A hybrid symbolic-numeric approach to exceptional sets of generically zero-dimensional systems

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ABSTRACT

Exceptional sets are the sets where the dimension of the fiber of a map is larger than the generic fiber dimension, which we assume is zero. Such situations naturally arise in kinematics, for example, when designing a mechanism that moves when the generic case is rigid. In 2008, Sommese and Wampler showed that one can use fiber products to promote such sets to become irreducible components. We propose an alternative approach using rank constraints on Macaulay matrices. Symbolic computations are used to construct the proper Macaulay matrices, while numerical computations are used to solve the rank-constraint problem. Various exceptional sets are computed, including exceptional RR dyads, lines on surfaces in $\mathbb{C}^3$, and exceptional planar pentads.

Keywords

exceptional sets, exceptional mechanisms, symbolic-numeric, Macaulay matrices, numerical algebraic geometry

1. INTRODUCTION

Consider the system of polynomials

$$F(x;p) = \begin{bmatrix} F_1(x;p) \\ \vdots \\ F_n(x;p) \end{bmatrix} \tag{1}$$

with variables $x \in \mathbb{C}^N$ and parameters $p \in \mathbb{C}^P$. Denote the solution set of $F = 0$ in $\mathbb{C}^{N+P}$ as $\mathcal{V}(F)$. The *fiber* over $p^* \in \mathbb{C}^P$, denoted $\mathcal{F}_{p^*} \subset \mathbb{C}^N$, is the solution set in $\mathbb{C}^N$ of the system of polynomial equations $F(x;p^*) = 0$. That is, if $\pi_x(x;p) = x$ and $\pi_p(x;p) = p$, then

$$\mathcal{F}_{p^*} = \pi_x(\pi_p^{-1}(p^*) \cap \mathcal{V}(F)). \tag{2}$$

We present a symbolic-numeric approach for describing the irreducible components of the *exceptional set*, which is the closure of the set of parameters p^* where $\dim \mathcal{F}_{p^*} > 0$ in the case when the generic fiber consists of finitely many points.

The main motivation for computing exceptional sets is the computation of exceptional mechanisms [22, 25, 29, 28], that is, computing mechanisms which have unexpected motion. For example, a general Stewart-Gough platform is rigid having 40 assembly modes. The subfamily consisting of Griffis-Duffy platforms exhibit motion [10, 17]. Another family of exceptional Stewart-Gough platforms was discovered in [8]. A full classification of the exceptional Stewart-Gough platforms remains an open problem [24].

If $\overline{\pi_p(\mathcal{V}(F))} \subsetneq \mathbb{C}^P$, then the generic fiber is empty. Thus, the exceptional set is simply

$$E(F) = \overline{\{p^* \in \mathbb{C}^P \mid \mathcal{F}_{p^*} \neq \emptyset\}} = \overline{\pi_p(\mathcal{V}(F))},$$

which is the set of parameters p^* for which $F(x;p^*) = 0$ has at least one solution. Computing the set of polynomials that vanish on $E(F)$ is the prime goal of classical elimination theory. Rather than computing defining equations, the prime goal of numerical elimination theory is to compute pseudo-witness sets [13] for the irreducible components of $E(F)$.

Our focus is on the case when $\overline{\pi_p(\mathcal{V}(F))} = \mathbb{C}^P$. Let

$$d_{gf}(F) = \min\{\dim \mathcal{F}_{p^*} \mid p^* \in \mathbb{C}^P, \mathcal{F}_{p^*} \neq \emptyset\}, \tag{3}$$

which is the *generic fiber dimension*, i.e., there is a nonempty Zariski open subset $U \subset \mathbb{C}^P$ such that $\dim \mathcal{F}_{p^*} = d_{gf}(F)$ for all $p^* \in U$. Our symbolic-numeric approach applies when $d_{gf}(F) = 0$ and computes the irreducible components of

$$E(F) = \overline{\{p^* \in \mathbb{C}^P \mid \dim \mathcal{F}_{p^*} > 0\}}. \tag{4}$$

Since we are solely focused on the dimension of the fibers and one may investigate each irreducible component of $\mathcal{V}(F)$ independently, we will additionally assume that $\mathcal{V}(F)$ is irreducible and has multiplicity 1 with respect to F, i.e., the ideal generated by F is prime. In particular, $\dim \mathcal{V}(F) = P$.

One approach for computing exceptional sets is presented in [25]. This method uses fiber products to "promote" the exceptional sets to irreducible components. That is, they reduce the problem of computing exceptional sets to computing irreducible components of polynomial systems of the form

$$\begin{bmatrix} F(x^{(1)};p) \\ \vdots \\ F(x^{(k)};p) \end{bmatrix}.$$

They provide bounds on k based on information about the fiber dimension of the exceptional sets one aims to compute.

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This fiber product approach is geometric which works by adding new variables to increase the dimension of the corresponding solution set. Our approach, summarized in Section 4, follows from an algebraic viewpoint and works by imposing rank conditions on matrices. First, we need to construct the matrices to impose the rank constraints. This step, described in Section 2, uses symbolic computations to compute a *parameterized h-basis* using linear algebra routines which could be performed in parallel. With this basis, in hand, we then impose rank constraints on the *Macaulay matrices*, described in Section 3. We impose the rank constraints using the method of [3] via numerical algebraic geometry to compute in parallel the rank-deficiency sets of the matrix of coefficients, i.e., the Macaulay matrix. The result of computing a parameterized h-basis is that the Macaulay matrix has entries which only depend on the parameters p. Several examples are presented in Section 5.

We note that there is another approach for computing exceptional sets for linkages is through the theory of bonds, e.g., see [1, 15, 16]. The main difference is that this combinatorial technique is applicable to linkages while the fiber product [25] and our new approach are general purpose methods.

Macaulay matrices have been used in related situations. For example, [19] uses rank conditions on Macaulay matrices to locate embedded components along with other singular sets. In [6], the left null spaces of Macaulay matrices are related to syzygies. Also, [20] uses Macaulay matrices to deflate, i.e., "regularize," singular isolated solutions (see also [7, 11, 30]). Macaulay matrices were also used to develop a numerical local dimension test in [4].

2. CONSTRUCTING A PARAMETERIZED BASIS

Let $F : \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^n$ be a polynomial system as in (1). As stated in the Introduction, we assume the following:

1. $\mathcal{V}(F) \subset \mathbb{C}^{N+P}$ is irreducible of dimension P and has multiplicity 1 with respect to F; and
2. $\overline{\pi_p(\mathcal{V}(F))} = \mathbb{C}^P$ where $\pi_p(x;p) = p$.

This implies that $d_{gf}(F) = 0$, i.e., for p^* in a nonempty Zariski open subset of $\mathbb{C}^P$, the fiber $\mathcal{F}_{p^*}$ of F over p^* is nonempty and contains finitely many points.

The following describes constructing a parameterized h-basis for $F(x;p)$. To do this, we first consider the nonparameterized case and then return to the parameterized setting.

2.1 Hilbert function

Suppose that $G : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system such that $\mathcal{V}(G) \subset \mathbb{C}^N$ is nonempty and has finitely many points. Thus, the polynomial system G generates a zero-dimensional ideal, namely

$$I = \langle G(x) \rangle \subset \mathbb{C}[x_1, \ldots, x_N].$$

We define the *Hilbert function of* G to be the Hilbert function of $\mathbb{C}[x_1, \ldots, x_N]/I$, i.e., $H_G : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ where

$$H_G(k) = \binom{N+k}{k} - \dim I_{\leq k}$$

with $I_{\leq k}$ being the vector space of polynomials in I of degree at most k. The binomial coefficient $\binom{N+k}{k}$ is simply the dimension of the vector space of polynomials in N variables of degree at most k. Since $0 < |\mathcal{V}(G)| < \infty$, we know that there exists $k^* \geq 0$, called the *index of regularity*, such that

$$1 = H_G(0) < H_G(1) < \cdots < H_G(k^*) = H_G(k^*+1) = \cdots.$$

We will often write the function H_G as the vector

$$[H_G(0), H_G(1), \ldots, H_G(k^*), H_G(k^*+1)]$$

with the equality of the last two entries signifying the stabilization of the Hilbert function.

Example 2.1. For the zero-dimensional system

$$G(x) = [x_1x_2 - x_3,\ x_1^2 - x_2,\ x_1 + x_2 + x_3 - 1]^T,$$

the Hilbert function of G is $[1, 3, 3]$.

The Hilbert function for G can be computed directly from a Gröbner basis for G using a graded ordering. Another approach is via [9] given numerical approximations of the points in $\mathcal{V}(G)$. This numerical approach uses the Macaulay dual space at each point in $\mathcal{V}(G)$, which we summarize next.

2.2 Macaulay dual space

The Macaulay dual space, also called the local inverse system, encodes the local multiplicity structure at a point with respect to a system of equations. Techniques for computing Macaulay dual spaces are described in [7, 11, 21, 23, 30].

Let $G : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $x^* \in \mathbb{C}^N$. For $\alpha \in (\mathbb{Z}_{\geq 0})^N$, let $x^\alpha = \prod_i x_i^{\alpha_i}$, $|\alpha| = \sum_i \alpha_i$, $\alpha! = \prod_i \alpha_i!$, and $\partial_\alpha[x^*] : \mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}$ be defined by

$$\partial_\alpha[x^*](h) = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} h}{\partial x^\alpha}\bigg|_{x=x^*}.$$

When the point x^* is clear from the context, we simply write ∂_α to mean $\partial_\alpha[x^*]$. Additionally, ∂_{x^α} will be used interchangeably with ∂_α. Consider the vector space

$$\mathcal{D}_{x^*} = \operatorname{span}_{\mathbb{C}} \left\{ \partial_\alpha[x^*] \;\middle|\; \alpha \in (\mathbb{Z}_{\geq 0})^N \right\}.$$

The *Macaulay dual space* of G at x^*, denoted $\mathcal{D}_{x^*}(G)$, is the vector subspace of $\mathcal{D}_{x^*}$ consisting of all elements that vanish on all polynomials in the ideal generated by G, namely

$$\mathcal{D}_{x^*}(G) = \left\{ \partial \in \mathcal{D}_{x^*} \;\middle|\; \partial\left(\sum_{i=1}^n h_i G_i\right) = 0, \forall h_i \in \mathbb{C}[x_1, \ldots, x_n] \right\}.$$

A *Macaulay dual basis* is a basis for the vector space $\mathcal{D}_{x^*}(G)$.

Let $\operatorname{mult}(G, x^*)$ be the multiplicity of G at x^*. The following theorem of Macaulay provides the relationship between $\operatorname{mult}(G, x^*)$ for isolated $x^* \in \mathcal{V}(G)$ and $\mathcal{D}_{x^*}(G)$.

Theorem 2.2. *Let $G : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and $x^* \in \mathcal{V}(G)$.*

1. *x^* is isolated in $\mathcal{V}(G)$ if and only if $\dim \mathcal{D}_{x^*}(G) < \infty$.*
2. *If $x^* \in \mathcal{V}(G)$ is isolated, $\dim \mathcal{D}_{x^*}(G) = \operatorname{mult}(G, x^*)$.*

Example 2.3. For the polynomial system

$$G = [x_2 - x_1^2,\ x_2^2]^T$$

one can easily verify that, for $x^* = (0, 0)$, we have

$$\mathcal{D}_{x^*}(G) = \operatorname{span}_{\mathbb{C}} \left\{ \partial_1,\ \partial_{x_1},\ \partial_{x_2} + \partial_{x_1^2},\ \partial_{x_1x_2} + \partial_{x_1^3} \right\}.$$

Hence, $\operatorname{mult}(G, x^*) = \dim \mathcal{D}_{x^*}(G) = 4$.

The Macaulay dual space is dual to the localization of the ideal of G at x^*. We can turn this into a global view of the ideal by performing computations at the origin for a homogeneous ideal. Computations using this observation, which included the ideas leading to this current paper, were described in an initial draft by the first author [12]. Before exploring this relationship in the following section, we quickly consider Macaulay dual spaces for homogeneous systems.

Suppose that G is homogeneous, that is,

$$G_i(\lambda x) = \lambda^{d_i} G_i(x)$$

for each $i = 1, \ldots, n$ and any $\lambda \neq 0$ where $d_i = \deg G_i$. Let $x^* = 0 \in \mathcal{V}(G) \subset \mathbb{C}^N$. For each $k \geq 0$, consider

$$\mathcal{D}_0^k = \operatorname{span}_{\mathbb{C}} \left\{ \partial_\alpha[0] \;\middle|\; \alpha \in (\mathbb{Z}_{\geq 0})^N, |\alpha| = k \right\}$$

and the corresponding subspace

$$\mathcal{D}_0^k(G) = \left\{ \partial \in \mathcal{D}_0^k \;\middle|\; \begin{array}{l} \partial\left(\sum_{i=1}^n h_i G_i\right) = 0, \\ \forall h_i \in \mathbb{C}[x_1, \ldots, x_n] \text{ homogeneous} \end{array} \right\}.$$

In fact, $H_G(k) - H_G(k-1) = \dim \mathcal{D}_0^k(G)$ where $H_G(-1) = 0$. For a homogeneous system G, $H_G^p(k) = H_G(k) - H_G(k-1)$ is called the *projective Hilbert function.*

The vector space $\mathcal{D}_0^k(G)$ can be identified with the (right) null space of the *order k Macaulay matrix* for G at 0, denoted $\mathcal{M}_0^k(G)$, which is constructed as follows. The rows of $\mathcal{M}_0^k(G)$ are indexed by tuples (i, β) where $i = 1, \ldots, n$ and $\beta \in (\mathbb{Z}_{\geq 0})^N$ with $|\beta| = k - \deg G_i$. In particular, the set of all corresponding $x^\beta G_i(x)$ forms a generating set of the vector space of all homogeneous polynomials in $\langle G(x) \rangle$ of degree k. The columns of $\mathcal{M}_0^k(G)$ are indexed by $\alpha \in (\mathbb{Z}_{\geq 0})^N$ such that $|\alpha| = k$. The $((i, \beta), \alpha)$ entry of $\mathcal{M}_0^k(G)$ is

$$(\mathcal{M}_0^k(G))_{((i,\beta),\alpha)} = \partial_\alpha[0]\left(x^\beta G_i(x)\right).$$

Example 2.4. For the homogeneous polynomial system

$$G = [x_0 x_2 - x_1^2,\ x_2^2]^T$$

one can easily verify that $H_G(k)$ is

$$1, 4, 8, 12, 16, 20, 24, \ldots$$

so that $H_G^p(k) = H_G(k) - H_G(k-1) = \dim \mathcal{D}_0^k(G)$ is

$$1, 3, 4, 4, 4, \ldots.$$

Since $\deg G_1 = \deg G_2 = 2$, $\mathcal{M}_0^0(G)$ and $\mathcal{M}_0^1(G)$ are 0×1 and 0×3 matrices, respectively, which, by definition, have a 1- and 3-dimensional null space respectively. Now, $\mathcal{M}_0^2(G)$ is

	$\partial_{x_0^2}$	$\partial_{x_0x_1}$	$\partial_{x_0x_2}$	$\partial_{x_1^2}$	$\partial_{x_1x_2}$	$\partial_{x_2^2}$
G_1	0	0	1	-1	0	0
G_2	0	0	0	0	0	1

which, as expected, has a 4-dimensional null space.

2.3 H-basis

As above, let $G : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $I = \langle G(x) \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$ be the ideal generated by G. One common approach in algebraic geometry is to study the closure of affine algebraic sets, i.e., solution sets of polynomial systems in $\mathbb{C}^N$, in projective space $\mathbb{P}^N$. Algebraically, this corresponds studying the homogenization of an ideal.

For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ of degree d, the *homogenization* of f, denoted $f^h \in \mathbb{C}[y_0, y_1, \ldots, y_N]$, is

$$f^h(y_0, \ldots, y_N) = y_0^d f\left(\frac{y_1}{y_0}, \ldots, \frac{y_N}{y_0}\right).$$

The homogenization of I is an ideal that is defined by homogenizing each element of I, namely

$$I^h = \{f^h \mid f \in I\} \subset \mathbb{C}[y_0, y_1, \ldots, y_N].$$

Since I is generated by G, it always the case that

$$\langle G^h(y) \rangle \subset I^h.$$

However, the following shows this containment may be strict.

Example 2.5. Let G be as in Example 2.1. Then,

$$G^h(y) = [y_1 y_2 - y_0 y_3,\ y_1^2 - y_0 y_2,\ y_1 + y_2 + y_3 - y_0]^T.$$

Since $\mathcal{V}(G) \subset \mathbb{C}^3$ consists of 3 points and $\mathcal{V}(G^h) \subset \mathbb{P}^3$ consists of 4 points, we know $\langle G^h(y) \rangle \subsetneq I^h$. This extra point lies at "infinity" with coordinates $[0, 0, 1, -1] \in \mathbb{P}^3$. In fact, the polynomial $y_1 y_3 - y_2^2$ is contained in I^h but not in $\langle G^h(y) \rangle$.

The polynomials $G_1(x), \ldots, G_n(x)$ is called an *h-basis* for the ideal $I = \langle G(x) \rangle$ if $G_1^h(y), \ldots, G_n^h(y)$ is a basis for I^h. A Gröbner basis with respect to any graded monomial ordering is an h-basis.

Example 2.6. The system G from Example 2.1 is not an h-basis, but it is easy to verify that the following is:

$$H(x) = [x_1 x_2 - x_3,\ x_1^2 - x_2,\ x_1 x_3 - x_2^2,\ x_1 + x_2 + x_3 - 1]^T.$$

In fact, $G_1(x), \ldots, G_n(x)$ form an h-basis for $I = \langle G(x) \rangle$ if and only if $H_G(k) = H_{G^h}^p(k)$ for all $k \geq 0$. Before turning to make this test effective, we consider a simple example.

Example 2.7. With G as in Example 2.1, $H_G(2) = 3$ but $H_{G^h}^p(2) = 4$ which again shows that G is not an h-basis.

Let I be a zero-dimensional ideal with index of regularity k^* and I^h be the homogenization of I. The maximum degree of a minimal generating set for I^h consisting of homogeneous polynomials is well-defined, called the *Noether number*, denoted $\beta(I^h)$. In this zero-dimensional case, $k^* + 1$ is precisely the Castelnuovo-Mumford regularity of I^h yielding $\beta(I^h) \leq k^* + 1$. Thus, one may use $k^* + 1$ in the following.

Proposition 2.8. *Suppose that $G : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system such that $\mathcal{V}(G)$ is nonempty and finite. Let $I = \langle G(x) \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$ and $I^h \subset \mathbb{C}[y_0, \ldots, y_N]$ be the homogenization of I. The polynomials in $G(x)$ form an h-basis for I if and only if*

$$H_G(k) = H_{G^h}^p(k) \text{ for } k = 0, 1, \ldots, \beta(I^h).$$

Proof. If $G(x)$ forms an h-basis for I, we immediately know that $H_G(k) = H_{G^h}^p(k)$ for all $k \geq 0$.

Conversely, we know $I^h = \left\langle I^h_{\leq \beta(I^h)} \right\rangle$ and $\langle G^h(y) \rangle \subset I^h$. By dimension counting, $\langle G^h(y) \rangle_k = I_k^h$ for $k = 0, 1, \ldots, \beta(I^h)$ where, for a homogeneous ideal J, J_k is the vector space of polynomials in J of degree k. Hence, $I^h = \langle G^h(y) \rangle$ so that $G(x)$ forms an h-basis for I. □

Corollary 2.9. *Let $G : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system such that $\mathcal{V}(G)$ nonempty and finite, $I = \langle G(x)\rangle \subset \mathbb{C}[x_1, \ldots, x_N]$, and $I^h \subset \mathbb{C}[y_0, \ldots, y_N]$ be the homogenization of I. If $\ell \geq \beta(I^h)$ such that $H^p_{G^h}(\ell) = H_G(\ell)$, then*

$$H_G(k) = H^p_{G^h}(k) \text{ for } k \geq \ell.$$

Proof. The statement follows immediately from the fact that, for $k \geq \beta(I^h)$, $I^h_{k+1} = \langle I^h_k\rangle_{k+1}$. $\square$

As demonstrated in Example 2.5, the reason why $\langle G^h(y)\rangle$ may be strictly contained in I^h is due to extraneous solutions at "infinity," i.e., solutions contained in the hyperplane defined by $y_0 = 0$. In order to remove such solutions, one simply saturates with respect to y_0. This is described next.

2.4 Quotients and saturation

Suppose that $J \subset \mathbb{C}[y_0, \ldots, y_N]$ is a homogeneous ideal, i.e., it has a basis consisting of homogeneous polynomials, and $f \in \mathbb{C}[y_0, \ldots, y_N]$ is a homogeneous polynomial. The *quotient* of J with respect to f is the ideal

$$J : f = \{a \in \mathbb{C}[y_0, \ldots, y_N] \mid a \cdot f \in J\}.$$

It is easy to verify that $(J : f) : f = J : f^2$ and

$$J = J : f^0 \subset J : f^1 \subset J : f^2 \subset \cdots .$$

The ascending chain condition yields there exists $\ell^* \geq 0$ with

$$J = J : f^0 \subsetneq J : f^1 \subsetneq \cdots \subsetneq J : f^{\ell^*} = J : f^{\ell^*+1} = \cdots .$$

The ideal $J : f^{\ell^*}$ is called the *saturation* of J with respect to f, which will be denoted $J : f^\infty$.

Example 2.10. For $G^h(y)$ defined as in Example 2.5,

$$y_0(y_1y_3 - y_2^2) \in \langle G^h(y)\rangle.$$

Thus, $y_1y_3 - y_2^2 \in \langle G^h(y)\rangle : y_0$. In fact,

$$\langle G^h(y)\rangle \subsetneq \langle G^h(y)\rangle : y_0 = \langle G^h(y)\rangle : y_0^\infty = \langle G(x)\rangle^h.$$

With the setup as above, let $d = \deg f$. For any $k \geq 0$, we can compute $\mathcal{D}^k_0(J : f)$ from $\mathcal{D}^{k+d}_0(J)$ using the linear operator $\Phi_f : \mathcal{D}_0 \to \mathcal{D}_0$ defined as follows:

$$\Phi_f(\partial)(g) = \partial(gf), \forall g \in \mathbb{C}[x_0, x_1, \ldots, x_N] \text{ homogeneous.} \quad (5)$$

When $f = y_i$, then Φ_f reduces to the operator Φ_i defined by [26, 27]. For the nonhomogeneous case, see [18].

Since Φ_f is a linear operator, Φ_f is described by how it performs on basis elements. If $|\alpha| = k+d$, Leibniz rule yields

$$\Phi_f(\partial_\alpha) = \sum_{\substack{\gamma\leq\alpha\\|\gamma|=d}} \partial_\gamma(f)\partial_{\alpha-\gamma}. \quad (6)$$

where $\gamma \leq \alpha$ means that $\gamma_i \leq \alpha_i$ for all i. This immediately shows that $\Phi_f(\mathcal{D}^{k+d}_0) \subset \mathcal{D}^k_0$.

The following relates $\mathcal{D}^k_0(J)$ and $\mathcal{D}^k_0(J \cap \langle f\rangle)$ under Φ_f.

Lemma 2.11. *Let $J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal and $f \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous polynomial. Then, for every $k \geq 0$,*

$$\Phi_f(\mathcal{D}^k_0(\langle f\rangle)) = \{0\} \quad \textit{and} \quad \Phi_f(\mathcal{D}^k_0(J)) = \Phi_f(\mathcal{D}^k_0(J \cap \langle f\rangle)).$$

Proof. Let $\partial \in \mathcal{D}^j_0(\langle f\rangle)$. For every $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$, $gf \in \langle f\rangle$ so that $\Phi_f(\partial)(g) = \partial(gf) = 0$ Hence, $\Phi_g(\partial) = 0$.

Assuming that $\mathcal{D}^k_0(J \cap \langle f\rangle) = \mathcal{D}^k_0(J) + \mathcal{D}^k_0(\langle f\rangle)$,

$$\begin{aligned}\Phi_f(\mathcal{D}^k_0(J \cap \langle f\rangle)) &= \Phi_f(\mathcal{D}^k_0(J) + \mathcal{D}^k_0(\langle f\rangle))\\ &= \Phi_f(\mathcal{D}^k_0(J)) + \Phi_f(\mathcal{D}^k_0(\langle f\rangle))\\ &= \Phi_f(\mathcal{D}^k_0(J)).\end{aligned}$$

Since $J, \langle f\rangle \supset J \cap \langle f\rangle$, it is clear from the definition that

$$\mathcal{D}^k_0(J), \mathcal{D}^k_0(\langle f\rangle) \subset \mathcal{D}^k_0(J \cap \langle f\rangle)$$

which immediately implies

$$\mathcal{D}^k_0(J) + \mathcal{D}^k_0(\langle f\rangle) \subset \mathcal{D}^k_0(J \cap \langle f\rangle)$$

Conversely, the inclusion-exclusion principle provides

$$\begin{aligned}\dim \mathcal{D}^k_0(J \cap \langle f\rangle) &= H^p_{J\cap\langle f\rangle}(k)\\ &= H^p_J(k) + H^p_{\langle f\rangle}(k) - H^p_{J+\langle f\rangle}(k).\end{aligned}$$

Since, from the definition, $\mathcal{D}^k_0(J + \langle f\rangle) = \mathcal{D}^k_0(J) \cap \mathcal{D}^k_0(\langle f\rangle)$,

$$\begin{aligned}&H^p_J(k) + H^p_{\langle f\rangle}(k) - H^p_{J+\langle f\rangle}(k)\\ &= \dim \mathcal{D}^k_0(J) + \dim \mathcal{D}^j_0(J) - \dim\left(\mathcal{D}^k_0(J) \cap D^k_0(\langle f\rangle)\right)\\ &= \dim\left(\mathcal{D}^k_0(J) + \mathcal{D}^k_0(\langle f\rangle)\right).\end{aligned}$$

Therefore, $\mathcal{D}^k_0(J \cap \langle f\rangle) = \mathcal{D}^k_0(J) + \mathcal{D}^k_0(\langle f\rangle)$. $\square$

We now aim to construct a one-sided inverse for Φ_f when $f \neq 0$. If $f = y_i$, a one-sided inverse of Φ_f was constructed in [30] as the linear operator defined by $\Psi_{y_i}(\partial_\beta) = \partial_{\beta+e_i}$. In particular, $\Phi_{y_i} \circ \Psi_{y_i}$ is the identity operator. The following generalizes the construction of such an operator.

Definition 2.12. Let $f \in \mathbb{C}[x_0, \ldots, x_N]$ be a nonzero homogeneous polynomial of degree d. Define the linear operator $\Psi_f : \mathcal{D}_0 \to \mathcal{D}_0$ as follows. Write $f = \sum_{|\alpha|=d} f_\alpha x^\alpha$ and let $\prec$ be a graded lexicographic ordering on $(\mathbb{Z}_{\geq 0})^{N+1}$ with

$$\alpha_0 = \min_{\prec}\{\alpha \mid |\alpha| = d \text{ and } f_\alpha \neq 0\}.$$

For any $\beta, \gamma \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\gamma| - |\beta| = d$, define

$$M(\beta, \gamma) = \begin{cases} f_{\gamma-\beta} & \text{if } \gamma \geq \beta,\\ 0 & \text{otherwise.}\end{cases}$$

For any $n \geq 0$ and $\beta \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\beta| = n$, define

$$\Psi_f(\partial_\beta) = \sum_{|\alpha|=n+d} c_\alpha(\beta)\partial_\alpha$$

where $c_\alpha(\beta)$ is 0 when $\alpha \ngeq \alpha_0$ and is

$$\frac{1}{f_{\alpha_0}}\left(\delta(\alpha - \alpha_0, \beta) - \sum_{\substack{|\gamma|=|\alpha|\\\gamma\succ\alpha}} M(\alpha - \alpha_0, \gamma)c_\gamma(\beta)\right)$$

when $\alpha \geq \alpha_0$ where $\delta(\zeta, \epsilon)$ is Kronecker's delta.

Lemma 2.13. *If $f \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a nonzero homogeneous polynomial, then $\Phi_f \circ \Psi_f$ is the identity operator.*

Proof. Fix $k \geq 0$ and $\beta \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\beta| = k$. Utilizing the notation from Definition 2.12, for any $\gamma \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\gamma| = k$, we claim

$$\delta(\gamma, \beta) = \sum_{|\alpha|=k+d} M(\gamma, \alpha)c_\alpha(\beta). \quad (7)$$

This is shown by splitting the summation as

$$\begin{aligned}\sum_{|\alpha|=k+d} M(\gamma,\alpha)c_\alpha(\beta) &= \sum_{\substack{|\alpha|=k+d\\ \alpha\prec\gamma+\alpha_0}} M(\gamma,\alpha)c_\alpha(\beta)\\ &\quad + M(\gamma,\gamma+\alpha_0)c_{\gamma+\alpha_0}(\beta)\\ &\quad + \sum_{\substack{|\alpha|=k+d\\ \alpha\succ\gamma+\alpha_0}} M(\gamma,\alpha)c_\alpha(\beta).\end{aligned}$$

Let α be such that $|\alpha| = k + d$ and $\alpha \prec \gamma + \alpha_0$. If $\alpha \not\geq \alpha_0$, $c_\alpha(\beta) = 0$. Otherwise, we must have $M(\gamma, \alpha) = 0$ by construction of α_0. In particular,

$$\sum_{\substack{|\alpha|=k+d\\ \alpha\prec\gamma+\alpha_0}} M(\gamma,\alpha)c_\alpha(\beta) = 0.$$

Since $M(\gamma, \gamma + \alpha_0) = f_{\alpha_0}$, the definition of $c_{\gamma+\alpha_0}(\beta)$ yields

$$M(\gamma,\gamma+\alpha_0)c_{\gamma+\alpha_0}(\beta) = \delta(\gamma,\beta) - \sum_{\substack{|\alpha|=k+d\\ \alpha\succ\gamma+\alpha_0}} M(\gamma,\alpha)c_\alpha(\beta)$$

which immediately yields (7).

The following using (6) and (7) completes the proof:

$$\begin{aligned}\Phi_f(\Psi_f(\partial_\beta)) &= \sum_{|\alpha|=k+d} c_\alpha(\beta)\Phi_g(\partial_\alpha)\\ &= \sum_{|\alpha|=k+d} c_\alpha(\beta)\sum_{\gamma\leq\alpha}\partial_{\alpha-\gamma}(g)\partial_\gamma\\ &= \sum_{|\alpha|=k+d} c_\alpha(\beta)\sum_{|\gamma|=k} M(\gamma,\alpha)\partial_\gamma\\ &= \sum_{|\gamma|=k}\left(\sum_{|\alpha|=k+d} M(\gamma,\alpha)c_\alpha(\beta)\right)\partial_\gamma\\ &= \sum_{|\gamma|=k}\delta(\gamma,\beta)\partial_\gamma = \partial_\beta.\end{aligned}$$

□

THEOREM 2.14. *Let $J \subset \mathbb{C}[x_0,\ldots,x_N]$ be a homogeneous ideal and $f \in \mathbb{C}[x_0,\ldots,x_N]$ be a nonzero homogeneous polynomial of degree d. Then, for every $k \geq 0$,*

$$D_0^k(J:f) = \Phi_f\left(D_0^{k+d}(J)\right) = \Phi_f\left(D_0^{k+d}(J\cap\langle f\rangle)\right). \quad (8)$$

PROOF. Let $\partial \in D_0^{k+d}(J)$. For any $g \in J : f$, we know $\Phi_f(\partial)(g) = \partial(gf) = 0$. Hence, $\Phi_f(\partial) \in D_0^k(J : f)$.

Conversely, let $\partial \in D_0^k(J : f)$. Suppose that $g \in J \cap \langle f\rangle$. Then, $h = \frac{g}{f} \in J : f$ and

$$\Psi_f(\partial)(g) = \Psi_f(\partial)(hf) = \Phi_f(\Psi_f(\partial))(h) = \partial(h) = 0.$$

Thus, $\Psi_f(\partial) \in D_0^{k+d}(J \cap \langle f\rangle)$ and

$$\partial = \Phi_f(\Psi_f(\partial)) \in \Phi_f(D_0^{k+d}(J\cap\langle f\rangle)).$$

□

In the zero-dimensional case of interest here, we can compute an h-basis from the given system G as follows.

First, we compute the Hilbert function H_G using [9], which also yields k^*. This requires computing numerical approximations of the finitely many points in $\mathcal{V}(G)$ which can be performed in parallel using homotopy continuation. Then, for each point $x^* \in \mathcal{V}(G)$, we independently compute $\mathcal{D}_{x^*}(G)$ which provides the necessary data for [9]. The null space computations required here can be performed in parallel.

Secondly, we compute $\mathcal{D}_0^k(G^h)$ for increasingly larger k, which can be performed using parallel null space computations, until we find ℓ such that

$$\dim\Phi_{y_0^\ell}(\mathcal{D}_0^{r+\ell}(G^h)) = H_G(r) \text{ for } r = 0,1,\ldots,k^*+1.$$

We again use parallel null space computations to compute a basis for the vector space of polynomials of degree r that are annihilated by $\Phi_{y_0^\ell}(\mathcal{D}_0^{r+\ell}(G^h))$ for $r = 0,1,\ldots,k^*+1$. Dehomogenizing the generators, i.e., setting $y_0 = 1$ and $y_i = x_i$ for $i = 1,\ldots,N$, yields an h-basis for $\langle G(x)\rangle$.

To help reduce the degrees under consideration, one can perform this computation iteratively by adding in the new generators as they are found.

EXAMPLE 2.15. In Example 2.10, we claimed

$$I^h = \langle G(x)\rangle^h = \langle G^h(y)\rangle : y_0.$$

Since $H_G = [1,3,3]$ with $k^* = 1$, the following table shows computing the projective Hilbert functions using dual spaces:

k	0	1	2	3
$\langle G^h(y)\rangle$	1	3	4	4
$\langle G^h(y)\rangle : y_0$	1	3	3	

which verifies this claim. The annihilators of the corresponding dual spaces yields that I^h is generated by 1 linear and 7 quadratic polynomials, with the minimal generating set consisting of 1 linear and 3 quadratic polynomials, say

$$y_1 + y_2 + y_3 - y_0,\ y_1y_2 - y_0y_3,\ y_1^2 - y_0y_2,\ y_1y_3 - y_2^2$$

which dehomogenizes to the h-basis $H(x)$ in Example 2.6.

2.5 The parameterized case

The aforementioned computations can be performed over the field generated by the coefficients. Moreover, with the setup described at the beginning of Section 2, there is a nonempty Zariski open subset $U \subset \mathbb{C}^P$ such that the Hilbert function of $F(x;p^*)$ is the same for all $p^* \in U$. In practice, we select a random point $p^* \in \mathbb{C}^P$, say using a complex Gaussian distribution, so that, with probability one, $p^* \in U$.

From the generic Hilbert function, we can compute a system $G(x;p)$, called a *parameterized h-basis* for $F(x;p)$, such that there is a Zariski open dense subset $Z \subset \mathbb{C}^P$ where $G(x;p^*)$ is an h-basis of $\langle F(x;p^*)\rangle$ for all $p^* \in Z$. To compute a parameterized h-basis, we could first compute an h-basis at a random $p^* \in \mathbb{C}^P$. Then, we could recreate the computations symbolically to yield a parameterized h-basis.

2.6 Example using RR dyad

We demonstrate computing a parameterized h-basis using the inverse kinematic problem of an RR dyad. As shown in Figure 1, the RR dyad consists of two legs of length ℓ_1 and ℓ_2 together with two pin joints. The mechanism is anchored at point O which, without loss of generality, we may assume is the origin. Given a point $P = (p_x, p_y)$, the problem is to find the angles θ_1 and θ_2 so that the end of the second leg is located at P. By treating $\sin(\theta_j)$ and $\cos(\theta_j)$ as indeterminants, namely, s_j and c_j, together with the Pythagorean theorem, we start with the polynomial system

$$f(c_1,c_2,s_1,s_2;\ell_1,\ell_2,p_x,p_y) = \begin{bmatrix}\ell_1c_1+\ell_2c_2-p_x\\ \ell_1s_1+\ell_2s_2-p_y\\ c_1^2+s_1^2-1\\ c_2^2+s_2^2-1\end{bmatrix}.$$

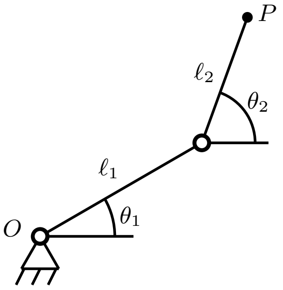

Figure 1: RR dyad

Since the physically meaningful parameters have $\ell_i > 0$, we can use the linear equations to eliminate c_2, s_2, namely

$$F(c_1, s_1; \ell_1, \ell_2, p_x, p_y) = \left[\begin{array}{c} s_1^2 + c_1^2 - 1 \\ (p_x - \ell_1 c_1)^2 + (p_y - \ell_1 s_1)^2 - \ell_2^2 \end{array} \right].$$

It is well-known that F has two solutions for general parameters with generic Hilbert function $[1, 2, 2]$ and index of regularity $k^* = 1$. In particular, there is a linear relationship between c_1 and s_1. With the homogenizing variable y_0, $\mathcal{M}_0^2(F^h)$ is the rank 2 matrix

	$\partial_{y_0^2}$	$\partial_{y_0 c_1}$	$\partial_{y_0 s_1}$	$\partial_{c_1^2}$	$\partial_{c_1 s_1}$	$\partial_{s_1^2}$
F_1	-1	0	0	1	0	1
F_2	$p_x^2 + p_y^2 - \ell_2^2$	$-2\ell_1 p_x$	$-2\ell_1 p_y$	ℓ_1^2	0	ℓ_1^2

whose null space corresponds with $\mathcal{D}_0^2(F^h)$. Then,

$$\Phi_{y_0}(\mathcal{D}_0^2(F^h)) = \operatorname{span}_{\mathbb{C}} \left\{ \begin{array}{c} p_y \partial_{c_1} - p_x \partial_{s_1}, \\ 2\ell_1 p_x \partial_{y_0} + (\ell_1^2 - \ell_2^2 + p_x^2 + p_y^2)\partial_{c_1} \end{array} \right\}$$

which annihilates the linear polynomial

$$2\ell_1 p_x c_1 + 2\ell_1 p_y s_1 - (\ell_1^2 - \ell_2^2 + p_x^2 + p_y^2) y_0.$$

In particular, a parameterized h-basis is

$$G(c_1, s_1; \ell_1, \ell_2, p_x, p_y) = \left[\begin{array}{c} s_1^2 + c_1^2 - 1 \\ 2\ell_1 p_x c_1 + 2\ell_1 p_y s_1 - (\ell_1^2 - \ell_2^2 + p_x^2 + p_y^2) \end{array} \right].$$

3. RANK-DEFICIENCY SETS

For $x \in \mathbb{C}^N$, suppose that $A(x) \in \mathbb{C}^{m \times n}$ whose entries are polynomial in x. For $r = 0, 1, \ldots, \min\{m, n\}$, consider

$$\mathcal{R}_r(A) = \{x \in \mathbb{C}^N \mid \operatorname{rank} A(x) \le r\}.$$

Each $\mathcal{R}_r(A)$ is an algebraic set, in particular, it is closed under both the Zariski and Euclidean topologies, since they are defined by the vanishing of the determinants of all (r+1) $\times$ $(r+1)$ submatrices of $A(x)$, i.e., the $(r+1)$-minors.

Instead of using determinants, we follow the approach of [3] which uses a null space approach. Without loss of generality, we may assume that $m \le n$. Then, $A(x)$ has rank r if and only if the left null space of $A(x)$ has dimension $m - r$. In particular, there is a Zariski open subset $U \subset \mathbb{C}^{m \times m}$ such that, for every $B \in U$,

$$\mathcal{R}_r(A) = \overline{\left\{ x \in \mathbb{C}^N \;\middle|\; \exists \Lambda \in \mathbb{C}^{r \times (m-r)} \text{ s.t. } A(x)^T \cdot B \cdot \left[\begin{array}{c} I \\ \Lambda \end{array} \right] = 0 \right\}}.$$

where I is the $(m - r) \times (m - r)$ identity matrix.

The particular case of interest is $\mathcal{R}_{m-1}(A)$, which we compute using the simplification

$$\mathcal{R}_{m-1}(A) = \overline{\{x \in \mathbb{C}^N \mid \exists \lambda \in \mathbb{P}^{m-1} \text{ s.t. } \lambda \cdot A(x) = 0\}}.$$

The advantage of using such an approach with numerical algebraic geometry is that the system of equations is naturally bihomogeneous and one avoids the exponentially many determinants of potentially high degree. Moreover, using the observation in [2] further helps to reduce the computation by only using slices in x rather than slices in (x, λ). Such an approach can also be combined with intersection via regeneration [14] to compute the rank-deficiency set.

4. ALGORITHM

We now describe our parallelizable hybrid symbolic-numeric approach for computing exceptional sets. Since it requires several random choices, say using a complex Gaussian distribution, this algorithm succeeds with probability one.

We assume, as input, $F : \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^n$ is a polynomial system as described in § 2. The output is a description of the irreducible components of $E(F)$ described in (4) which are presented via (pseudo)witness sets.

First, we compute a parameterized h-basis G using the parallelizable symbolic-numeric technique described in § 2. Let k^* be the generic index of regularity and β be the maximum degree of the generators, which is the generic Noether number. Let $d := \max\{k^*, \beta\}$.

Construct a submatrix $\mathcal{N}$ of $\mathcal{M}_0^d(G)$ which generically has full row rank, i.e., $m := H_{G^h}^p(d)$ which is the degree of a general fiber. Since we know that we have either $\le m$ points in the fiber or infinitely many, $E(F)$ is contained in $\mathcal{R}_{m-1}(\mathcal{N})$. Thus, we compute the irreducible components of $\mathcal{R}_{m-1}(\mathcal{N})$ via the parallelizable approach described in § 3.

Each irreducible component of $\mathcal{R}_{m-1}(\mathcal{N})$ is either contained in $E(F)$ or arose due to the choice of rows when constructing $\mathcal{N}$. In the latter case, we construct another submatrix, which we also call $\mathcal{N}$, possibly using a different parameterized h-basis G, that, on this irreducible set, $\mathcal{N}$ has generically rank m. We repeat the process with the newly constructed $\mathcal{N}$. Since the superfluous components decrease in dimension each time, this process terminates after finitely many loops yielding the irreducible components of $E(F)$.

4.1 Example using RR dyad

Let F with parameterized h-basis G be as defined in Section 2.6. In this case, $\mathcal{N} = \mathcal{M}_0^2(G)$ is the following 4×6 matrix that generically has full rank:

$$\left[\begin{array}{cccccc} -1 & 0 & 0 & 1 & 0 & 1 \\ Z & 2\ell_1 p_x & 2\ell_1 p_y & 0 & 0 & 0 \\ 0 & Z & 0 & 2\ell_1 p_x & 2\ell_1 p_y & 0 \\ 0 & 0 & Z & 0 & 2\ell_1 p_x & 2\ell_1 p_y \end{array} \right]$$

where $Z = \ell_2^2 - \ell_1^2 - p_x^2 - p_y^2$. Using `Bertini` [5], we find that $\mathcal{R}_3(\mathcal{N})$ consists of 3 components:

$$\mathcal{V}(\ell_1, \ell_2 - p_x^2 - p_y^2) \cup \mathcal{V}(p_x, p_y, \ell_1 + \ell_2) \cup \mathcal{V}(p_x, p_y, \ell_1 - \ell_2).$$

It is easy to verify that each of these three are indeed contained in $E(F)$ with only the last component containing physically meaningful parameters, i.e., $\ell_i > 0$.

We note that the reduction from f to F in Section 2.6 was based on the fact that physically meaningful parameters have $\ell_2 > 0$, which permitted us to easily demonstrate the algorithm. Of course, one can repeat this process using $\ell_1 > 0$, i.e., eliminating c_1 and s_1, or simply using f.

5. EXAMPLES

The following examples used `Bertini` [5] to compute numerical irreducible decompositions running on a node having four AMD Opteron 6378 2.4 GHz processors, a total of 64 cores, with 128 GB memory. Supplementary files for the following examples are available at `www.nd.edu/~jhauenst/exceptional`.

5.1 Lines on surfaces

As a comparison with the fiber product approach of [25], we use the formulation presented in [25, § 4] to demonstrate the approaches to compute rulings of a quadric and lines on a cubic in $\mathbb{C}^3$. In particular, even though such a formulation is not the most efficient way for computing these, they provide good test cases. To that end, suppose that $g : \mathbb{C}^3 \to \mathbb{C}$ is a polynomial that defines the surface. They represent a line $\mathcal{L} \subset \mathbb{C}^3$ via $(u, v) \in (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{C}^3$ where $\mathcal{L} = \{ut + v \mid t \in \mathbb{C}\}$. In fact, each line $\mathcal{L}$ is actually represented by a 2-dimensional family since (u, v) and $(\lambda u, v+\mu u)$ generate the same line for any $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$. Due to this, we simplify the setup by writing u_3 as a general affine linear combination of u_1 and u_2, and v_3 as a general affine linear combination of u_1, u_2, v_1, and v_2.

With this setup, we aim to compute (u, v) such that the corresponding line $\mathcal{L}$ is contained in $\mathcal{V}(g)$. This can be accomplished by computing the exceptional set of

$$F(x; u, v) = \begin{bmatrix} g(x) \\ u \times (x - v) \end{bmatrix}.$$

Here, $\times$ denotes the cross product of vectors in $\mathbb{C}^3$.

Rulings of a quadric

We first consider the hyperboloid of one sheet defined by

$$g(x) = x_1^2 + x_2^2 - x_3^2 - 1.$$

It is well-known this hyperboloid and most quadric surfaces in $\mathbb{C}^3$ are doubly ruled. That is, we are looking for curves of lines contained in $\mathcal{V}(g)$. For a general linear polynomial $\ell(u, v)$, we follow [25] and consider the third fiber product along with the linear ℓ, say

$$F_3(x^{(1)}, x^{(2)}, x^{(3)}; u, v) = \begin{bmatrix} F(x^{(1)}; u, v) \\ F(x^{(2)}; u, v) \\ F(x^{(3)}; u, v) \\ \ell(u, v) \end{bmatrix}.$$

Since the fiber product works by increasing the dimension, we need to compute a numerical irreducible decomposition of $\mathcal{V}(F_3)$. Using `Bertini`, in 11 seconds, we found that $\mathcal{V}(F_3)$ consists of 12 irreducible components of dimension 3. Since four have $x^{(i)} = x^{(j)}$ for some $i \neq j$, there are 8 components of interest, each of which is a linear space supported over one parameter point. The 8 parameter points decompose into witness point sets for the two rulings, which, in this formulation, correspond with curves in (u, v) of degree 4.

We now use the approach of Section 4. The generic Hilbert function is $[1, 2, 2]$ with g and any two of the polynomials in $u \times (x - v)$ forming a parameterized h-basis, say

$$G(x; u, v) = \begin{bmatrix} g(x) \\ u_2(x_3 - v_3) - u_3(x_2 - v_2) \\ u_3(x_1 - v_1) - u_1(x_3 - v_3) \end{bmatrix}.$$

Since the 9×10 matrix $\mathcal{M}_0^2(G)$ generically has rank 8, we take $\mathcal{N}$ to be the 8×10 submatrix where we remove the row corresponding to x_3G_3. As above, we restrict to $\ell(u, v) = 0$. Using [2] with a natural 3-homogeneous setup using `Bertini`, we compute in 4 seconds 148 points: 140 have some $u_i = 0$ which arose from the construction and 8 points as above corresponding to the two rulings.

Lines on a cubic

We next consider the following variant of the Clebsch cubic obtained from `www.singsurf.org/parade/Cubics.php`:

$$\begin{aligned} g(x) \;=\; & 16x_1^3 + 16x_2^3 - 31x_3^3 + 24x_1^2x_3 - 48x_1^2x_2 \\ & - 48x_1x_2^2 + 24x_2^2x_3 - 54\sqrt{3}x_3^2 - 72x_3. \end{aligned}$$

It is well-known that most cubic surfaces in $\mathbb{C}^3$ contain 27 lines with $\mathcal{V}(g)$ having all 27 lines being real. That is, if we use real coefficients for the linear combinations describing u_3 and v_3, each of the 27 lines corresponds to a real point (u, v). In the cubic case, one needs the fourth fiber product, which we denote by F_4. As above, since the fiber product works by increasing the dimension, we need to compute a numerical irreducible decomposition of $\mathcal{V}(F_4)$. `Bertini`, in 5.33 minutes, computes 41 irreducible components. Since 14 have $x^{(i)} = x^{(j)}$ for some $i \neq j$, there are 27 components of interest, each of which is a linear space supported over one real parameter point corresponding to a line on the cubic.

We now use the approach of Section 4. The generic Hilbert function is $[1, 2, 3, 3]$ and we take a similar parameterized h-basis G as above. Since the 21×20 matrix $\mathcal{M}_0^3(G)$ generically has rank 17, we take $\mathcal{N}$ to be the 17×20 submatrix where we remove the four rows corresponding to $x_ix_3G_3$ for $i = 0, \ldots, 3$. Since the column corresponding to the coefficient of $x_1x_3^2$ has only one nonzero entry, namely u_2, we either have $u_2 = 0$ which one can easily verify cannot yield a line on the Clebsch cubic or the 16×19 matrix removing this column and corresponding row is rank deficient. In this latter case, we use [2] with a natural 3-homogeneous setup and `Bertini` to compute 3474 distinct points in 2.5 minutes: 3447 have some $u_i = 0$ which arose from the construction and 27 corresponding to the 27 lines on the Clebsch cubic.

5.2 Planar pentads

We conclude with a demonstration of our approach for verifying the existence of the double parallelograms which are moving planar pentads, e.g., see [29]. We start with the following setup:

$$\begin{aligned} \text{variables} \quad & \theta = (\theta_1, \theta_1', \theta_2, \theta_2', \theta_3, \theta_3', \theta_4, \theta_4'), \\ \text{parameters} \quad & p = (u_3, u_3', u_4, u_4', v_0, v_0', v_4, v_4'), \end{aligned}$$

$$f(\theta; p) = \begin{bmatrix} \theta_i\theta_i' - 1 \quad \text{for } i = 1, \ldots, 4 \\ u_1\theta_1 + u_2\theta_2 + u_4\theta_4 + 1 \\ v_0 + u_1\theta_1 + u_3\theta_3 + v4\theta4 \\ u_1'\theta_1' + u_2'\theta_2' + u_4'\theta_4' + 1 \\ v_0' + u_1'\theta_1' + u_3'\theta_3' + v_4'\theta_4' \end{bmatrix}$$

where

$$\begin{aligned} u_1 &= -u_3 - v_0 - v_4 \\ u_1' &= -u_3' - v_0' - v_4' \\ u_2 &= u_3 - u_4 + v_0 + v_4 - 1 \\ u_2' &= u_3' - u_4' + v_0' + v_4' - 1. \end{aligned}$$

Since the physically meaningful solutions have all parameters nonzero, we can eliminate $\theta_2, \theta_2', \theta_3, \theta_3'$ to obtain

$$F(\theta; p) = \begin{bmatrix} \theta_1\theta_1' - 1 \\ \theta_4\theta_4' - 1 \\ (u_1\theta_1 + u_4\theta_4 + 1)(u_1'\theta_1' + u_4'\theta_4' + 1) - u_2u_2' \\ (v_0 + u_1\theta_1 + v_4\theta_4)(v_0' + u_1'\theta_1' + v_4'\theta_4') - u_3u_3' \end{bmatrix}.$$

The Hilbert function for F is generically $[1, 5, 6, 6]$ with a parameterized h-basis, say G, consisting of 9 quadratics that one readily computes using the parallelizable symbolic approach of Section 2.

Since $\mathcal{N} := \mathcal{M}_0^2(G)$ is a 9×15 matrix that generically has rank 9, we need to compute $\mathcal{R}_8(\mathcal{N})$. By simply factoring the resulting polynomials in $\lambda \cdot \mathcal{N}$, we can remove factors corresponding to the vanishing of parameters which are not physically meaningful. We then solve the resulting system using `Bertini` by cascading down the possible dimensions of the set of exceptional parameters. The verification that $\mathcal{R}_8(\mathcal{N})$ contains no sets of dimensions 7, 6, and 5 took 5, 14, and 80 seconds, respectively. In dimension 4, the only component with all lengths nonzero is a linear space which corresponds to the double parallelograms. We used an intersection based approach via [14] to compute this exceptional set in a total of 225 seconds.

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