Certifying solutions to square systems of polynomial-exponential equations

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Abstract

Smale's α-theory certifies that Newton iterations will converge quadratically to a solution of a square system of analytic functions based on the Newton residual and all higher order derivatives at the given point. Shub and Smale presented a bound for the higher order derivatives of a system of polynomial equations based in part on the degrees of the equations. For a given system of polynomial-exponential equations, we consider a related system of polynomial-exponential equations and provide a bound on the higher order derivatives of this related system. This bound yields a complete algorithm for certifying solutions to polynomial-exponential systems, which is implemented in `alphaCertified`. Examples are presented to demonstrate this certification algorithm.

Key words and phrases. certified solutions, alpha theory, polynomial system, polynomial-exponential systems, numerical algebraic geometry, alphaCertified

1 Introduction

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a square system of polynomial-exponential functions if f is polynomial in both the variables $x_1, \ldots, x_n$ and finitely many exponentials of the form $e^{\beta x_i}$ where $\beta \in \mathbb{C}$. That is, there exists a polynomial system $P : \mathbb{C}^{n+m} \to \mathbb{C}^n$, analytic functions $g_1, \ldots, g_m : \mathbb{C} \to \mathbb{C}$, and integers $\sigma_1, \ldots, \sigma_m \in \{1, \ldots, n\}$ such that

$$f(x_1, \ldots, x_n) = P(x_1, \ldots, x_n, g_1(x_{\sigma_1}), \ldots, g_m(x_{\sigma_m}))$$

where each g_i satisfies some linear homogeneous partial differential equation (PDE) with complex coefficients. In particular, for each $i = 1, \ldots, m$, there exists a positive integer r_i and a linear function $\ell_i : \mathbb{C}^{r_i+1} \to \mathbb{C}$ such that $\ell_i(g_i, g_i', \ldots, g_i^{(r_i)}) = 0$.

Consider the square polynomial-exponential system $\mathcal{F} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ where

$$\mathcal{F}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \begin{bmatrix} P(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ y_1 - g_1(x_{\sigma_1}) \\ \vdots \\ y_m - g_m(x_{\sigma_m}) \end{bmatrix}. \quad (1)$$

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Since the projection map $(x, y) \mapsto x$ defines a bijection between the solutions of $\mathcal{F}(x, y) = 0$ and $f(x) = 0$, we will only consider certifying solutions to square systems of polynomial-exponential equations of the form $\mathcal{F}(x, y) = 0$.

For a square system $g : \mathbb{C}^n \to \mathbb{C}^n$ of analytic functions, a point $x \in \mathbb{C}^n$ is an *approximate solution* of $g = 0$ if Newton iterations applied to x with respect to g quadratically converge immediately to a solution of $g = 0$. The certificate returned by our approach that a point is an approximation solution of $\mathcal{F} = 0$ is an α-theoretic certificate. In short, α-theory, which started for systems of analytic equations in [11], provides a rigorous mathematical foundation for the fact that if the Newton residual at the point is small and the higher order derivatives at the point are controlled, then the point is an approximate solution. For polynomial systems, by exploiting the fact that there are only finitely many nonzero derivatives, Shub and Smale [10] provide a bound on all of the higher order derivatives. For polynomial-exponential systems, our approach uses the structure of $\mathcal{F}$ together with the linear functions ℓ_i to bound the higher order derivatives.

Systems of polynomial-exponential functions naturally arise in many applications including engineering, mathematical physics, and control theory, to name a few. On the other hand, such functions are typical solutions to systems of linear partial differential equations with constant coefficients. Systems, including ubiquitous functions like $\sin(x)$, $\cos(x)$, $\sinh(x)$, and $\cosh(x)$, can be equivalently reformulated as systems of polynomial-exponential functions, since these functions can be expressed as polynomials involving $e^{\beta x}$ for suitable $\beta \in \mathbb{C}$. Since computing all solutions to such systems is often nontrivial, methods for approximating and certifying some solutions for general systems is very important, especially in the aforementioned applications.

In the rest of this section, we introduce the needed concepts from α-theory. Section 2 formulates the bounds for the higher order derivatives of polynomial-exponential systems and presents a certification algorithm for polynomial-exponential systems. In Section 3, we discuss methods for generating numerical approximations to solutions of polynomial-exponential systems. Section 4 describes the implementation of the certification algorithm in `alphaCertified` as well as demonstrating the algorithms on a collection of examples. Appendix A demonstrates the input, command-line execution, and output of `alphaCertified` for a polynomial-exponential system from Section 4.1. Files for all of the examples are available at `www.nd.edu/~jhauenst/PolyExp`.

1.1 Smale's α-theory

We provide a summary of the elements of α-theory used in the remainder of the article as well as in `alphaCertified`. Hence, this section closely follows [5, § 1] expect "polynomial" is replaced by "analytic." We focus on *square* systems, which are systems with the same number of variables and functions, with more details provided in [2].

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a system of analytic functions with zeros $\mathcal{V}(f) = \{\xi \in \mathbb{C}^n \mid f(\xi) = 0\}$ and $Df(x)$ be the Jacobian matrix of f at x. For a point $x \in \mathbb{C}^n$, the point $N_f(x)$ is called the *Newton iteration of f at x* where the map $N_f : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$N_f(x) = \begin{cases} x - Df(x)^{-1} f(x) & \text{if } Df(x) \text{ is invertible,} \\ x & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{N}$, let $N_f^k(x)$ be the k^{th} Newton iteration of f at x, that is,

$$N_f^k(x) = \underbrace{N_f \circ \cdots \circ N_f}_{k \text{ times}}(x).$$

The following defines an approximate solution of f to be a point which converges quadratically in the standard Euclidean norm on $\mathbb{C}^n$ to a point in $\mathcal{V}(f)$.

Definition 1.1 Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic system. A point $x \in \mathbb{C}^n$ is an *approximate solution* of $f = 0$ with *associated solution* $\xi \in \mathcal{V}(f)$ if, for every $k \in \mathbb{N}$,

$$\|N_f^k(x) - \xi\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \xi\|.$$

Clearly, every solution of $f = 0$ is an approximate solution of $f = 0$. Additionally, when $Df(x)$ is not invertible, then a point x is an approximate solution of $f = 0$ if and only if $x \in \mathcal{V}(f)$. When $Df(x)$ is invertible, the results of α-theory provide a certificate that x is an approximate solution of $f = 0$. This certificate is based on $\alpha(f, x)$, $\beta(f, x)$, and $\gamma(f, x)$, namely

$$\begin{array}{rcl} \alpha(f,x) & = & \beta(f,x) \cdot \gamma(f,x), \\ \beta(f,x) & = & \|x - N_f(x)\| = \|Df(x)^{-1} f(x)\|, \text{ and} \\ \gamma(f,x) & = & \sup\limits_{k \geq 2} \left\| \dfrac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{\frac{1}{k-1}} \end{array} \tag{2}$$

where $D^k f(x)$ is the k^{th} derivative of f (see [8, Chap. 5]).

When $Df(x)$ is not invertible, we define $\beta(f, x)$ as zero and $\gamma(f, x)$ as infinity. The constant $\alpha(f, x)$ is then the indeterminate form $0 \cdot \infty$ which is defined based on the value of $f(x)$. If $f(x) = 0$, then $\alpha(f, x)$ is defined as zero, otherwise $\alpha(f, x)$ is defined as infinity.

The following lemma, which is a conclusion of Theorem 2 of [2, Chap. 8], shows that, when x is an approximate solution of $f = 0$, the distance between x and its associated solution can be bounded in terms of $\beta(f, x)$. Moreover, this bound can be used to produce a certificate that two approximate solutions have distinct associated solutions.

Lemma 1.2 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic system. If $x \in \mathbb{C}^n$ is an approximate solution of $f = 0$ with associated solution ξ, then*

$$\|x - \xi\| \leq 2\beta(f, x).$$

Moreover, if $x_1, x_2 \in \mathbb{C}^n$ are approximate solutions of $f = 0$ with associated solutions ξ_1, ξ_2, respectively, then $\xi_1 \neq \xi_2$ provided that

$$\|x_1 - x_2\| > 2(\beta(f, x_1) + \beta(f, x_2)).$$

Proof. Both results immediately follow from the triangle inequality. In particular,

$$\|x - \xi\| \leq \|x - N_f(x)\| + \|N_f(x) - \xi\| \leq \beta(f, x) + \frac{1}{2}\|x - \xi\|$$

yields $\|x - \xi\| \leq 2\beta(f, x)$. Additionally,

$$\|x_1 - x_2\| \leq \|x_1 - \xi_1\| + \|\xi_1 - \xi_2\| + \|\xi_2 - x_2\| \leq 2(\beta(f, x_1) + \beta(f, x_2)) + \|\xi_1 - \xi_2\|$$

yields that $\xi_1 \neq \xi_2$ when $\|x_1 - x_2\| > 2(\beta(f, x_1) + \beta(f, x_2))$. □

The following theorem, called an α-theorem, is a version of Theorem 2 of [2, Chap. 8] which shows that the value of $\alpha(f, x)$ can be used to produce a certificate that x is an approximate solution of $f = 0$.

Theorem 1.3 *If $f : \mathbb{C}^n \to \mathbb{C}^n$ is an analytic system and $x \in \mathbb{C}^n$ with*

$$\alpha(f, x) < \frac{13 - 3\sqrt{17}}{4} \approx 0.157671,$$

then x is an approximate solution of $f = 0$.

The following theorem, called a robust α-theorem that is a version of Theorem 4 and Remark 6 of [2, Chap. 8], shows that the value of $\alpha(f, x)$ and $\gamma(f, x)$ can be used to produce a certificate that x and another point y have the same associated solution.

Theorem 1.4 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic system and $x \in \mathbb{C}^n$ with $\alpha(f, x) < 0.03$. If $y \in \mathbb{C}^n$ such that*

$$\|x - y\| < \frac{1}{20\gamma(f, x)},$$

then x and y are both approximate solutions of $f = 0$ with the same associated solution.

Let $\pi_{\mathbb{R}} : \mathbb{C}^n \to \mathbb{R}^n$ be the real projection map defined by $\pi_{\mathbb{R}}(x) = \dfrac{x + \overline{x}}{2}$ where $\overline{x}$ is the complex conjugate of x. If f is an analytic system such that $N_f(\overline{x}) = \overline{N_f(x)}$ for all x such that $Df(x)$ is invertible, then N_f defines a real map, i.e., $N_f(\mathbb{R}^n) \subset \mathbb{R}^n$. In particular, if x is an approximate solution of $f = 0$ with associated solution ξ, then $\overline{x}$ is also an approximate solution of $f = 0$ with associated solution $\overline{\xi}$ and $\beta(f, x) = \beta(f, \overline{x})$. The following proposition, which is a summary of the approach in [5, § 2.1], can be used to determine if the associated solution of an approximation solution is real.

Proposition 1.5 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial system such that $N_f(\overline{x}) = \overline{N_f(x)}$ for all $x \in \mathbb{C}^n$ such that $Df(x)$ is invertible. Let $x \in \mathbb{C}^n$ be an approximate solution of $f = 0$ with associated solution ξ.*

1. *If $\|x - \pi_{\mathbb{R}}(x)\| > 2\beta(f, x)$, then $\xi \notin \mathbb{R}^n$.*
2. *If $\alpha(f, x) < 0.03$ and $\|x - \pi_{\mathbb{R}}(x)\| < \dfrac{1}{20\gamma(f, x)}$, then $\xi \in \mathbb{R}^n$.*

Proof. Since $\|x - \overline{x}\| = 2\|x - \pi_{\mathbb{R}}(x)\|$ and $\beta(f, x) = \beta(f, \overline{x})$, Item 1 follows by concluding $\xi \neq \overline{\xi}$ using Lemma 1.2. Item 2 follows from Theorem 1.4 together with $\pi_{\mathbb{R}}(x) \in \mathbb{R}^n$ and $N_f(\mathbb{R}^n) \subset \mathbb{R}^n$. $\square$

1.2 Bounding higher order derivatives

The constant $\gamma(f, x)$ defined in (2) yields information regarding the higher order derivatives of f evaluated at x. Even though, for polynomial systems, $\gamma(f, x)$ is actually a maximum of finitely many values, it is often computationally difficult to compute exactly. However, in the polynomial case, it can be bounded above based in part on the degrees of the polynomials [10]. Due to the nature of polynomial-exponential systems, this bound will be used in our algorithm presented in Section 2 for certifying solutions to polynomial-exponential systems.

Let $g : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d where $g(x) = \sum_{|\rho| \le d} a_\rho x^\rho$ and

$$\|g\|^2 = \frac{1}{d!} \sum_{|\rho| \le d} \rho! \cdot (d - |\rho|)! \cdot |a_\rho|^2$$

is the standard unitarily invariant norm on the homogenization of g. For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f(x) = [f_1(x), \ldots, f_n(x)]^T$, we have

$$\|f\|^2 = \sum_{i=1}^{n} \|f_i\|^2.$$

For a point $x \in \mathbb{C}^n$, define $\|x\|_1^2 = 1 + \|x\|^2 = 1 + \sum_{i=1}^{n} |x_i|^2$.

The following is an affine version of Propositions 1 and 3 from [10].

Proposition 1.6 *If $g : \mathbb{C}^n \to \mathbb{C}$ is a polynomial of degree d, then, for all $x \in \mathbb{C}^n$ and $k \geq 1$,*

$$|g(x)| \leq \|g\| \cdot \|x\|_1^d \quad \text{and} \quad \|D^k g(x)\| \leq d \cdot (d-1) \cdots (d-k+1) \cdot \|g\| \cdot \|x\|_1^{d-k}.$$

Let $k \geq 2$. Lemma 3 of [10] yields

$$\left(\frac{d \cdot (d-1) \cdots (d-k+1)}{d^{1/2} \cdot k!} \right)^{\frac{1}{k-1}} \leq \frac{d^{1/2}(d-1)}{2} \leq \frac{d^{3/2}}{2}.$$

Additionally, since $\|x\|_1 \geq 1$, we know $\|x\|_1^{d-1} \geq \|x\|_1^{d-k}$. These facts together with Proposition 1.6 yield

$$\begin{aligned}
\left\| \frac{D^k g(x)}{k!} \right\|^{\frac{1}{k-1}} &\leq \left(\frac{d^{1/2} \cdot \|D^k g(x)\|}{d^{1/2} \cdot k!} \right)^{\frac{1}{k-1}} \\
&\leq d^{\frac{1}{2(k-1)}} \left(\frac{d \cdot (d-1) \cdots (d-k+1) \cdot \|g\| \cdot \|x\|_1^{d-k}}{d^{1/2} k!} \right)^{\frac{1}{k-1}} \\
&\leq \left(d^{1/2} \cdot \|x\|_1^{d-k} \cdot \|g\| \right)^{\frac{1}{k-1}} \left(\frac{d \cdot (d-1) \cdots (d-k+1)}{d^{1/2} \cdot k!} \right)^{\frac{1}{k-1}} \\
&\leq \frac{d^{3/2}}{2\|x\|_1} \left(d^{1/2} \cdot \|x\|_1^{d-1} \cdot \|g\| \right)^{\frac{1}{k-1}}
\end{aligned}$$

which we summarize in the following proposition.

Proposition 1.7 *If $g : \mathbb{C}^n \to \mathbb{C}$ is a polynomial of degree d, then, for all $x \in \mathbb{C}^n$ and $k \geq 2$,*

$$\left\| \frac{D^k g(x)}{k!} \right\|^{\frac{1}{k-1}} \leq \frac{d^{3/2}}{2\|x\|_1} \left(d^{1/2} \cdot \|x\|_1^{d-1} \cdot \|g\| \right)^{\frac{1}{k-1}}.$$

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial system with $\deg f_i = d_i$. Define $D = \max d_i$ and

$$\mu(f, x) = \max\{1, \ \|f\| \cdot \|Df(x)^{-1} \Delta_{(d)}(x)\|\} \tag{3}$$

assuming $Df(x)$ is invertible where

$$\Delta_{(d)}(x) = \begin{bmatrix} d_1^{1/2} \cdot \|x\|_1^{d_1 - 1} & & \\ & \ddots & \\ & & d_n^{1/2} \cdot \|x\|_1^{d_n - 1} \end{bmatrix}. \tag{4}$$

Since $\mu(f, x) \geq 1$, $\mu(f, x)^{\frac{1}{k-1}} \leq \mu(f, x)$ for any $k \geq 2$.

The following version of Proposition 3 of [10, § I-3] yields an upper bound for $\gamma(f, x)$.

Proposition 1.8 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial system with $\deg f_i = d_i$ and $D = \max d_i$. For any $x \in \mathbb{C}^n$ such that $Df(x)$ is invertible,*

$$\gamma(f, x) \leq \frac{\mu(f, x) \cdot D^{3/2}}{2 \cdot \|x\|_1}.$$

Proof. For $k \geq 2$, we have

$$\begin{aligned}
\left\| \frac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{\frac{1}{k-1}} &\leq \left(\|f\| \cdot \|Df(x)^{-1} \Delta_{(d)}(x)\| \right)^{\frac{1}{k-1}} \left\| \frac{\Delta_{(d)}(x)^{-1} D^k f(x)}{\|f\| \cdot k!} \right\|^{\frac{1}{k-1}} \\
&\leq \mu(f, x) \left(\sum_{i=1}^{n} \frac{\|f_i\|^2}{\|f\|^2} \left(\frac{d_i^{3/2}}{2 \cdot \|x\|_1} \right)^{2(k-1)} \right)^{\frac{1}{2(k-1)}} \\
&\leq \frac{\mu(f, x) D^{3/2}}{2 \cdot \|x\|_1}.
\end{aligned}$$

□

2 Certifying solutions

Since the bound provided in Proposition 1.8 does not apply to a polynomial-exponential system $\mathcal{F}$, we develop a new bound based on the solutions of linear homogeneous partial differential equations. With this bound, algorithms for certifying approximate solutions, distinct associated solutions, and real associated solutions of [5] apply to $\mathcal{F}$.

Consider $g(x) = e^{\beta x}$ for some $\beta \in \mathbb{C}$. Clearly, for any $k \geq 0$, $|g^{(k)}(x)| = |\beta|^k \cdot |g(x)|$. By letting $B(x) = |g(x)|$ and $C = \max\{1, |\beta|\}$, we have

$$|g^{(k)}(x)| \leq C^k \cdot B(x). \tag{5}$$

The following lemma shows that a similar bound holds in general.

Lemma 2.1 *Let $c_0, \ldots, c_{r-1} \in \mathbb{C}$, $\ell(x_0, \ldots, x_r) = x_r - \sum_{i=0}^{r-1} c_i x_i$, and $g : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\ell(g, g', \ldots, g^{(r)}) = 0$ and r is minimal with such a property. If*

$$B(x) = \max\{|g(x)|, |g'(x)|, \ldots, |g^{(r-1)}(x)|\} \quad \text{and} \quad C = \max\{1, |c_0|, \ldots, |c_{r-1}|\},$$

then, for any $x \in \mathbb{C}$ and $k \geq 0$, we have

$$|g^{(k)}(x)| \leq \begin{cases} B(x) & \text{if } k < r \\ (2 \cdot C)^{k-r} \cdot r \cdot B(x) \cdot C\cdot & \text{if } k \geq r. \end{cases}$$

In particular, $|g^{(k)}(x)| \leq (2 \cdot C)^{k-1} \cdot r \cdot B(x) \cdot C = 2^{k-1} \cdot r \cdot C^k \cdot B(x)$.

Proof. We know $g^{(r)} = \sum_{i=0}^{r-1} c_i g^{(i)}(x)$. For any $k > r$, by differentiation, we know

$$g^{(k)} = \sum_{i=0}^{r-1} c_i g^{(i+k-r)}(x).$$

We will now proceed by induction starting at $k = r$. In particular,

$$|g^{(r)}(x)| \leq \sum_{i=0}^{r-1} |c_i| \cdot |g^{(i)}(x)| \leq B(x) \cdot C \sum_{i=0}^{r-1} 1 = r \cdot B(x) \cdot C.$$

For $k > r$ with $p = k - r$, we have

$$\begin{array}{rcl} |g^{(k)}(x)| & \leq & \displaystyle\sum_{i=0}^{r-1} |c_i| \cdot |g^{(i+p)}(x)| \leq C \left(\sum_{i=0}^{\max\{r-1-p,0\}} |g^{(i+p)}(x)| + \sum_{i=\max\{0,r-p\}}^{r-1} |g^{(i+p)}(x)| \right) \\ & \leq & C \left(r \cdot B(x) + r \cdot B(x) \cdot C \displaystyle\sum_{i=r-p}^{r-1} (2 \cdot C)^{i+p-r} \right) \\ & \leq & r \cdot B(x) \cdot C^2 \left(1 + C^{p-1} \displaystyle\sum_{i=0}^{p-1} 2^i \right) \\ & \leq & 2^p \cdot r \cdot B(x) \cdot C^{p+1} \\ & = & (2 \cdot C)^{k-r} \cdot r \cdot B(x) \cdot C. \end{array}$$

The remaining statement follows from the fact that $C \geq 1$ and $r \geq 1$. □

The following lemma will also be used to deduce our bound.

Lemma 2.2 *If $\delta_0 \geq 0$ and $\alpha_1, \delta_1, \ldots, \alpha_m, \delta_m \geq 1$, then*

$$\sup_{k \geq 2} \left(\delta_0^{2(k-1)} + 2^{2(k-1)} \sum_{i=1}^{m} \left(\alpha_i^k \delta_i \right)^2 \right)^{\frac{1}{2(k-1)}} \leq \delta_0 + 2 \sum_{i=1}^{m} \alpha_i^2 \delta_i.$$

Proof. Fix $k \geq 2$. Since $2(k-1) \geq 2$ and $4(k-1) \geq 2k$, we know $\alpha_i^{4(k-1)} \geq \alpha_i^{2k}$ and $\delta_i^{2(k-1)} \geq \delta_i^2$ for $i = 1, \ldots, m$. The lemma now follows since

$$\begin{array}{rcl} \left(\delta_0 + 2 \displaystyle\sum_{i=1}^{m} \alpha_i^2 \delta_i \right)^{2(k-1)} & \geq & \delta_0^{2(k-1)} + 2^{2(k-1)} \left(\displaystyle\sum_{i=1}^{m} \alpha_i^2 \delta_i \right)^{2(k-1)} \\ & \geq & \delta_0^{2(k-1)} + 2^{2(k-1)} \displaystyle\sum_{i=1}^{m} \alpha_i^{4(k-1)} \delta_i^{2(k-1)} \\ & \geq & \delta_0^{2(k-1)} + 2^{2(k-1)} \displaystyle\sum_{i=1}^{m} \alpha_i^{2k} \delta_i^2. \end{array}$$

□

Throughout the remainder of this section, we assume that $\mathcal{F} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ is a polynomial-exponential system such that there exists a polynomial system $P : \mathbb{C}^{n+m} \to \mathbb{C}^n$, analytic functions $g_1, \ldots, g_m : \mathbb{C} \to \mathbb{C}$, and integers $\sigma_1, \ldots, \sigma_m \in \{1, \ldots, n\}$ such that

$$\mathcal{F}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \begin{bmatrix} P(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ y_1 - g_1(x_{\sigma_1}) \\ \vdots \\ y_m - g_m(x_{\sigma_m}) \end{bmatrix}. \quad (6)$$

Also, for $i = 1, \dots, n$, we define $d_i = \deg P_i$ and $D = \max d_i$.

We assume that each g_i satisfies some nonzero linear homogeneous PDE with complex coefficients. For each $i = 1, \dots, m$, let r_i be the smallest positive integer such that there exists a nonzero linear function $\ell_i : \mathbb{C}^{r_i+1} \to \mathbb{C}$ with $\ell_i(g_i, g_i', \dots, g_i^{(r_i)}) = 0$. By construction, the coefficient of z_{r_i} in $\ell_i(z_0, z_1, \dots, z_{r_i})$ must be nonzero. Upon rescaling ℓ_i, we will assume that this coefficient is one, that is, we have

$$\ell_i(z_0, z_1, \dots, z_{r_i}) = z_{r_i} - c_{i,r_i-1} z_{r_i-1} - \cdots - c_{i,0} z_0 \tag{7}$$

which yields $g_i^{(r_i)} = \sum_{j=0}^{r_i-1} c_{i,j} g_i^{(j)}$. We note that the minimal integer r_i with such a property is called the *order* of g_i.

For example, for nonzero $\lambda, \mu \in \mathbb{C}$, if $g_1(x) = e^{\lambda x}$, $g_2(x) = \cos(\mu x)$, and $g_3(x) = x \sin(x)$, then the order of g_i is $1, 2$, and 4, respectively. The corresponding differential equations are

$$\frac{\partial g_1}{\partial x} - \lambda g_1 = 0, \quad \frac{\partial^2 g_2}{\partial x^2} + \mu^2 g_2 = 0, \quad \text{and} \quad \frac{\partial^4 g_3}{\partial x^4} + 2\frac{\partial^2 g_3}{\partial x^2} + g_3 = 0$$

with linear functions

$$\ell_1(z_0, z_1) = z_1 - \lambda z_0, \quad \ell_2(z_0, z_1, z_2) = z_2 + \mu^2 z_0, \quad \text{and} \quad \ell_3(z_0, z_1, z_2, z_3, z_4) = z_4 + 2z_2 + z_0.$$

The bound obtained in Proposition 1.8 depends upon $\mu(f, x)$ defined in (3) for polynomial systems. We extend this to polynomial-exponential systems by defining

$$\mu(\mathcal{F}, (x, y)) = \max \left\{ 1, \left\| D\mathcal{F}(x, y)^{-1} \begin{bmatrix} \Delta_{(d)}(x, y)\|P\| & \\ & I_m \end{bmatrix} \right\| \right\} \tag{8}$$

assuming that $D\mathcal{F}(x, y)$ is invertible. The matrix $\Delta_{(d)}(x, y)$ is the $n \times n$ diagonal matrix defined in (4) and I_m is the $m \times m$ identity matrix. We note that (8) reduces to (3) when $m = 0$.

The following theorem yields a bound for $\gamma(\mathcal{F}, (x, y))$.

Theorem 2.3 *For $i = 1, \dots, m$ and $z \in \mathbb{C}$, define*

$$B_i(z) = \max\{|g_i(z)|, \dots, |g_i^{(r_i-1)}(z)|\} \quad \text{and} \quad C_i = \max\{1, |c_{i,0}|, \dots, |c_{i,r_i-1}|\}.$$

Then, for any $(x, y) \in \mathbb{C}^{n+m}$ such that $D\mathcal{F}(x, y)$ is invertible,

$$\gamma(\mathcal{F}, (x, y)) \le \mu(\mathcal{F}, (x, y)) \left(\frac{D^{3/2}}{2\|(x, y)\|_1} + 2 \sum_{i=1}^{m} C_i^2 \max\{1, r_i \cdot B_i(x_{\sigma_i})\} \right). \tag{9}$$

Proof. Let $\mathcal{M} = \begin{bmatrix} \Delta_{(d)}(x, y)\|P\| & \\ & I_m \end{bmatrix}$ and $k \ge 2$. We have

$$\begin{aligned} \left\| \frac{D\mathcal{F}(x, y)^{-1} D^k \mathcal{F}(x, y)}{k!} \right\| &\le \left\| D\mathcal{F}(x, y)^{-1} \mathcal{M} \right\| \left\| \frac{\mathcal{M}^{-1} D^k \mathcal{F}(x, y)}{k!} \right\| \\ &\le \mu(\mathcal{F}, (x, y)) \left\| \frac{\mathcal{M}^{-1} D^k \mathcal{F}(x, y)}{k!} \right\|. \end{aligned}$$

By Proposition 1.7 and Lemma 2.1,

$$\begin{aligned}\left\|\frac{\mathcal{M}^{-1}D^k\mathcal{F}(x,y)}{k!}\right\|^2 &= \sum_{i=1}^n\left\|\frac{D^kP_i(x,y)}{d_i^{1/2}\cdot\|(x,y)\|_1^{d_i-1}\cdot\|P\|\cdot k!}\right\|^2+\sum_{i=1}^m\left\|\frac{D^kg_i(x_{\sigma_i})}{k!}\right\|^2\\ &\leq \sum_{i=1}^n\frac{\|P_i\|^2}{\|P\|^2}\left(\frac{d_i^{3/2}}{2\|(x,y)\|_1}\right)^{2(k-1)}+\sum_{i=1}^m\left(2^{k-1}\cdot r_i\cdot C_i^k\cdot B_i(x_{\sigma_i})\right)^2\\ &\leq \left(\frac{D^{3/2}}{2\|(x,y)\|_1}\right)^{2(k-1)}+2^{2(k-1)}\sum_{i=1}^m\left(r_i\cdot C_i^k\cdot B_i(x_{\sigma_i})\right)^2.\end{aligned}$$

This yields

$$\begin{aligned}\gamma(\mathcal{F},(x,y)) &= \sup_{k\geq 2}\left\|\frac{D\mathcal{F}(x,y)^{-1}D^k\mathcal{F}(x,y)}{k!}\right\|^{\frac{1}{k-1}}\\ &\leq \mu(\mathcal{F},(x,y))\sup_{k\geq 2}\left(\left(\frac{D^{3/2}}{2\|(x,y)\|_1}\right)^{2(k-1)}+2^{2(k-1)}\sum_{i=1}^m\left(r_i\cdot C_i^k\cdot B_i(x_{\sigma_i})\right)^2\right)^{\frac{1}{2(k-1)}}\\ &\leq \mu(\mathcal{F},(x,y))\sup_{k\geq 2}\left(\left(\frac{D^{3/2}}{2\|(x,y)\|_1}\right)^{2(k-1)}+\right.\\ &\qquad\qquad \left.2^{2(k-1)}\sum_{i=1}^m\left(C_i^k\max\{1,r_i\cdot B_i(x_{\sigma_i})\}\right)^2\right)^{\frac{1}{2(k-1)}}.\end{aligned}$$

The result now follows from Lemma 2.2. □

Remark 2.4 When $m=0$, the bounds provided in Theorem 2.3 and Proposition 1.8 agree.

The following is an algorithm to certify approximate solutions of $\mathcal{F}=0$.

Procedure $B=$ **CertifySoln**$(\mathcal{F},z)$

Input A polynomial-exponential system $\mathcal{F}:\mathbb{C}^{n+m}\to\mathbb{C}^{n+m}$ and a point $z\in\mathbb{C}^{n+m}$.

Output A boolean which is *True* if z can be certified as an approximate solution of $\mathcal{F}=0$, otherwise, *False*.

Begin

1. If $\mathcal{F}(z)=0$, return *True*, otherwise, if $D\mathcal{F}(z)$ is not invertible, return *False*.
2. Set $\beta:=\|D\mathcal{F}(z)^{-1}\mathcal{F}(z)\|$ and γ to be the upper bound for $\gamma(\mathcal{F},z)$ provided in Theorem 2.3.
3. If $\beta\cdot\gamma<\dfrac{13-3\sqrt{17}}{4}$, return *True*, otherwise return *False*.

The algorithms **CertifyDistinctSoln** and **CertifyRealSoln** from [5] apply to polynomial-exponential systems using the bound provided in Theorem 2.3. The algorithm **CertifyDistinctSoln** determines if two approximate solutions have distinct associated solutions. The algorithm **CertifyRealSoln** applies to polynomial-exponential systems $\mathcal{F}$ such that $N_{\mathcal{F}}(\mathbb{R}^{n+m})\subset\mathbb{R}^{n+m}$ and determines if the associated solution to a given approximate solution is real.

We conclude this section with a refinement of Theorem 2.3 applied to polynomial-exponential systems depending on exp, sin, cos, sinh, and cosh. This refinement uses the following lemma.

Lemma 2.5 *If $\lambda_0, \ldots, \lambda_m \geq 0$ and $\mu_1, \ldots, \mu_m \geq 2$, then*

$$\sup_{k\geq 2}\left(\lambda_0^{2(k-1)} + \sum_{i=1}^{m}\left(\frac{\mu_i\lambda_i^{k-1}}{k!}\right)^2\right)^{\frac{1}{2(k-1)}} \leq \lambda_0 + \frac{1}{2}\sum_{i=1}^{m}\mu_i\lambda_i.$$

Proof. Fix $k \geq 2$. Since $2(k-1) \geq 2$ and $\mu_i \geq 2$, we know $\left(\frac{\mu_i}{2}\right)^{2(k-1)} \geq \left(\frac{\mu_i}{2}\right)^2$. The lemma follows from

$$\begin{aligned}
\left(\lambda_0 + \frac{1}{2}\sum_{i=1}^{m}\mu_i\lambda_i\right)^{2(k-1)} &\geq \lambda_0^{2(k-1)} + \left(\sum_{i=1}^{m}\frac{\mu_i\lambda_i}{2}\right)^{2(k-1)} \\
&\geq \lambda_0^{2(k-1)} + \sum_{i=1}^{m}\left(\frac{\mu_i}{2}\right)^{2(k-1)}\lambda_i^{2(k-1)} \\
&\geq \lambda_0^{2(k-1)} + \sum_{i=1}^{m}\frac{\mu_i^2\lambda_i^{2(k-1)}}{2^2} \\
&\geq \lambda_0^{2(k-1)} + \sum_{i=1}^{m}\left(\frac{\mu_i\lambda_i^{(k-1)}}{k!}\right)^2.
\end{aligned}$$

□

Let $a, b, c, e, h \in \mathbb{Z}_{\geq 0}$, $\delta_i, \epsilon_j, \zeta_k, \eta_p, \kappa_q \in \mathbb{C}$, and $\sigma_i, \tau_j, \phi_k, \chi_p, \psi_q \in \{1, \ldots, n\}$. The following considers the following polynomial-exponential system

$$\mathcal{G}(x_1, \ldots, x_n, u_1, \ldots, u_a, v_1, \ldots, v_b, w_1, \ldots, w_c, y_1, \ldots, y_d, z_1, \ldots, z_e) =$$
$$\left[\begin{array}{cc}
\multicolumn{2}{c}{P(x_1, \ldots, x_n, u_1, \ldots, u_a, v_1, \ldots, v_b, w_1, \ldots, w_c, y_1, \ldots, y_d, z_1, \ldots, z_e)} \\
u_i - \exp(\delta_i x_{\sigma_i}), & i = 1, \ldots, a \\
v_j - \sin(\epsilon_j x_{\tau_j}), & j = 1, \ldots, b \\
w_k - \cos(\zeta_k x_{\phi_k}), & k = 1, \ldots, c \\
y_p - \sinh(\eta_p x_{\chi_p}), & p = 1, \ldots, e \\
z_q - \cosh(\kappa_q x_{\psi_q}), & q = 1, \ldots, h
\end{array}\right]. \tag{10}$$

Corollary 2.6 *Let $\mathcal{G}$ be defined as in (10) where $P : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system with $N = n + a + b + c + e + h$, $d_i = \deg P_i$ and $D = \max d_i$. For any λ, $\theta \in \mathbb{C}$, define*

$$\begin{aligned}
&A(\lambda, \theta) = \max\{|\lambda|, |\lambda^2 \exp(\lambda\theta)/2|\}, \\
&B(\lambda, \theta) = \max\{|\lambda|, |\lambda^2 \sin(\lambda\theta)/2|, |\lambda^2 \cos(\lambda\theta)/2|\}, \quad \textit{and} \\
&C(\lambda, \theta) = \max\{|\lambda|, |\lambda^2 \sinh(\lambda\theta)/2|, |\lambda^2 \cosh(\lambda\theta)/2|\}.
\end{aligned}$$

Then, for any $X = (x, u, v, w, y, z) \in \mathbb{C}^N$ such that $D\mathcal{G}(X)$ is invertible,

$$\begin{aligned}
\gamma(\mathcal{G}, X) \leq \mu(\mathcal{G}, X)\Bigg(\frac{D^{3/2}}{2\|X\|_1} &+ \sum_{i=1}^{a} A(\delta_i, x_{\sigma_i}) + \sum_{j=1}^{b} B(\epsilon_j, x_{\tau_j}) + \sum_{k=1}^{c} B(\zeta_k, x_{\phi_k}) \\
&+ \sum_{p=1}^{e} C(\eta_p, x_{\chi_p}) + \sum_{q=1}^{h} C(\kappa_q, x_{\psi_q})\Bigg).
\end{aligned} \tag{11}$$

Proof. Let $k \geq 2$. The following table lists the bounds on the higher derivatives together with associated quantities λ and μ used when applying Lemma 2.5.

$g(x)$	bound for $\lvert g^{(k)}(x)\rvert$	λ	μ
$\exp(\theta x)$	$\lvert\theta^k \exp(\theta x)\rvert$	$\lvert\theta\rvert$	$\max\{2, \lvert\theta \exp(\theta x)\rvert\}$
$\sin(\theta x)$ $\cos(\theta x)$	$\lvert\theta^k\rvert \max\{\lvert\sin(\theta x)\rvert, \lvert\cos(\theta x)\rvert\}$	$\lvert\theta\rvert$	$\max\{2, \lvert\theta \sin(\theta x)\rvert, \lvert\theta \cos(\theta x)\}$
$\sinh(\theta x)$ $\cosh(\theta x)$	$\lvert\theta^k\rvert \max\{\lvert\sinh(\theta x)\rvert, \lvert\cosh(\theta x)\rvert\}$	$\lvert\theta\rvert$	$\max\{2, \lvert\theta \sinh(\theta x)\rvert, \lvert\theta \cosh(\theta x)\}$

The result now follows immediately by modifying the proof of Theorem 2.3 incorporating the bounds presented in this table together with Lemma 2.5. Based on Lemma 2.5, the functions A, B, and C are one-half of the product of the entries in the λ and μ columns. $\square$

3 Approximating solutions

In order to certify that a point is an approximate solution of $\mathcal{F} = 0$, where $\mathcal{F}$ is a polynomial-exponential system, one needs to first have a candidate point. In some applications, candidate points arise naturally from the formulation of the problem. One systematic approach to yield candidate points is to replace each analytic function g_i by a polynomial g_i^p and solve the resulting polynomial system, namely

$$\mathcal{F}^p(x_1, \ldots, x_n, y_1, \ldots, y_m) = \begin{bmatrix} P(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ y_1 - g_1^p(x_{\sigma_1}) \\ \vdots \\ y_m - g_m^p(x_{\sigma_m}) \end{bmatrix}. \tag{12}$$

When the degree of the polynomial approximations are sufficiently large, the numerical solutions of $\mathcal{F}^p = 0$ are candidates for being approximate solutions of $\mathcal{F} = 0$. In Section 3.1, we discuss using regeneration [4] to solve $\mathcal{F}^p = 0$.

If a numerical solution of $\mathcal{F}^p = 0$ is not an approximate solution of $\mathcal{F} = 0$, one can try to apply Newton's method for $\mathcal{F}$ directly to these points to possibly yield an approximate solution of $\mathcal{F} = 0$. Another approach is to construct a homotopy between $\mathcal{F}^p$ and $\mathcal{F}$, and numerically approximate the endpoint of the path starting with a solution of $\mathcal{F}^p = 0$. We note that neither method is guaranteed to yield an approximate solution of $\mathcal{F} = 0$.

3.1 Regeneration and polynomial-exponential systems

Regeneration [4] solves a polynomial system by using solutions to related, but easier to solve, polynomial systems. In particular, we will utilize the linear product [14] structure of $\mathcal{F}^p$ in (12).

Suppose that g is a univariate polynomial of degree d. The polynomial $y - g(x)$ has a linear product structure of

$$\langle x, y, 1\rangle \times \underbrace{\langle x, 1\rangle \times \cdots \times \langle x, 1\rangle}_{d-1 \text{ times}}.$$

That is, $y - g(x)$ is a finite sum of polynomials of the form $L_1(x, y) \cdots L_d(x, y)$ where

$$L_1(x, y) = ay + b_1 x + c_1 \quad \text{and, for } i = 2, \ldots, d, \quad L_i(x, y) = b_i x + c_i$$

for some $a, b_i, c_i \in \mathbb{C}$.

For $i = 1, \ldots, m$, let $r_i = \deg g_i^p$ and $a_i, b_{i,1}, \ldots, b_{i,r_i} \in \mathbb{C}$. Similar to the algorithms proposed in [4], we note that the following arguments and proposed algorithm depend on the genericity of a_i and $b_{i,j}$. Define

$$L_{i,1}(x, y) = a_i y + b_{i,1}x + 1 \text{ and, for } j = 2, \ldots, r_i \quad L_{i,j}(x, y) = b_{i,j}x + 1.$$

Let $\nu = (\nu_1, \ldots, \nu_m)$ such that $1 \le \nu_i \le r_i$. Consider the polynomial systems $\mathcal{Q}_\nu : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ defined by

$$\mathcal{Q}_\nu(x_1, \ldots, x_n, y_1, \ldots, y_m) = \begin{bmatrix} P(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ L_{1,\nu_1}(x_{\sigma_1}, y_1) \\ \vdots \\ L_{m,\nu_m}(x_{\sigma_m}, y_m) \end{bmatrix}. \tag{13}$$

For $\mathbf{1} = (1, \ldots, 1)$, we first compute the solutions of $\mathcal{Q}_\mathbf{1} = 0$. We note that in practice, $\mathcal{Q}_\mathbf{1}$ is solved by working intrinsically on the linear space defined by $L_{1,\nu_1}(x_{\sigma_1}, y_1) = \cdots = L_{m,\nu_m}(x_{\sigma_m}, y_m) = 0$. Numerical approximations of these solutions can be obtained using standard numerical solving methods for square polynomial systems (see [12, 15]) including, for example, polyhedral homotopies [7] or basic regeneration [4].

In order to compute the nonsingular isolated solutions of $\mathcal{F}^p = 0$, we need to compute the nonsingular isolated solutions of $\mathcal{Q}_\nu = 0$ for all possible ν. By the theory of coefficient-parameter homotopies [9], the nonsingular isolated solutions of $\mathcal{Q}_\nu = 0$ can be obtained by using a homotopy from $\mathcal{Q}_\mathbf{1}$ to $\mathcal{Q}_\nu$ starting with the nonsingular isolated solutions of $\mathcal{Q}_\mathbf{1} = 0$. We note that if $i \neq j$ such that $\sigma_i = \sigma_j$ and $\nu_i, \nu_j > 1$, then $\mathcal{Q}_\nu = 0$ has no solutions.

After solving $\mathcal{Q}_\nu = 0$ for all possible ν, we thus have all nonsingular isolated solutions of

$$\mathcal{P}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \begin{bmatrix} P(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ \prod_{j=1}^{r_1} L_{1,j}(x_{\sigma_1}, y_1) \\ \vdots \\ \prod_{j=1}^{r_m} L_{m,j}(x_{\sigma_m}, y_m) \end{bmatrix} = 0. \tag{14}$$

The final step is to use a homotopy deforming $\mathcal{P}$ to $\mathcal{F}^p$ starting with the nonsingular isolated solutions of $\mathcal{P} = 0$. The finite endpoints of this homotopy form a superset of the isolated nonsingular solutions of $\mathcal{F}^p = 0$.

4 Implementation details and examples

The certification of polynomial-exponential systems is implemented in `alphaCertified` [6]. The systems must be of the form $\mathcal{G}$ in (10) where the coefficients of P as well as the constant in the argument of exp, sin, cos, sinh, and cosh must be rational complex numbers. The bound for γ presented in (11) is computed. Due to the nature of exponential functions, the computations are performed using arbitrary precision floating point arithmetic. Since floating point errors arising from the internal computations are not fully controlled, the results of `alphaCertified` for polynomial-exponential systems are said to be *soft certified*. See Appendix A and [5, 6] for more details regarding input syntax, internal computations, and output.

In the following examples, we used `Bertini` [1] and `alphaCertified` on a 2.4 GHz Opteron 250 processor running 64-bit Linux with 8 GB of memory. All files for running these examples can be found at `www.nd.edu/~jhauenst/PolyExp`.

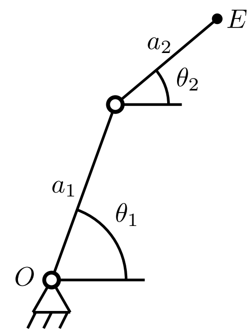

Figure 1: RR dyad

4.1 A rigid mechanism

Consider the algebraic kinematics problem [15] of the inverse kinematics of the RR dyad. The RR dyad, which is displayed in Figure 1, consists of two legs of fixed length, say a_1 and a_2, which are connected by a pin joint. The mechanism is anchored with a pin joint at the point O, which we take as the origin. Given a point $E = (e_1, e_2)$, the problem is compute the angles θ_1 and θ_2 so that the end of the second leg is at E. That is, we want to solve $f(\theta_1, \theta_2) = 0$ where

$$f(\theta_1, \theta_2) = \begin{bmatrix} a_1 \cos(\theta_1) + a_2 \cos(\theta_2) - e_1 \\ a_1 \sin(\theta_1) + a_2 \sin(\theta_2) - e_2 \end{bmatrix}.$$

The polynomial-exponential system $\mathcal{G} : \mathbb{C}^6 \to \mathbb{C}^6$ of the form (10) is

$$\mathcal{G}(\theta_1, \theta_2, y_1, y_2, y_3, y_4) = \begin{bmatrix} a_1 y_3 + a_2 y_4 - e_1 \\ a_1 y_1 + a_2 y_2 - e_2 \\ y_1 - \sin(\theta_1) \\ y_2 - \sin(\theta_2) \\ y_3 - \cos(\theta_1) \\ y_4 - \cos(\theta_2) \end{bmatrix}. \tag{15}$$

Since θ_i only appears in f as arguments of the sine and cosine functions, we can compute solutions of $f = 0$ by using the solutions of a related polynomial system. In particular, consider the polynomial system $g : \mathbb{C}^4 \to \mathbb{C}^4$ obtained by replacing $\sin(\theta_i)$ and $\cos(\theta_i)$ with s_i and c_i, respectively, and adding the Pythagorean identities, namely

$$g(s_1, s_2, c_1, c_2) = \begin{bmatrix} a_1 c_1 + a_2 c_2 - e_1 \\ a_1 s_1 + a_2 s_2 - e_2 \\ s_1^2 + c_1^2 - 1 \\ s_2^2 + c_2^2 - 1 \end{bmatrix}.$$

Given a solution of $g = 0$, solutions of $f = 0$ are generated using either the arcsin or arccos functions. Moreover, it is easy to verify that, for general $a_i, e_i \in \mathbb{C}$, $g = 0$ has two solutions and thus $f = 0$ has two 2π-periodic families of solutions.

Consider the inverse kinematics problem with $a_1 = 3$, $a_2 = 2$, and $E = (1, 3.5)$. We used `Bertini` to numerically approximate the two solutions of $g = 0$. For demonstration, consider the two digit rational approximations of the solutions

$$X_1 = \frac{1}{100}(65, 77, 76, -64) \quad \text{and} \quad X_2 = \frac{1}{100}(95, 32, -30, 95).$$

The certified upper bounds for $\alpha(g, X_i)$ computed by `alphaCertified` using exact rational arithmetic and rounded to four digits are 0.0736 and 0.0788, respectively. Hence, X_1 and X_2 are both approximate solutions of $g = 0$. Furthermore, `alphaCertified` certified that the associated solutions are distinct and real.

k	$\beta(\mathcal{G}, N_{\mathcal{G}}^k(Z_1))$	$\beta(\mathcal{G}, N_{\mathcal{G}}^k(Z_2))$
0	$4.94 \cdot 10^{-3}$	$5.26 \cdot 10^{-3}$
1	$7.46 \cdot 10^{-9}$	$6.29 \cdot 10^{-9}$
2	$1.21 \cdot 10^{-17}$	$8.86 \cdot 10^{-18}$
3	$3.65 \cdot 10^{-35}$	$2.01 \cdot 10^{-35}$
4	$3.56 \cdot 10^{-70}$	$1.10 \cdot 10^{-70}$
5	$3.56 \cdot 10^{-140}$	$3.41 \cdot 10^{-141}$
6	$3.50 \cdot 10^{-280}$	$3.21 \cdot 10^{-282}$
7	$3.44 \cdot 10^{-560}$	$2.90 \cdot 10^{-564}$

Table 1: Newton residuals for $\mathcal{G}$

We now consider two corresponding approximations to solutions of $\mathcal{G} = 0$ namely

$$Z_1 = (0.711, 2.261, 0.65, 0.77, 0.76, -0.64) \text{ and } Z_2 = (1.874, 0.324, 0.95, 0.32, -0.30, 0.95). \quad (16)$$

The upper bounds for $\alpha(\mathcal{G}, Z_i)$ computed by `alphaCertified` using 96-bit floating point arithmetic and rounded to four digits are 0.1265 and 0.1355, respectively. In order to reduce the effect of roundoff errors, we also used 1024-bit floating point arithmetic and obtained the same four digit value. Hence, `alphaCertified` has soft certified that Y_1 and Y_2 are both approximate solutions of $\mathcal{G} = 0$. Furthermore, `alphaCertified` has soft certified that the associated solutions are distinct and real. Table 1 lists the Newton residuals computed by `alphaCertified` using 4096-bit precision which demonstrates the quadratic convergence of Newton's method.

By using Euler's formula, we could alternatively use the polynomial-exponential system $\mathcal{G}' : \mathbb{C}^6 \to \mathbb{C}^6$ of the form (10) where

$$\mathcal{G}'(\theta_1, \theta_2, x_1, x_2, y_1, y_2) = \begin{bmatrix} a_1 x_1 + a_2 x_2 - e_1 + i e_2 \\ a_1 y_1 + a_2 y_2 - e_1 - i e_2 \\ x_1 y_1 - 1 \\ x_2 y_2 - 1 \\ y_1 - \exp(i\theta_1) \\ y_2 - \exp(i\theta_2) \end{bmatrix}$$

and $i = \sqrt{-1}$. Consider the two points

$$W_1 = (0.711, 2.261, 0.758 - 0.653i, -0.637 - 0.771i, 0.758 + 0.653i, -0.637 + 0.771i) \text{ and}$$
$$W_2 = (1.874, 0.324, -0.299 - 0.954i, 0.948 - 0.318i, -0.299 + 0.954i, 0.948 + 0.318i).$$

The upper bounds for $\alpha(\mathcal{G}', W_i)$ computed by `alphaCertified` using both 96-bit and 1024-bit floating point arithmetic and rounded to four digits are 0.1492 and 0.1422, respectively. In particular, `alphaCertified` soft certified that W_1 and W_2 are both approximate solutions of $\mathcal{G}' = 0$ with distinct associated solutions.

Finally, consider the polynomial system obtained by replacing the sine and cosine functions in f with a third and second degree truncated Taylor series approximation, respectively, centered at the origin, namely

$$f^p(\theta_1, \theta_2) = \begin{bmatrix} a_1(1 + \theta_1^2/2) + a_2(1 + \theta_2^2/2) - e_1 \\ a_1(\theta_1 + \theta_1^3/6) + a_2(\theta_2 + \theta_2^3/6) - e_2 \end{bmatrix}.$$

The system of equations $f^p = 0$ has six solutions and yield six solutions of $f = 0$ upon deforming f^p to f. These six solutions split into two groups of three based on the values of $\sin(\theta_i)$ and $\cos(\theta_i)$ corresponding to the two families of solutions of $f = 0$.

4.2 A compliant mechanism

In [13], Su and McCarthy study a polynomial-exponential system modeling a compliant four-bar linkage displayed in [13, Fig. 4]. Upon solving a related polynomial system and applying Newton's method, they conclude based on the numerical results that a specific compliant four-bar linkage has two stable configurations. We will first use `alphaCertified` to certify that their numerical approximations of the two stable configurations are indeed approximate solutions. Afterwards, we will use the approaches of Section 3 to recompute these two stable configurations.

The polynomial-exponential system $f : \mathbb{C}^5 \to \mathbb{C}^5$ modeling a compliant four-bar linkage is

$$f(\alpha, \theta_1, \theta_2, \nu_1, \nu_2) = \left[\begin{array}{c} R(\alpha)(W_2 - W_1) + G_1 + r_1 cs(\theta_1) - G_2 - r_2 cs(\theta_2) \\ R(\alpha)(W_2 - W_1)\nu_1 + r_1 cs(\theta_1) - r_2 cs(\theta_2)\nu_2 \\ k_1(\alpha - \alpha^0 - \theta_1 + \theta_1^0)(\nu_1 - 1) + k_2(\alpha - \alpha^0 - \theta_2 + \theta_2^0)(\nu_1 - \nu_2) \end{array} \right]$$

where

$$R(\alpha) = \left[\begin{array}{cc} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{array} \right] \text{ and } cs(\theta) = \left[\begin{array}{c} \cos(\theta) \\ \sin(\theta) \end{array} \right].$$

We note that each of the first two lines in f consists of two functions. Additionally, f is not algebraic since X, $\sin(X)$, and $\cos(X)$ all appear in f when X is either α, θ_1, or θ_2.

The values for the specific linkage under consider are

$$W_1 = \left[\begin{array}{c} -112.632 \\ -45.053 \end{array} \right], W_2 = \left[\begin{array}{c} 112.632 \\ -45.053 \end{array} \right], G_1 = \left[\begin{array}{c} 0 \\ 0 \end{array} \right], G_2 = \left[\begin{array}{c} 100 \\ 0 \end{array} \right], r_1 = r_2 = 250,$$

$$k_1 = 29250, k_2 = 5824.29, \theta_1^0 = 1.4486, \theta_2^0 = 0.925, \text{ and } \alpha^0 = -0.2169.$$

with numerical approximations for the stable configurations

$$\begin{array}{l} A_1 = (-0.216933, 1.448567, 0.924966, 0.610174, 1.094669) \text{ and} \\ A_2 = (-1.516473, 0.131930, -0.875993, 1.570656, 1.668379). \end{array}$$

The polynomial-exponential system $\mathcal{G} : \mathbb{C}^{11} \to \mathbb{C}^{11}$ of the form (10) is

$$\mathcal{G}(\alpha, \theta_1, \theta_2, \nu_1, \nu_2, y_1, \ldots, y_6) = \left[\begin{array}{c} R(y_1, y_2)(W_2 - W_1) + G_1 + r_1 cs(y_3, y_4) - G_2 - r_2 cs(y_5, y_6) \\ R(y_1, y_2)(W_2 - W_1)\nu_1 + r_1 cs(y_3, y_4) - r_2 cs(y_5, y_6)\nu_2 \\ k_1(\alpha - \alpha^0 - \theta_1 + \theta_1^0)(\nu_1 - 1) + k_2(\alpha - \alpha^0 - \theta_2 + \theta_2^0)(\nu_1 - \nu_2) \\ y_1 - \sin(\alpha) \\ y_2 - \cos(\alpha) \\ y_3 - \sin(\theta_1) \\ y_4 - \cos(\theta_1) \\ y_5 - \sin(\theta_2) \\ y_6 - \cos(\theta_2) \end{array} \right]$$

where

$$R(y_1, y_2) = \left[\begin{array}{cc} y_2 & -y_1 \\ y_1 & y_2 \end{array} \right] \text{ and } cs(w, z) = \left[\begin{array}{c} z \\ w \end{array} \right].$$

Let $B_i = (A_i, Y_i)$ where

$$\begin{array}{l} Y_1 = (-0.215236, 0.976562, 0.992539, 0.121925, 0.798600, 0.601862) \text{ and} \\ Y_2 = (-0.998525, 0.0542970, 0.131547, 0.991310, -0.768180, 0.640235). \end{array}$$

The upper bounds for $\alpha(\mathcal{G}, B_i)$ computed by `alphaCertified` using both 96-bit and 1024-bit floating point arithmetic and rounded to four digits are 0.0166 and 0.0427, respectively. In

	bound for		approximation of		bound for	
F	$\alpha(F, B_1)$	$\alpha(F, B_2)$	$\beta(F, B_1)$	$\beta(F, B_2)$	$\gamma(F, B_1)$,	$\gamma(F, B_2)$
$\mathcal{G}$	$1.66 \cdot 10^{-2}$	$4.27 \cdot 10^{-2}$	$8.08 \cdot 10^{-7}$	$1.06 \cdot 10^{-6}$	$2.05 \cdot 10^{4}$	$4.02 \cdot 10^{4}$
$\mathcal{G}'$	11.9	42.5	$8.08 \cdot 10^{-7}$	$1.06 \cdot 10^{-6}$	$1.47 \cdot 10^{7}$	$4.00 \cdot 10^{7}$

Table 2: Values obtained for $\mathcal{G}$ and $\mathcal{G}'$ at B_1 and B_2

particular, `alphaCertified` has soft certified that B_1 and B_2 are both approximate solutions of $\mathcal{G} = 0$. Furthermore, `alphaCertified` has soft certified that the associated solutions are distinct and real.

The formulation of the polynomial-exponential system can have an adverse effect on certifying solutions. For example, consider the polynomial-exponential system $\mathcal{G}' : \mathbb{C}^{11} \to \mathbb{C}^{11}$ obtained by replacing the 7^{th}, 9^{th}, and 11^{th} functions of $\mathcal{G}$ with

$$y_1^2 + y_2^2 - 1,\ y_3^2 + y_4^2 - 1,\ \text{and}\ \ y_5^2 + y_6^2 - 1.$$

Clearly, every solution of $\mathcal{G} = 0$ must also be a solution of $\mathcal{G}' = 0$. Table 2 compares the bounds for α and γ and the value of β for $\mathcal{G}$ and $\mathcal{G}'$ at B_1 and B_2 computed by `alphaCertified`. This table shows that the bounds computed for $\alpha(\mathcal{G}', B_i)$ and $\gamma(\mathcal{G}', B_i)$ are three orders of magnitude larger than the bounds computed for $\alpha(\mathcal{G}, B_i)$ and $\gamma(\mathcal{G}, B_i)$. In particular, due to the larger bounds, `alphaCertified` is unable to certify that B_1 and B_2 are approximate solutions of $\mathcal{G}' = 0$. If we replace B_i with $N_{\mathcal{G}'}(B_i)$, then `alphaCertified` is able to soft certify that the resulting points are approximate solutions of $\mathcal{G}' = 0$ using both 96-bit and 1024-bit precision.

We now consider solving a polynomial system obtained by replacing the sine and cosine functions with a fifth and fourth degree truncated Taylor series approximation, respectively, centered at the origin. Let the polynomial system $P : \mathbb{C}^{11} \to \mathbb{C}^5$ consists of the first five functions in $\mathcal{G}$. In particular, P consists of two linear and three quadratic polynomials and thus has total degree of the polynomial $\mathcal{Q}_\nu$ defined in (13) has total degree $2^3 = 8$.

Since we are using fifth and fourth degree polynomial approximations for the sine and cosine functions, respectively, we have $r_i = 5$ if i is odd and $r_i = 4$ if i is even. We picked random $a_i, b_{i,j} \in \mathbb{C}$ for $i = 1, \ldots, 6$ and $j = 1, \ldots, r_i$ and used `Bertini` to solve each $\mathcal{Q}_\nu = 0$. In total, this produced numerical approximations to 356 nonsingular isolated solutions of $\mathcal{P} = 0$ where $\mathcal{P}$ is defined in (14).

The tracking of the 356 paths from $\mathcal{P}$ to the polynomial approximation, $\mathcal{G}^p$, of $\mathcal{G}$ produced 120 points which became the start points for the homotopy deforming $\mathcal{G}^p$ to $\mathcal{G}$. This homotopy yielded 93 numerical approximations to solutions of $\mathcal{G} = 0$. By using both 96-bit and 1024-bit floating point arithmetic, `alphaCertified` soft certified that each of these 93 points are indeed approximate solutions with distinct associated solutions Moreover, this computation soft certified that 65 have real associated solutions, two of which are the two stable configurations computed in [13].

5 Conclusion

One key to certification using α-theory is the ability to compute a bound on γ, which is defined in terms of all higher order derivatives. For polynomial systems, where there are only finitely many nonzero derivatives, Shub and Smale developed the bound presented in Proposition 1.8. This bound is based on first order derivatives, coefficients and degrees of the polynomials, and the point of interest. Theorem 2.3 extends this bound to polynomial-exponential systems and is implemented in `alphaCertified`.

The computationally expensive part of computing the bound on γ is the linear algebra computations required to compute μ as defined in (8). Thus, the restriction on the size of the systems for which the bound could be computed arises from the linear algebra algorithms used. Even though large systems could be investigated, the fact that this produces an upper bound of γ means that β will need to be smaller in order to certify an approximate solution. Therefore, the use of this bound may induce additional computational cost via higher precision.

Since the certification approach presented for polynomial-exponential systems is based on the quadratic convergence of Newton's method and α-theory, we limit our focus to certifying nonsingular solutions to square systems. Even though Newton's method near singular solutions can have a variety of behavior, e.g., see [3], one can attempt the certification method at any point.

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A Using `alphaCertified`

As a demonstration of using `alphaCertified`, we consider the polynomial-exponential system $\mathcal{G}$ in (15) where $a_1 = 3$, $a_2 = 2$, and $E = (1, 3.5)$ along with the points Z_i in (16).

A.1 Input

We describe the three required files: input system, points, and configurations.

Input system

In order to describe the system $\mathcal{G}$, which is of the required form (10), we first list the total number of variables, 6, and the number of polynomials, 2. With this setup, `alphaCertified` assumes that the last four variables will be defined in terms of the first two variables, which are described after the polynomials. Since the system is assumed to be exact, the real and imaginary parts of all numbers listed in this file must be rational.

Each polynomial is represented as a sum of monomials. Thus, we list the total number of monomials in the polynomial (both have 3 terms) followed by a description of each monomial. A monomial is described by the entries of the exponent vector followed by the real and imaginary parts of the coefficient.

The relations for the last four polynomials are described by the variable number for which the analytic function depends upon, a string indicating which analytic function ("X" for exp(), "S" for sin(), "C" for cos(), "SH" for sinh(), and "CH" for cosh()), and the real and imaginary parts of the corresponding constant.

Figure 2 lists the contents of this file, which we name `inputSystem`, along with comments.

Points

Since the number of variables was described in the input system, we only need to list the number of points, 2, followed by floating-point representation of the real and imaginary parts of the coordinates for each of the points Z_1 and Z_2.

Figure 3 lists the contents of this file, which we name `points`.

Configurations

The last file indicates the settings and algorithms for `alphaCertified` to run. For polynomial-exponential systems, we need to utilize floating-point arithmetic and can set the precision.

Figure 4 lists the contents of this file, which we name `config`, which instructs `alphaCertified` to use 1024-bit floating-point arithmetic while executing the default certification procedures in `alphaCertified`. We refer the reader to [6] for more details on settings and algorithms.

`6 2`	number of variables and number of polynomials
`3`	number of terms in first polynomial: $3y_3 + 2y_4 - 1$
`0 0 0 0 1 0 3 0`	$3y_3$
`0 0 0 0 0 1 2 0`	$2y_4$
`0 0 0 0 0 0 -1 0`	-1
`3`	number of terms in second polynomial: $3y_1 + 2y_2 - 7/2$
`0 0 1 0 0 0 3 0`	$3y_1$
`0 0 0 1 0 0 2 0`	$2y_2$
`0 0 0 0 0 0 -7/2 0`	$-7/2$
`1 S 1 0`	$y_1 - \sin(\theta_1)$
`2 S 1 0`	$y_2 - \sin(\theta_2)$
`1 C 1 0`	$y_3 - \cos(\theta_1)$
`2 C 1 0`	$y_4 - \cos(\theta_2)$

Figure 2: `inputSystem` and a line-by-line description of the file

```
2

0.711 0
2.261 0
0.65 0
0.77 0
0.76 0
-0.64 0

1.874 0
0.324 0
0.95 0
0.32 0
-0.30 0
0.95 0
```

Figure 3: `points`

```
ARITHMETICTYPE: 1;
PRECISION: 1024;
```

Figure 4: `config`

A.2 Execution

For simplicity, we follow Linux syntax and assume that a binary file for `alphaCertified` along with the three files constructed above are in the same folder. With this setup, we execute

$\gg$ `./alphaCertified inputSystem points config`

A.3 Output

The output of `alphaCertified` is contained in a summary of the results printed to the screen, contained in Figure 5, and several files. Figure 6 contains the portions of the human readable file `summary` created during the execution of `alphaCertified`. These values are also printed in the machine readable file `constantValues`.

```
  alphaCertified v1.3.0 (October 16, 2013)
  Jonathan D. Hauenstein and Frank Sottile
       GMP v4.3.2 & MPFR v3.1.2

Please note that all coefficients must be complex rational numbers.

alphaCertified is using the polynomial-exponential certification algorithms.

Analyzing 2 points using 1024-bit floating point arithmetic.

Isolating 2 approximate solutions.

Classifying 2 distinct approximate solutions.

Floating point (1024 bits) soft certification results:

Number of points tested:             2
Certified approximate solutions:     2
Certified distinct solutions:        2
Certified real distinct solutions:   2
```

Figure 5: Summary printed to the screen

```
alpha < 1.265465288439055e-1    alpha < 1.355028294876322e-1
beta ~= 4.938677034638513e-3    beta ~= 5.257805074083256e-3
gamma < 2.562356840836994e1     gamma < 2.577174839659842e1
```

Figure 6: Portions of `summary` corresponding to Z_1 and Z_2, respectively