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The average rate of change for continuous time models

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The average rate of change (ARC) is a concept that has been misunderstood in the applied longitudinal data analysis literature, where the slope from the straight-line change model is often thought of as though it were the ARC. The present article clarifies the concept of ARC and shows unequivocally the mathematical definition and meaning of ARC when measurement is continuous across time. It is shown that the slope from the straight-line change model generally is not equal to the ARC. General equations are presented for two measures of discrepancy when the slope from the straight-line change model is used to estimate the ARC in the case of continuous time for any model linear in its parameters, and for three useful models nonlinear in their parameters.

The analysis of change is important in many fields for assessing the effects of the passage of time on some dependent variable. Time-varying and time-invariant covariates can be incorporated into the analysis in an effort to understand and model interindividual differences in change. Many times, analysis of change procedures are important with or without experimental manipulation. Modern conceptualizations of the analysis of change regard intraindividual change to be the starting point for longitudinal data analysis (e.g., Collins, 1996; Mehta & West, 2000; Raudenbush, 2001; Rogosa, Brandt, & Zimowski, 1982; Rogosa & Willett, 1985). Thus, before aggregating over individuals in a multilevel model framework, a prerequisite for modeling change parameters as dependent variables is that the change parameters be themselves meaningful.[1] The present article focuses on a single individual trajectory, since specifying the individual-level model is a necessary but not sufficient condition of a meaningful model for a collection of individuals for a phenomenon that is repeatedly measured.

Some research questions demand a reasonably large time span between measurement occasions and can reasonably expect to obtain only a relatively small number of repeated observations. For example, researchers studying academic achievement over a school year cannot expect to obtain a large number of measurements based on comprehensive examinations. This is due in part to the logistics of collecting comprehensive measurements, as well as the relatively slow change in achievement. Researchers studying topics such as marital satisfaction, depression, employee satisfaction, employee motivation, and so forth generally fall into similar situations. However, other research questions can be addressed with instruments that measure the variable of interest continuously, or nearly so, or at least with a relatively large number of measurement occasions. For example, heart rate, electrical activity of the heart, blood flow to various regions of the brain, eye-gaze position and amount of movement, body movement, and respiration can be measured literally or essentially continuously. Because behavioral and biological systems are inextricably linked, more and more research is cutting across traditional behavioral/psychological and medical/physiological research topics, and a growing list of journal titles suggests that scientific progress can be and is being made by bridging various aspects of behavior/psychology and medicine/physiology. As formerly disparate fields continue to blend, ways of collecting data will continue to evolve, some of which will consist of measurements that are taken continuously or nearly so. As such, new opportunities will emerge for studying the nature of behavior and biological systems, as well as, and perhaps most importantly, the interaction of the two.

Kelley and Maxwell (2008) discussed the average rate of change (ARC) generally and derived measures of discrepancy between the ARC and the regression coefficient from the straight-line change model for a discrete number of time points. The ARC describes the average or typical rate of change over some time interval of interest for a particular trajectory and is thus a parsimonious measure that can potentially describe a complicated process, regardless of the functional form of change. Although the concept of the ARC in a longitudinal context is appealing and seems to be straightforward, the technical underpinnings have not received much formal attention (cf. Kelley & Maxwell, 2008; Seigel, 1975). The regression coefficient from the straight-line change model has often been the way in which such a succinct description of change over time has been attempted. Although using a single

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value as a descriptor of a potentially complicated process of change has an intuitive appeal, the present work will demonstrate that the regression coefficient from the straight-line change model is generally not equal to the ARC for a given trajectory. Aggregating across individuals in a multilevel modeling context when the focus of interest is the overall ARC is thus generally problematic and will tend to lead to biased estimates. The purpose of the present article is to extend the work of Kelley and Maxwell to the case of continuous time. In so doing, the limiting case of continuous time models can be developed and examined, so that the discrepancy between the slope from the straight-line change model and the ARC can be better understood.

Mathematical Form of the ARC

Kelley and Maxwell (2008) detailed the mathematical underpinnings of the ARC, which we summarize here. The rate of change of a nonvertical straight line that passes through two points, (a_1, Y_1) and (a_T, Y_T), is the slope of the line, where a_t represents some basis of time (e.g., a monotonic rescaling) and Y_t is a continuous function of time, $Y_t = f(a_t)$, at the tth measurement occasion ($t = 1, T$). The slope of the line connecting two points is the change in Y_t divided by the change in time:

$$\text{Slope} = \frac{f(a_T) - f(a_1)}{a_T - a_1} = \frac{Y_T - Y_1}{a_T - a_1} = \frac{\Delta Y}{\Delta a}, \tag{1}$$

where $f(a_t)$ is the dependent variable Y_t, ΔY is the change in the dependent variable, and Δa is the change in time.

In the limit as Δa approaches zero, Equation 1 yields the instantaneous rate of change when evaluated at a specific time value:

$$\frac{dY_t}{da} = \lim_{\Delta a \to 0} \frac{f(a_t + \Delta a) - f(a_t)}{\Delta a} = f'(a), \tag{2}$$

where dY_t/da is the derivative of Y_t with respect to a, which will be represented as $f'(a)$.

The mean value for a continuous function that is differentiable over the interval a_1 to a_T is given as

$$\bar{f}_c = \frac{1}{a_T - a_1} \int_{a_1}^{a_T} f(a)da, \tag{3}$$

where $\bar{f}_c$ represents the mean value of the function of interest, which is $f'(a)$ for the ARC (Finney, Weir, & Giordano, 2001, p. 352; Stewart, 1998, p. 470). Since Equation 3 yields the mean of a continuous differentiable function, and Equation 2 is a special case of a continuous function, combining the two equations will yield the mean of the derivatives (i.e., instantaneous rates of change) of the function from a_1 to a_T.

When the limiting equation for a derivative is combined with the mean of a continuous function, it can be shown that the mean of derivatives, which is literally the ARC, can be written as

$$\text{ARC} = \overline{f'(a_t)} = \frac{1}{a_T - a_t} \int_{a_t}^{a_T} f'(a_t)da \tag{4A}$$

$$= \frac{f(a_T) - f(a_t)}{a_T - a_t} \tag{4B}$$

$$= \frac{\Delta Y}{\Delta a}. \tag{4C}$$

As can be seen in Equation 4C, the mathematical definition of the ARC is the change in Y_t divided by the change in time during some specified interval. Equation 4C is well-known in analytic calculus (e.g., Finney et al., 2001, pp. 86–88; Stewart, 1998, pp. 146–147 and 208), where, regardless of the function, the mean of all of the derivatives evaluated over a specified continuous interval must equal $\Delta Y/\Delta a$. In the context of longitudinal data analysis, the mathematics underlying the ARC are not generally well known (cf. Kelley & Maxwell, 2008; Seigel, 1975), which has led to some confusion in the applied longitudinal data analysis literature. As a single measure describing overall change, the ARC holds promise. The problem, however, is that in an attempt to convey an estimate of the ARC, researchers have used the slope from the straight-line change model. As Kelley and Maxwell showed in the case of discrete time, the slope from the straight-line change model generally is not equal to the ARC. As monitoring instruments increasingly allow for more measurement occasions to be obtained in the same time interval—so much so that some are essentially continuous and others are approaching continuous—a discussion of the ARC in the context of continuous time is appropriate and is provided here.

Discrepancy Between the Regression Coefficient From the Straight-Line Change Model and the ARC

The discrepancy between the regression coefficient and the ARC will be quantified by two parameters: the *bias* and the *discrepancy factor*. For fixed values of time, the bias is operationally defined by Equation 5 (see below), where β_{SLCM} is the general representation of the slope from the straight-line change model (i.e., an individual's ordinary least squares regression slope), Y_t is conditional on the true functional form of change, and E[·] represents the expected value of the random variable in brackets. For fixed values of time, the second parameter that describes the discrepancy is the discrepancy factor and is operationally defined as

$$\Psi = \frac{\text{E}\left[\beta_{\text{SLCM}} \mid f(a)\right]}{\dfrac{\text{E}\left[(Y_T - Y_1) \mid f(T)\right]}{a_T - a_1}} = \frac{\beta_{\text{SLCM}}}{\text{ARC}}, \tag{6}$$

$$\text{B} = \text{E}\left[\beta_{\text{SLCM}} \mid f(a)\right] - \frac{\text{E}\left[(Y_T - Y_1) \mid f(a)\right]}{a_T - a_1} = \beta_{\text{SLCM}} - \text{ARC} \tag{5}$$

where, again, Y_t is conditional on the true functional form of change.

In situations where $\mathrm{B} = 0$ (implying $\Psi = 1$), interpreting β_{SLCM} as if it were the ARC yields no inconsistency in research conclusions or interpretation. However, when $\mathrm{B} \neq 0$ (and by implication $\Psi \neq 1$), conceptualizing β_{SLCM} as the ARC may be problematic and can potentially lead to misinformed conclusions regarding intraindividual change, interindividual change, and group differences in change. Although at times interpretation of B may be more straightforward than interpretation of Ψ, it is also potentially arbitrary due to the potential rescaling of time and/or the dependent variable. We include both so that, depending on the particular situation, either or both may be used.

Examining the Bias in the ARC When Time Is Continuous

In the case of continuously measured time values, the ordinary slope from the straight-line change model generalizes, with the use of integration rather than summation, to

$$\beta_{\mathrm{SLCM_C}} = \frac{\int_{a_1}^{a_T} (a_t - \mu_a)(Y_t - \mu_Y)\,da}{\int_{a_1}^{a_T} (a_t - \mu_a)^2\,da}, \qquad (7)$$

where $\beta_{\mathrm{SLCM_C}}$ is the regression coefficient for the straight-line change model when time is continuous. Equation 7 can be rewritten as the integral of a sum after expanding the numerator and the denominator:

$$\beta_{\mathrm{SLCM_C}} = \frac{\int_{a_1}^{a_T} (Y_t a_t - Y_t \mu_a - a_t \mu_Y + \mu_a \mu_Y)\,da}{\int_{a_1}^{a_T} (a_t^2 - 2a_t \mu_a + \mu_a^2)\,da}. \qquad (8)$$

Because the integral of a sum is the sum of the integrals, Equation 8 can be rewritten as Equation 9, shown below.

Realizing that $\mu_a = (a_T + a_1)/2$ and that

$$\mu_Y \int_{a_1}^{a_T} (a_t)\,da = \frac{\mu_Y (a_T^2 - a_1^2)}{2}$$

in the situation of continuous time, the last two components in the numerator of Equation 9 are equal and of opposite sign, leading to a simplification of the numerator because the two components cancel. Alternatively, a second perspective for understanding why the two components cancel in the numerator of Equation 9 can be seen by rewriting the last two components as

$$\mu_Y \int_{a_1}^{a_T} (a_t - \mu_a)\,da.$$

Because

$$\int_{a_1}^{a_T} (a_t - \mu_a)\,da$$

is the first moment about the mean, this quantity must always equal zero (Stuart & Ord, 1994, chap. 3). In the following subsections describing situations where time is continuous, the reduced form of Equation 9,

$$\beta_{\mathrm{SLCM_C}} = \frac{\int_{a_1}^{a_T} (Y_t a_t)\,da - \mu_a \int_{a_1}^{a_T} (Y_t)\,da}{\int_{a_1}^{a_T} (a_t^2)\,da - 2\mu_a \int_{a_1}^{a_T} (a_t)\,da + \mu_a^2 (a_T - a_1)}, \qquad (10)$$

will be applied to linear and then to nonlinear models.

When Y_t Can Be Written As a Linear Function of Time

Any functional form can be represented by a power series, such that the sum of squared deviations between the values of the true function and the values approximated by the power series can be made to be infinitesimally small by adding enough polynomial powers and coefficients (Finney et al., 2001, chap. 8; Stewart, 1998, section 8.6). A power series in the longitudinal context is a limiting sum of coefficients multiplied by positive integer powers of time. Such a power series is given as

$$f(a_t) = \lim_{M \to \infty} \sum_{m=0}^{M} (\lambda_m a_t^m), \qquad (11)$$

where λ_m is the coefficient ($-\infty < \lambda_m < \infty$) for the mth power ($m = 0, \ldots, M$).

Although a power series is infinite by definition, known functional forms can be represented by finite sums. In general, the following finite sum can be used to impose or approximate some known or unknown functional form of change and is more general than the power series, since the powers of time are not limited to nonnegative integers (as is the definition of a polynomial change model), but can take on any real values:

$$f(a_t) = \sum_{k=1}^{K} (\lambda_k a_t^{\Upsilon_k}), \qquad (12)$$

$$\beta_{\mathrm{SLCM_C}} = \frac{\int_{a_1}^{a_T} (Y_t a_t)\,da - \mu_a \int_{a_1}^{a_T} (Y_t)\,da - \mu_Y \int_{a_1}^{a_T} (a_t)\,da + \mu_a \mu_Y (a_T - a_1)}{\int_{a_1}^{a_T} (a_t^2)\,da - 2\mu_a \int_{a_1}^{a_T} (a_t)\,da + \mu_a^2 (a_T - a_1)} \qquad (9)$$

where Υ_k ($-\infty < \Upsilon_k < \infty$) represents the kth ($k = 1, \ldots, K$; $1 \leq K < \infty$) power. The intercept of a particular change curve is the sum of the λ_ks whose Υ_k is zero. In the special case where $a \in [0, a_T]$, the intercept is $\sum \lambda_k 0^{\Upsilon_k}$, which, strictly speaking, is an indeterminate form when $\Upsilon_k = 0$. However, due to l'Hôpital's rule, which uses derivatives to evaluate the converging limit of a function that would otherwise be indeterminate under standard algebraic rules, the quantity $0^0 \equiv 1$ by standard conventions (Finney et al., 2001, section 7.6; Stewart, 1998, section 4.5). When evaluating the equations given in this section by computer, care should be taken to ensure that the particular program defines 0^0 as 1 (rather than, e.g., returning an error message). General results emerge for B and Ψ by realizing that functional forms of change can generally be represented by Equation 12. The following section makes use of this fact when examining B and Ψ for any model linear in its parameters.

Y_t will be replaced by Equation 12 so that the results will be in the most general form of models linear in their parameters. Replacing Y_t in Equation 10 with the finite sum of Equation 12 yields

$$\beta_{\mathrm{SLCM_C}} = \frac{\int_{a_1}^{a_T}\left(\sum_{k=1}^{K}\lambda_k a_t^{\Upsilon_k+1}\right)da - \mu_a\int_{a_1}^{a_T}\left(\sum_{k=1}^{K}\lambda_k a_t^{\Upsilon_k}\right)da}{\int_{a_1}^{a_T}\left(a_t^2\right)da - 2\mu_a\int_{a_1}^{a_T}\left(a_t\right)da + \mu_a^2\left(a_T - a_1\right)}. \quad (13)$$

Carrying out the integration and replacing μ_a with its definition [$\mu_a = (a_T + a_1)/2$] yields Equation 14, shown below. After simplifying both the numerator and the denominator, the general equation for the regression coefficient from the straight-line change model when Y_t can be written in the form of Equation 12 and when time is continuous is given by Equation 15, shown below. It is useful to note that Equation 15 does not constrain the values of a_1 or a_T, the number of components defining Y_t (i.e., K), or the values of Υ_k and λ_k.

The ARC when Y_t is defined as a sum of K coefficients multiplied by powers of time can be written as the following:

$$\mathrm{ARC} = \frac{\sum_{k=1}^{K}\lambda_k\left(a_T^{\Upsilon_k} - a_1^{\Upsilon_k}\right)}{a_T - a_1}. \quad (16)$$

Because the slope (Equation 15) and the ARC (Equation 16) have been defined when Y_t is expressed as a special case of Equation 12, general expressions emerge for B and Ψ. The general bias for the present situation is found by substituting Equations 15 and 16 into Equation 5 (see Equation 17, below).

The general discrepancy factor is then found by substituting Equations 15 and 16 into Equation 6 (see Equation 18, below).

It can be shown that when $f(a_t)$ is defined as a linear or a quadratic change curve, $\beta_{\mathrm{SLCM_C}}$ = ARC and B from Equation 17 is zero (and thus $\Psi = 1$). Thus, in the case of continuous time, if the function governing change is a straight-line change model or a quadratic change model, no problems arise when $\beta_{\mathrm{SLCM_C}}$ is interpreted as the ARC. However, for the general case of any linear model other than a straight-line or quadratic model, $\mathrm{B} \neq 0$ (and thus $\Psi \neq 1$). Thus, interpreting $\beta_{\mathrm{SLCM_C}}$ as if it were the ARC potentially leads to misleading conclusions. Although a

$$\beta_{\mathrm{SLCM_C}} = \frac{\sum_{k=1}^{K}\frac{\lambda_k\left(a_T^{\Upsilon_k+2} - a_1^{\Upsilon_k+2}\right)}{\Upsilon_k+2} - \frac{a_T + a_1}{2}\sum_{k=1}^{K}\frac{\lambda_k\left(a_T^{\Upsilon_k+1} - a_1^{\Upsilon_k+1}\right)}{\Upsilon_k+1}}{\frac{a_T^3 - a_1^3}{3} - \frac{\left(a_T + a_1\right)\left(a_T^2 - a_1^2\right)}{2} + \frac{\left(a_T + a_1\right)^2\left(a_T - a_1\right)}{4}}. \quad (14)$$

$$\beta_{\mathrm{SLCM_C}} = \frac{6}{\left(a_T - a_1\right)^3}\sum_{k=1}^{K}\left(\frac{\lambda_k\left[\Upsilon_k\left(a_T^{\Upsilon_k+2} - a_1^{\Upsilon_k+2} + a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right) + 2\left(a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right)\right]}{\left(\Upsilon_k+2\right)\left(\Upsilon_k+1\right)}\right). \quad (15)$$

$$\mathrm{B} = \frac{6}{\left(a_T - a_1\right)^3}\sum_{k=1}^{K}\left(\frac{\lambda_k\left[\Upsilon_k\left(a_T^{\Upsilon_k+2} - a_1^{\Upsilon_k+2} + a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right) + 2\left(a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right)\right]}{\left(\Upsilon_k+2\right)\left(\Upsilon_k+1\right)}\right) - \frac{\sum_{k=1}^{K}\lambda_k\left(a_T^{\Upsilon_k} - a_1^{\Upsilon_k}\right)}{a_T - a_1}. \quad (17)$$

$$\Psi = \frac{6}{\left(a_T - a_1\right)^2\sum_{k=1}^{K}\lambda_k\left(a_T^{\Upsilon_k} - a_1^{\Upsilon_k}\right)}\sum_{k=1}^{K}\left(\frac{\lambda_k\left[\Upsilon_k\left(a_T^{\Upsilon_k+2} - a_1^{\Upsilon_k+2} + a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right) + 2\left(a_T a_1^{\Upsilon_k+1} - a_1 a_T^{\Upsilon_k+1}\right)\right]}{\left(\Upsilon_k+2\right)\left(\Upsilon_k+1\right)}\right). \quad (18)$$

formal proof has not been provided for the most general case, analytic and empirical evaluation of the equations for a wide variety of models linear in their parameters when time is continuous yields $\text{B} \neq 0$ for nontrivial parameter combinations. However, what is clear is that, in general, $\beta_{\text{SLCM}_\text{C}} \neq \text{ARC}$ unless the functional form is linear, quadratic, or some combination of linear and quadratic. Thus, interpreting the slope from the straight-line change model as though it is the ARC generally leads to biased estimates of the ARC.

Often in applied longitudinal research the initial value of time is represented as zero ($a_1 = 0$). This is especially true in experimental studies when Y_1 represents a baseline measure of some attribute (pretest) before treatment begins. Another reason why a_1 many times equals zero is because time is often scaled such that the intercept represents the initial (starting) value. In the special case where a_1 is replaced by zero, Equations 17 and 18 can be simplified. The simplified slope when the initial value of time (or scaled time) is zero can be written as

$$\beta_{\text{SLCM}_\text{C}} = 6\sum_{k=1}^{K}\left(\frac{\lambda_k a_T^{\Upsilon_k - 1}\Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)}\right). \quad (19)$$

The ARC for such a series defined by Equation 12 can be written as

$$\text{ARC} = \frac{\left(\sum_{k=1}^{K}\lambda_k a_T^{\Upsilon_k}\right) - \beta_0}{a_T}, \quad (20)$$

where β_0 is the intercept of the particular change curve. Recall that the intercept is simply the sum of the coefficients whose Υ_k equals zero. If no Υ_k equals zero when $a = 0$, then β_0 itself equals zero and the change curve goes through the origin.

The general expression for B when $a \in [0, a_T]$ is obtained by subtracting the right-hand side (RHS) of Equation 20 from the RHS of Equation 19:

$$\text{B} = 6\sum_{k=1}^{K}\left(\frac{\lambda_k a_T^{\Upsilon_k - 1}\Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)}\right) - \frac{\left(\sum_{k=1}^{K}\lambda_k a_T^{\Upsilon_k}\right) - \beta_0}{a_T}. \quad (21)$$

The general expression for Ψ in this situation is obtained by dividing Equation 19 by Equation 20:

$$\Psi = \frac{6}{\left(\sum_{k=1}^{K}\lambda_k a_T^{\Upsilon_k}\right) - \beta_0}\sum_{k=1}^{K}\left(\frac{\lambda_k a_T^{\Upsilon_k}\Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)}\right). \quad (22)$$

Of course, since Equations 21 and 22 are special cases of Equations 17 and 18, it holds true that when the functional form of change is a linear or quadratic change curve, the regression coefficient for the straight-line change model and the ARC are equivalent. Again, however, as the equations in this section show, it is generally the case that $\beta_{\text{SLCM}_\text{C}} \neq \text{ARC}$. The exact values of B and/or P can be found with the appropriate equation(s) from the present section when time is continuous. The next section deals with the case where $f(a_t)$ equals each of the nonlinear models previously discussed.

When Y_t Conforms to Certain Nonlinear Functions of Time

Fitting a statistical model linear in its parameters to longitudinal data is generally straightforward. As the phenomenon under study grows increasingly more complex, the order of the polynomial change model can be increased accordingly, until the predicted scores reasonably correspond with the observed scores. Nonlinear models of the same complex phenomenon can often be more interpretable and parsimonious, and are generally more valid beyond the observed range of data, when compared with linear models (Pinheiro & Bates, 2000). Furthermore, it is often the case that the parameters in nonlinear models can be easily interpreted, whereas once a polynomial model is beyond quadratic, the meaning of the higher order parameters typically offers little meaningful interpretation. An example of such a difference between nonlinear and linear models relates to asymptotes.

In polynomial change models, asymptotic values cannot generally be modeled for the asymptote to hold beyond the range of the observed data. Thus, researchers who make use of polynomial trends must accept that their model will necessarily fail at some point beyond the range of the data actually collected. Such scenarios can potentially lead to inadequate models where impossible values are predicted.

To demonstrate problems that arise when data truly follow nonlinear functional forms yet are modeled by straight-line change models, three nonlinear change models will be presented so that later the bias and discrepancy factor can be developed for each. The selected nonlinear models are the asymptotic regression change curve, the Gompertz change curve, and the logistic change curve. Although a wide variety of nonlinear models exist, these models of change were chosen because they are especially helpful for applied research. A brief introduction to each is given here based on the descriptions found in Kelley and Maxwell (2008).

The Asymptotic Regression Change Curve

The general asymptotic regression change curve—often referred to as the negative exponential change model—describes a family of potential regression models where the dependent variable approaches some limiting value as time increases. A general asymptotic regression equation for a single trajectory was given by Stevens (1951) as

$$Y_t = \alpha + \beta\rho^{a_t} + \varepsilon_t, \quad (23)$$

where α is the asymptotic value approached as $a \to \infty$, β is the change in Y_t from $a = 0$ to $a \to \infty$ (i.e., β represents total change in Y_t), and ρ ($0 < \rho < 1$) is a scalar that defines the factor by which the deviation between Y_t and α is reduced for each unit change of time, thus reflecting the rate at which $Y_t \to \alpha$. Equation 23 can be equivalently written as

$$Y_t = \alpha + \beta\exp(-\gamma a_t) + \varepsilon_t, \quad (24)$$

where $\gamma = -\log(\rho)$ $(0 < \gamma < \infty)$ and can be thought of as a scaling parameter (Stevens, 1951).

The Gompertz Change Curve

The Gompertz change model is a nonlinear model that is often used in the biological sciences. The asymmetric sigmoidal form of the Gompertz change curve offers an option for those who seek to model certain types of nonlinear trends. The general three-parameter Gompertz change model for a single trajectory can be written as

$$Y_t = \alpha \exp\left[-\exp\left(\beta - \gamma a_t\right)\right] + \varepsilon_t, \tag{25}$$

where α is the asymptote as $a \to \infty$. The parameters β and γ define the point of inflection on the abscissa at $a = \beta/\gamma$. The point of inflection on the ordinate is at $Y = \alpha/\exp(1)$, which is approximately 37% of the asymptotic change (Ratkowsky, 1983, chap. 4 and pp. 163–167; Winsor, 1932).

The Logistic Change Curve

The logistic change model is another nonlinear sigmoidal model that provides another option for modeling change over time in the behavioral sciences. The general three-parameter logistic change model for a single trajectory can be written as

$$Y_t = \frac{\alpha}{1 + \exp\left(\beta - \gamma a_t\right)} + \varepsilon_t, \tag{26}$$

where α is the asymptote as $a \to \infty$. The parameters β and γ define the point of inflection on the abscissa at $a = \beta/\gamma$. The point of inflection on the ordinate is at $Y = \alpha/2$, 50% of the asymptotic change (chap. 4 and pp. 167–169 of Ratkowsky, 1983; Winsor, 1932).

Nonlinear Models for the Analysis of Change

In this section, β_{SLCMC} and ARC are derived for the asymptotic change curve (Equation 24), the Gompertz change curve (Equation 25), and the logistic change curve (Equation 26). General equations are presented for β_{SLCMC} and ARC for these nonlinear models, thus allowing one to compute B by subtraction and/or Ψ by division, as needed. The derivations proceed in a manner analogous to (albeit not as detailed as, for space considerations) the way they did for the derivations presented in the previous section for models linear in their parameters.

The discrepancy in the asymptotic regression change model. The regression coefficient for the straight-line change model applied to change that follows an asymptotic regression (also termed a *negative exponential*) model in the case of continuous time is given in Equation 27, shown at the bottom of this page, where subscripts will be used—AR in this case for asymptotic regression—to identify the particular nonlinear change model. The ARC for the asymptotic regression model, obtained by substituting Equation 24 into Equation 4C, is given as the following:

$$\text{ARC}_{\text{AR}} = \frac{\beta\left[\exp\left(-\gamma a_T\right) - \exp\left(-\gamma a_1\right)\right]}{a_T - a_1}. \tag{28}$$

The value of B for the asymptotic regression model is thus obtained by subtracting the RHS of Equation 28 from the RHS of Equation 27, and Ψ is obtained by dividing the RHS of Equation 27 by the RHS of Equation 28.

The discrepancy in the Gompertz change model. The regression coefficient for the straight-line change model applied to change that follows a Gompertz change model in the case of continuous time is obtained by first expressing G as in Equation 29 (see below), where Ei is the exponential integral. The exponential integral is defined as

$$\text{Ei}(q, x) = \int_{g=1}^{\infty} \frac{\exp(-xg)}{g^q}\, dg, \tag{30}$$

with q being a nonnegative integer and x some algebraic expression (Abramowitz & Stegun, 1965). Given G, the slope for the Gompertz change model is equal to the following:

$$\beta_{\text{SLCM}_{\text{C}_{\text{GC}}}} = G\frac{6\alpha}{\gamma\left(a_T - a_1\right)^4}, \tag{31}$$

where the subscript GC denotes the Gompertz change model. The ARC for the Gompertz change model is given in Equation 32 (below). The value of B for the Gompertz change model is thus obtained by subtracting the RHS of Equation 32 from the RHS of Equation 31, and Ψ is obtained by dividing the RHS of Equation 31 by the RHS of Equation 32.

The discrepancy in the logistic change model. The regression coefficient for the straight-line change model applied to change that follows a logistic change model in the case of continuous time is obtained by defining L_1, L_2, L_3, and L_4 (see Equations 33–36, next page). In L_3 the dilogarithm function is required. The function dilog (Lewin, 1981) is defined as Equation 37:

$$\beta_{\text{SLCM}_{\text{C}_{\text{AR}}}} = 6\beta\exp\left[-\gamma\left(a_T + a_1\right)\right]\frac{a_1\left[\exp\left(\gamma a_1\right) + \exp\left(\gamma a_T\right)\right] - a_T\left[\exp\left(\gamma a_1\right) + \exp\left(\gamma a_T\right)\right] + 2\left[\exp\left(\gamma a_T\right) - \exp\left(\gamma a_1\right)\right]/\gamma}{\gamma\left(a_T - a_1\right)^3} \tag{27}$$

$$G = -\int_{a_1}^{a_T}\left[a_T + a_1 - 2a\right]\left\{\exp\left[-\exp\left(\beta - \gamma a\right)\right]\gamma\left(a_T - a_1\right) + \text{Ei}\left[1, \exp\left(\beta - \gamma a_T\right)\right] - \text{Ei}\left[1, \exp\left(\beta - \gamma a_1\right)\right]\right\}da \tag{29}$$

$$\text{ARC}_{\text{GC}} = \frac{\alpha\left\{\exp\left[-\exp\left(\beta - a_T\gamma\right)\right] - \exp\left[-\exp\left(\beta - a_1\gamma\right)\right]\right\}}{a_T - a_1} \tag{32}$$

$$\text{dilog}(x) = \int_{g=1}^{x} \frac{\log(g)}{1-g}\,dg. \tag{37}$$

The four logistic components are then combined with the other necessary parameters in the following manner:

$$\beta_{\text{SLCM}_{\text{C}_{\text{LC}}}} = 6\alpha \frac{L_1 - L_2 + 2(L_3 + L_4)/\gamma}{\gamma(a_T - a_1)^3}, \tag{38}$$

where LC denotes logistic change. The ARC for the logistic change model is given in Equation 39 (see below). The value of B for the logistic change model is thus obtained by subtracting the RHS of Equation 39 from the RHS of Equation 38, and Ψ is obtained by dividing the RHS of Equation 38 by the RHS of Equation 39.

Although it would be advantageous to show generally whether it is possible for $\beta_{\text{SLCMC}_{\text{AR}}} - \text{ARC}_{\text{AR}} = 0$, $\beta_{\text{SLCMC}_{\text{GC}}} - \text{ARC}_{\text{GC}} = 0$, and/or $\beta_{\text{SLCMC}_{\text{LC}}} - \text{ARC}_{\text{LC}} = 0$, at the present time no mathematically tractable solution was obtainable due to the complications that arise with the nonlinear functional forms used. Analytic and empirical investigations have shown that for nontrivial cases, the regression coefficient from the straight-line change model and the ARC are not generally equal. For any specific situation, given the equations provided, the exact value of B and Ψ can be determined.

Examples of the Discrepancies

Although general equations are presented for the bias and discrepancy factors, it can be difficult to discern whether the bias and discrepancy factors amount to any meaningful deviations between the ARC and the slope from the straight-line change model. Figures 1, 2, and 3 show plots of asymptotic regression, Gompertz, and logistic change models, respectively, for 15 different combinations in the case of continuous time for β and γ values when $T \in [0, 1]$ and α is fixed at 5. The purpose of the figures is to show the reader a variety of nonlinear functional forms with a variety of parameter values, to illustrate how the change models discussed in the present work generalize to a variety of trajectories that might be useful in applied research. In addition to illustrating the trajectories themselves, the particular parameter values governing the curves have been included atop the particular plot. Within each of the plots is the value of β_{SLCM}, ARC, B, and Ψ. From the plots it can be seen that B is sometimes positive (i.e., when $\beta_{\text{SLCM}} > \text{ARC}$, implying that $\Psi > 1$) but in other situations it is negative (i.e., when $\beta_{\text{SLCM}} < \text{ARC}$, implying that $\Psi < 1$).

It is important to note that B and Ψ for the 45 different scenarios examined are specific to the selected parameters and the chosen time interval. The exact values of B and Ψ are arbitrary to a large extent, since modification of the parameters will change the B and Ψ values. However, the particular examples of change curves provided in Figures 1, 2, and 3 are thought to consist of a variety of realistic change curves. The straight line within each plot represents the predicted Y scores given time (i.e., the regression line) for the straight-line change model, whereas the nonlinear trend represents the true change for the particular situation.

Although it is difficult to say what a large discrepancy would be, a discrepancy factor as small as 0.376 (bottom left of Figure 2) and one as large as 1.43 (bottom right of Figure 2) seem to be very problematic. Certainly, commonly used statistics would be regarded as problematic if their expected values were 0.376 times smaller or 1.43 times larger than their corresponding population values. Furthermore, the smallest and largest discrepancy factors shown in the figures (i.e., the 0.376 and the 1.43 noted above) would have been surpassed had different parameter values been used. Thus, the figures are meant to supplement the mathematical derivations with examples showing a variety of change curves and the corresponding bias and discrepancy factor of each.

Discussion

Confusion exists in the literature regarding the definition and interpretation of the ARC. Because many monitoring systems are now capable of recording information continuously or near continuously over time, it is important to consider the effects of estimating and interpreting the slope from a straight-line change model as the ARC. As is shown in the present article, there is generally a bias when using the slope from the straight-line change model as if it were the ARC.

Three straightforward, sufficient conditions can be described such that there is no discrepancy when using the straight-line change model to estimate the ARC when time is continuous:

$$L_1 = a_T\left\{\log\left[1+\exp(\beta - a_1\gamma)\right] - \log\left[\exp(\beta - a_1\gamma)\right] + \log\left[\exp(\beta - a_T\gamma)\right] + \log\left[1+\exp(\beta - a_T\gamma)\right] - 2\beta + a_T\gamma\right\} \tag{33}$$

$$L_2 = a_1\left\{\log\left[1+\exp(\beta - a_T\gamma)\right] - \log\left[\exp(\beta - a_T\gamma)\right] + \log\left[\exp(\beta - a_1\gamma)\right] + \log\left[1+\exp(\beta - a_1\gamma)\right] - 2\beta + a_1\gamma\right\} \tag{34}$$

$$L_3 = \text{dilog}\left\{\left[\exp(a_1\gamma) + \exp(\beta)\right]\exp(-a_1\gamma)\right\} - \text{dilog}\left\{\left[\exp(a\gamma) + \exp(\beta)\right]\exp(-a_T\gamma)\right\} \tag{35}$$

$$L_4 = \beta\left\{\log\left[\exp(\beta - a_1\gamma)\right] - \log\left[\exp(\beta - a_T\gamma)\right]\right\} \tag{36}$$

$$\text{ARC}_{\text{LC}} = \frac{\alpha\left[\exp(\beta - a_T\gamma) - \exp(\beta - a_1\gamma)\right]}{\left[1+\exp(\beta - a_T\gamma)\right]\left[1+\exp(\beta - a_1\gamma)\right](a_1 - a_T)} \tag{39}$$

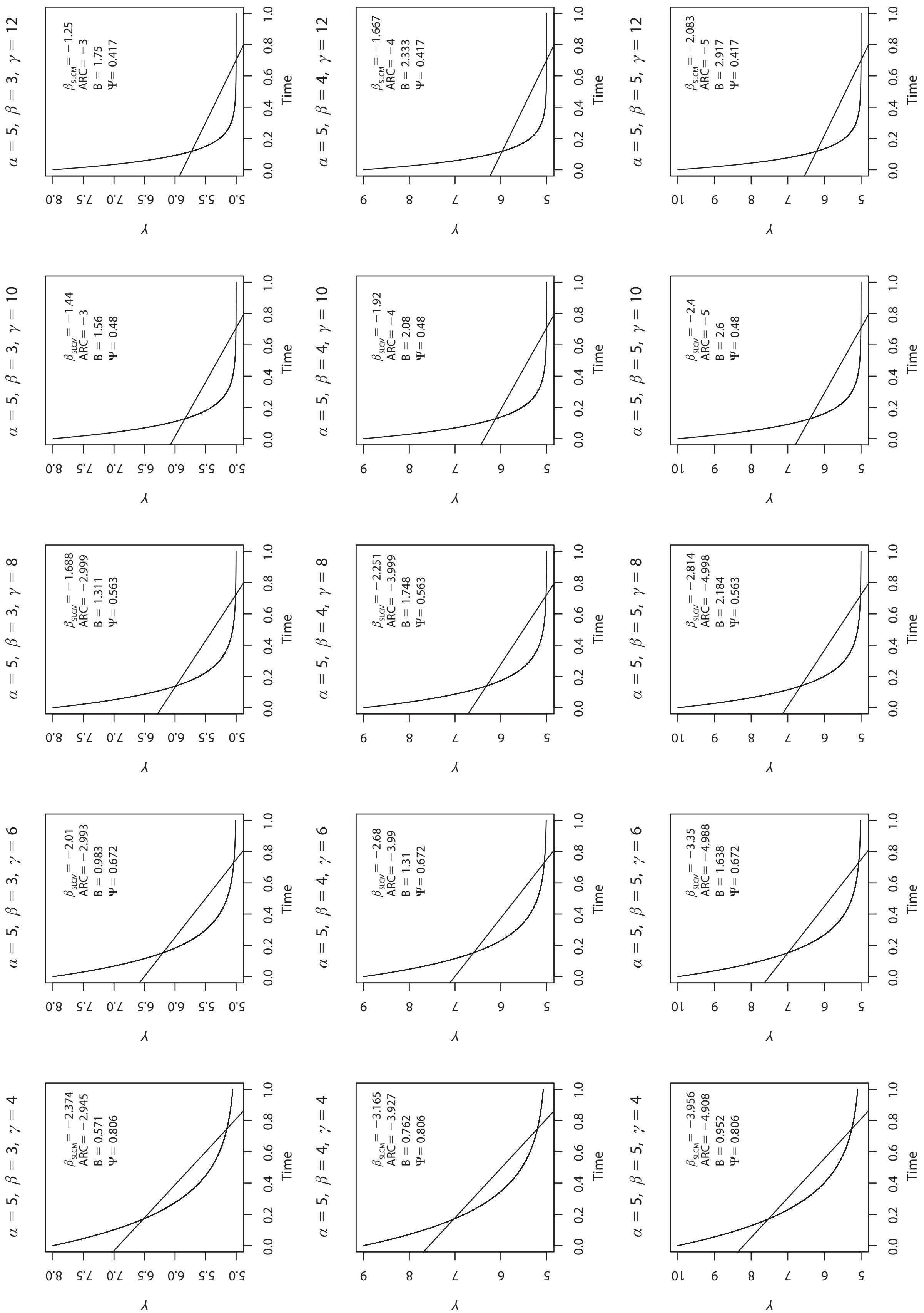

Figure 1. Illustration of the straight-line change model fit to a variety of asymptotic change curves along with β_{SLCM}, ARC, B, and Ψ, given the parameters that are specified.

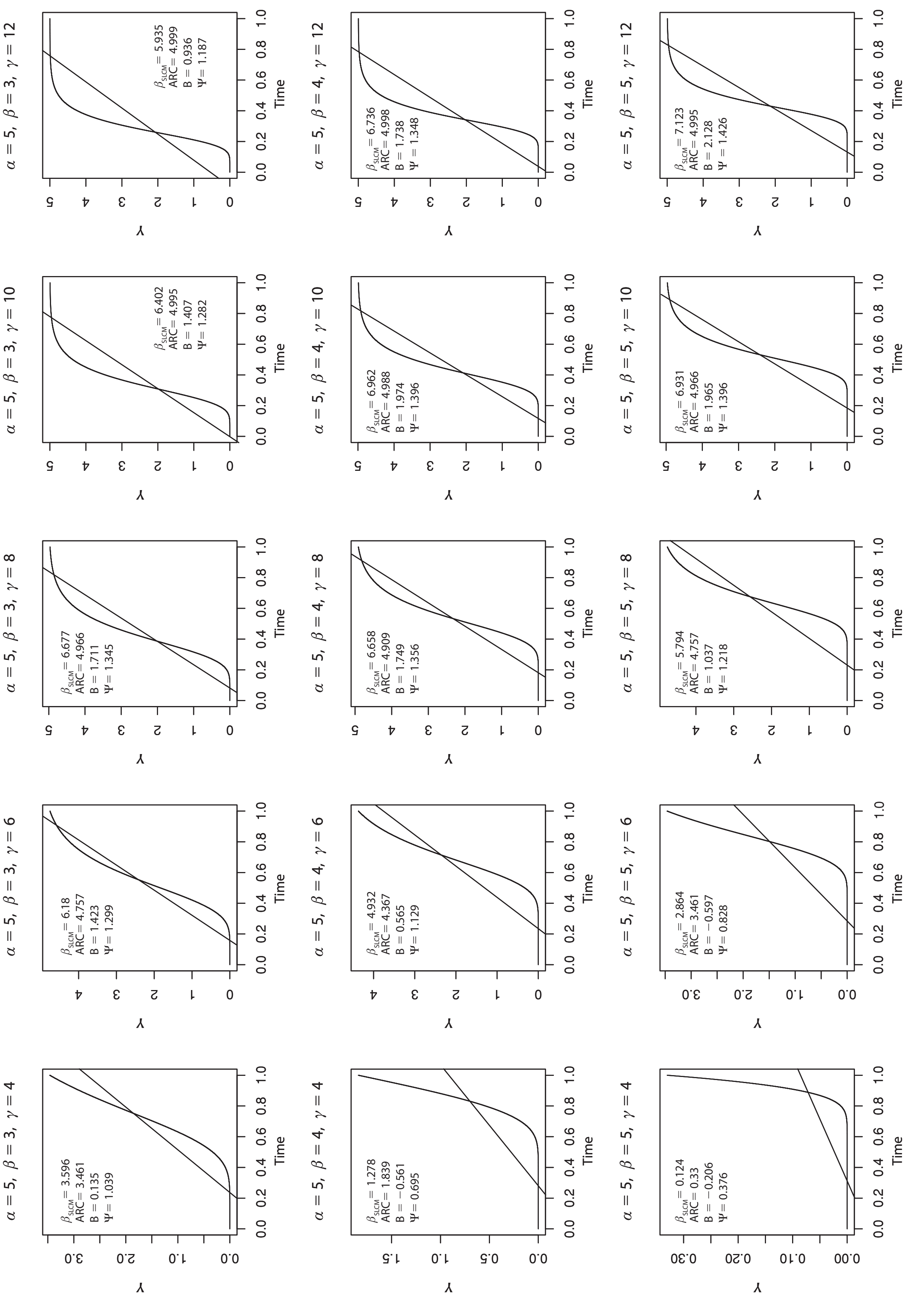

Figure 2. Illustration of the straight-line change model fit to a variety of Gompertz change curves along with β_{SLCM}, ARC, B, and Ψ, given the parameters that are specified.

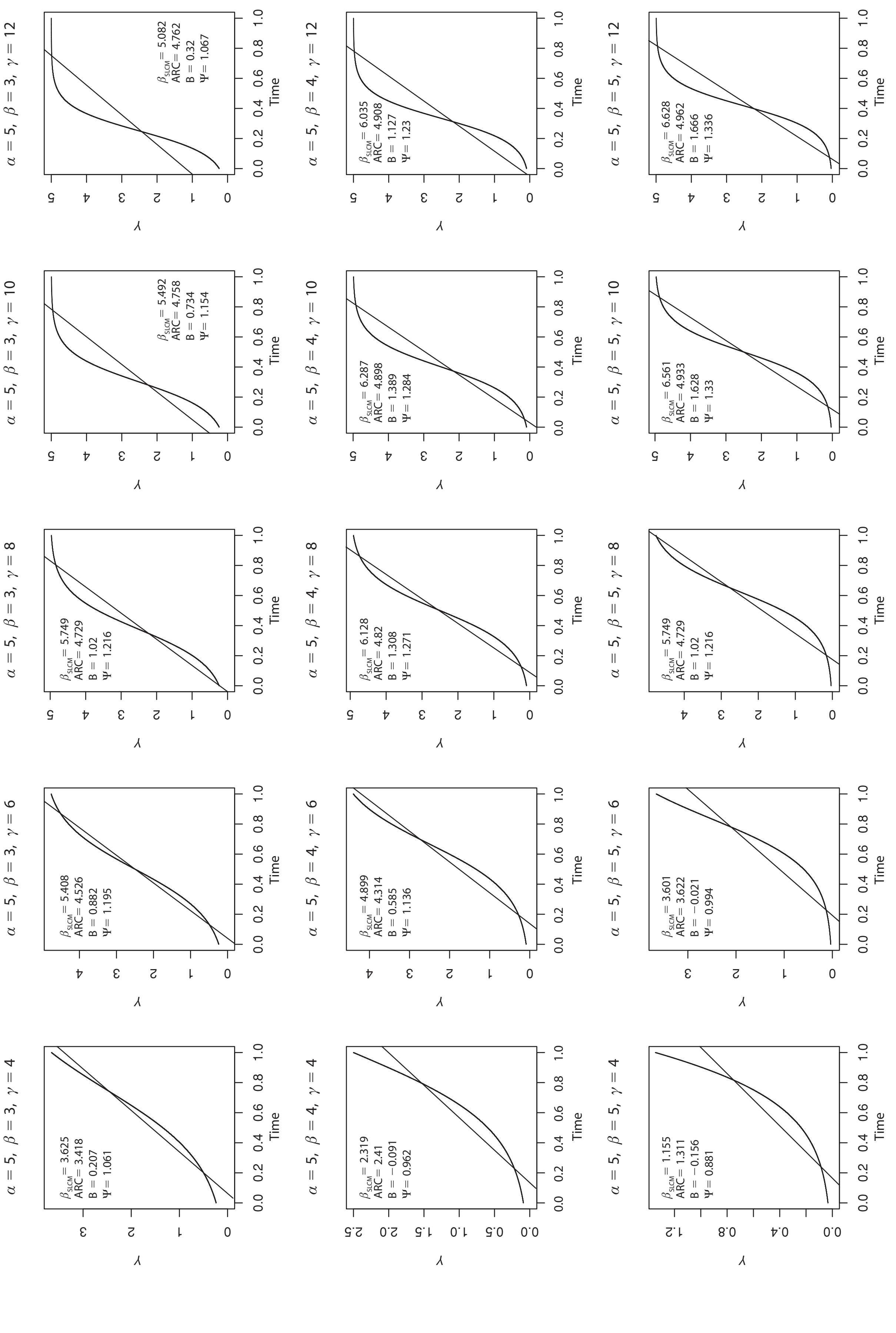

Figure 3. Illustration of the straight-line change model fit to a variety of logistic change curves along with β_{SLCM}, ARC, B, and Ψ, given the parameters that are specified.

1. The true functional form of change consists of only a linear component.

2. The true functional form of change consists of only a quadratic component.

3. The true functional form of change consists of only some combination of linear and quadratic components.

Of course, Conditions 1 and 2 are special cases of Condition 3 when the quadratic and linear components are zero. Thus, as this article has shown, the slope from the straight-line change model and the ARC are not generally equal to one another for an individual trajectory when time is measured continuously. This is not to say that no other functions can have an ARC that equals the slope from the straight-line change model, but generally it is the case. Certainly, special cases of other functions can be made so that the slope from the straight-line change model and the ARC are equal. However, such is generally not the case, and in most circumstances there will be some degree of bias. This article has shown that the bias between the ARC and the slope from the straight-line change model can be positive or negative and small or large, potentially yielding misleading conclusions regarding change over time. It can be shown (e.g., Kelley & Maxwell, 2008) that when the bias is nonzero and all other things are equal, the larger the number of time points, the larger the discrepancy between the slope from the straight-line change model and the ARC when the bias is nonzero. Thus, it is especially important to understand the relationship between the slope from the straight-line change model and the ARC in the case of continuous or nearly continuous time, since in such situations the discrepancy between the slope from the straight-line change models reaches its maximum for any given scenario.

AUTHOR NOTE

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NOTE

1. Multilevel models are also equivalent or closely related to random effects models, hierarchical (non)linear models, latent change curves, and mixed effects models. Thus, regardless of the verbiage given to such models, the issues discussed in the present article are equally applicable.

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