

A symbolic test for (i,j)-Uniformity in reduced zero-schemes

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Abstract

Let Z denote a finite collection of points in projective n -space and let \mathbf{I} denote the homogeneous ideal of Z . The points in Z are said to be in (i, j) -uniform position if every cardinality i subset of Z imposes the same number of conditions on forms of degree j . The points are in uniform position if they are in (i, j) -uniform position for all values of i and j . We present a symbolic algorithm that, given \mathbf{I} , can be used to determine if the points in Z are in (i, j) -uniform position. In addition it can be used to determine if the points in Z are in uniform position, in linearly general position and in general position. The algorithm uses the Chow Form of various d -Uple embeddings of Z and derivatives of these forms. The existence of the algorithm provides an answer to a question of Kreuzer.

Key words: Uniform Position, Chow Variety, Chow Form, General Position, Zero-scheme, Points

1 Introduction

Let k be a field, either of characteristic zero or of sufficiently large characteristic (see below). Let $\mathbf{R} = k[x_0, x_1, \dots, x_n]$ and let $\mathbf{P}^n = \text{Proj}(\mathbf{R})$. The

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set of all degree 1 monomials in \mathbf{R} forms a basis for the k -vector space of all forms of degree 1 in \mathbf{R} . In terms of this basis, to each linear form (up to scalar multiples), $L = a_0x_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n \in \mathbf{R}$, we can associate the point $[a_0 : a_1 : \cdots : a_n] \in \mathbf{P}^n$. In addition, we can associate to L the hyperplane in \mathbf{P}^n determined by $L = 0$. This gives the well-known bijection between hyperplanes in \mathbf{P}^n and points in \mathbf{P}^{n*} (similarly, between points in \mathbf{P}^n and hyperplanes in \mathbf{P}^{n*}). In a similar way, letting $N_d = \binom{n+d}{d} - 1$, we have the well-known bijection between degree d hypersurfaces in \mathbf{P}^n and points in \mathbf{P}^{N_d} .

In this paper we are concerned with the question of what can be said about the uniformity of a set of points in projective space. The notion of uniformity appears in a variety of settings. For instance, on the theoretical side, it is central to the study of curves in projective space, via hyperplane sections and Castelnuovo theory (cf. for instance [3]). On the applied side, it has been shown to be important in coding theory ([2]).

Given a homogeneous ideal, it is possible to determine if the ideal corresponds to a zero-dimensional object. In this case, by taking the radical, it is possible to form the largest ideal that defines this zero-dimensional object set-theoretically. However, due to Galois theory it is in general not possible to determine the precise coordinates of the points (although there exist numerical methods to approximate them). Without this information it would seem at first glance to be difficult to determine what kind of uniformity properties the set of points enjoys. We were thus a bit surprised at the level of detail that can be extracted by symbolic methods and by the simplicity of the approach.

Let P be a point in \mathbf{P}^n . For each fixed value of d , the collection of all degree d hypersurfaces that contain P determines a hyperplane, $H_{P,d} \subset \mathbf{P}^{N_d}$ (the degree d Chow Form of P). Thus, to a collection of r distinct points $Z = \bigcup_{i=1}^r P_i \subset \mathbf{P}^n$, we can associate an arrangement of r distinct hyperplanes $\Gamma_t = \bigcup_{i=1}^r H_{i,d} \subset \mathbf{P}^{N_d}$ (one for each point in Z). With this correspondence, questions about finite collections of points in \mathbf{P}^n can be rephrased as questions about hyperplane arrangements in \mathbf{P}^{N_d} for various choices of d . In this paper we will utilize this correspondence to obtain a symbolic algorithm for determining when a collection of points satisfies various types of uniformity conditions. Because of the use of derivatives in some of the algorithms, we make the further assumption that $\text{char}(k) > r$.

2 Ingredients

In what follows, Z will denote a reduced set of points in \mathbf{P}^n , \mathbf{I}_Z will denote the homogeneous ideal of Z and \mathcal{I}_Z will denote the associated ideal sheaf of

Z . There exists a symbolic algorithm for finding the radical of an ideal [1]. In light of this, if $\mathbf{I} \subset k[x_0, x_1, \dots, x_n]$ is an ideal which defines Z set theoretically, we can determine the unique radical ideal $\mathbf{I}_Z \subset k[x_0, x_1, \dots, x_n]$ which defines the reduced zero-scheme Z .

2.1 Uniform Position and General Position

If \mathcal{E} is a sheaf on \mathbf{P}^n then $H^0(\mathbf{P}^n, \mathcal{E})$ will denote the vector space of all global sections of \mathcal{E} and $h^0(\mathbf{P}^n, \mathcal{E})$ will denote the dimension of $H^0(\mathbf{P}^n, \mathcal{E})$. Hence, $H^0(\mathbf{P}^n, \mathcal{I}_Z(t))$ represents the vector space of all forms of degree t whose associated hypersurfaces contain the scheme Z , and $h^0(\mathbf{P}^n, \mathcal{I}_Z(t))$ represents the dimension of this vector space. This information is equivalent to knowing the *Hilbert function* of Z ,

$$h_Z(t) = \dim(\mathbf{R}/\mathbf{I}_Z)_t = \binom{n+t}{t} - h^0(\mathbf{P}^n, \mathcal{I}_Z(t)).$$

Definition 1 Let Z denote a reduced collection of points in \mathbf{P}^n and let $|Z|$ denote the number of points in Z . Then Z is said to be in **(\mathbf{i}, \mathbf{j}) -uniform position** if for every pair of subsets $X, Y \subseteq Z$ with $|X| = |Y| = i$ it holds that $h^0(\mathbf{P}^n, \mathcal{I}_X(j)) = h^0(\mathbf{P}^n, \mathcal{I}_Y(j))$. Furthermore, Z is said to be in **uniform position** if it is (i, j) -uniform for all integers i and j .

Remark 2 In [5], [6] Kreuzer introduced the concept of (i, j) -uniformity for a set of points. In his definition, Z is (i, j) -uniform if whenever X and Y are two sets of points obtained from Z by removing i points then the Hilbert functions of the corresponding ideals agree in the j^{th} spot. Our definition states that Z is (i, j) -uniform if whenever X and Y are two sets of points obtained by selecting i points from Z then the Hilbert functions of the corresponding ideals agree in the j^{th} spot. The difference is superficial in that knowing (i, j) -uniformity for all values of i and j provides exactly the same information regardless of which definition is used. The definition we have chosen provides a notational advantage in the course of the paper.

Definition 3 Let Z denote a finite collection of points in \mathbf{P}^n and let $|Z|$ denote the number of points in Z . Then Z is said to be in **\mathbf{d} -general position** if for every subset $X \subseteq Z$, it holds that $h^0(\mathbf{P}^n, \mathcal{I}_X(d)) = \max\{0, \binom{n+d}{d} - |X|\}$. Z is said to be in **general position** if it is in \mathbf{d} -general position for all $\mathbf{d} > 0$. Z will be said to be in **linearly general position** if it is in 1-general position.

Remark 4 Let Z be a set of points. If Z has regularity r and if $\mathbf{d} \geq r - 1$ then Z is in \mathbf{d} -general position. As a consequence, to determine if a set of points is in general position, it is only necessary to determine if it is in \mathbf{d} -general position for $0 < \mathbf{d} < r - 1$. There is a symbolic algorithm for determining the

regularity of a scheme. Thus, if we have a finite step algorithm to determine if a scheme is in d -general position for any given d , then we have a finite step algorithm to determine if it is in general position. Similarly, if we have a finite step algorithm to determine if a scheme is in (i, j) -uniform position then we have a finite step algorithm to determine if a scheme is in uniform position.

2.2 Chow Forms

Let Z be an equidimensional variety in \mathbf{P}^n of codimension d . A general linear space of dimension $d-1$ will not intersect Z . Consider the collection of all linear spaces of dimension $d-1$ which do intersect Z . These form a codimension 1 subvariety of $GR(n, d-1)$ (the Grassmannian of $(d-1)$ -dimensional linear spaces in \mathbf{P}^n). This codimension 1 subvariety is called the Chow Variety of Z . If Z decomposes into s irreducible components then so will the Chow Variety of Z .

Points in \mathbf{P}^n have codimension n . To compute the Chow Variety of a set of points in \mathbf{P}^n we will use linear spaces of dimension $n-1$ (i.e. hyperplanes). The Grassmann variety of hyperplanes in \mathbf{P}^n is a dual projective space \mathbf{P}^{n*} . The correspondence is simply

$$A_0x_0 + A_1x_1 + \cdots + A_nx_n \Leftrightarrow [A_0 : A_1 : \cdots : A_n].$$

The Chow Variety of a set, Z , of s distinct points in \mathbf{P}^n will be a hypersurface of degree s in \mathbf{P}^{n*} . This hypersurface will be the union of s distinct hyperplanes (the hyperplanes corresponding to the points of Z in the bijection described in the introduction) and will be defined by a single equation. This equation is the Chow Form of the set of points. The points can be recovered from the Chow Form. Thus, in some sense, the Chow Form contains all of the information of the original set of distinct points.

Proposition 5 *The Chow Form of a set of points, Z , can be computed symbolically from the ideal \mathbf{I}_Z , without knowing the coordinates of the points.*

Proof. We first introduce a polynomial, $H = A_0x_0 + A_1x_1 + \cdots + A_nx_n$, representing a general hyperplane. We would like to determine algebraic conditions on the coefficients, A_0, \dots, A_n , that force H to intersect Z . To do this, we need to form the ideal $\mathbf{J} = \mathbf{I}_Z + H \subseteq k[x_0, \dots, x_n, A_0, \dots, A_n]$ and determine conditions on A_0, \dots, A_n such that the resulting ideal (viewed as a subset of $k[x_0, \dots, x_n]$) defines a non-empty scheme.

Let us denote by \mathbf{P}_X^n the projective space with coordinate ring $\mathbf{R} = k[x_0, \dots, x_n]$, by \mathbf{P}_A^n the projective space with coordinate ring $\mathbf{S} := k[A_0, \dots, A_n]$, and by

\mathbf{P}^{2n+1} the projective space with coordinate ring $\mathbf{T} := k[x_0, \dots, x_n, A_0, \dots, A_n]$. Let $Q \subset \mathbf{P}^{2n+1}$ be the quadric hypersurface defined by H . Note that in \mathbf{P}^{2n+1} , \mathbf{P}_X^n is defined by $A_0 = \dots = A_n = 0$ and \mathbf{P}_A^n is defined by $x_0 = \dots = x_n = 0$. Also,

$$Z \subset \mathbf{P}_X^n \subset Q \subset \mathbf{P}^{2n+1} \quad \text{and} \quad \mathbf{P}_A^n \subset Q \subset \mathbf{P}^{2n+1}.$$

The generators of $\mathbf{I}_Z \subset \mathbf{R}$ can also be viewed in \mathbf{T} . Let \bar{Z} denote the subscheme of \mathbf{P}^{2n+1} defined by $\mathbf{T} \cdot \mathbf{I}_Z \subset \mathbf{T}$. \bar{Z} is determined geometrically as follows: for each point P of Z , let Λ_P be the linear space spanned by P and \mathbf{P}_A^n . Then $\bar{Z} = \bigcup_{P \in Z} \Lambda_P$. Note that $\dim \bar{Z} = n + 1$.

We first claim that the hyperquadric Q does not contain any component Λ_P of \bar{Z} . Indeed, if it did, this would say that every hyperplane H of \mathbf{P}_X^n vanishes at the point P (choose arbitrary values for the A_i), which is clearly nonsense. So \mathbf{J} corresponds to a proper hypersurface section of \bar{Z} . \bar{Z} is arithmetically Cohen-Macaulay since Z is arithmetically Cohen-Macaulay. As a result, \mathbf{J} is unmixed.

Fix again a component Λ_P of \bar{Z} . This contains \mathbf{P}_A^n , which is also contained in Q . So on this component, Q cuts out the union of \mathbf{P}_A^n and some linear variety λ_P of dimension n . We have that λ_P meets \mathbf{P}_X^n at P and meets \mathbf{P}_A^n in an $(n - 1)$ -dimensional linear space V_P (since \mathbf{P}_A^n and λ_P are both hyperplanes in the $(n + 1)$ -dimensional linear space Λ_P). Note that V_P is precisely the hyperplane in \mathbf{P}_A^n dual to P . As such, $V_P \neq V_{P'}$ if and only if $P \neq P'$. V_P is defined by a linear form L_P in $k[A_0, \dots, A_n]$. The product of the L_P is the Chow Form of Z .

From this discussion it follows that the ideal $\mathbf{J} = \mathbf{I}_Z + (H)$ defines a subscheme of \mathbf{P}^{2n+1} that consists of the union of the λ_P and some subscheme supported on \mathbf{P}_A^n . Removing this latter component and restricting the result to \mathbf{P}_A^n gives the Chow form. \square

This suggests two algorithms for determining the Chow form of Z . In the first algorithm, we isolate the union of the λ_P , and then we intersect with the plane defined by (x_0, \dots, x_n) to determine the Chow Form. Note again that $I_{\mathbf{P}_A^n} = (x_0, \dots, x_n)$.

Algorithm 1 for computing the Chow Form of a reduced zero-scheme:

- Start with an ideal \mathbf{I} which defines Z set theoretically.
- Compute the radical of \mathbf{I} to obtain the ideal \mathbf{I}_Z .
- Form the ideal $\mathbf{J} := \mathbf{I}_Z + (H)$ where $H = A_0x_0 + A_1x_1 + \dots + A_nx_n$.

- For each i , saturate \mathbf{J} with respect to the ideal (x_i) to obtain a new ideal \mathbf{L}_i . In other words, $\mathbf{L}_i = \mathbf{J} : (x_i)^\infty$.
- Form the ideal $\mathbf{C}_i = \mathbf{L}_i + (x_0, \dots, x_n)$ and view the result in $k[A_0, \dots, A_n]$. This yields a principle ideal $\mathbf{C}'_i = (F_i)$.
- Let $(F) = (F_0) \cap (F_1) \cap \dots \cap (F_n)$.
- F is the Chow form of Z .

Remark 6 *If no point of Z lies in the hyperplane in \mathbf{P}_X^n defined by x_i then we do not need to iterate the above algorithm. We are finished after that particular x_i .*

In the second algorithm we project the entire scheme defined by \mathbf{J} to the projective space $\text{Proj}(k[x_i, A_0, \dots, A_n])$. Note that this projective space is the $(n+1)$ -dimensional linear space spanned by \mathbf{P}_A^n and the point in \mathbf{P}_X^n defined by $x_0 = \dots = \hat{x}_i = \dots = x_n = 0$.

We now show how to recover the Chow form from this projection, and we will give the precise algorithm shortly. The discussion in the proof of Proposition 5 shows that the projection gives an ideal defining the union of two n -dimensional (hence hypersurface) subschemes of $\text{Proj}(k[x_i, A_0, \dots, A_n])$. One is supported on \mathbf{P}_A^n , hence is defined by x_i^ℓ for some ℓ . The second subscheme is a union of n -dimensional linear spaces, coming as the image under the projection of those λ_P for which P does not lie in the hyperplane of \mathbf{P}_X^n defined by $x_i = 0$. But this projection leaves fixed the intersection of λ_P and \mathbf{P}_A^n . Therefore the second scheme is defined by a form F_i only in the variables A_0, \dots, A_n , which is the Chow form of those points of Z that do not lie in the hyperplane of \mathbf{P}_X^n defined by $x_i = 0$. Since there are no points of Z that lie on $x_i = 0$ for every $0 \leq i \leq n$, we can determine the Chow form of Z by finding the least common multiple of the Chow forms that show up in each projection.

Algorithm 2 for computing the Chow Form of a reduced zero-scheme:

- Start with an ideal \mathbf{I} which defines Z set theoretically.
- Compute the radical of \mathbf{I} to obtain the ideal \mathbf{I}_Z .
- Form the ideal $\mathbf{J} := \mathbf{I}_Z + (H)$ where $H = A_0x_0 + A_1x_1 + \dots + A_nx_n$.
- For each i , compute $\mathbf{P}_i := \mathbf{J} \cap k[x_i, A_0, \dots, A_n]$.
- $\mathbf{P}_i = (x_i^{l_i} F_i)$ where $F_i \in k[A_0, \dots, A_n]$. Determine F_i .
- Let $(F) = (F_0) \cap (F_1) \cap \dots \cap (F_n)$.
- F is the Chow form of Z .

Remark 7 *If there exists an i such that no points of Z lie on the hyperplane*

$x_i = 0$ then $F = F_i$. This simple observation can save a substantial amount of time in the two algorithms.

Example 8 Let $\mathbf{I} = (x^2 + y^2 + z^2, x^3 + y^3 + z^3) \subseteq \mathbf{C}[x, y, z]$. \mathbf{I} is a complete intersection and defines 6 reduced points in $\mathbf{P}_{\mathbf{C}}^2$. Form the ideal $\mathbf{J} = (x^2 + y^2 + z^2, x^3 + y^3 + z^3, Ax + By + Cz) \subseteq \mathbf{C}[x, y, z, A, B, C]$. A quick check shows that none of the six points lie on the hyperplane $z = 0$. If we were to use the second algorithm, we would compute $\mathbf{P}_z = \mathbf{J} \cap \mathbf{C}[z, A, B, C]$ (and determine that $\mathbf{P}_z = (z^4 F)$). Alternatively, we can compute $\mathbf{C}' = (F)$ by the first algorithm. In either case we find

$$\begin{aligned} F &= A^6 + B^6 + C^6 \\ &+ \frac{3}{2}(A^4 B^2 + A^2 C^4 + B^4 C^2 + A^2 B^4 + A^4 C^2 + B^2 C^4) \\ &+ 3(A^4 BC + AB^4 C + ABC^4) \\ &+ 3(A^3 B^2 C + AB^3 C^2 + A^2 BC^3 + A^3 BC^2 + A^2 B^3 C + AB^2 C^3) \\ &- (A^3 B^3 + A^3 C^3 + B^3 C^3). \end{aligned}$$

F is the Chow Form of the zero-scheme defined by \mathbf{I} . F factors as a product of 6 linear forms in $\mathbf{C}[A, B, C]$ (corresponding to the Chow Forms of the six individual points). The 6 linear forms determine 6 hyperplanes (lines) in \mathbf{P}^{2*} . These hyperplanes are the Chow Varieties of the six individual points.

2.3 Degree d Chow Forms and d -Uple Embeddings

The Chow form describes precisely which hyperplanes contain at least one of the points of Z . In the sequel we will be very interested in the analogous problem for hypersurfaces of degree d . The algorithm to achieve this is strongly related to the one in the previous section.

Let $\mathbf{R} = k[x_0, \dots, x_n]$ and let $\mathbf{P}^n = \text{Proj}(\mathbf{R})$. For a given $d > 0$, let v_0, \dots, v_{N_d} be a monomial basis for the vector space of forms of degree d in \mathbf{R} (where $N_d = \binom{n+d}{d} - 1$). The d -Uple embedding of \mathbf{P}^n into \mathbf{P}^{N_d} is the image of the map which sends $\mathbf{a} = [a_0 : a_1 : \dots : a_n]$ to $[v_0(\mathbf{a}) : v_1(\mathbf{a}) : \dots : v_{N_d}(\mathbf{a})]$ (i.e. evaluate each of the monomials at \mathbf{a}).

Let Z denote a scheme in \mathbf{P}^n . Let \mathbf{I}_Z denote its saturated ideal. To determine the ideal of the image of the d -Uple embedding of Z , we can use the following well-known algorithm.

Algorithm for computing the d -Uple embedding of a scheme:

- Start with the ideal $\mathbf{I}_Z \subseteq k[x_0, x_1, \dots, x_n]$.
- Let v_0, v_1, \dots, v_{N_d} be a monomial basis for the forms of degree d .
- Let $\mathbf{S} = k[x_0, x_1, \dots, x_n, X_0, X_1, \dots, X_{N_d}]$ with the degrees of x_0, x_1, \dots, x_n set equal to 1 and the degrees of X_0, X_1, \dots, X_{N_d} set equal to d . Let $\mathbf{J} = (X_0 - v_0, X_1 - v_1, \dots, X_{N_d} - v_{N_d}) + \mathbf{I}_Z \subseteq \mathbf{S}$.
- Let $\mathbf{L} = \mathbf{J} \cap k[X_0, X_1, \dots, X_{N_d}]$.
- \mathbf{L} is the ideal of the d -Uple embedding of Z .

With this algorithm in hand, we now define:

Definition 9 *The degree d Chow Form of a reduced zeroscheme, Z , is the Chow Form of the d -Uple embedding of Z .*

The d -uple embedding embeds \mathbf{P}^n into \mathbf{P}^{N_d} as a non-degenerate, smooth projectively normal subvariety X of dimension n . A hyperplane section of X corresponds to a hypersurface in \mathbf{P}^n of degree d . Let P be a point of \mathbf{P}^n and let $P' \in \mathbf{P}^{N_d}$ be the image of the d -uple embedding of P . Then $h^0(\mathbf{P}^n, \mathcal{I}_P(d)) = h^0(\mathbf{P}^{N_d}, \mathcal{I}_{P'}(1)) = N_d$. Thus P corresponds to a hyperplane in $(\mathbf{P}^{N_d})^*$. This hyperplane in $(\mathbf{P}^{N_d})^*$ parameterizes the degree d hypersurfaces of \mathbf{P}^n that contain P . For any reduced zeroscheme Z in \mathbf{P}^n , then, the degree d Chow form of Z parameterizes the degree d hypersurfaces of \mathbf{P}^n that contain at least one point of Z .

Thus, to compute the degree d Chow Form of a zeroscheme, we need only compute a d -uple embedding of the zeroscheme and then apply the algorithm for computing the Chow Form. Alternatively, we can skip the computation of the d -Uple embedding if we modify the algorithm for computing the Chow Form. All that has to be done is to replace $H = A_0x_0 + A_1x_1 + \dots + A_nx_n$ with $H = A_0v_0 + A_1v_1 + \dots + A_{N_d}v_{N_d}$.

By computing the degree d Chow Forms of a set of points, we have transformed questions about the set of points into questions about hyperplane arrangements in \mathbf{P}^{N_d} for various d . We now need to understand how the geometry of these hyperplane arrangements relates to the geometry of the original set of points and we need to understand how to extract this information from the hyperplane arrangement.

2.4 Geometry of the Hyperplane Arrangements

Let $\mathbf{R} = k[x_0, \dots, x_n]$. Let $Z = \bigcup_{i=1}^r P_i \subset \mathbf{P}^n$ be a reduced set of r points with saturated ideal \mathbf{I}_Z . Let $G_d = \prod_{i=1}^r L_{i,d}$ be the degree d Chow form of Z . Let $\Gamma_d = \bigcup_{i=1}^r H_{i,d}$ be the union of hyperplanes in $(\mathbf{P}^{N_d})^*$ defined by G_d . $H_{i,d}$ is the projective space parameterizing the linear system of hypersurfaces of degree d passing through P_i .

Consider two hyperplanes $H_{i,d}$ and $H_{j,d}$ corresponding to two points P_i and P_j . The points in the intersection of $H_{i,d}$ and $H_{j,d}$ correspond to the degree d hypersurfaces in \mathbf{P}^n which contain both P_i and P_j . Considering all pairs of hyperplanes in Γ_d , we see that the singular locus of Γ_d corresponds to the collection of all degree d hypersurfaces in \mathbf{P}^n which contain 2 or more points of Z .

The singular locus of Γ_d is determined set theoretically by the ideal of all first partial derivatives of G_d . More generally, we now show that the locus of all degree d hypersurfaces which contain t or more points of Z is determined set theoretically by the ideal of all $t - 1$ partial derivatives of G_d .

Proposition 10 *Let $F = L_1 L_2 \cdots L_r$ be a product of linear forms over a field of characteristic $> r$ or characteristic zero. Let $P \in \mathbf{P}^n$ be a point. Let L_{i_1}, \dots, L_{i_k} be the factors of F that vanish at P . Then*

- (a) *The i^{th} derivatives of F vanish at P , for $1 \leq i \leq k - 1$;*
- (b) *There is at least one k^{th} derivative of F that does not vanish at P .*

Proof. We have $F = L_1 L_2 \cdots L_r$, where each of the L_i is a linear form. Note that any second or higher derivative of an L_i is 0. Hence any j^{th} derivative $\frac{\partial^j F}{\partial x_0^{j_0} \cdots \partial x_n^{j_n}}$ is a sum of terms of the form $M_1 \cdots M_r$ where each M_i is either equal to the corresponding L_i or else is a suitable first derivative of the corresponding L_i . There are exactly j such first derivatives among the M_i in each term of $\frac{\partial^j F}{\partial x_0^{j_0} \cdots \partial x_n^{j_n}}$. Then (a) follows since for $j \leq k - 1$ each term must have at least one of L_{i_1}, \dots, L_{i_k} undifferentiated. Similarly, for (b) we can choose a k^{th} derivative that differentiates each of L_{i_1}, \dots, L_{i_k} yielding a non-zero constant. \square

Corollary 11 *The locus of all degree d hypersurfaces that contain t or more points of Z is determined set-theoretically by the ideal of all $(t - 1)^{\text{st}}$ partial derivatives of the degree d Chow form.*

Proof. The desired locus is precisely the locus of points where t or more of the hyperplanes $H_{i,d}$ meet. By Proposition 10 this locus is determined by the $(t - 1)^{\text{st}}$ derivatives of G_d . \square

With this preparation we now turn to the description of the geometry of Γ_d and how it relates to geometric properties of Z . We will talk about (s, d) -uniform position rather than (i, j) -uniform position simply for convenience of exposition.

Lemma 12 a. Γ_d is reduced.

b. If $m \leq d + 1$ then any m components meet in a linear space of codimension m .

Proof. The first statement is clear. The second follows from the fact that when $m \leq d + 1$, m points impose independent conditions on forms of degree d . \square

Proposition 13 Let $V_d = \bigcap_{i=1}^r H_{i,d}$. Then

- a. The value of the Hilbert function in degree d is $h_{R/I_Z}(d) = \text{codim } V_d$.
- b. The points of Z impose independent conditions on forms of degree d if and only if $\text{codim } V_d = r$.
- c. The points of Z are in (s, d) -uniform position if and only if for every choice $\{P_{i_1}, \dots, P_{i_s}\}$ of s points of Z , either

$$\text{codim} \bigcap_{j=1}^s H_{i_j,d} = s \quad \text{or} \quad \text{codim} \bigcap_{j=1}^s H_{i_j,d} = \text{codim } V_d.$$

Proof. The number of independent forms of degree d vanishing on $Z \subset \mathbf{P}^n$ is equal to the number of independent linear forms vanishing on the d -uple embedding of Z . This number is $1 + \dim V_d$. Parts a. and b. follow immediately from this observation (see also the following example).

For c., recall that (s, d) -uniform position means that for every subset Y of Z of cardinality $s \leq r$, we must have

$$h_{R/I_Y}(d) = \min\{|Y|, h_{R/I_Z}(d)\}.$$

In other words, for each Y with $|Y| = s$, either Y imposes independent conditions on forms of degree d or else $(I_Y)_d = (I_Z)_d$. (Note that it can't happen that some Y impose independent conditions and others don't but satisfy $(I_Y)_d = (I_Z)_d$, so there is no ambiguity in the statement of c.) This means that every choice of s of the hyperplanes $H_{i,d}$ satisfies one of the two conditions given in c. \square

Example 14 Let $Z \subset \mathbf{P}^2$ consist of 20 points on a conic. We consider the 2-uple embedding of Z in $\mathbf{P}^{6-1} = \mathbf{P}^5$. This consists of 20 points lying in a single hyperplane. Then V_2 is a single point in $(\mathbf{P}^5)^*$ (dual to this hyperplane), and $h_{R/I_Z}(2) = 5$.

In order to implement these ideas, we now give an interpretation in terms of derivatives. Let D_d^i be the ideal defined by the i -th partial derivatives of G_d . Note that

$$G_d = D_d^0 \subset D_d^1 \subset \cdots \subset D_d^{r-1} \subset D_d^r = (1)$$

(by Euler's theorem).

Thanks to Corollary 11 and Proposition 13 we can make the following observations, which lead immediately to algorithms for (s, d) -uniform position and its special cases (e.g. linear general position, uniform position, etc.)

- a. D_d^{r-1} defines the linear variety V_d , so the Hilbert function $h_{R/I_Z}(d)$ is just the codimension of the variety defined by D_d^{r-1} .
- b. The points of Z impose independent conditions on forms of degree d if and only if the linear variety defined by D_d^{r-1} has codimension r .
- c. $Z \subseteq \mathbf{P}^n$ is in linearly general position if and only if one of the following is true:
 - (i) $r \geq n + 1$ and the variety defined by D_1^r has codimension $n + 1$;
 - (ii) $r < n + 1$ and the variety defined by D_1^{r-1} has codimension r .
- d. $Z \subseteq \mathbf{P}^n$ is in d -general position if and only if one of the following is true:
 - (i) $r \geq \binom{n+d}{d}$ and the variety defined by $D_d^{N_d}$ has codimension $\binom{n+d}{d}$;
 - (ii) $r < \binom{n+d}{d}$ and the variety defined by D_d^{r-1} has codimension r .
- e. Z is in general position if and only if Z is in d -general position for all $1 \leq d \leq t$ where t is the regularity of Z .
- f. Z is in (s, d) -uniform position if and only if one of the following is true:
 - (i) The variety defined by D_d^{s-1} has codimension s ;
 - (ii) The variety defined by D_d^{s-1} has the same codimension as the variety defined by D_d^{r-1} .
- g. Z is in uniform position if and only if for every $s \leq r$ and for every d with $1 \leq d \leq t$ where t is the regularity of Z , one of the following is true:
 - (i) The variety defined by D_d^{s-1} has codimension s ;
 - (ii) The variety defined by D_d^{s-1} has the same codimension as the variety defined by D_d^{r-1} .

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