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Delineating the Average Rate of Change in Longitudinal Models

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The average rate of change is a concept that has been misunderstood in the literature. This article attempts to clarify the concept and show unequivocally the mathematical definition and meaning of the average rate of change in longitudinal models. The slope from the straight-line change model has at times been interpreted as if it were always the average rate of change. It is shown, however, that this is generally not the case and holds true in only a limited number of situations. General equations are presented for two measures of discrepancy when the slope from the straight-line change model is used to estimate the average rate of change. The importance of fitting an appropriate individual change model is discussed, as are the benefits provided by models nonlinear in their parameters for longitudinal data. An empirical data set is used to illustrate the analytic developments.

Keywords: *average rate of change; longitudinal data analysis; analysis of change; growth modeling; nonlinear growth models; nonlinear change models; functional form of growth; functional form of change*

Modern conceptualizations of the analysis of change regard intraindividual change as the starting point for longitudinal data analysis (e.g., Collins, 1996; Mehta & West, 2000; Raudenbush, 2001; Rogosa, Brandt, & Zimowski, 1982; Rogosa & Willett, 1985). It is by first focusing on the individual that broad generalizations over individuals can or cannot be made. The description of intraindividual change can be given in numerous ways and is limited only by the research design and the researcher's creativity in forming and testing models. For example, by focusing on one individual trajectory, the unknown functional form of change can be described as any combination of linear, quadratic, exponential, logistic, or even as a dampened or undampened oscillating function. The adequacy of the particular model chosen, however, depends in large part on

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the true functional form of change and the number of measurement occasions. Given that such a vast array of possibilities exists for describing intraindividual change, a measure of change that can describe all possible functional forms of change by way of a single descriptive statistic would have great practical value for the numerical description that it could provide. A quantity known as the *average rate of change* (ARC) has been conceptualized as such a measure in the literature.

The major purpose of this article is to delineate the meaning and interpretation of the ARC, as well as what the ARC is not, for this measure to be better understood by researchers who study phenomena that change over time. A better understanding of the ARC will allow researchers to realize when the ARC addresses the question of interest and when it should be avoided. Although the ARC provides a single measure of overall change, many facets of change are ignored by this global measure, such as the process of change. A thorough delineation of the ARC is given in this article to clarify concepts that have been misunderstood in the methodological and the substantive literatures.

Our delineation of the ARC begins at an intuitive level and progresses to a formal mathematical description. The primary emphasis throughout the article concerns a single trajectory, as describing individual trajectories is a necessary condition before describing a collection of individual trajectories.

Implicitly or explicitly, the ARC is often a central focus for many longitudinal research projects. Attempts are often made to succinctly describe the *average* or *typical* amount of change that occurs within some time interval. The regression coefficient from the straight-line change model has often been the way in which such a succinct description of change has been attempted. Although potentially beneficial, the remainder of the article shows that the regression coefficient from the straight-line change model is generally not equal to the ARC for a given trajectory and treating it as such will generally yield biased estimates of the ARC.

The ARC is a parsimonious measure that describes the overall trend of a trajectory, regardless of the functional form of change. Although the concept of the ARC is appealing and seems to be straightforward, the technical underpinnings have not received much formal attention (cf. Seigel, 1975). The majority of attention that the ARC has received, however, is often misguided and surrounded by confusion and misinterpretation. It is believed that a thorough delineation of the ARC will help researchers to understand the dynamic and static relationships that exist among sets of variables over time.

Derivation of the Average Rate of Change

The rate of change of a nonvertical straight line that passes through two sets of points, (a_1, Y_1) and (a_T, Y_T), is the slope of the line, where a_t represents some basis of time and Y_t is a continuous function of time, $Y_t = f(a_t)$, at the tth measurement occasion ($t = 1, T$).[1] The slope of the line connecting two points is the change in Y_t divided by the change in time:

$$Slope = \frac{f(a_T) - f(a_1)}{a_T - a_1} = \frac{Y_T - Y_1}{a_T - a_1} = \frac{\Delta Y}{\Delta a}, \tag{1}$$

where $f(a_t)$ is the dependent variable Y_t, ΔY is the change in the dependent variable, and Δa is the change in time.

Equation 1 is closely related to the derivative. In the limit as Δa approaches zero, Equation 1 yields the *instantaneous* rate of change when evaluated at a specific time value. The derivative of a function can be written as

$$\frac{dY}{da} = \lim_{\Delta a \to 0} \frac{f(a_t + \Delta a) - f(a_t)}{\Delta a} = f'(a), \tag{2}$$

where $\frac{dY}{da}$ is read as the derivative of Y_t with respect to a, which is represented as $f'(a)$ to make explicit that the derivative of the function is contingent on time. The next step is to relate the ARC to $f'(a)$ over time.

Deriving the mean of an infinite number of derivatives, because the true functional form of change is generally assumed to exist continuously over time, requires integration. The mean value theorem for integrals states that over a closed interval, an integrable function assumes its mean value at least once within the interval. The particular mean value for a continuous function that is differentiable over the interval a_1 to a_T is given as

$$\overline{f_c} = \frac{1}{a_T - a_1} \int_{a_1}^{a_T} f(a)da, \tag{3}$$

where $\overline{f_c}$ represents the mean value of the function $f(\cdot)$, in this case a continuous function that is differentiable (Finney, Weir, & Giordano, 2001, p. 352; Stewart, 1998, p. 470). Thus, after the function has been integrated, the value is then divided by the length of the interval to obtain the mean value of the function. The $f(\cdot)$ in Equation 3 of interest for the ARC is $f'(a)$. Because Equation 3 yields the mean of a continuous differentiable function and Equation 2 is a special case of a continuous function, combining the two equations will yield the mean instantaneous rate of change of the function from a_1 to a_T.

A corollary of the fundamental theorem of calculus implies that the derivative of an integrated function is the original function itself (Kline, 1977, p. 258). Thus, when Equations 2 and 3 are combined, the mean of the derivatives, which is literally the ARC, can be written as the following:

$$\text{ARC} = \overline{f'(a_t)} = \frac{1}{a_T - a_1} \int_{a_1}^{a_T} f'(a_t)da \tag{4}$$

$$= \frac{f(a_T) - f(a_1)}{a_T - a_1} \quad (5)$$

$$= \frac{\Delta Y}{\Delta a}. \quad (6)$$

As can be seen in Equation 6, the mathematical definition of the ARC is the change in Y_t divided by the change in time during some specified interval. The resultant formulation of Equation 6 is well known in analytic calculus (e.g., Finney, Weir, & Giordano, 2001, pp. 86–88; Stewart, 1998, pp. 146–147, 208), where regardless of the function, the mean of all of the derivatives evaluated over a specified continuous interval must equal $\frac{\Delta Y}{\Delta a}$. Notice that the true functional form of change was never specified. Equation 6 holds regardless of whether the functional form is known or unknown, as only the initial and final pairs of points are required. Although the ARC was defined in the case where time was continuous, the same formulation holds true in the more typical case where the occasions of measurement are discrete. More explicitly,

$$\text{ARC} = \frac{(Y_T - Y_1)|f(a_t)}{a_T - a_1} \quad (7)$$

when time is discrete, where $(Y_T - Y_1)|f(a_t)$ implies the difference between the final and initial value of Y given a particular functional form for the specific time basis. This holds true because $a_T - a_1 = \Delta a$ and $Y_T - Y_1 = \Delta Y$, regardless of whether time is continuous or discrete and regardless of whether the true functional form governing change is known or unknown. Even though Equation 6 is well known in the field of analytic calculus, in the context of longitudinal data analysis the mathematics underlying the ARC have not been well delineated. Because of the lack of technical attention given to the ARC, yet its intuitive appeal as the mean instantaneous rate of change, the ARC has often been misunderstood in practice. The major purpose of this article is to clarify misconceptions about the ARC that persist, both implicitly and explicitly, in the analysis of change literature.

Statistical Models of Individual Change

Before delineating the ARC in the context of longitudinal data analysis, a necessary digression provides an overview of statistical models useful for describing individual trajectories. This digression provides a broad context for the ARC as well as elucidating a variety of change models not often discussed or considered in applications of longitudinal data analysis within the behavioral, educational, and social sciences.

Throughout the article, Y_{it} is the dependent variable for the ith individual $(i = 1, \ldots, N)$ at the tth time point $(t = 1, \ldots, T)$. Unless otherwise specified, it is assumed that the occasions of measurement are equally spaced, have the same starting point, and have a constant τ with no missing data, where τ is defined as the change in time from time t to $t+1(\tau = a_{t+1} - a_t)$. Such a data structure implies that all N individuals have the same starting value, no missing data, T measurement occasions, and constant τ both within and across individuals. Thus, all of the N individuals have a common set (i.e., vector) of time values. Data are required to follow this measurement scheme because in this special case simplified derivations follow without loss of generality.

Relationship Between Straight-Line Change Models and the Average Rate of Change

Because of the hierarchical structure of longitudinal data (scores over time nested within persons, who in turn may be nested within group, etc.), statistical models that take into consideration the nonindependence of the hierarchically structured data are required (see Davidian & Giltinan, 1995; Goldstein, 2003; Pinheiro & Bates, 2000; or Raudenbush & Bryk, 2002, for a thorough treatment of these issues). The most common method of analyzing an individual's trajectory is with a multilevel model (MLM) using a polynomial functional form.[2] Change models linear in their parameters allow various polynomial and nonpolynomial trends to be specified and then tested against other competing models linear in their parameters. Given an observed set of data, provided a sufficient number of polynomial trends are specified (at the expense of parsimony and degrees of freedom), the change model can be made to accurately represent the data. This desirable property, combined with the relative ease of calculation, has made the MLM of polynomial change essentially the model of choice for analyzing change in the behavioral, educational, and social sciences.[3]

The general MLM linear in its parameters for the ith individual's set of scores can be given as

$$\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{U}_i + \boldsymbol{\varepsilon}_i, \tag{8}$$

where $\boldsymbol{\beta}$ is the vector of unknown fixed-effect population parameters linked to the vector $\mathbf{Y}_i$ by the design matrix $\mathbf{X}_i$, $\mathbf{U}_i$ is a matrix of unknown unique individual effects linked to $\mathbf{Y}_i$ by the design matrix $\mathbf{Z}_i$, and $\boldsymbol{\varepsilon}_i$ is a vector of errors generally assumed to be normally distributed about a mean of zero with a constant variance across time (Laird & Ware, 1982). This general MLM formulation allows for the desired polynomial function(s) of time to be included in the model, as well as other time-varying and fixed covariates. Furthermore, models having the form of Equation 8 offer great flexibility in terms of model testing, model comparisons, and parameter estimation.

A straight-line change model for individual i can be represented as

$$Y_{it} = \pi_{0i} + \pi_{1i}a_{it} + \varepsilon_{it}, \tag{9}$$

where π_{0i} is the intercept and π_{1i} is the slope for the ith individual. The parameters of Equation 9 can themselves be modeled as dependent variables, but we will focus on the unconditional case where the intercept and slope consist of a fixed and a unique effect for each of the individuals in the following manner:

$$\pi_{0i} = \beta_{00} + u_{0i}, \tag{10}$$

$$\pi_{1i} = \beta_{10} + u_{1i}, \tag{11}$$

where β_{00} is the theoretical mean of the individual intercepts, β_{10} is the theoretical mean of the individual slopes across the population of individuals, and u_{0i} and u_{1i} represent the unique (random) effects associated with the ith individual's intercept and slope parameter, respectively.

In the context of change models where each of the N individuals share a common design matrix for the unique effects ($\mathbf{Z}_i$ from Equation 8 equals $\mathbf{Z}$ for all N individuals), the means of the ordinary least squares (OLS) regression coefficients calculated for each individual are equivalent to the estimated fixed effects of the MLM model (see Laird & Ware, 1982). In the context of the straight-line change model of Equation 9, a common design matrix of the unique effects implies a common set (i.e., vector) of time values across each of the N individuals. Thus, β_{00} and β_{10}, from Equations 10 and 11, respectively, will be equivalent to the theoretical mean of the OLS estimates across individuals. That is, if OLS regression analyses were performed for each of the N individuals, the mean of the estimated intercepts and slopes would correspond to the estimated fixed effects calculated via the MLM change model. Because the estimated fixed effects of the MLM model are equal to the mean OLS estimates in the case of fully balanced data, to make the discussion more comprehensible and generalizable, the remainder of the article focuses specifically on the OLS estimates of a single trajectory. The MLM regression model of straight-line change for a specific individual thus simplifies to the following OLS formulation:

$$Y_t = \beta_0 + \beta_1 a_t + \varepsilon_t, \tag{12}$$

where the intercept is

$$\beta_0 = \mu_Y - \beta_1 \mu_a, \tag{13}$$

and the slope from the straight-line change model is

$$\beta_1 = \frac{\sum_{t=1}^{T} (Y_t - \mu_Y)(a_t - \mu_a)}{\sum_{t=1}^{T} (a_t - \mu_a)^2} = \beta_{SLCM}, \tag{14}$$

where μ_Y and μ_a represent the population means of the dependent variable and time, respectively, and β_{SLCM} is the regression coefficient for the straight-line change model.[4] Notice that no i subscripts are needed in Equation 12 (and thus Equations 13 and 14) because $N = 1$.

When straight-line models are used in the context of the analysis of change, an implicit assumption for descriptive and inferential purposes is that β_{SLCM} provides a meaningful measure of change. If the relationship between time and the dependent variable of interest is something other than a straight line, use of β_{SLCM} for individual trajectories may lead to incorrect conclusions. When using statistical methods that treat β_{SLCM} as a dependent variable, such as MLMs, growth curve models, or two-stage analyses, the results of such statistical procedures may be misleading, as the chosen measure of change (β_{SLCM}) may not accurately reflect the particular phenomenon under study as it changes and/or evolves over time. Two situations arise when using β_{SLCM} as an estimate of the ARC: (a) when change is governed by a straight line and (b) when change is governed by something other than a straight line. A major goal of the article is to delineate the β_{SLCM} and the ARC so that it can be shown that the two are fundamentally different and are not generally equal to one another. The remainder of the article illustrates that conceptualizing β_{SLCM} as a measure of the ARC potentially leads to incorrect conclusions, not only for an individual trend, but also for examining group differences across individuals.

Models Nonlinear in Their Parameters for the Analysis of Change

Statistical models that are linear in their parameters are generally straightforward to fit given a set of observed data. As the phenomenon under study grows increasingly more complex, the order of the polynomial change model can be increased accordingly until the predicted scores reasonably correspond with the observed scores. Models nonlinear in their parameters of the same complex phenomenon can often be more interpretable and parsimonious and are generally more valid beyond the observed range of data when compared with models linear in their parameters (Pinheiro & Bates, 2000, p. 273). Furthermore, it is often the case that the parameters in models nonlinear in their parameters can be easily interpreted, whereas once a polynomial model is beyond quadratic, the meaning of the higher order parameters typically offers little physical or substantive interpretation. An example of such a difference between models nonlinear and linear in their parameters models relates to asymptotes.

Polynomial change models are generally unable to model an asymptote beyond the range of the observed data. Thus, researchers who use polynomial trends must accept the fact that their model will necessarily fail at some point beyond the range of the data actually collected. Such scenarios can potentially lead to inadequate models where impossible values are implied by the model. Illustrations of three models nonlinear in their parameters are provided that are thought to be especially helpful for phenomena in the behavioral, educational, and social sciences. The selected models nonlinear in their parameters are the asymptotic regression change curve, the Gompertz change curve, and the logistic change curve. Although a wide variety of models nonlinear in their parameters exist, these models of change were chosen because they seem especially useful for behavioral, educational, and social science research. A brief introduction to each of these models is provided.

The Asymptotic Regression Change Curve

The general asymptotic regression change curve—sometimes referred to as the negative exponential or exponential growth/decay model—describes a family of regression models where the dependent variable approaches some limiting value as time increases. A general asymptotic regression equation for a single trajectory is given by Stevens (1951) as

$$Y_t = \alpha + \beta\rho^{a_t} + \varepsilon_t, \tag{15}$$

where α is the asymptotic value approached as $a \to \infty$, β is the change in Y_t from $a = 0$ to $a \to \infty$ (i.e., β represents total change in Y_t), and ρ $(0 < \rho < 1)$ is a scalar that defines the factor by which the deviation between Y_t and α is reduced for each unit change of time, thus reflecting the rate at which $Y_t \to \alpha$. Equation 15 can be equivalently written as

$$Y_t = \alpha + \beta \exp(-\gamma a_t) + \varepsilon_t, \tag{16}$$

where $\gamma = -\log(\rho)$ $(0 < \gamma < \infty)$, and can be thought of as a scaling parameter (Stevens, 1951). The top left plot in Figure 1 is provided as an illustrative example of change curves typical of the asymptotic regression model.[5]

The Gompertz Change Curve

The Gompertz curve is nonlinear in its parameters and has been used most often in the biological sciences. The asymmetric sigmoidal ("S" shape) form of the Gompertz change curve offers an option for those who seek to model certain types of sigmoidal trends. The general three-parameter Gompertz model for a single trajectory can be written as

$$Y_t = \alpha \exp(-\exp(\beta - \gamma a_t)) + \varepsilon_t, \tag{17}$$

where α is the positive asymptote as $T \to \infty$ when γ is positive or the negative asymptote as $T \to -\infty$ when γ is negative. The parameters β and γ define the point of inflection on the abscissa at $a = \frac{\beta}{\gamma}$. The point of inflection on the ordinate is at $Y = \frac{\alpha}{\exp(1)}$, which is approximately 37% of the asymptotic value (Ratkowsky, 1983, chap. 4, pp. 163–167; Winsor, 1932). The top right plot in Figure 1 is

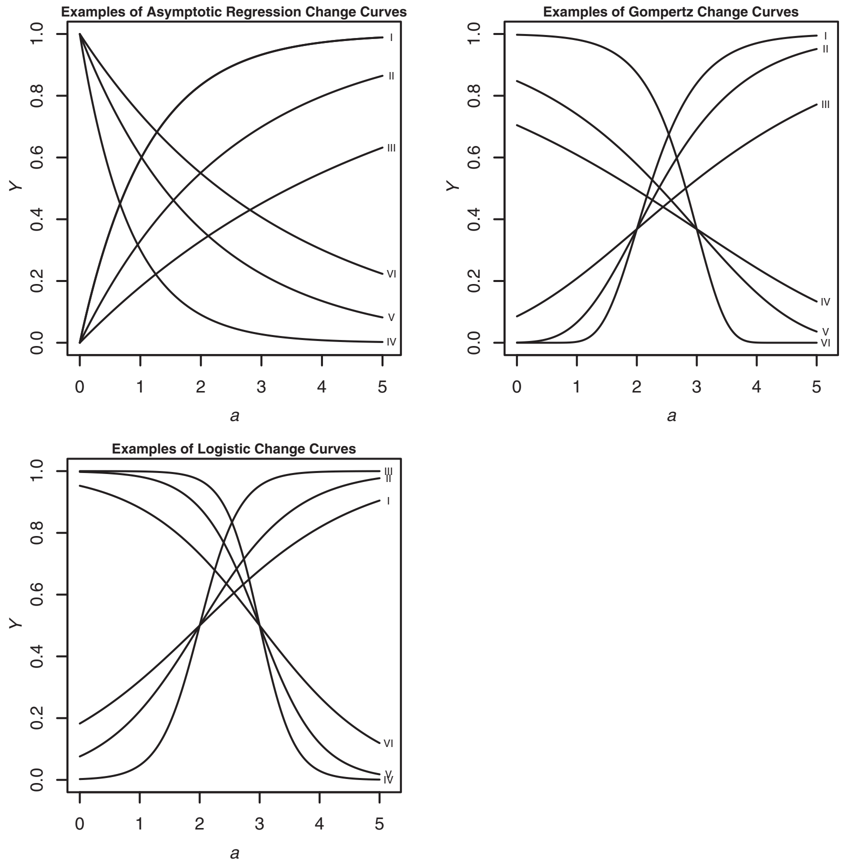

FIGURE 1. *Illustration of models nonlinear in their parameters that are potentially appropriate for modeling phenomena over time in the behavioral, educational, and social sciences.*

provided as an illustrative example of change curves typical of the Gompertz change model.[6]

The Logistic Change Curve

The logistic change model is nonlinear in its parameters and another option for sigmoidal change where the sigmoidal form is symmetric. The general three-parameter logistic change model for a single trajectory can be written as

$$Y_t = \frac{\alpha}{1 + \exp\left(\frac{\beta - a_t}{\gamma}\right)} + \varepsilon_t, \tag{18}$$

where α is the positive asymptote as $T \to \infty$ when γ is positive or the negative asymptote as $T \to -\infty$ when γ is negative. The parameters β and γ define the point of inflection on the abscissa and the curvature, respectively. The point of inflection on the ordinate is at $Y = \frac{\alpha}{2}$, 50% of the asymptotic change (Pinheiro & Bates, 2000; Ratkowsky, 1983, chap. 4, pp. 167–169). The bottom left plot in Figure 1 is provided as an illustrative example of change curves typical of the logistic change model.[7]

Models Nonlinear in Their Parameters for the Behavioral, Educational, and Social Sciences

Given the three types of models nonlinear in their parameters that have been introduced, it is beneficial to relate their functional forms to phenomena encountered in the behavioral, educational, and social sciences. When limits on some behavior, ability, or measure of performance exist, models nonlinear in their parameters will likely offer more realistic representations of reality than do models linear in their parameters.[8] As Cudeck (1996) states, referring to human behavior, "many responses are inherently nonlinear and cannot be treated by a linear mixed [i.e., multilevel] model" (p. 372).

Van Geert (1991) provides a powerful argument for taking seriously the notion of applying models nonlinear in their parameters and states that a variant of the logistic change function "applies to all—or at least a very significant majority—of the variables involved in cognitive growth processes" (p. 45). Van Geert contends that cognitive processes occur under the constraints of limited resources and that these constraints need to be explicit in models of change. Models that are linear in their parameters, the ones most commonly used in the behavioral, educational, and social sciences, are usually untenable models for the phenomenon of interest as there is no constraint on growth or decline. For example, as time increases, the model-implied predictions may be erratic and unrealistic. Such untenable characteristics of unconstrained models are evidenced by models linear in their parameters that continuously "grow" or "decay" as time increases.

In the context of latent variable models, Browne and du Toit (1991) present three different model formulations for data on learning with the goal of isolating interindividual differences in intraindividual learning characteristics and to discern the effects of a covariate on this relationship. Browne and du Toit use the Gompertz change curve of Equation 17 for each of the model formulations, but state that the exponential (a special case of Equation 15) and logistic curves (Equation 18) may also be suitable (p. 56). Using such models nonlinear in their parameters seems reasonable in the sense that learning is not an unlimited cognitive process (van Geert, 1991), but yet instead one that changes little after the task has been nearly mastered and tends to level off at some asymptotic value (Browne & du Toit, 1991, pp. 57–59).

Although numerous types of change in the behavioral, educational, and social sciences likely follow functional forms nonlinear in their parameters, the straight-line change model seems to be used more than any other change model. In fact, Mehta and West (2000) state that "linear growth [i.e., the straight-line change model] is virtually the only form of change that is investigated by substantive researchers" (p. 40). Other reasons exist, but one reason that the straight-line change model is so often used is because change is then described by one parameter, the slope. However, the adequacy of the straight-line change model when the true functional form is something other than a straight line is often suspect and can lead researchers astray when attempting to understand change. For example, making statements about the "average" rate or "typical" amount of change is often appealing. Statements about the "average" rate or "typical" amount of change are often based on the estimated slope from the straight-line change model. The next section explores the relationship between the slope from the straight-line change model and the ARC.

The Regression Coefficient From the Straight-Line Change Model and the Average Rate of Change

The slope from the straight-line change model implied by Equation 14 has been labeled and/or interpreted as the ARC for an individual trajectory. Evidence of this is available by examining how some authors determine and use the term *average rate of change*. It is thus important to clarify the technical meaning of the ARC so that substantive researchers do not (a) ignore searching for the true functional form of change, (b) "fall back" on the straight-line change model, and/or (c) interpret biased estimates of the ARC or mean ARC across individuals.

A commonly used but potentially confusing statement regarding the ARC occurs when the average rate of change is presented and interpreted in MLMs. This "average rate of change," however, is generally not the ARC examined in this article. When fitting the straight-line change model in the context of MLM, each individual is typically allowed a unique value for their slope over time, as well as a unique intercept. As previously stated, the expected value (i.e., the mean) of each parameter across all individuals is known as a fixed effect. Recall that the fixed effect for the slope is represented in Equation 11 by β_{10}. In straight-line change models, this parameter is often referred to as the average rate of change (e.g., Laird & Wang, 1990, p. 405; Raudenbush & Bryk, 2002, p. 184; Raudenbush & Xiao-Feng, 2001, p. 387) because it is literally the mean of all individual slope (i.e., rate of change) estimates. Authors who use the term average rate of change when referring to the fixed effect are not wrong, *provided* the average rate of change is not interpreted as the grand mean of the instantaneous rate of change for the individual trajectories over time, but as the mean of the individual slopes. As will be shown momentarily, in general $\beta_{SLCM} \neq \text{ARC}$ and $\beta_{10} \neq \mu_{\text{ARC}}$, where μ_{ARC} is the population mean of the individual ARC values.

In summary, β_{10} (Equation 11) is the mean slope across all individuals in some population; however, β_{10} generally does not represent the mean ARC across individuals, nor does β_{SLCM} (Equation 14) generally represent the ARC for an individual (as will be shown in the next section). The belief that the slope from the straight-line change model is always equal to the ARC is explicit in some work and implicit in the interpretations of many others. The overall group effect for the rate of change, although it is an averaged value, is not generally a measure of the overall ARC across individuals.

The Discrepancy Between the Regression Coefficient From the Straight-Line Change Model and the Average Rate of Change

We describe the potential discrepancy between the regression coefficient and the ARC by two parameters. For fixed values of time, the first parameter that describes the discrepancy is the *bias,* which is operationally defined as

$$\mathrm{B} = E[\beta_{SLCM} | f(a)] - \frac{E[(Y_T - Y_1) | f(a)]}{a_T - a_1} = \beta_{SLCM} - \mathrm{ARC}, \tag{19}$$

where Y_t is conditional on the true functional form of change and $E[\cdot]$ represents the expected value of the random variable in brackets. Bias, as defined here, is analogous to its definition in an estimation context, whereas the formal definition of bias is the expected value of the difference between an estimator and the parameter it estimates (e.g., Rozeboom, 1966). For fixed values of time, the second parameter that describes the discrepancy is the *discrepancy factor* and is operationally defined as

$$\Psi = \frac{E[\beta_{SLCM} | f(a)]}{\dfrac{E[(Y_T - Y_1) | f(a)]}{a_T - a_1}} = \frac{\beta_{SLCM}}{\mathrm{ARC}}, \tag{20}$$

where again Y_t is conditional on the true functional form of change.

In situations where $\mathrm{B} = 0$ (implying $\Psi = 1$), interpreting β_{SLCM} as the ARC yields no inconsistency in research conclusions or interpretation. However, when $\mathrm{B} \neq 0$ (by implication $\Psi \neq 1$), conceptualizing β_{SLCM} as the ARC may be problematic and can potentially lead to misinformed conclusions regarding intraindividual change and interindividual change, as well as group differences in change. Although there is a one-to-one correspondence between B and Ψ, both values are helpful for interpretation. Depending on the particular situation, a seemingly small bias could have a large discrepancy factor, or vice versa. Furthermore, the bias is not invariant with respect to transformation. Although the value of B is often more straightforward to interpret than Ψ, it is also potentially arbitrary owing to rescaling time and/or the dependent variable. Therefore, it is helpful to base the developments in the article on both forms of discrepancy. For example, suppose $\beta_{SLCM} = 100$ and $\mathrm{ARC} = 90$, yet after rescaling $\beta_{SLCM} = 1$ and $\mathrm{ARC} = .90$. Although there is a dramatic drop in B (10 compared with .10), the Ψ is left unchanged (1.11).

Before examining B and Ψ, it is first helpful to realize that any functional form can generally be represented by a power series, such that the sum of squared deviations between values of the true function and the values approximated by the power series can be made to be infinitesimally small by adding enough polynomial powers and coefficients (Finney et al., 2001, chap. 8; Stewart, 1998, section 8.6). A power series in the longitudinal context is a limiting sum of coefficients multiplied by positive integer powers of time. Such a power series is given as

$$f(a_t) = \lim_{M \to \infty} \sum_{m=0}^{M} \left(\lambda_m a_t^m\right), \tag{21}$$

where λ_m is the coefficient ($-\infty < \lambda_m < \infty$) for the mth power ($m = 0, \ldots, M$).

Although a power series is infinite by definition, known functional forms can be represented by finite sums. In general, the following finite sum can be used to impose or approximate some known or unknown functional form of change and is more general than the power series, as the powers of time are not limited to nonnegative integers (as is the definition of a polynomial change model), but can take on any real values:

$$f(a_t) = \sum_{k=1}^{K} \left(\lambda_k a_t^{\Upsilon_k}\right), \tag{22}$$

where $\Upsilon_k(-\infty < \Upsilon_k < \infty)$ represents the kth ($k = 1, \ldots, K;\ 1 \leq K < \infty$) power.[9] General results emerge for B and Ψ by realizing that functional forms of change can generally be represented by Equation 22. The following section uses this fact when examining B and Ψ for any model linear in its parameters.

Examining the Bias in the Average Rate of Change

In the context of continuous time models, Kelley and Maxwell (2006) give expressions for B and Ψ for the most general case and show that $\text{B} = 0$ (implying $\Psi = 1$) in general only when the functional form of change consists of only linear and/or quadratic components. From a methodological perspective, the work on continuous time models is very interesting; however, in practice time is nearly always measured at discrete occasions. Thus, it is important to examine B and Ψ when time is limited to a finite number of measurement occasions. For finite occasions of measurement, the general equation for the bias can be written as

$$\text{B} = \frac{\sum_{t=1}^{T} (Y_t - \mu_Y)(a_t - \mu_a)}{\sum_{t=1}^{T} (a_t - \mu_a)^2} - \frac{Y_T - Y_1}{a_T - a_1}. \tag{23}$$

The equation for Ψ in the general case is given by

$$\Psi = \frac{\frac{\sum_{t=1}^{T}(Y_t - \mu_Y)(a_t - \mu_a)}{\sum_{t=1}^{T}(a_t - \mu_a)^2}}{\frac{(Y_T - Y_1)}{(a_T - a_1)}} = \frac{(a_T - a_1)\sum_{t=1}^{T}(Y_t - \mu_Y)(a_t - \mu_a)}{(Y_T - Y_1)\sum_{t=1}^{T}(a_t - \mu_a)^2}. \tag{24}$$

For equally spaced occasions of measurement, Equations 23 and 24 can be simplified by realizing that all of the values of time can be written in terms of a_1 and a_T. Knowing T, a_1, and a_T, the remaining $T-2$ values of time can be expressed as

$$a_t = a_1 + (t-1)\frac{a_T - a_1}{T-1}. \tag{25}$$

Combining Equation 25 with Equation 23 and Equation 24 allows B and Ψ to be derived for arbitrary values of T in situations with equally spaced measurement occasions. Given T equally spaced occasions of measurement, B can be written as

$$\mathrm{B}_T = \frac{\sum_{t=1}^{T} Y_t\left[\left(a_1 + (t-1)\frac{a_T - a_1}{T-1}\right) - \mu_a\right]}{\sum_{t=1}^{T}\left[\left(a_1 + (t-1)\frac{a_T - a_1}{T-1}\right) - \mu_a\right]^2} - \frac{(Y_T - Y_1)}{(a_T - a_1)}, \tag{26}$$

whereas Ψ can be written as

$$\Psi_T = \frac{(a_T - a_1)\sum_{t=1}^{T} Y_t\left[\left(a_1 + (t-1)\frac{a_T - a_1}{T-1}\right) - \mu_a\right]}{(Y_T - Y_1)\sum_{t=1}^{T}\left[\left(a_1 + (t-1)\frac{a_T - a_1}{T-1}\right) - \mu_a\right]^2}. \tag{27}$$

From the previous results and the derivations given in the appendix, it can be shown that when the true functional form follows a quadratic change model and the a_t values are symmetric about μ_a (and thus μ_a is equivalent to the median value of time), the ARC (Equation A3) and β_{SLCM} (Equation A13) are equivalent:

$$\mathrm{ARC}_{QCM} = \beta_{SLCM|QCM} = \beta_1 + \beta_2(a_T + a_1), \tag{28}$$

where ARC_{QCM} and $\beta_{SLCM|QCM}$ are the ARC and β_{SLCM} when data are governed by a quadratic functional form with β_1 and β_2 representing the linear and quadratic coefficients, respectively, in a quadratic change model. It is important to remember that Equation 28 was proven only for situations where the true change was governed by a quadratic change model and when the values of a_t were symmetric about μ_a. The equality in Equation 28 holds for population values as well as sample values. An obvious special case of a_t being symmetric about μ_a is when time is equally spaced. In fact, Siegel (1975) shows similar findings for quadratic change curves that have equally spaced time points, but does not relate $\beta_{SLCM|QCM}$ to other functional forms or to multilevel models (as they were not fully developed at that time). Thus, when a_t is symmetric about μ_a and when change is governed by a quadratic change model (a special case being a straight-line change model when $\beta_2 = 0$), $ARC_{QCM} = \beta_{SLCM|QCM}$. In general, this equality does not hold for other functional forms of change and equally spaced measurement occasions. Special cases of other functional forms lead to β_{SLCM} that equal ARC, but generally the two concepts are not equal to one another and represent fundamentally different quantities.[10]

By setting $B = 0$ and $\Psi = 1$, general results for the specific value of T can be obtained showing when β_{SLCM} and ARC are equal. Table 1 shows the general B and Ψ for 2 to 12 equally spaced measurements with arbitrary a_1 and a_T values. Notice that regardless of the true functional form of change, for two or three equally spaced values of time, the slope from the straight-line growth model and the ARC are always equivalent. Thus, for $T = 2$ or $T = 3$, one need not worry about any discrepancy that may arise if β_{SLCM} is labeled and interpreted as the ARC, provided the time points are equally spaced. However, for $T \geq 4$, the ARC generally does not generally equal β_{SLCM}.

Table 1 can be used in at least two ways. Suppose that some functional form of growth, the initial value of time, and T are known for equally spaced values of time. The value of B and Ψ can then be determined with the expressions given in Table 1 ($2 \leq T \leq 12$). For any case where B does not equal zero (implying $\Psi \neq 1$), the extent of the discrepancy will be known for the functional form of interest. In the event that the true functional form of growth is known exactly, the expected value for β_{SLCM} could be "corrected" by scaling the expected β_{SLCM} for B to equal 0 (equivalently, $\Psi = 1$), such that the scaled regression coefficient could be used as an unbiased estimate of the ARC.[11] Furthermore, for a given T, B could be set to 0 (or Ψ set to 1) to discern under what circumstances the expected β_{SLCM} will equal ARC. For example, when $T = 4$, $\beta_{SLCM} = \text{ARC}$ whenever $Y_1 - Y_4 = 3(Y_2 - Y_3)$. Thus, there is 0 bias whenever $-Y_1 + 3Y_2 - 3Y_3 + Y_4 = 0$.

Note that the coefficients in the bias equation when $T = 4$(-1, 3, -3, and 1 for Y_1 through Y_4, respectively) have an interpretation beyond that of the present context. In fact, the coefficients (or the coefficients scaled by a constant) correspond to orthogonal polynomial coefficients in the context of trend analysis. Specifically, for four levels of a quantitative factor in an analysis of variance context, the coefficients in the bias equation for $T = 4$ correspond to the

Table 1

General Equations for B and Ψ for 2 to 12 Equally Spaced Time Points With Arbitrary Initial and End Points

T	Bias (B)	Discrepancy Factor (Ψ)
2	0	1
3	0	1
4	$\frac{(Y_1-Y_4)+3(Y_3-Y_2)}{10(T_4-T_1)}$	$\frac{3}{10}\frac{3(Y_1-Y_4)+(Y_2-Y_3)}{Y_1-Y_4}$
5	$\frac{(Y_1-Y_5)+2(Y_4-Y_2)}{5(T_5-T_1)}$	$\frac{2}{5}\frac{2(Y_5-Y_1)+(Y_4-Y_2)}{Y_5-Y_1}$
6	$\frac{2(Y_1-Y_6)+3(Y_5-Y_2)+(Y_4-Y_3)}{7(T_6-T_1)}$	$\frac{5(Y_6-Y_1)+3(Y_5-Y_2)+(Y_4-Y_3)}{7(Y_6-Y_1)}$
7	$\frac{5(Y_1-Y_7)+6(Y_6-Y_2)+3(Y_5-Y_3)}{14(T_7-T_1)}$	$\frac{3}{14}\frac{3(Y_7-Y_1)+2(Y_6-Y_2)+(Y_5-Y_3)}{Y_7-Y_1}$
8	$\frac{5(Y_1-Y_8-Y_7-Y_2)+3(Y_6-Y_3)+(Y_5-Y_4)}{12(T_8-T_1)}$	$\frac{7(Y_8-Y_1)+5(Y_7-Y_2)+3(Y_6-Y_3)+(Y_5-Y_4)}{12(Y_8-Y_1)}$
9	$\frac{7(Y_1-Y_9)+6(Y_8-Y_2)+4(Y_7-Y_3)+2(Y_6-Y_4)}{15(T_9-T_1)}$	$\frac{2}{15}\frac{4(Y_9-Y_1)+3(Y_8-Y_2)+2(Y_7-Y_3)+(Y_6-Y_4)}{Y_9-Y_1}$
10	$\frac{28(Y_1-Y_{10})+21(Y_9-Y_2)+15(Y_8-Y_3)+9(Y_7-Y_4)+3(Y_6-Y_5)}{55(T_{10}-T_1)}$	$\frac{3}{55}\frac{9(Y_{10}-Y_1)+7(Y_9-Y_2)+5(Y_8-Y_3)+3(Y_7-Y_4)+(Y_6-Y_5)}{Y_{10}-Y_1}$
11	$\frac{6(Y_1-Y_{11})+4(Y_{10}-Y_2)+3(Y_9-Y_3)+2(Y_8-Y_4)+(Y_7-Y_5)}{11(T_{11}-T_1)}$	$\frac{5(Y_{11}-Y_1)+4(Y_{10}-Y_2)+3(Y_9-Y_3)+2(Y_8-Y_4)+(Y_7-Y_5)}{11(Y_{11}-Y_1)}$
12	$\frac{15(Y_1-Y_{12})+9(Y_{11}-Y_2)+7(Y_{10}-Y_3)+5(Y_9-Y_4)+3(Y_8-Y_5)+(Y_7-Y_6)}{26(T_{12}-T_1)}$	$\frac{11(Y_{12}-Y_1)+9(Y_{11}-Y_2)+7(Y_{10}-Y_3)+5(Y_9-Y_4)+3(Y_8-Y_5)+(Y_7-Y_6)}{26(Y_{12}-Y_1)}$

coefficients for the test of the cubic trend (tables of which can often be found in experimental design books). When $T = 4$, fitting a third-degree polynomial growth model ensures a perfect fit. For the expected value of β_{SLCM} to be equal to the ARC when $T = 4$, there must be no cubic trend. That is, the coefficients for the bias from Table 1 multiplied by the appropriate expected dependent variables must equal zero. Because any strictly linear trend or a combination of linear and/or quadratic trends (when measurement occasions are equally spaced) yields an unbiased estimate of the ARC as measured from β_{SLCM}, the sum of the coefficients multiplied by the appropriate Y_t will equal zero. In cases where $T = 4$ (and indeed when $T > 4$) with no cubic trend, β_{SLCM} will not be an unbiased estimate of the ARC.

A similar interpretation for values of $T > 4$ also exists, namely that when a strictly linear trend exists, when occasions of measurement are equally spaced and a quadratic trend exists, or when occasions of measurement are equally spaced and a combination of linear and quadratic trends exists, the sum of the orthogonal polynomials multiplied by the appropriate Y_t will be zero for all trends greater than quadratic. Thus, when measurement occasions are equally spaced, the sum of the orthogonal polynomials multiplied by the appropriate Y_t is zero for trends greater than quadratic, implying that only a linear and/or a quadratic trend exists, β_{SLCM} will exactly equal ARC. The point is that when a strictly linear trend exists, when measurement occasions are equally spaced and a strictly quadratic trend exists, or some combination of linear and quadratic trends exists for equally spaced measurement occasions, the bias in Table 1 will be zero. When there are higher order trends beyond quadratic, the equations in Table 1 will generally, if not always, differ from 0 for B and from 1 for Ψ, implying that β_{SLCM} will differ from the ARC.

Given the general bias in using β_{SLCM} as an estimator of ARC for a single trajectory, there is no reason to believe that taking the mean β_{SLCM} would lead to the mean ARC. In fact, assuming all individuals have the same fixed-effect parameters conditional on appropriate covariant(s) and/or grouping variable(s), which is assumed in standard multilevel models (cf. Muthén, 2001; Muthén et al., 2002),

$$E[\beta_{SLCM}|\mathbf{X}] \neq E[\text{ARC}|\mathbf{X}], \tag{29}$$

where $\mathbf{X}$ is the appropriate conditioning factor(s) (e.g., group membership, values of covariates, etc.). Thus, conceptualizing the mean ARC as the slope from a multilevel model will generally lead to the mean ARC being biased. This is the case because E[ARC] is simply the mean of the ARC (the expected value of an expected value is simply the mean of that value). Furthermore, if the mean of a single trajectory is biased, and because $E[\beta_{SLCM}|\mathbf{X}]$ is a constant in any particular situation, there is no reason to believe that positive and negative biases would cancel to yield unbiased estimates.

Application to Empirical Data

Gardner's (1958) data, which were used by Tucker (1960) to illustrate novel methods for dealing with change, are used here to illustrate empirically some of the analytic results. The data consist of 24 participants, where each participant responded to 420 presentations of one of four letters. The task was to identify the next letter that was to be presented. The "target" letter had a .70 probability of being presented, and the three distracter letters each had a probability of .10. After some number of trials, and what would be optimal to maximize the chances of a correct response, participants tended to choose the target letter with a high probability (if not always). Many of the trajectories resembled the asymptomatic regression or the logistic change curve. A logistic change curve was used to fit the data in part owing to learning theory and in part owing to visual inspection.[12]

Using R (R Development Core Team, 2007) and the nlme package (Pinheiro, Bates, DebRoy, & Sarkar, 2007), a multilevel logistic change curve was fit to the data. The fixed-effect change curve, where each parameter also had an associated unique effect for each individual and assuming a homogeneous and independent error structure, was

$$Y_t' = \frac{16.06}{1 + \exp\left(\frac{2.09 - a_t}{1.62}\right)}, \tag{30}$$

where Y_t' is the predicted score at the tth time point and each of the coefficients was statistically significant ($p < .01$ for each fixed effect).

Recall that this article did not discuss ways to estimate the ARC for an individual or for a group of individuals. Nevertheless, the mean ARC across individuals was estimated for purposes of the example in three ways: (a) The first method used the difference scores of the empirical Bayes estimates obtained from fitting the logistic change curve, (b) the second method used the simple difference scores divided by the change in time, and (c) the third method used the difference score based on the parameter estimates from Equation 30 as if they were the parameters themselves. The slope from the straight-line change model for the first two methods was estimated for each individual in the standard way, and for the third method the slope was based on the model-implied trajectory using the fixed effects estimated from the original time basis. For simplicity, we focus only on Ψ. The estimated Ψ was found by taking the mean of the 24 individual Ψ values for the first two methods and by dividing the slope by the estimated ARC for the final method. Whereas the intermediate time points were implicitly used in the first and third methods (as they are based on the parameter estimates themselves), all intermediate time points are ignored when calculating the simple difference score in the second method.

The values for $\hat{\Psi}_1$, $\hat{\Psi}_2$, and $\hat{\Psi}_3$ were .585, .617, and .681, respectively, for Methods 1, 2, and 3. Although the third method is not as discrepant as the first two, using the slope from the straight-line change model underestimates the

ARC in this situation by more than 30%. Thus, given the three estimates of Ψ, at best, using the slope from the straight-line change model underestimates the ARC by more than 30% for these data, which is unacceptable by essentially any standard of estimation quality. Although at present the optimal way to estimate ARC, Ψ, and B is not known, the three methods used here are each reasonable and yield consistent results.

Discussion

The ARC has been, both implicitly and explicitly, conceptualized as the regression coefficient from the straight-line change model in the methodological literature as well as in applications of straight-line change models in substantive research. However, as this article has shown, the slope from the straight-line change model and the average rate of change are not generally equal to one another for an individual trajectory. The bias between the two values can be positive or negative, potentially yielding misleading conclusions regarding change over time in the context of longitudinal data analysis. One or more of the following four sufficient conditions being met implies that the regression coefficient from the straight-line change model yields an unbiased estimate of the ARC:

1. The true functional form of change consists of only a linear component.
2. The true functional form of change consists only of some combination of linear and/or quadratic components with occasions of measurement symmetric about the mean of time.
3. Change is described by two time points.
4. Change is described by three symmetric (and thus equally spaced as $T = 3$) time points.

Special cases of Condition 2 are when time is equally spaced and change is governed by a completely linear, completely quadratic, or some combination of linear and quadratic change. Of course, Condition 3 and Condition 4 can be considered special cases of Condition 1 or Condition 2, but they also hold true when the functional form of change is some arbitrary or unknown function. The reason Condition 3 and Condition 4 hold true is because there are no intermediate time points to alter β_{SLCM}. Because $a_2 - \mu_a$ is in the numerator of β_{SLCM} when $T = 3$, Y_2 receives a weight of zero because $a_2 - \mu_a$. Thus, when $T = 2$ or $T = 3$ and time points are symmetric about μ_a, β_{SLCM} reduces to $(Y_T - Y_1)/(a_T - a_1)$. Although not proven here, the discrepancy between the true β_{SLCM} and the ARC will increase as T increases; the case where time is continuous yields the greatest amount of bias. This fact is contrary to conventional wisdom because it is generally thought that as more time points are included, better estimates are obtained. Actually, better estimates are obtained for β_{SLCM}, but because this quantity does not generally equal the ARC, better estimates for the ARC are not obtained. When there are many time points, the precision of β_{SLCM} is improved, but when it is used to estimate the ARC, what is obtained is a precise estimate of a biased quantity.

Although not proved here, when the time interval is fixed, the reliability of the ARC estimated by way of the difference score ($Y_T - Y_1$) divided by the change in time ($a_T - a_1$) is equal to the reliability of the difference score; this also provides an unbiased estimate of the ARC (Rogosa et al., 1982). When estimating the ARC by way of Equation 6, all intermediate time points are ignored. Ignoring data is not generally advisable, and using intermediate time points in an informative way could lead to increased reliability and accuracy when estimating the ARC. Thus, future work could investigate various procedures for efficiently and accurately estimating the ARC. One reasonable way to estimate the ARC for an individual is to fit the correct functional form of change and use the predicted values of Y_{1i} and Y_{Ti}—which are denoted Y'_{1i} and Y'_{Ti}, respectively—in place of the numerator of Equation 6. Such a method implicitly uses the specified functional form and intermediate values for calculating Y'_{1i} and Y'_{Ti}. For more than one individual, the predicted scores used can be the empirical Bayes estimates from a MLM context, which many times have desirable properties.

We believe that the present article adds meaningfully to the analysis of change literature by clarifying and extending the current understanding of the ARC. By better understanding the ARC, researchers may realize that their questions are or are not appropriately addressed by such a measure. The ARC may not be very helpful when the process of change is of interest. However, if a description of the mean of the instantaneous rates of change across time is of interest, the ARC will provide such a measure. It is believed that a better understanding of the ARC, and what it is not, will help researchers better describe and understand change over time.

Appendix
Proof That $\beta_{SLCM} = \mathrm{ARC}_{QCM}$ When Y Is a Quadratic Function of Equally Spaced Time Points

A proof that β_{SLCM} equals ARC_{QCM} when Y is a quadratic function of equally spaced time points can be seen by first realizing the average rate of change (ARC) for a quadratic change model (i.e., $Y_t = \beta_0 + \beta_1 a_t + \beta_2 a_t^2$) can be written as

$$\mathrm{ARC}_{QCM} = \frac{Y_T - Y_1}{a_T - a_1} = \frac{(\beta_0 + \beta_1 a_T + \beta_2 a_T^2) - (\beta_0 + \beta_1 a_1 + \beta_2 a_1^2)}{a_T - a_1}. \quad \text{(A1)}$$

Equation A1 can first be reduced (by canceling the β_0s) and then rewritten:

$$\mathrm{ARC}_{QCM} = \frac{\beta_1 (a_T - a_1) + \beta_2 (a_T^2 - a_1^2)}{a_T - a_1}. \quad \text{(A2)}$$

The quantity $(a_T - a_1)$ can be factored in the numerator, which cancels with the denominator leading to the following:

$$\mathrm{ARC}_{QCM} = \beta_1 + \beta_2(a_T + a_1). \tag{A3}$$

Thus, Equation A3 is the reduced form of the ARC when the true functional form of change is governed by a quadratic change model.

The regression coefficient for the straight-line change model can be written as

$$\beta_{SLCM} = \frac{\sum_{t=1}^{T} Y_t a_t - \mu_a \sum_{t=1}^{T} Y_t}{\sum_{t=1}^{T} a_t^2 - \frac{\left(\sum_{t=1}^{T} a_t\right)^2}{T}}, \tag{A4}$$

which after substituting $\beta_0 + \beta_1 a_t + \beta_2 a_t^2$ for Y_t and defining the denominator of Equation A4 as the sum of squares of a (SSa) can be rewritten as

$$\beta_{SLCM} = \frac{\sum_{t=1}^{T} (\beta_0 + \beta_1 a_t + \beta_2 a_t^2) a_t - \mu_a \sum_{t=1}^{T} (\beta_0 + \beta_1 a_t + \beta_2 a_t^2)}{SSa}. \tag{A5}$$

After distributing a_t and μ_a in the numerator of Equation A4 and then distributing the summation, Equation A5 can be rewritten as

$$\beta_{SLCM} = \frac{(\beta_0 \sum_{t=1}^{T} a_t + \beta_1 \sum_{t=1}^{T} a_t^2 + \beta_2 \sum_{t=1}^{T} a_t^3) - (\beta_0 \mu_a T + \beta_1 \mu_a \sum_{t=1}^{T} a_t + \beta_2 \mu_a \sum_{t=1}^{T} a_t^2)}{SSa}. \tag{A6}$$

Because $\sum_{t=1}^{T} a_t / T = \mu_a$, the terms involving β_0 in each quantity of the numerator reduce, and Equation A6 can be rewritten as

$$\beta_{SLCM} = \frac{\beta_1 (\sum_{t=1}^{T} a_t^2 - \mu_a \sum_{t=1}^{T} a_t) + \beta_2 (\sum_{t=1}^{T} a_t^3 - \mu_a \sum_{t=1}^{T} a_t^2)}{SSa}. \tag{A7}$$

Substituting $\sum_{t=1}^{T} a_t/T$ for μ_a in Equation A7 yields

$$\beta_{SLCM} = \frac{\beta_1 SSa + \beta_2\left(\sum_{t=1}^{T} a_t^3 - \frac{\sum_{t=1}^{T} a_t \sum_{t=1}^{T} a_t^2}{T}\right)}{SSa}. \tag{A8}$$

Realizing that $\sum_{t=1}^{T} (a_t - \mu_a)^3$ is necessarily zero when a_t is symmetric about μ_a, it is helpful to expand this quantity

$$\sum_{t=1}^{T} (a_t - \mu_a)^3 = \sum_{t=1}^{T} a_t^3 - 3\mu_a \sum_{t=1}^{T} a_t^2 + 3\mu_a^2 \sum_{t=1}^{T} a_t - T\mu_a^3 = 0, \tag{A9}$$

and then solve for $\sum_{t=1}^{T} a_t^3$ so that $\sum_{t=1}^{T} a_t^3$ can be replaced in Equation A8 with the following reexpression:

$$\sum_{t=1}^{T} a_t^3 = \mu_a\left(3\sum_{t=1}^{T} a_t^2 - 3\mu_a \sum_{t=1}^{T} a_t + T\mu_a^2\right). \tag{A10}$$

Rewriting Equation A8 by replacing $\sum_{t=1}^{T} a_t^3$ with the right-hand side of Equation A10 yields

$$\beta_{SLCM} = \frac{\beta_1 SSa + \beta_2\left(\mu_a\left(3\sum_{t=1}^{T} a_t^2 - 3\mu_a \sum_{t=1}^{T} a_t + T\mu_a^2\right) - \frac{\sum_{t=1}^{T} a_t \sum_{t=1}^{T} a_t^2}{T}\right)}{SSa}. \tag{A11}$$

Factoring the μ_a from the second quantity in the numerator of Equation A11 and then reducing the remaining terms in the quantity yields

$$\beta_{SLCM} = \frac{\beta_1 SSa + 2\mu_a \beta_2 SSa}{SSa}. \tag{A12}$$

After factoring out SSa and realizing that because time is symmetric about the mean of time $\mu_a = (a_1 + a_T)/2$, Equation A12 reduces to

$$\beta_{SLCM} = \beta_1 + \beta_2(a_T + a_1). \tag{A13}$$

Therefore, ARC_{QCM} from Equation A3 and β_{SLCM} from Equation A13 are equivalent:

$$\beta_{SLCM} = \mathrm{ARC}_{QCM} \tag{A14}$$

when Y is governed by linear and/or quadratic components and the time points are equally spaced.

Notes

1. The "basis of time" can be any variable that is a function of time. For example, the time basis could be time itself, grade level, age, the identifier of a particular measurement occasion, and so forth. The term *time* is used generically throughout this article rather than using *basis of time*, but *time* is meant to be regarded much more generally.

2. Because latent growth curves (Bollen & Curran, 2006; McArdle & Epstein, 1987; Meredith & Tisak, 1990) can be formulated to be equivalent to multilevel models (MLMs) linear in their parameters, the discussion also applies to applications of latent growth curve analysis. See Bauer (2003), Curran (2003), and Willett and Sayer (1994) for a discussion of the relations that exist between MLMs linear in their parameters and latent growth curves.

3. However, by adding additional polynomial trends to a change model, the sum of squared deviations between the predicted scores and the observed scores will necessarily decrease (or at the very least stay the same). In fact, as the number of polynomial trends approach the number of time points, the sum of squared deviations between predicted and observed scores approaches zero (T waves of data can be perfectly fit with $T-1$ polynomial trends). One wants to avoid overparameterization in change models, otherwise the model will account for measurement error in addition to the true relationship (Box, 1984).

4. The Y_t in Equation 14 is technically $\mathrm{E}[Y_t]$ because β_1 is a population parameter. Such an expectation of Y_t is assumed throughout the work for notational ease.

5. The asymptotic regression Curves I, II, and III have an α value of 1, whereas Curves IV, V, and VI have an α value of 0. Curves I, II, and III have a β value of -1, whereas Curves IV, V, and VI have a β value of 1. The γ values for Curves I through VI are, respectively, 0.9, 0.4, 0.2, 1.2, 0.5, and 0.3.

6. All Gompertz curves have an α value of 1. Curves I, II, and III have β values of 2, whereas Curves IV, V, and VI have β values of 3. The γ values for Curves I through VI are, respectively, 1.75, 1, 0.45, -0.35, -0.60, and -2.

7. All logistic curves have an α value of 1. Curves I, II, and III have β values of 2, whereas Curves IV, V, and VI have β values of 3. The γ values for Curves I through VI are, respectively, 0.75, 1.25, 3, -3.5, -2, and -1.

8. Of course, the phenomena of interest may display local linearity, where over the time interval of interest change is well approximated by a straight line. In such a situation the use of a model nonlinear in its parameters is not as compelling. However, it is especially important when using a model linear in its parameters to avoid extrapolation beyond the range of observed data.

9. The intercept of a particular change curve is the sum of the λ_ks whose Υ_k is zero. In the special case where $a \in [0, a_T]$ the intercept is $\sum_{k=1}^{k} \lambda_k 0^{\Upsilon_k}$, which strictly speaking is an indeterminate form when $\Upsilon_k = 0$. However, because of l'Hôpital's Rule, which uses derivatives to evaluate the converging limit of a function that would otherwise be indeterminate under standard algebraic rules, the quantity $0^0 \equiv 1$ by standard conventions (Finney, Weir, & Giordano, 2001, section 7.6; Stewart, 1998, section 4.5). When evaluating the equations given in this section by computer, care should be taken to ensure the particular program defines 0^0 as 1 (rather than returning an error message).

10. For example, any functional form with three equally spaced measurement occasions has a β_{SLCM} that equals average rate of change (ARC).

11. Of course, this second potential use for Table 1 would imply that the true functional form of growth was known exactly. If this were true, it would be better to fit the correct functional form of growth to begin with rather than fitting the straight-line growth model for "interpretational ease."

12. Gardner's learning data are available from the MBESS (Kelley, 2007), R (R Development Core Team, 2007) package with the command "data(Gardner.LD)" after MBESS has been loaded with the command "library(MBESS)".

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