq 28 + 2 - 3 = 27$. In any event, if $r \geq \operatorname{reg} C$ is the selected bound, [17, Alg. 2.4] requires computing $HF_C(r - 1)$. Additionally, [17, Alg. 2.4] also requires computing $HF_{C\cap F}(r-1)$ where F is a general form of degree at most $r-1$. Using $r = 27$ from [14], one at least needs to compute $HF_C(26)$ and $HF_{C\cap F}(26)$ where F is a general form of degree 26, that is, $C \cap F$ is a zero-dimensional scheme of degree $28 \cdot 26 = 728$. Two advantages of using Corollary 3.3 is that the zero-dimensional scheme under consideration arises as a general hyperplane section of C rather than a general hypersurface section of C of possibly high degree and that $HF_C(t)$ is only needed up to $\rho_W + 1$ with $\rho_W + 1 \leq \operatorname{reg} C \leq r$.

5.4 An application from physics

A question arising in theoretical physics is the nature of the vacuum space in the Minimal Supersymmetric Standard Model. This gives rise to a family of problems that can be written as polynomial images of algebraic sets [11]. The following considers one such problem.

Let $F : \mathbb{C}^{16} \to \mathbb{C}^{16}$ and $\pi : \mathbb{C}^{16} \to \mathbb{C}^{25}$ be the polynomial systems defined in Appendix A.1. Consider the algebraic set $A = \overline{\pi(\mathcal{V}(F))} \subset \mathbb{C}^{25}$. Using the approach presented in [16] and `Bertini`, A has 11 irreducible components, namely $Y_1, \ldots, Y_8$ each of dimension 5 and degree 6, and 3 three-dimensional linear spaces. We take $Y_1, \ldots, Y_4$ as the self-conjugate ones whereas Y_6 and Y_8 are conjugate to Y_5 and Y_7, respectively. For $j = 1, \ldots, 8$, let $X_j \subset \mathbb{P}^{25}$ be the closure of the image of Y_j under the map $\mathbb{C}^{25} \hookrightarrow \mathbb{P}^{25}$ defined by $x \to (1, x)$.

We first investigate the arithmetically Cohen-Macaulayness of each X_j. After selecting a random linear space $\mathcal{L} \subset \mathbb{P}^{25}$ of codimension 4 and random hyperplane $\mathcal{H} \subset \mathbb{P}^{25}$, we computed the following for each $C_j = X_j \cap \mathcal{L}$ and $W_j = C_j \cap \mathcal{H}$:

$$HF_{W_j} = 1, 5, 6, 6, \quad HF_{C_j} = 1, 6, 12, 18, \quad \Delta HF_{C_j} = 1, 5, 6, 6.$$

Thus, Corollary 3.3 and Theorem 4.1 yield C_j and X_j are aCM for each $j = 1, \ldots, 8$. In particular, $\operatorname{reg} X_j = \operatorname{reg} C_j = \operatorname{reg} W_j = 3$, $\rho_{C_j} = 1$, $\rho_{X_j} = -3$, and (11) yields $g_{C_j} = 1$.

Using Prop. 4.4, we can describe the minimal generators of X_j via W_j. For $k > \operatorname{reg} X_j = 3$, we know $d_k(X_j) = 0$. By treating $W_j \subset \mathcal{L} \cap \mathcal{H}$, $HF_{W_j}(1) = \binom{20+1}{1} - 5 = 16$ implies that $d_1(X_j) = d_1(W_j) = 16$. By additionally restricting to this 16 dimensional linear space, we know $d_2(X_j) = d_2(W_j) = 9$ since $HF_{W_j}(2) = \binom{4+2}{2} - 9 = 6$. Moreover, since these quadratics generate a 29 dimensional space of cubics with $HF_{W_j}(3) = \binom{4+3}{3} - 29 = 6$, $d_3(X_j) = d_3(W_j) = 0$. Therefore, each X_j is minimally generated over $\mathbb{C}$ by 16 linear and 9 quadratic polynomials with

$$HP_{C_j}(t) = 6t \qquad HS_{C_j}(t) = (1 + 4t + t^2)/(1 - t)^2$$
$$HF_{X_j} = 1, 10, 46, 146, 371, \ldots \qquad HS_{X_j}(t) = (1 + 4t + t^2)/(1 - t)^6$$
$$HP_{X_j}(t) = 1/20 \cdot t^5 + 1/2 \cdot t^4 + 23/12 \cdot t^3 + 7/2 \cdot t^2 + 91/30 \cdot t + 1 = \tfrac{3\cdot t^2 + 12\cdot t + 10}{60} \prod_{j=1}^{3} (t + j).$$

Next, we investigate the $\mathbb{R}$-irreducible components $X_5 \cup X_6$ and $X_7 \cup X_8$. Since

$$HF_{W_j \cup W_{j+1}} = 1, 9, 12, 12, \quad HF_{C_j \cup C_{j+1}} = 1, 10, 24, 36, \quad \Delta HF_{C_j \cup C_{j+1}} = 1, 9, 14, 12$$

for $j = 5$ and $j = 7$, $X_5 \cup X_6$ and $X_7 \cup X_8$ are not aCM.

Finally, we consider the arithmetically Cohen-Macaulayness of $X = X_1 \cup \cdots \cup X_8$ using $C = C_1 \cup \cdots \cup C_8$ and $W = W_1 \cup \cdots \cup W_8$. Since $HF_W(1) = 11 \neq 13 = \Delta HF_C(1)$, X is not aCM.

5.5 A nonreduced scheme

In this example, we demonstrate computations on a scheme with components of multiplicity 2. Let $F : \mathbb{C}^{17} \to \mathbb{C}^{17}$ be the polynomial system defined in Appendix A.2, which arises from the same physics application mentioned in Section 5.4. We investigate both the algebraic set $V = \mathcal{V}(F)$ and the scheme S defined by F.

Following the approach presented in [16] and using `Bertini`, we find that V has irreducible decomposition consisting of 3 components of dimension 8 with multiplicity 2 and 1 component of dimension 3 with multiplicity 1. The multiplicity 2 components have degrees 8, 8, and 26. Let V_1 and V_2 denote the components of degree 8, and let $V_3 = V_1 \cup V_2$. For $j = 1, 2, 3$, let $X_j \subset \mathbb{P}^{17}$ be the closure of the image of V_j under the map $\mathbb{C}^{17} \hookrightarrow \mathbb{P}^{17}$ defined by $x \to (1, x)$.

After selecting a random linear space $\mathcal{L} \subset \mathbb{P}^{17}$ of codimension 7 and random hyperplane $\mathcal{H} \subset \mathbb{P}^{17}$, we consider $C_j = X_j \cap \mathcal{L}$ and $W_j = C_j \cap \mathcal{H}$ for each j. We compute:

$$\begin{array}{lll} HF_{W_1} = 1,5,8,8, & HF_{C_1} = 1,6,14,22, & \Delta HF_{C_1} = 1,5,8,8, \\ HF_{W_2} = 1,5,8,8, & HF_{C_2} = 1,6,14,22, & \Delta HF_{C_2} = 1,5,8,8, \\ HF_{W_3} = 1,5,11,15,16,16, & HF_{C_3} = 1,6,17,32,48,64, & \Delta HF_{C_3} = 1,5,11,15,16,16. \end{array}$$

Thus, Corollary 3.3 and Theorem 4.1 yield that C_j and X_j are aCM for $j = 1, 2, 3$. We compute $\operatorname{reg} X_1 = \operatorname{reg} X_2 = 3$, $\rho_{X_1} = \rho_{X_2} = -6$, $g_{C_1} = g_{C_2} = 3$, $\operatorname{reg} X_3 = 5$, $\rho_{X_3} = -4$, and $g_{C_3} = 17$. The strategy outlined in Section 4 provides

$$\begin{array}{ll} HF_{X_1}(t) = 1, 13, 84, 372, \ldots, & HS_{X_1}(t) = (1 + 4t + 3t^2)/(1 - t)^9, \\ HF_{X_2}(t) = 1, 13, 84, 372, \ldots, & HS_{X_2}(t) = (1 + 4t + 3t^2)/(1 - t)^9, \\ HF_{X_3}(t) = 1, 13, 87, 403, 1462, 4446, \ldots, & HS_{X_3}(t) = (1 + 4t + 6t^2 + 4t^3 + t^4)/(1 - t)^9, \\ \multicolumn{2}{l}{HP_{X_1}(t) = 1/5040 \cdot t^8 + \cdots + 443/140 \cdot t + 1 = \frac{t^2+5\cdot t+7}{5040} \prod_{j=1}^{6}(t + j),} \\ \multicolumn{2}{l}{HP_{X_2}(t) = 1/5040 \cdot t^8 + \cdots + 443/140 \cdot t + 1 = \frac{t^2+5\cdot t+7}{5040} \prod_{j=1}^{6}(t + j),} \\ \multicolumn{2}{l}{HP_{X_3}(t) = 1/2520 \cdot t^8 + \cdots + 263/84 \cdot t + 1 = \frac{(t^2+5\cdot t+7)\cdot(t^2+5\cdot t+15)}{2520} \prod_{j=1}^{4}(t + j).} \end{array}$$

Next, we consider the scheme S. For the sets X_j, curve sections C_j, and witness point sets W_j, we denote the corresponding projective schemes by $\widehat{X}_j$, $\widehat{C}_j$, and $\widehat{W}_j$. Using Algorithm 1, we have:

$$\begin{array}{lll} HF_{\widehat{W}_1} = 1,6,13,16,16, & HF_{\widehat{C}_1} = 1,7,20,36,52, & \Delta HF_{\widehat{C}_1} = 1,6,13,16,16, \\ HF_{\widehat{W}_2} = 1,6,13,16,16, & HF_{\widehat{C}_2} = 1,7,20,36,52, & \Delta HF_{\widehat{C}_2} = 1,6,13,16,16, \\ HF_{\widehat{W}_3} = 1,6,16,26,31,32,32, & HF_{\widehat{C}_3} = 1,7,23,49,80,112,144, & \Delta HF_{\widehat{C}_3} = 1,6,16,26,31,32,32. \end{array}$$

In particular, $\widehat{X}_1$, $\widehat{X}_2$, and $\widehat{X}_3$ are aCM schemes with $\operatorname{reg} \widehat{X}_1 = \operatorname{reg} \widehat{X}_2 = 4$, $\rho_{\widehat{X}_1} = \rho_{\widehat{X}_2} = -5$, $g_{\widehat{C}_1} = g_{\widehat{C}_2} = 13$, $\operatorname{reg} \widehat{X}_3 = 6$, $\rho_{\widehat{X}_3} = -3$, and $g_{\widehat{C}_3} = 49$. Following Section 4, we have

$$\begin{array}{ll} HF_{\widehat{X}_1}(t) = 1, 14, 97, 456, 1662, \ldots, & HS_{\widehat{X}_1}(t) = (1 + 5t + 7t^2 + 3t^3)/(1 - t)^9, \\ HF_{\widehat{X}_2}(t) = 1, 14, 97, 456, 1662, \ldots, & HS_{\widehat{X}_2}(t) = (1 + 5t + 7t^2 + 3t^3)/(1 - t)^9, \\ HF_{\widehat{X}_3}(t) = 1, 14, 100, 490, 1865, 5908, 16272, \ldots, & HS_{\widehat{X}_3}(t) = \frac{1+5t+10t^2+10t^3+5t^4+t^5}{(1-t)^9}, \\ \multicolumn{2}{l}{HP_{\widehat{X}_1}(t) = 1/2520 \cdot t^8 + \cdots + 453/140 \cdot t + 1 = \frac{t^3+7\cdot t^2+20\cdot t+21}{2520} \prod_{j=1}^{5}(t + j),} \\ \multicolumn{2}{l}{HP_{\widehat{X}_2}(t) = 1/2520 \cdot t^8 + \cdots + 453/140 \cdot t + 1 = \frac{t^3+7\cdot t^2+20\cdot t+21}{2520} \prod_{j=1}^{5}(t + j),} \\ \multicolumn{2}{l}{HP_{\widehat{X}_3}(t) = 1/1260 \cdot t^8 + \cdots + 337/105 \cdot t + 1 = \frac{(t+2)\cdot(t^4+8\cdot t^3+39\cdot t^2+92\cdot t+105)}{1260} \prod_{j=1}^{3}(t + j).} \end{array}$$

6 Conclusion

A fundamental goal of computational algebraic geometry is to compute information about a scheme, even when the ideal precisely defining the scheme is unknown. We developed an effective test, which can be performed using numerical algebraic geometric techniques for deciding the arithmetically Cohen-Macaulayness of a scheme. If the scheme is aCM, additional information such as the Castelnuovo-Mumford regularity, index of regularity, Hilbert series, and Hilbert polynomial can be computed directly from a (pseudo)witness point set. Also, a numerical algebraic geometric approach for computing the arithmetic genus of any curve is presented (see [5] for a numerical approach to compute the geometric genus). The effectiveness of our methods is demonstrated by performing computations related to schemes arising in various applications.

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References

[1] E. Allman. Open problem: Determine the ideal defining $\mathrm{Sec}^4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. Available at `http://www.dms.uaf.edu/~eallman/Papers/salmonPrize.pdf`.

[2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini: Software for numerical algebraic geometry. Available at `http://bertini.nd.edu`.

[3] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. *Numerically solving polynomial systems with Bertini*, volume 25 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[4] D.J. Bates and L. Oeding. Toward a salmon conjecture. *Exp. Math.*, 20(3):358–370, 2011.

[5] D.J. Bates, C. Peterson, A.J. Sommese, and C.W. Wampler. Numerical computation of the genus of an irreducible curve within an algebraic set. *J. Pure Appl. Algebra*, 215(8):1844–1851, 2011.

[6] D. Bayer and M. Stillman. A criterion for detecting m-regularity. *Invent. Math.*, 87(1):1–11, 1987.

[7] F. Cioffi, M.G. Marinari, and L. Ramella. Regularity bounds by minimal generators and Hilbert function. *Collect. Math.*, 60(1):89–100, 2009.

[8] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150. Springer, New York, NY, 1995.

[9] S. Friedland and E. Gross. A proof of the set-theoretic version of the salmon conjecture. *J. Algebra*, 356:374–379, 2012.

[10] E. Gorla. The general hyperplane section of a curve. *Trans. Am. Math. Soc.*, 358(2):819–869, 2006.

[11] J. Gray, Y.-H. He, V. Jejjala, and B.D. Nelson. Exploring the vacuum geometry of $\mathcal{N} = 1$ gauge theories. *Nucl. Phys. B*, 750(1):1–27, 2006.

[12] D.R. Grayson and M.E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2`.

[13] Z.A. Griffin, J.D. Hauenstein, C. Peterson, and A.J. Sommese. Numerical computation of the hilbert function and regularity of a zero dimensional scheme. In S.M. Cooper and S. Sather-Wagstaff, editors, *Connections Between Algebra, Combinatorics, and Geometry*, volume 76 of *Springer Proc. in Math. & Stat.*, pages 235–250. Springer New York, 2014.

[14] L. Gruson, R. Lazarsfeld, and C. Peskine. On a theorem of Castelnuovo, and the equations defining space curves. *Invent. Math.*, 72(3):491–506, 1983.

[15] W. Hao, A.J. Sommese, and Z. Zeng. Algorithm 931: an algorithm and software for computing multiplicity structures at zeros of nonlinear systems. *ACM Trans. Math. Software*, 40(1):Art. 5, 16, 2013.

[16] J.D. Hauenstein, Y.-H. He, and D. Mehta. Numerical elimination and moduli space of vacua. *J. High Energy Phys.*, to appear, 2013.

[17] J.D. Hauenstein, J.C. Migliore, C. Peterson, and A.J. Sommese. Numerical computation of the dimensions of the cohomology of twists of ideal sheaves. *Contemp. Math.*, 496:235–242, 2009.

[18] J.D. Hauenstein and A.J. Sommese. Witness sets of projections. *Appl. Math. Comput.*, 217(7):3349–3354, 2010.

[19] J.D. Hauenstein and A.J. Sommese. Membership tests for images of algebraic sets by linear projections. *Appl. Math. Comput.*, 219(12):6809–6818, 2013.

[20] J.D. Hauenstein and C.W. Wampler. Isosingular sets and deflation. *Found. Comp. Math.*, 13(3):371–403, 2013.

[21] C. Huneke and B. Ulrich. General hyperplane sections of algebraic varieties. *J. Algebraic Geom.*, 2(3):487–505, 1993.

[22] A. Iarrobino and V. Kanev. *Power Sums, Gorenstein Algebras, and Determinantal Loci*, volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[23] R. Krone. Numerical algorithms for dual bases of positive-dimensional ideals. *J. Algebra. Appl.*, 12(06):1350018, 2013.

[24] J.C. Migliore. *Introduction to Liaison Theory and Deficiency Modules*, volume 165. Birkhäuser Boston, Boston, MA, 1998.

[25] A.J. Sommese and C.W. Wampler. *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific Publishing, Hackensack, NJ, 2005.

[26] H.J. Stetter. *Numerical Polynomial Algebra*. SIAM, Philadelphia, PA, 2004.

[27] Z. Zeng. The closedness subspace method for computing the multiplicity structure of a polynomial system. *Contemp. Math.*, 496:347–362, 2009.

A Physics systems

A.1 First problem

The polynomial systems $F : \mathbb{C}^{16} \to \mathbb{C}^{16}$ and $\pi : \mathbb{C}^{16} \to \mathbb{C}^{25}$ used in Section 5.4 are defined by

$$
\begin{aligned}
F_1 &= 6x_{11}x_4 + 2x_{13}x_4 - 6x_{10}x_5 - 2x_{12}x_5 - 4x_5x_8 + 4x_4x_9\\
F_2 &= 6x_{10}x_3 + 2x_{12}x_3 - 3x_{14}x_6 - x_{15}x_6 - 4x_{16}x_6 + 4x_3x_8 + 3x_2x_{10} + 3x_2x_{12} + 8x_2x_8 + 2x_1x_{10} + 5x_1x_{12} + 10x_1x_8\\
F_3 &= 10x_1x_5 + 8x_2x_5 + 4x_3x_5\\
F_4 &= 6x_{11}x_3 + 2x_{13}x_3 - 3x_{14}x_7 - x_{15}x_7 - 4x_{16}x_7 + 2x_1x_{11} + 5x_1x_{13} + 10x_1x_9 + 3x_2x_{11} + 3x_2x_{13} + 8x_2x_9 + 4x_3x_9\\
F_5 &= 10x_1x_4 + 8x_2x_4 + 4x_3x_4\\
F_6 &= 2x_1x_5 + 3x_2x_5 + 6x_3x_5\\
F_7 &= 2x_1x_4 + 3x_2x_4 + 6x_3x_4\\
F_8 &= 5x_1x_5 + 3x_2x_5 + 2x_3x_5\\
F_9 &= 5x_1x_4 + 3x_2x_4 + 2x_3x_4\\
F_{10} &= 2x_5x_{10} + 5x_5x_{12} + 10x_5x_8 - 2x_4x_{11} - 5x_4x_{13} - 10x_4x_9\\
F_{11} &= 3x_5x_{10} + 3x_5x_{12} + 8x_5x_8 - 3x_4x_{11} - 3x_4x_{13} - 8x_4x_9\\
F_{12} &= 3x_{14}x_5 + x_{15}x_5 + 4x_{16}x_5\\
F_{13} &= 2x_{14} + 6x_{14}^2 + 3x_{15} + 4x_{14}x_{15} + 8x_{15}^2 + 10x_{16} + 16x_{14}x_{16} + x_{15}x_{16} + 3x_{16}^2 - 3x_5x_6 + 3x_4x_7\\
F_{14} &= 3x_{14}x_4 + x_{15}x_4 + 4x_{16}x_4\\
F_{15} &= 3x_{14} + 2x_{14}^2 + 6x_{15} + 16x_{14}x_{15} + 3x_{15}^2 + 2x_{16} + x_{14}x_{16} + 4x_{15}x_{16} + 4x_{16}^2 - x_5x_6 + x_4x_7\\
F_{16} &= 10x_{14} + 8x_{14}^2 + 2x_{15} + x_{14}x_{15} + 2x_{15}^2 + 4x_{16} + 6x_{14}x_{16} + 8x_{15}x_{16} + 27x_{16}^2 - 4x_5x_6 + 4x_4x_7
\end{aligned}
$$

$$
\begin{array}{lcllcl}
\pi_1 &=& x_{14} & \pi_{14} &=& x_1x_{11}x_{12} - x_1x_{10}x_{13}\\
\pi_2 &=& x_{15} & \pi_{15} &=& x_2x_{11}x_{12} - x_2x_{10}x_{13}\\
\pi_3 &=& x_{16} & \pi_{16} &=& x_3x_{11}x_{12} - x_3x_{10}x_{13}\\
\pi_4 &=& x_7x_8 - x_6x_9 & \pi_{17} &=& x_1x_5x_8 - x_1x_4x_9\\
\pi_5 &=& x_7x_{10} - x_6x_{11} & \pi_{18} &=& x_2x_5x_8 - x_2x_4x_9\\
\pi_6 &=& x_7x_{12} - x_6x_{13} & \pi_{19} &=& x_3x_5x_8 - x_3x_4x_9\\
\pi_7 &=& x_5x_6 - x_4x_7 & \pi_{20} &=& x_1x_5x_{10} - x_1x_4x_{11}\\
\pi_8 &=& x_1x_9x_{10} - x_1x_8x_{11} & \pi_{21} &=& x_2x_5x_{10} - x_2x_4x_{11}\\
\pi_9 &=& x_2x_9x_{10} - x_2x_8x_{11} & \pi_{22} &=& x_3x_5x_{10} - x_3x_4x_{11}\\
\pi_{10} &=& x_3x_9x_{10} - x_3x_8x_{11} & \pi_{23} &=& x_1x_5x_{12} - x_1x_4x_{13}\\
\pi_{11} &=& x_1x_9x_{12} - x_1x_8x_{13} & \pi_{24} &=& x_2x_5x_{12} - x_2x_4x_{13}\\
\pi_{12} &=& x_2x_9x_{12} - x_2x_8x_{13} & \pi_{25} &=& x_3x_5x_{12} - x_3x_4x_{13}\\
\pi_{13} &=& x_3x_9x_{12} - x_3x_8x_{13} & & &
\end{array}
$$

A.2 Second problem

The polynomial system $F : \mathbb{C}^{17} \to \mathbb{C}^{17}$ used in Section 5.5 is defined by

$$
\begin{aligned}
F_1 &= x_{10}x_{14} + 9x_{12}x_{14} + 9x_{10}x_{15} + 4x_{12}x_{15} + 9x_{10}x_{16} + 8x_{12}x_{16} + 9x_4 + 10x_{17}x_4 + 9x_{14}x_8 + 9x_{15}x_8 + 5x_{16}x_8\\
F_2 &= 3x_1x_5 + 2x_2x_5 + 8x_3x_5 + 9x_{14}x_7 + 9x_{15}x_7 + 5x_{16}x_7\\
F_3 &= -x_{11}x_6 - 9x_{13}x_6 + x_{10}x_7 + 9x_{12}x_7 + 9x_7x_8 - 9x_6x_9\\
F_4 &= x_{11}x_{14} + 9x_{13}x_{14} + 9x_{11}x_{15} + 4x_{13}x_{15} + 9x_{11}x_{16} + 8x_{13}x_{16} + 9x_5 + 10x_{17}x_5 + 9x_{14}x_9 + 9x_{15}x_9 + 5x_{16}x_9\\
F_5 &= 3x_1x_4 + 2x_2x_4 + 8x_3x_4 + 9x_{14}x_6 + 9x_{15}x_6 + 5x_{16}x_6\\
F_6 &= 9x_1x_5 + 8x_2x_5 + 8x_3x_5 + x_{14}x_7 + 9x_{15}x_7 + 9x_{16}x_7\\
F_7 &= 9x_1x_4 + 8x_2x_4 + 8x_3x_4 + x_{14}x_6 + 9x_{15}x_6 + 9x_{16}x_6\\
F_8 &= 4x_1x_5 + 4x_2x_5 + 9x_3x_5 + 9x_{14}x_7 + 4x_{15}x_7 + 8x_{16}x_7\\
F_9 &= 4x_1x_4 + 4x_2x_4 + 9x_3x_4 + 9x_{14}x_6 + 4x_{15}x_6 + 8x_{16}x_6\\
F_{10} &= -9x_{11}x_6 - 4x_{13}x_6 + 9x_{10}x_7 + 4x_{12}x_7 + 9x_7x_8 - 9x_6x_9\\
F_{11} &= -9x_{11}x_6 - 8x_{13}x_6 + 9x_{10}x_7 + 8x_{12}x_7 + 5x_7x_8 - 5x_6x_9\\
F_{12} &= 30x_{17}^2 - 10x_5x_6 + 10x_4x_7\\
F_{13} &= 8x_{10}x_2 + 4x_{12}x_2 + 8x_{10}x_3 + 9x_{12}x_3 - 9x_6 + 10x_{17}x_6 + 2x_2x_8 + 8x_3x_8 + 9x_1x_{10} + 4x_1x_{12} + 3x_1x_8\\
F_{14} &= 8x_5x_{10} + 4x_5x_{12} + 2x_5x_8 - 8x_4x_{11} - 4x_4x_{13} - 2x_4x_9\\
F_{15} &= 8x_5x_{10} + 9x_5x_{12} + 8x_5x_8 - 8x_4x_{11} - 9x_4x_{13} - 8x_4x_9\\
F_{16} &= 9x_5x_{10} + 4x_5x_{12} + 3x_5x_8 - 9x_4x_{11} - 4x_4x_{13} - 3x_4x_9\\
F_{17} &= -8x_{11}x_2 - 4x_{13}x_2 - 8x_{11}x_3 - 9x_{13}x_3 + 9x_7 + 10x_7x_{17} - 2x_2x_9 - 8x_3x_9 - 9x_1x_{11} - 4x_1x_{13} - 3x_1x_9
\end{aligned}
$$