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CERTIFIABLE NUMERICAL COMPUTATIONS IN SCHUBERT CALCULUS

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Abstract. Traditional formulations of geometric problems from the Schubert calculus, either in Plücker coordinates or in local coordinates provided by Schubert cells, yield systems of polynomials that are typically far from complete intersections and (in local coordinates) typically of degree exceeding two. We present an alternative primal-dual formulation using parametrizations of Schubert cells in the dual Grassmannians in which intersections of Schubert varieties become complete intersections of bilinear equations. This formulation enables the numerical certification of problems in the Schubert calculus.

1. Introduction

Numerical nonlinear algebra provides algorithms that certify numerically computed solutions to a system of polynomial equations, provided that the system is square—the number of equations is equal to the number of variables. To use these algorithms for certifying results obtained through numerical computation in algebraic geometry requires that we use equations which exhibit our varieties as complete intersections. While varieties are rarely global complete intersections, it suffices to have a *local* formulation in the folowing sense: The variety has an open dense set which our equations exhibit as a complete intersection in some affine space. Here, we use a primal-dual formulation of Schubert varieties to formulate all problems in Schubert calculus on a Grassmannian as complete intersections, and indicate how this extends to all classical flag manifolds.

The Schubert calculus of enumerative geometry has come to mean all problems which involve determining the linear subspaces of a vector space that have specified positions with respect to other fixed, but general, linear subspaces. It originated in work of Schubert [20] and others to solve geometric problems and was systemized in the 1880's [21, 22, 23]. Most work has been concerned with understanding the number of solutions to problems in the Schubert calculus, particularly finding [15], proving [24, 30], and generalizing the Littlewood-Richardson rule. As a rich and well-understood class of geometric problems, the Schubert calculus is a laboratory for the systematic study of new phenomena in enumerative geometry [28]. This study requires that Schubert problems be modeled and solved on a computer.

Symbolic methods, based on Gröbner bases and elimination theory, are well-understood and quite general. They are readily applied to solving Schubert problems—their use was

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central to uncovering evidence for the Shapiro Conjecture [27] as well as formulating its generalizations [5, 7, 19]. An advantage of symbolic methods is that they are exact—a successful computation is a proof that the outcome is as claimed. This exactness is also a limitation, particularly for Gröbner bases. The output of a Gröbner basis computation contains essentially all the information of the object computed, and this is the reason for the abysmal complexity of Gröbner bases [16], including that of zero-dimensional ideals [6].

Besides fundamental complexity, another limitation on Gröbner bases is that they do not appear to be parallelizable. This matters since the predictions of Moore's Law are now fulfilled through increased processor parallelism, and not by increased processor speed. Numerical methods based upon homotopy continuation [26] offer an attractive parallelizable alternative. A drawback to these numerical methods is that they do not intrinsically come with a proof that their output is as claimed, and for the Schubert calculus, standard homotopies perform poorly since standard upper bounds on the number of solutions, e.g., total degree and mixed volume, drastically over estimate the true number of solutions. The Pieri homotopy [11] and Littlewood-Richardson homotopy [29] offer optimal homotopy methods, but these are limited to Schubert calculus on the Grassmannian.

A numerical approximation to a solution of a system of polynomial equations may be refined using Newton's method and we call each such refinement a Newton iteration. Smale analyzed the convergence of repeated Newton iterations, when the system is square [25]. The name, α-theory, for this study refers to a constant α which depends upon the approximate solution x_0 and system f of polynomials [1, Ch. 8]. Smale showed that there exists $\alpha_0 > 0$ such that if $\alpha < \alpha_0$, then Newton iterations starting at x_0 will converge quadratically to a solution x of the system f. That is, the number of significant digits doubles with each Newton iteration. We note that the value of α_0 computed by Smale was 0.130707, but one can take $\alpha_0 = (13-3\sqrt{17})/4 \approx 0.157671$ [1, § 8.3]. With α-theory, we may use numerical methods in place of symbolic methods in many applications, e.g., counting the number of real solutions [9], while retaining the certainty of symbolic methods. While there has been some work studying the convergence of Newton iterations when the system is overdetermined [2], certification for solutions is only known to be possible for square systems.

Using a determinantal formulation, Schubert problems are prototypical overdetermined polynomial systems. Our main result is Theorem 3.5 which states their exists a natural reformulation of these systems as complete intersections using bilinear equations, thereby enabling the certification of approximate solutions.

In the next section, we give the usual determinantal formulation of intersections of Schubert varieties, and present local coordinates which simplify calculation in the Schubert calculus. In Section 3, we reformulate Schubert problems as complete intersections by solving a dual problem in a larger space, exchanging high-degree determinantal equations for bilinear equations. Finally, in Section 4, we combine competing formulations and discuss generalizations of our formulation.

2. Schubert Calculus

The solutions to a problem in the Schubert calculus are the points of an intersection of Schubert varieties in a Grassmannian. These intersections are formulated as systems of polynomial equations in local coordinates for the Grassmannian, which we now present.

Fix positive integers $k < n$ and let V be a complex vector space of dimension n. The set of all k-dimensional linear subspaces of V, denoted $\mathrm{Gr}(k,V)$, is the *Grassmannian of k-planes in V*. An ordered basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ for V yields an identification of V with $\mathbb{C}^n$ and leads to a system of local coordinates for $\mathrm{Gr}(k,V)$ given by matrices $X \in \mathbb{C}^{k\times(n-k)}$. In particular, the k-plane associated to a matrix X is the row space of the matrix $[X : I_k]$ where I_k is the $k\times k$ identity matrix. If $X = (x_{i,j})_{i=1,\ldots,k}^{j=1,\ldots,n-k}$, then this row space is the span of the vectors $h_i = \sum_{j=1}^{n-k} \mathbf{e}_j x_{i,j} + \mathbf{e}_{n-k+i}$ for $i = 1,\ldots,k$.

Let $\binom{[n]}{k}$ denote the set of sublists of $[n] := (1,2,\ldots,n)$ of cardinality k. A *Schubert (sub)variety* $X_\beta F_\bullet \subset \mathrm{Gr}(k,V)$ is given by the data of a *Schubert condition* $\beta \in \binom{[n]}{k}$ and a *(complete) flag* $F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n = V$ of linear subspaces with $\dim F_i = i$ where

$$X_\beta F_\bullet \;:=\; \{H \in \mathrm{Gr}(k,V) \mid \dim(H \cap F_{\beta_i}) \geq i,\ \text{for } i = 1,\ldots,k\}\,. \tag{2.1}$$

In particular, the Schubert variety $X_\beta F_\bullet$ is the set of k-planes satisfying the Schubert condition β with respect to the flag $F_\bullet$.

There are two standard formulations for a Schubert variety $X_\beta F_\bullet$, one as an implicit subset of $\mathrm{Gr}(k,V)$ given by a system of equations, and the other explicity, as a parametrized subset of $\mathrm{Gr}(k,V)$. For the first formulation, observe that the flag $F_\bullet$ may be given by an ordered basis $\mathbf{f}_1,\ldots,\mathbf{f}_n$ for V, where F_ℓ is the linear span of $\mathbf{f}_1,\ldots,\mathbf{f}_\ell$. Writing this basis $\{\mathbf{f}_i\}$ in terms of the basis $\{\mathbf{e}_j\}$ gives a matrix which we also write as $F_\bullet$. The space F_ℓ is the linear span of first ℓ rows of the matrix $F_\bullet$. The submatrix of $F_\bullet$ consisting of the first ℓ rows will also be written as F_ℓ.

In these local coordinates $[X : I_k]$, the Schubert variety $X_\beta F_\bullet$ is defined by

$$\operatorname{rank}\begin{bmatrix} X \;:\; I_k \\ F_{\beta_i}\end{bmatrix} \;\leq\; \beta_i + k - i \quad \text{for } i = 1,\ldots,k\,. \tag{2.2}$$

These rank conditions are equivalent to the vanishing of determinantal equations since the condition $\operatorname{rank}(M) \leq a-1$ is equivalent to the vanishing of all $a\times a$ minors (determinants of $a\times a$ submatrices) of M. These determinants are polynomials in the entries of X of degree up to $\min\{k, n-k\}$, and there are

$$\sum_{i=1}^{k} \tbinom{n}{\beta_i+k-i+1}\tbinom{k+\beta_i}{\beta_i+k-i+1}$$

of them. If $\beta = (n-k\,, n-k+2\,, n-k+3\,,\ldots,n)$, we write $\beta = \square$ and since the determinant is the only minor required to vanish, $X_\square F_\bullet$ is a hypersurface in $\mathrm{Gr}(k,V)$. In all other cases, there are linear dependencies among the minors, but any maximal linearly independent subset S of minors remains overdetermined, i.e., $\#(S) > \operatorname{codim}_{\mathrm{Gr}(k,V)} X_\beta F_\bullet$.

For the second formulation, consider the *coordinate flag $E_\bullet$* whose ordered basis is $\mathbf{e}_1,\ldots,\mathbf{e}_n$. The Schubert variety $X_\beta E_\bullet$ has a system of local coordinates similar to those for $\mathrm{Gr}(k,V)$. Consider the set of $k \times n$ matrices $M_\beta = (m_{i,j})$ whose entries satisfy

$$m_{i,\beta_j} \;=\; \delta_{i,j}\,, \qquad m_{i,j} \;=\; 0 \text{ if } j > \beta_i\,,$$

and the remaining entries are unconstrained. These unconstrained entries identify M_β with $\mathbb{C}^{\sum_i(\beta_i - i)}$. The association of a matrix in M_β to its row space yields a parametrization of an open subset of the Schubert variety $X_\beta E_\bullet$ that defines local coordinates.

Example 2.1. *When $k=3$ and $n=7$ and $\beta=(2,5,7)$ we have*

$$M_{257} \;=\; \begin{bmatrix} m_{11} & 1 & 0 & 0 & 0 & 0 & 0 \\ m_{21} & 0 & m_{23} & m_{24} & 1 & 0 & 0 \\ m_{31} & 0 & m_{33} & m_{34} & 0 & m_{36} & 1 \end{bmatrix}.$$

Lemma 2.2. *The association of a matrix in M_β to its row space identifies M_β with a dense open subset of $X_\beta E_\bullet$. If $F_\bullet$ is a complete flag given by a $n \times n$ matrix $F_\bullet$, then the association of a matrix H in M_β to the row space of the product $HF_\bullet$ identifies M_β with a dense open subset of $X_\beta F_\bullet$.*

Proof. The first statement is the assertion that M_β gives local coordinates for $X_\beta E_\bullet$, which is classical [4, p. 147]. The second statement follows from the observation that if $g \in GL(n,\mathbb{C})$ is an invertible linear transformation, a k-plane H lies in $X_\beta E_\bullet$ if and only if Hg lies in $(X_\beta E_\bullet)g = X_\beta(E_\bullet g)$. The lemma follows as the transformation g with $F_\bullet = E_\bullet g$ is given by the matrix $F_\bullet$. ☐

Counting parameters gives a formula for the codimension of $X_\beta F_\bullet$ in $\mathrm{Gr}(k,V)$ namely

$$|\beta| \;:=\; \operatorname{codim} X_\beta F_\bullet \;=\; k(n-k) - \sum_i (\beta_i - i)\,.$$

There is a smaller system of local coordinates M_β^γ which explicitly parametrizes an intersection of two Schubert varieties. Let $E'_\bullet$ be the coordinate flag opposite to $E_\bullet$ in which $E'_\ell := \langle \mathbf{e}_n, \ldots, \mathbf{e}_{n+1-\ell}\rangle$. Let $\beta,\gamma \in \binom{[n]}{k}$. By the definition of a Schubert variety (2.1) and of the flags $E_\bullet$ and $E'_\bullet$, the intersection $X_\beta E_\bullet \cap X_\gamma E'_\bullet$ is nonempty if and only if we have $n+1-\gamma_{k+1-i} \leq \beta_i$ for each $i=1,\ldots,k$. When this holds, the intersection has a system of local coordinates given by the row space of $k \times n$ matrices $M_\beta^\gamma = (m_{i,j})$ in which

$$m_{i,j} \;:=\; 0 \;\text{ if }\; j \notin [n+1-\gamma_{k+1-i}, \beta_i] \;\text{ and }\; m_{i,\beta_i} \;:=\; 1\,, \;\text{ for }\; i = 1,\ldots,k\,.$$

The unconstrained entries of M_β^γ identify it with the affine space $\mathbb{C}^{k(n-k)-|\beta|-|\gamma|}$.

Example 2.3. *When $k=3$, $n=7$, $\beta=(2,5,7)$, and $\gamma=(3,5,7)$ we have*

$$M_{257}^{357} \;=\; \begin{bmatrix} m_{11} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{23} & m_{24} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{35} & m_{36} & 1 \end{bmatrix}.$$

Lemma 2.4. *The association of a matrix in M_β^γ to its row space identifies M_β^γ with a dense open subset of $X_\beta E_\bullet \cap X_\gamma E'_\bullet$.*

Remark 2.5. *The lemma is classical. When $F^1_\bullet$ and $F^2_\bullet$ are in linear general position, there is a choice of basis for $\mathbb{C}^n$ for which $F^1_\bullet = E_\bullet$ and $F^2_\bullet = E'_\bullet$, so we often make the assumption that $F^1_\bullet = E_\bullet$ and $F^1_\bullet = E'_\bullet$.*

A *Schubert problem* on $\mathrm{Gr}(k, V)$ is a list of Schubert conditions $\boldsymbol{\beta} = (\beta^1, \ldots, \beta^\ell)$ with $\sum_{i=1}^{\ell} |\beta^i| = k(n-k)$. Given a Schubert problem $\boldsymbol{\beta}$ and flags $F^1_\bullet, \ldots, F^\ell_\bullet$, the intersection

$$X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet \tag{2.3}$$

is an *instance* of a Schubert problem $\boldsymbol{\beta}$. When the flags are general, the intersection (2.3) is transverse [12]. The points in the intersection are the *solutions* to this instance of the Schubert problem, and their number $N(\boldsymbol{\beta})$ may be calculated using algorithms based on the Littlewood-Richardson rule.

Example 2.6. *Suppose that $k = 2$, $n = 6$, and $\boldsymbol{\beta} = (\beta, \beta, \beta, \beta)$ where $\beta = (3, 6)$. Since $|(3, 6)| = 2$ and $2 + 2 + 2 + 2 = 2(6 - 2) = \dim \mathrm{Gr}(2, \mathbb{C}^6)$, $\boldsymbol{\beta}$ is a Schubert problem on $\mathrm{Gr}(2, \mathbb{C}^6)$. One can verify that $N(\boldsymbol{\beta}) = 3$.*

We wish to solve instances (2.3) of a Schubert problem $\boldsymbol{\beta}$ formulated as a system of equations given by the rank conditions (2.2). Rather than use the local coordinates $[X : I_k]$ for the Grassmannian, which has $k(n-k)$ variables, we may use M_{β^1} as local coordinates for $X_{\beta^1} F^1_\bullet$, which gives $k(n-k) - |\beta^1|$ variables. When $F^1_\bullet$ and $F^2_\bullet$ are in linear general position, we may use $M^{\beta^2}_{\beta^1}$ as local coordinates for $X_{\beta^1} F^1_\bullet \cap X_{\beta^2} F^2_\bullet$, which gives only $k(n-k) - |\beta^1| - |\beta^2|$ variables. These smaller sets of local coordinates often lead to more efficient computation.

Example 2.7. *For the Schubert problem of Example 2.6, if we assume that $F^1_\bullet = E_\bullet$ and $F^2_\bullet = E'_\bullet$, then we may use the local coordinates M^{36}_{36}. In these local coordinates, the essential rank conditions (2.2) on the Schubert variety $X_{36} F_\bullet$ are equivalent to the vanishing of all full-sized (5×5) minors of the 5×6 matrix whose first two rows are M^{36}_{36} and last three are F_3. In particular, we have $2 \cdot 6 = 12$ equations of degree at most 2 in four variables, which have three common solutions. The maximal linearly independent set of equations consists of six equations, which is still overdetermined.*

3. Primal-Dual Formulation of Schubert Problems

Large computational experiments [5, 7, 27] have successfully used symbolic computation to solve billions of instances of Schubert problems, producing compelling conjectures, some of which have since been proved [3, 10, 17, 18]. These experiments required certified symbolic methods in characteristic zero and were constrained by the limits of computability imposed by the complexity of Gröbner basis computation. Roughly, Schubert problems with more than 100 solutions or whose formulation involves more than 16 variables are infeasible, and a typical problem at the limit of feasibility has 30 solutions in 9 variables.

Numerical methods offer the best route for studying larger Schubert problems. This led to the development of specialized numerical algorithms for Schubert problems, such as the Pieri homotopy algorithm [11], which was used to study a problem with 17589 solutions [14]. It is also driving the development [29] and implementation [13] of the

Littlewood-Richardson homotopy, based on Vakil's geometric Littlewood-Richardson rule [31, 32]. Regeneration [8] offers another numerical approach for Schubert problems.

As explained in Section 2, traditional formulations of Schubert problems typically lead to *overdetermined* (more equations than variables) systems of polynomials of degree $\min\{k, n-k\}$, expressed in whichever of the systems $[X : I_k]$, M_β, or M_β^γ of local coordinates is relevant. We present an alternative formulation of Schubert varieties and Schubert problems complete intersections of bilinear equations involving more variables.

Recall that V is a vector space equipped with a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Let V^* be its dual vector space and $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ be the corresponding dual basis. For every $k = 1, \dots, n-1$, the association of a k-plane $H \subset V$ to its annihilator $H^\perp \subset V^*$ defines a cannonical isomorphism between the Grassmannian $\mathrm{Gr}(k, V)$ and its *dual Grassmannian* $\mathrm{Gr}(n-k, V^*)$. To every Schubert variety $X_\beta F_\bullet \subset \mathrm{Gr}(k, V)$ we have $\perp(X_\beta F_\bullet) := \{H^\perp \mid H \in X_\beta F_\bullet\}$, which is a subset of $\mathrm{Gr}(n-k, n)$. To identify $\perp(X_\beta F_\bullet)$, we make some definitions.

Each flag $F_\bullet$ on V has a corresponding *dual flag* $F_\bullet^\perp$ on V^*,

$$F_\bullet^\perp \;:\; (F_{n-1})^\perp \subset (F_{n-2})^\perp \subset \cdots \subset (F_1)^\perp \subset V*\,,$$

which is a flag since $\dim(F_i) + \dim(F_{n-i})^\perp = n$. For $\beta \in \binom{[n]}{k}$, a subset of $[n]$ of cardinality k, consider $\beta^\perp := (j \mid n+1-j \in [n] \smallsetminus \beta) \in \binom{[n]}{n-k}$. The map $\beta \mapsto \beta^\perp$ is a bijection.

Lemma 3.1. *For a Schubert variety $X_\beta F_\bullet \subset \mathrm{Gr}(k, V)$, we have $\perp(X_\beta F_\bullet) = X_{\beta^\perp} F_\bullet^\perp$.*

Note that $X_\beta F_\bullet = \perp(X_{\beta^\perp} F_\bullet^\perp)$. We call $X_\beta F_\bullet$ and $X_{\beta^\perp} F_\bullet^\perp$ *dual Schubert varieties*.

Proof. Observe that if $F_\bullet$ is a flag and H a linear subspace, then $\dim H \cap F_b \geq a$ implies that $\dim H \cap F_{b+1} \geq a$. Thus the definition (2.1) of Schubert variety is equivalent to

$$X_\beta F_\bullet \;:=\; \{H \in \mathrm{Gr}(k, V) \;\mid\; \dim(H \cap F_i) \geq \#\{\beta \cap [i]\},\ \text{for } i = 1, \dots, n\}\,.$$

For every $H \in \mathrm{Gr}(k, V)$ and all $i = 1, \dots, n$, the following are equivalent:

$$\begin{aligned}
&\dim H \cap F_i \;\geq\; \#(\beta \cap [i]) \\
&\qquad \Leftrightarrow\; \dim(\mathrm{Span}\{H, F_i\}) \;\leq\; k + i - \#(\beta \cap [i]) \;=\; i + \#(\beta \cap \{i+1, \dots, n\}) \\
&\qquad \Leftrightarrow\; \dim(\mathrm{Span}\{H, F_i\}^\perp) \;\geq n - i - \#(\beta \cap \{i+1, \dots, n\}) \\
&\qquad \Leftrightarrow\; \dim(H^\perp \cap F_{n-i}^\perp) \;\geq n - i - \#(\beta \cap \{i+1, \dots, n\})\,.
\end{aligned}$$

Since $n - i - \#(\beta \cap \{i+1, \dots, n\}) = \#(\beta^\perp \cap [n-i])$, the lemma follows from (2.1). □

Let $\Delta : \mathrm{Gr}(k, V) \to \mathrm{Gr}(k, V) \times \mathrm{Gr}(n-k, V^*)$ be the *dual diagonal* map sending $H \in \mathrm{Gr}(k, V)$ to $(H, H^\perp) \in \mathrm{Gr}(k, V) \times \mathrm{Gr}(n-k, V^*)$.

Lemma 3.2. *We have an isomorphism $X_\beta F_\bullet \cong \Delta(X_\beta F_\bullet)$, and $\Delta(X_\beta F_\bullet)$ is locally defined by $k(n-k)$ bilinear equations in the coordinates $(X, M_{\beta^\perp})$ for $\mathrm{Gr}(k, V) \times X_{\beta^\perp} F_\bullet^\perp$.*

Proof. The local coordinates $[X : I_k]$ for $\mathrm{Gr}(k, V)$ induce dual coordinates $[I_{n-k} : Y]$ such that $\Delta(\mathrm{Gr}(k, V)) \subset \mathrm{Gr}(k, V) \times \mathrm{Gr}(n-k, V^*)$ is defined by the $k(n-k)$ bilinear forms given by $[X : I_k][I_{n-k} : Y]^T = 0$. Lemma 3.1 ensures that $\Delta(X_\beta F_\bullet) \subset \mathrm{Gr}(k, V) \times X_{\beta^\perp} F_\bullet^\perp$. By applying a change of basis in the second factor $\mathrm{Gr}(n-k, V^*)$, points of $\mathrm{Gr}(k, V) \times X_{\beta^\perp} F_\bullet^\perp$

are given locally by $([X : I_k], M_{\beta^\perp})$. The change of basis acts linearly on the bilinear forms producing $k(n-k)$ different bilinear forms.

Since the projection of $\Delta(X_\beta F_\bullet)$ onto the second factor is $X_{\beta^\perp}$, the coordinates $M_{\beta^\perp}$ in the second factor parametrize only points of $\Delta(X_\beta F_\bullet) \subset \mathrm{Gr}(k,V) \times \mathrm{Gr}(n-k,V^*)$. Thus $\Delta(X_\beta F_\bullet)$ is locally defined by the $k(n-k)$ bilinear equations. □

Remark 3.3. *We have shown that $X_\beta F_\bullet$ may be defined using $k(n-k)$ equations in $2k(n-k)-|\beta|$ variables. We call this the dual formulation of $X_\beta F_\bullet$. We have the option of choosing between this and the primal formulation, in which we solve the Schubert problem directly in coordinates $[X : I_k]$.*

We may extend the map Δ to a larger dual diagonal map

$$\Delta^\ell : \mathrm{Gr}(k,V) \to \mathrm{Gr}(k,V) \times \underbrace{\mathrm{Gr}(n-k,V^*) \times \cdots \times \mathrm{Gr}(n-k,V^*)}_{\ell \text{ factors}},$$

defined by $H \mapsto (H, H^\perp, \ldots, H^\perp)$. Then, the points of $\Delta^\ell(X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet)$, and thus $X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet$, can be computed by solving a system of equations which define each $X_{\beta^i} F^i_\bullet$ using either a primal formulation in the first factor or a dual formulation in one of the subsequent factors.

Lemma 3.4. *Let $\boldsymbol{\beta}$ be a Schubert problem. The set $\Delta^\ell(X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet)$ is locally defined by $\ell k(n-k)$ bilinear equations in the coordinates $(X, M_{\beta^{1\perp}}, \ldots, M_{\beta^{\ell\perp}})$ for*

$$\mathrm{Gr}(k,V) \times X_{\beta^{1\perp}} F^{1\perp}_\bullet \times \cdots \times X_{\beta^{\ell\perp}} F^{\ell\perp}_\bullet .$$

Proof. Using local coordinates $[X : I_k], [Y^1 : I_{n-k}], \ldots, [Y^\ell : I_{n-k}]$, there are $\ell k(n-k)$ bilinear forms defining $\Delta^\ell(\mathrm{Gr}(k,V))$ in $\mathrm{Gr}(k,V) \times \mathrm{Gr}(n-k,V^*) \times \cdots \times \mathrm{Gr}(n-k,V^*)$ given by pairing the first factor with each of the others. The set $\Delta^\ell(X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet)$ may be calculated by its primal formulation by solving equations defining $X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet$ in the first factor and then using the $\ell k(n-k)$ bilinear forms to determine the values of the other coordinates. By applying Lemma 3.2 one may equivalently find points in $X_{\beta^1} F^1_\bullet$ via its dual formulation in the second factor. Following the proof of Lemma 3.2, $\Delta^\ell(X_{\beta^1} F^1_\bullet \cap \cdots \cap X_{\beta^\ell} F^\ell_\bullet)$ is given by the system of equations defining

$$\left(\bigcap_{i=2}^{\ell} X_{\beta^i} F^i_\bullet \times X_{\beta^{1\perp}} F^{1\perp}_\bullet \times \prod_{i=2}^{\ell} \mathrm{Gr}(n-k,V^*) \right) \cap \Delta(\mathrm{Gr}(k,V))$$

in local coordinates $(X, M_{\beta^{1\perp}}, \ldots, [Y^\ell : I_{n-k}])$. Similarly, recasting the problem using dual formulations for each $X_{\beta^i} F^i_\bullet$ in the $(i+1)$-st factor, we obtain a system of bilinear forms on

$$\mathrm{Gr}(k,V) \times X_{\beta^{1\perp}} F^{1\perp}_\bullet \times \cdots \times X_{\beta^{\ell\perp}} F^{\ell\perp}_\bullet ,$$

in local coordinates $(X, M_{\beta^{1\perp}}, \ldots, M_{\beta^{\ell\perp}})$. □

Theorem 3.5. *Any sufficiently general instance of a Schubert problem $\boldsymbol{\beta}$ may be naturally reformulated as a complete intersection in the coordinates $(X, M_{\beta^{1\perp}}, \ldots, M_{\beta^{\ell\perp}})$.*

Proof. A sufficiently general instance of $\boldsymbol{\beta}$ is zero-dimensional with $N(\boldsymbol{\beta})$ solutions. The result follows from Lemma 3.4, as there are $\ell k(n-k)$ equations in $\ell k(n-k)$ variables. □

Theorem 3.5 provides a natural formulation for an instance of a Schubert problem as a square system. This rectifies the fundamental obstruction to using numerical methods in place of certified symbolic methods for solving Schubert problems. Morever, this method of reformulation may be applied to other geometric intersections involving varieties with suitable parametrizations.

The following increases the efficiency of the dual formulation.

Corollary 3.6. *Any sufficiently general instance of a Schubert problem $\boldsymbol{\beta}$ given by the intersection of ℓ Schubert varieties may be naturally reformulated as a complete intersection in $\lfloor \frac{\ell}{2} \rfloor k(n-k)$ variables.*

Proof. When ℓ is even one may reduce the number of equations and variables by equivalently parametrizing

$$(X_{\beta^1}F^1_\bullet \cap X_{\beta^2}F^2_\bullet) \times (X_{\beta^{3\perp}}F^{3\perp}_\bullet \cap X_{\beta^{4\perp}}F^{4\perp}_\bullet) \times \cdots \times (X_{\beta^{\ell-1\perp}}F^{\ell-1\perp}_\bullet \cap X_{\beta^{\ell\perp}}F^{\ell\perp}_\bullet)\,,$$

using local coordinates $(M^{\beta^2}_{\beta^1}, M^{\beta^4}_{\beta^3}, \ldots, M^{\beta^\ell}_{\beta^{\ell-1}})$. When ℓ is odd, the last factor is simply $X_{\beta_{\ell\perp}}F^{\ell\perp}_\bullet$, and the local coordinates are $(M^{\beta^2}_{\beta^1}, M^{\beta^4}_{\beta^3}, \ldots, M^{\beta^{\ell-1}}_{\beta^{\ell-2}}, M_{\beta^\ell})$. $\square$

4. Specialization and Generalization.

The previous section formulated Schubert problems as a square system, enabling one to certify output from numerical methods. In many cases, it is possible to eliminate some variables without the system becoming overdetermined.

Recall that $X_{\square}F_\bullet$ is a hypersurface defined by one equation. Given a Schubert problem $\boldsymbol{\beta} = (\square, \ldots, \square, \beta^m, \ldots, \beta^\ell)$, one has a square system when using the primal formulation for the intersection of the first $m+1$ Schubert varieties in local coordinates $M^{\beta^{m+1}}_{\beta^m}$. While this generally introduces equations of higher degree, it reduces the number of variables.

Example 4.1. *We denote $\beta = (n{-}k{-}1\,, n{-}k{+}2\,, n{-}k{+}3\,, \ldots, n)$ by $\beta = \square\square\square$. Consider the Schubert problem*

$$\boldsymbol{\beta} = (\square, \square, \square, \square, \square, \square, \square\square\square, \square\square\square, \square\square\square, \square\square\square)$$

in $\mathrm{Gr}(3,9)$*. The primal formulation* (2.2) *consists of 26 linearly independent determinants of degree at most 3 in $M^{\square\square\square}_{\square\square\square}$. By using the primal formulation for the intersection*

$$X_{\square}F^1_\bullet \cap \cdots \cap X_{\square}F^6_\bullet \cap X_{\square\square\square}F^7_\bullet \cap X_{\square\square\square}F^8_\bullet$$

and the dual formulation for the intersection

$$X_{\square\square\square^\perp}F^{9\perp}_\bullet \cap X_{\square\square\square^\perp}F^{10\perp}_\bullet,$$

this problem is reduced to a square system consisting of 18 bilinear equations and 6 determinants in the 24 variables $(M^{\square\square\square}_{\square\square\square}, M^{\square\square\square}_{\square\square\square})$. The number of solutions is $N(\boldsymbol{\beta}) = 437$.

We now have several techniques with which to apply standard homotopy methods to solve and certify Schubert problems in the Grassmannian. Other techniques have been optimized to perform similar computations, but our approach is applicable to many other situations. Local coordinates, similar to those used in our arguments, exist for flag varieties of types A, B, C, and D. Therefore, our approach applies to the Schubert calculus of all classical flag varieties as well. We have implemented these techniques in Schubert problems on Grassmannians, and we hope to do this in the future for other flag varieties.

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