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Diagnosis for Covariance Structure Models by Analyzing the Path

Ke-Hai Yuan, Chrystyna D. Kouros, and Ken Kelley
University of Notre Dame

When a covariance structure model is misspecified, parameter estimates will be affected. It is important to know which estimates are systematically affected and which are not. The approach of analyzing the path is both intuitive and informative for such a purpose. Different from path analysis, analyzing the path uses path tracing and elementary numerical analysis to identify affected parameters when a 1-way or 2-way arrow in a path diagram is omitted. It not only characterizes how a misspecification affects model parameters but also facilitates a good understanding of the relation among different parts of the model. This article introduces and studies this technique and, for commonly used models, provides detailed analysis to identify the directions of change for various model parameters. Examples based on real data show that the technique of analyzing the path can reliably predict the direction of change in parameter estimates even when the true model is unknown. Conditions that interfere with the results are also discussed and advice is provided for its proper application.

Structural equation modeling (SEM) plays an important role in understanding the relations among multivariate data (Bollen, 1989; MacCallum & Austin, 2000). In a typical application of SEM, one has a substantively justified model, which is quite likely unacceptable when statistically tested. The model modification index (Sörbom, 1989), Lagrange multiplier (LM) test (Chou & Bentler, 1990), or

Correspondence should be addressed to Ke-Hai Yuan, Department of Psychology, University of Notre Dame, Notre Dame, IN 46556. E-mail: kyuan@nd.edu

automated specification search procedures (Marcoulides & Drezner, 2001, 2003) can be used to locate paths[1] that may contribute to substantial improvement in model fit.[2] However, model modification cannot guarantee that the resulting model will reproduce the population. Of course, if a causal relationship exists, a valuable model has to reflect the actual relationship among latent and manifest variables. Such an issue involves both the substantive knowledge of the data and statistical tools for model evaluation. In this paper, we will develop a tool to study parameter biases in a substantially justified but misspecified model. A model will be regarded as correctly specified if it reproduces the population covariance matrix when the parameters are at a set of numbers called the population values.

Although a substantive model may barely fit the data, the model can still be accepted based on a few of many fit indexes in standard software. After accepting a model, the parameter estimates will be used for description or inference. For example, researchers interested in a mediation model obtain evidence about the strong mediation effect by testing whether the direct effect (a parameter estimate) is significantly different from zero (Baron & Kenny, 1986; Cole & Maxwell, 2003). Similarly, in the standardized solution, error variance estimates may be used to describe the reliabilities of the measurements. However, when a model is misspecified, parameter estimates and their derived statistics may be biased. Although for a given data set it might be difficult to find a better model that is closer to the data, as measured by a discrepancy function, and theoretically justified we still need to understand the effect of model misspecification, especially the possible biases a misspecification might bring. Only after a good understanding of the possible biases in the model can we apply the SEM methodology more scientifically and reach valid conclusions. In the applied literature many parameter estimates from grossly misspecified models are unduly elaborated, not because better models do not exist but because the SEM literature lacks effective techniques for fully understanding the consequence of a misspecified model. The goal of this article is to provide intuitive and valuable tools for model diagnosis so that SEM can be a better research methodology.

The effect of model misspecification has been studied by many authors (Bentler & Chou, 1993; Bollen & Ting, 2000; Cole & Maxwell, 2003; Cragg, 1968; Farley & Reddy, 1987; Gallini, 1983; Hausman, 1978; Luijben, Boomsma, & Molenaar, 1988; Raykov & Penev, 2002; Saris, Satorra, & Sörbom, 1987;

[1]We call any one-way or two-way arrow in a path diagram a *path*. A path represents either a fixed or a free parameter in the model. A nonzero path corresponds to a nonzero value of the parameter.

[2]Saris, Satorra, and Sörbom (1987), Luijben and Boomsma (1988), and Kaplan (1990) suggested that the model modification index or LM test should be used in conjunction with expected parameter change (EPC) or standardized EPC. Bentler (1990) and Bollen (1990) provided alternative views about EPC/standardized EPC.

Silvia & MacCallum, 1988; Sörbom, 1975; Yuan & Bentler, 2004). For a covariance structural model $\mathbf{M}(\boldsymbol{\theta})$, Yuan, Marshall, and Bentler (2003) observed that the set of parameter estimates $\hat{\boldsymbol{\theta}}$ is a continuous function of the sample covariance matrix $\mathbf{S}$ defined through an estimating equation. Let this function be $\hat{\boldsymbol{\theta}} = \mathbf{g}(\mathbf{S})$. As the sample size N increases, $\mathbf{S}$ and $\hat{\boldsymbol{\theta}}$ converge in probability to their population values $\boldsymbol{\Sigma}$ and $\boldsymbol{\theta}_*$, respectively; consequently $\boldsymbol{\theta}_* = \mathbf{g}(\boldsymbol{\Sigma})$. The model is correctly specified if $\boldsymbol{\Sigma} = \mathbf{M}(\boldsymbol{\theta})$ for certain $\boldsymbol{\theta}$, denote it as $\boldsymbol{\theta}_0$, then $\boldsymbol{\theta}_0 = \mathbf{g}(\boldsymbol{\Sigma}_0)$, where $\boldsymbol{\Sigma}_0 = \mathbf{M}(\boldsymbol{\theta}_0)$. Thus, the population value of $\hat{\boldsymbol{\theta}}$ corresponding to a correctly specified model is $\boldsymbol{\theta}_0$ and to a misspecified model is $\boldsymbol{\theta}_*$. The same model can be misspecified when $\boldsymbol{\Sigma}$ is different from $\boldsymbol{\Sigma}_0$. Let $\theta = g(\boldsymbol{\Sigma})$ be a component of $\mathbf{g}$. When $\boldsymbol{\Sigma}$ is close to $\boldsymbol{\Sigma}_0$, by applying the mean value theorem in calculus, we have

$$\Delta\theta = \theta_* - \theta_0 \approx \sum_{ij} a_{ij}(\sigma_{ij} - \sigma_{ij0}), \tag{1}$$

where a_{ij} is the partial derivative of g with respect to the ijth element of $\boldsymbol{\Sigma} = (\sigma_{ij})$. Equation 1 implies that when the ijth element of $\boldsymbol{\Sigma}$ is perturbed or slightly changed, the magnitude and sign of a_{ij} decide how θ_* is going to differ from θ_0. Let the bias in a parameter estimate $\hat{\theta}$ be defined as $\theta_* - \theta_0$. When $a_{ij} > 0$, a positive perturbation on σ_{ij} will cause a positive bias in $\hat{\theta}$; when $a_{ij} = 0$, changes in σ_{ij} will have no effect on θ_* or lead to a zero bias in $\hat{\theta}$. The formula in Equation 1 is rigorous but does not permit a thorough understanding of the relation of all the model parameters. Yuan et al. (2003) also briefly introduced the idea of analyzing the path, which provides similar information as Equation 1 but enables better comprehension of the relation of parameters within a model. Unfortunately, the authors only studied a single case when an error covariance is excluded from a confirmatory factor model. This article thoroughly studies the approach of analyzing the path and extends it to other types of models as well as to excluding paths other than error covariances. We also show that analyzing the path can equally apply when variables are excluded from a model. We hope that, unlike Equation 1, the technique developed here will be accessible to psychometricians as well as applied researchers.

Analyzing the path to identify the effect of model misspecification on parameter estimates is different from path analysis, where certain rules are necessary to identify causal effects among a set of related variables (see Boker & McArdle, 2005; Li, 1975; Loehlin, 2004; Wright, 1920). Analyzing the path studies how parameters change in response to a misspecification in the model by applying elementary numerical analysis to model-implied covariances. It only requires a basic knowledge of covariance algebra or path tracing (see Bollen, 1989, pp. 21–22; Loehlin, 2004, pp. 26–27). When a nonzero path between two variables is excluded, the model is not adequate in explaining the covariances of the corresponding indicators. Existing paths connecting the manifest or latent variables need to explain this ignored covariance. Typically, the shortest path connecting the two responds strongly. Other paths may also be affected due to their sharing parameters. Suppose a path with a positive loading between x_1 and x_2 is ignored and the model-implied covariance by the existing path is $\theta_1\theta_2\theta_3$. Then, to properly explain $\mathrm{Cov}(x_1, x_2)$, at least one of θ_1, θ_2, and θ_3 has to be greater than its population value corresponding to the correctly specified model. Typically, the parameters θ_1, θ_2, and θ_3 will also appear in the model-implied variances and covariances of the other variables in the model. In such a case, it is possible that one of θ_1, θ_2, and θ_3 becomes much greater, whereas the other two become slightly greater or even smaller to adjust the other parts of the model. In short, if a direct path between two variables is excluded, the other paths in the model indirectly connecting these two variables will need to "make up" for the ignored covariance, resulting in potential biases in parameter estimates of other paths in the model. The technique of analyzing the path is useful in determining how the different paths in the model may be affected.

Example 1

The data set of Wheaton, Muthén, Alwin, and Summers (1977) contains two indicators of alienation, anomie, and powerlessness, which were measured at both 1967 and 1971. The measured background variables are education and occupational status index. This data set has been used as an example in SEM software manuals and textbooks (e.g., Bentler, 1995; Bollen, 1989; Jöreskog & Sörbom, 1993). The full model for the six variables as presented in the literature is represented by Figure 1. The substantive interest is the stability of alienation across time as represented by the path coefficient γ_3. Notice that, in Figure 1, e_3 and e_5 are correlated and so are e_4 and e_6, which are justified by the nature of longitudinal data. However, correlated errors are not encouraged in general (see MacCallum, Roznowski, & Necowitz, 1992). It is interesting to know the effect of excluding the error covariances on γ_3. Of course, one may fit the models with and without the correlated errors to Wheaton et al.'s (1977) data and compare the parameter estimates. Due to sampling error however, the change in parameter estimates may not give us a clear picture of the population value. Also, comparing the estimates does not facilitate a good understanding of the interrelation of all the paths in Figure 1. The technique of analyzing the path will allow us to predict the direction of parameter change before actually running the model. It also allows us to thoroughly understand the interrelation among all the model parameters. We return to this example after presenting the technique of analyzing the path.

Although there exist many methods for parameter estimation (Yuan & Bentler, 2007b), the normal distribution-based maximum likelihood (ML) is the default

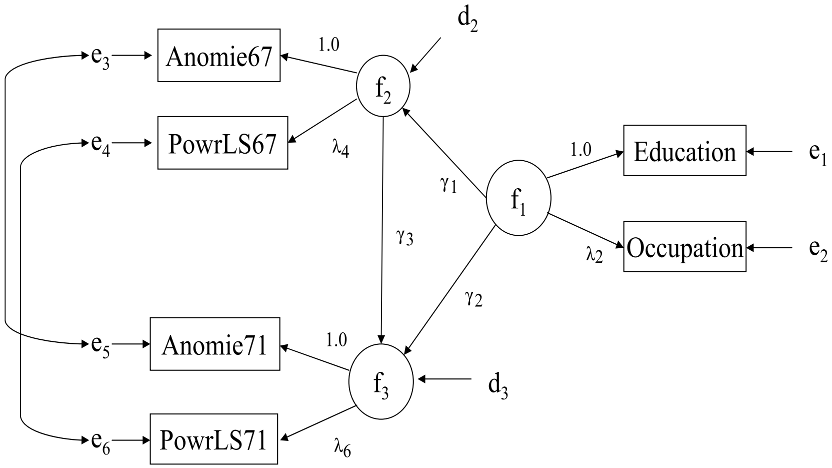

FIGURE 1 The path diagram for modeling the stability of alienation.

procedure in essentially all SEM programs (Amos, EQS, LISREL, M*plus*, Mx, SAS Calis, sem) and also the most commonly used procedure in practice. We mainly consider biases in the ML estimates. Throughout the text we denote $\mathbf{S} = (s_{ij})$ as the unbiased sample counterparts of $\boldsymbol{\Sigma}$ and $\mathbf{M} = (m_{ij})$ as the model-implied covariance matrix. For covariances involving latent variables, we use subscripts to distinguish them from the covariances between manifest variables as in $m_{x_1 f_1}$ and $\sigma_{e_1 e_2}$. We use λ, ϕ, ψ, γ, and φ for the parameter of factor loadings, factor variances and covariances, error variances and covariances, path coefficients between latent variables, and the variances of disturbances, respectively. Their estimates will be $\hat{\lambda}$, $\hat{\phi}$, $\hat{\psi}$, $\hat{\gamma}$, and $\hat{\varphi}$. The population value of the parameter $\boldsymbol{\theta}$ corresponding to model $\mathbf{M}(\boldsymbol{\theta})$ will be denoted as $\boldsymbol{\theta}_*$ in general and as $\boldsymbol{\theta}_0$ when $\boldsymbol{\Sigma} = \mathbf{M}(\boldsymbol{\theta}_0)$. For example, λ_{10} is the value for factor loading λ_1 in a correctly specified model and λ_{1*} is the value of λ_1 in a misspecified model. When the model is misspecified, $\hat{\boldsymbol{\theta}}$ will approach $\boldsymbol{\theta}_*$ as the sample size N increases. For a parameter θ, its estimate $\hat{\theta}$ will be different from θ_0 in general, the systematic difference or bias is $\theta_* - \theta_0$, and the difference due to sampling errors or a finite sample size is $\hat{\theta} - \theta_*$. Because $\hat{\theta}$ always contains sampling error, even for a correctly specified model, we discuss the change in θ_* when a model is misspecified. We always compare θ_* to θ_0. A greater θ_* means $\theta_* - \theta_0 > 0$ and a smaller one means $\theta_* - \theta_0 < 0$. Thus, a greater θ_* implies $\hat{\theta}$ contains a positive bias, whereas a smaller θ_* implies $\hat{\theta}$ contains a negative bias. The biases in $\hat{m} = m(\hat{\boldsymbol{\theta}})$ will be $m_* - \sigma$; for example, $m_{12*} - \sigma_{12}$ is the bias in $\hat{m}_{12}$.

An intact parameter, variance, or covariance means the value of the parameter, variance, or covariance remains the same when the model changes.

Our aim is to illustrate the technique of analyzing the path using simple confirmatory factor models and SEM models so that readers will master the technique and effectively apply it in model diagnosis. The next section studies the effect of model misspecification on parameter biases in two confirmatory factor models where analyzing the path is straightforward. After that, we study the same effect in two structural equation models where the model may involve direct and indirect effects. Then, the analysis becomes relatively more complicated and we also use basic covariance algebra to facilitate the analysis. Finally, we apply the developed technique to study the change in γ_3 in Figure 1 and two other models based on real data that have been previously reported in the literature. Because the commonly used path diagrams and models being estimated correspond to unstandardized parameters, we mainly study the biases in unstandardized parameter estimates. Biases in standardized coefficients as well as conditions that may interfere with the results predicted by analyzing the path are discussed in the concluding section. The analysis is closely related to path diagrams, but only a limited number of them are presented to save space. Readers may draw additional path diagrams, using dashed one-way or two-way arrows to represent omitted paths, which will facilitate the understanding of the analysis.

CONFIRMATORY FACTOR MODELS

We consider a one-factor model and a three-factor model. The same analysis can be applied to other factor models. We only explicitly discuss cases when the excluded path bears a positive loading. With a negative loading, parameters will change in the opposite direction of those corresponding to a positive loading, as implied by Equation 1.

A One-Factor Model

In a one-factor model, excluded paths can only occur between errors.[3] Suppose two errors are correlated and the model is correctly specified when the error covariance is explicitly modeled. Due to the ideal for unidimensional measurements (see Anderson & Gerbing, 1988), the model is set without any correlated errors. We study the effect of such a misspecification by first considering the

[3]Paths between indicators and errors may also exist (see Bentler, 1995, p. 103). These are seldom employed in practice and we do not consider them here.

case where the factor variance is set to 1 (i.e., $\phi_{11} = m_{f_1 f_1} = 1$), and then considering setting a path loading to 1 (i.e., $\lambda_1 = 1$). Both specifications are commonly used for model identification.

In Figure 2, the variance of the factor is set equal to 1 ($\phi_{11} = 1$) and the errors e_1 and e_2 are positively correlated in the population ($\sigma_{e_1 e_2} > 0$); however the path $e_1 \leftrightarrow e_2$ is excluded. The model needs to explain the extra association between x_1 and x_2. The only path for such an association is $x_1 \leftarrow f_1 \rightarrow x_2$ with the model-implied covariance given by $m_{12} = \lambda_1 \lambda_2$. Thus, $\lambda_{1*}\lambda_{2*}$ has to be greater than the population counterpart $\lambda_{10}\lambda_{20}$ corresponding to the correctly specified model. Due to the parallel positions of λ_1 and λ_2 in Figure 2, we must have $\lambda_{1*} > \lambda_{10}$ and $\lambda_{2*} > \lambda_{20}$. In the estimation of the factor model represented by Figure 2, the distance between all the pairs of (s, m) is minimized simultaneously. In estimating λ_1 and λ_2, the system of equations also estimates λ_3, λ_4, and ψ_{11} to ψ_{44}. With the greater λ_{1*} and λ_{2*}, it is necessary to have $\lambda_{3*} < \lambda_{30}$ to explain σ_{13} and σ_{23}. Similarly, $\lambda_{4*} < \lambda_{40}$ needs to hold to explain σ_{14} and σ_{24}. Ideally, it needs $\lambda_{3*}\lambda_{4*} = \lambda_{30}\lambda_{40}$ to fully explain σ_{34}, which is unlikely due to the misspecification. Because the unique variances in Figure 2 are free parameters, they can take any values to explain the variances σ_{ii}, $i = 1$ to 4. Because $\lambda_{1*} > \lambda_{10}$, $\lambda_{2*} > \lambda_{20}$, $\lambda_{3*} < \lambda_{30}$, and $\lambda_{4*} < \lambda_{40}$, we have $\psi_{11*} < \psi_{110}$, $\psi_{22*} < \psi_{220}$, $\psi_{33*} > \psi_{330}$ and $\psi_{44*} > \psi_{440}$. Actually, in the ML procedure, λ_{i*} and ψ_{ii*} satisfy $\lambda_{i*}^2 + \psi_{ii*} = \sigma_{ii}$. A greater λ_{i*} implies

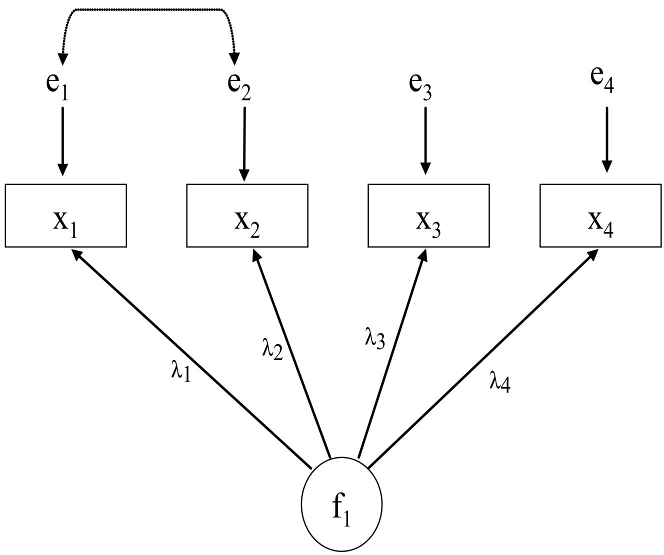

FIGURE 2 The path diagram for a one-factor model with $\phi_{11} = 1$.

a smaller ψ_{ii*} and vice versa.[4] The results of the analysis for Figure 2 are summarized by

λ_{1*}	λ_{2*}	λ_{3*}	λ_{4*}	ψ_{11*}	ψ_{22*}	ψ_{33*}	ψ_{44*}
+	+	−	−	−	−	+	+

where + means greater and − means smaller. In summary, when two errors in Figure 2 are positively correlated in the population, the factor loadings corresponding to the variables will be positively biased, and the error variances will be negatively biased. The other factor loadings will be negatively biased and error variances will be positively biased.

When $\lambda_1 = 1$ is used for model identification as in Figure 3, the λ_{i0} or λ_{i*} will be generally different from those in Figure 2 for the given σ's, but they are still uniquely defined (see Steiger, 2002). When $\sigma_{e_1e_2} > 0$ and $e_1 \leftrightarrow e_2$ is not included, the path $x_1 \leftarrow f_1 \rightarrow x_2$ has to explain the extra association using $m_{12} = \lambda_2\phi_{11}$. Thus, $\lambda_{2*}\phi_{11*} > \lambda_{20}\phi_{110}$. Notice that $m_{23} = \lambda_2\lambda_3\phi_{11}$ and $m_{24} = \lambda_2\lambda_4\phi_{11}$. The greater $\lambda_{2*}\phi_{11*}$ leads to $\lambda_{3*} < \lambda_{30}$ and $\lambda_{4*} < \lambda_{40}$. Due to $m_{13} = \lambda_3\phi_{11}$, $m_{14} = \lambda_4\phi_{11}$ and smaller λ_{3*} and λ_{4*}, ϕ_{11*} and ϕ_{110} satisfy $\phi_{11*} > \phi_{110}$. It might seem that λ_{2*} can be greater or smaller. Comparing $m_{34} = \lambda_3\lambda_4\phi_{11}$ with $m_{23} = \lambda_2\lambda_3\phi_{11}$ and $m_{24} = \lambda_2\lambda_4\phi_{11}$, where the positions of λ_2, λ_3, and λ_4 are parallel in modeling the intact σ_{34}, σ_{23}, and σ_{24}, λ_{2*} will most likely be smaller. Because m_{12*} will increase more than m_{13*} and m_{14*}, $\lambda_{2*} - \lambda_{20}$ may not be as small as $\lambda_{3*} - \lambda_{30}$ or $\lambda_{4*} - \lambda_{40}$. It is mainly ϕ_{11*} that will take the greater association between x_1 and x_2. Because $\phi_{11*} > \phi_{110}$, ψ_{11*} will be smaller. Our analysis cannot predict the directions of change in ψ_{22*}, ψ_{33*}, and ψ_{44*}.

When $\sigma_{e_2e_3} > 0$ and $e_2 \leftrightarrow e_3$ is excluded in Figure 3, the path $x_2 \leftarrow f_1 \rightarrow x_3$ has to explain the extra association using $m_{23} = \lambda_2\lambda_3\phi_{11}$, at least one of λ_{2*}, λ_{3*}, and ϕ_{11*} should be greater. Notice that ϕ_{11} is involved in every model-implied covariance, hence a greater ϕ_{11}^* will not solve the problem. Due to their parallel position, λ_{2*} and λ_{3*} will both be greater. Greater λ_{2*} and λ_{3*} need a smaller ϕ_{11*} to explain σ_{12} by $m_{12*} = \lambda_{2*}\phi_{11*}$ and σ_{13} by $m_{13*} = \lambda_{3*}\phi_{11*}$. The parameter λ_4 is involved in the model-implied covariances m_{14}, m_{24}, and m_{34}, which require λ_{4*} be greater, smaller, and smaller, respectively. So λ_{4*} will most likely become smaller. Smaller ϕ_{11*} and λ_{4*} imply greater ψ_{11*} and ψ_{44*}. Notice that $\lambda_{2*}\lambda_{3*}\phi_{11*}$ is greater and the positions of λ_2 and λ_3 are parallel in reacting to the misspecification, hence $\lambda_{2*}^2\phi_{11*}$ and $\lambda_{3*}^2\phi_{11*}$ should both be

[4]For commonly used models, the diagonal elements of the residual covariance matrix are all zero (Shapiro & Browne, 1990). When the diagonal elements of the residual covariance matrix are not zero, a greater factor loading may not imply a smaller error variance.

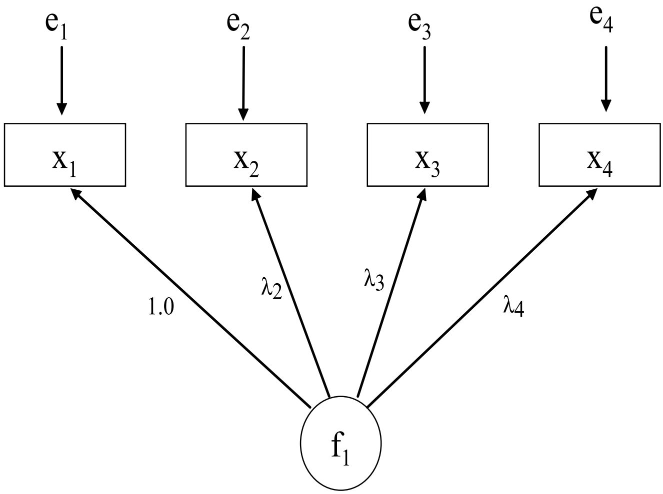

FIGURE 3 The path diagram for a one-factor model with ϕ_{11} being a free parameter.

greater and ψ_{22*} and ψ_{33*} should become smaller. The analysis for Figure 3 is summarized as

θ_*	ϕ_{11*}	λ_{2*}	λ_{3*}	λ_{4*}	ψ_{11*}	ψ_{22*}	ψ_{33*}	ψ_{44*}
$\sigma_{e_1e_2} > 0$	+	−	−	−	−	?	?	?
$\sigma_{e_2e_3} > 0$	−	+	+	−	+	−	−	+

where ? implies our analysis cannot determine the direction of the change.

For the simple case of a one-factor model, analyzing the path clearly shows the effect of excluding error covariances on other parameters in the model. The parameter estimates corresponding to + will have positive biases and those corresponding to − will have negative biases.

A Confirmatory Three-Factor Model

There are two kinds of misspecifications for a three-factor model. One is when error covariances are excluded, and the other is when factor loadings are excluded. We analyze the effect of excluding correlated errors first and then the effect of excluding factor loadings. We only consider the case when all the factor variances are set at 1.0 ($\phi_{jj} = 1$) in analyzing the model. Cases when one, two, or three λ_is are fixed for identification can be analyzed similarly. The indicators for a factor are called a *cluster.*

Figure 4 is a path diagram for a confirmatory factor model with three factors and nine indicators. Yuan et al. (2003) analyzed the effect of $\sigma_{e_1e_2} > 0$ on parameter estimates in this model. The analysis is similar to the one-factor model in Figure 2. We summarize their results without repeating the analysis. When $\sigma_{e_1e_2} > 0$ in Figure 4, λ_{1*} and λ_{2*} will be greater; λ_{3*} will be smaller; ψ_{11*} and ψ_{22*} will be smaller; ψ_{33*} will be greater; and ϕ_{12*} and ϕ_{13*} tend to be smaller. The parameters λ_{4*} to λ_{9*}, ϕ_{23*}, and ψ_{44*} to ψ_{99*} will equal their counterparts in a correctly specified model. Due to the parallel positions of the variables, the effect of any $\sigma_{e_ie_j} > 0$ within a cluster in Figure 4 can be obtained similarly (see Yuan et al., 2003, pp. 247–248).

For correlated errors corresponding to indicators for different factors, we only consider $\sigma_{e_1e_4} > 0$ in Figure 4 due to the parallel positions of the variables. The only path that can explain the greater σ_{14} in Figure 4 is $x_1 \leftarrow f_1 \leftrightarrow f_2 \rightarrow x_4$. The λ_is need to account for the within-cluster associations, so ϕ_{12*} has to be greater. Because $m_{14*} = \lambda_{1*}\lambda_{4*}\phi_{12*}$ needs to be greater, m_{25*}, m_{26*}, m_{35*}, and m_{36*} do not need to be; λ_{1*} and λ_{4*} will be greater. Because the pairwise

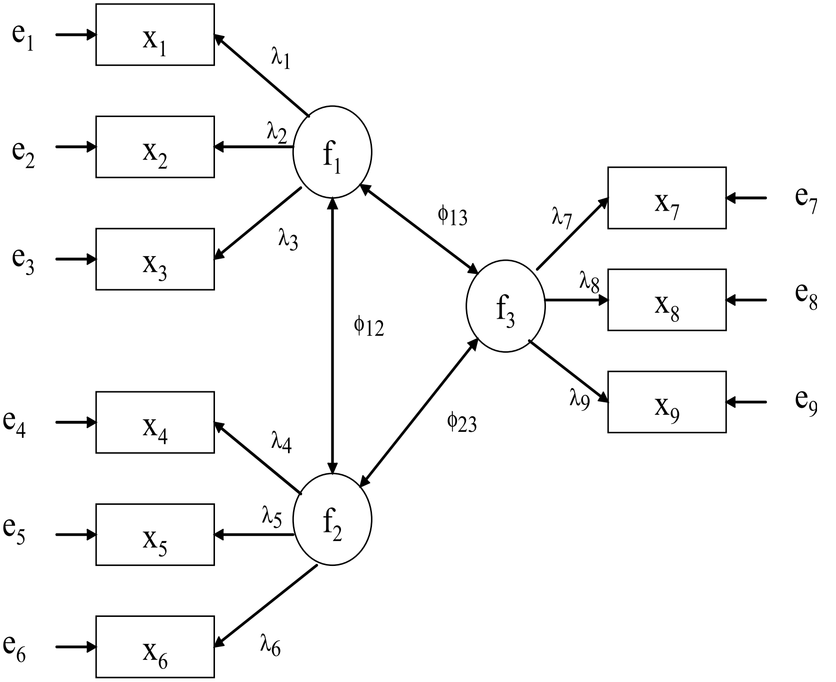

FIGURE 4 The path diagram for a three-factor model with $\phi_{jj} = 1$.

population covariances within clusters (x_1, x_2, x_3) and (x_4, x_5, x_6) remain the same, factor loadings λ_{2*}, λ_{3*}, λ_{5*}, and λ_{6*} have to be smaller to adjust for the greater λ_{1*} and λ_{4*}. The parameter ψ_{ii} needs to adjust for the difference between the model-implied variance; σ_{ii}, ψ_{11*}, and ψ_{44*} have to be smaller; and ψ_{22*}, ψ_{33*}, ψ_{55*}, and ψ_{66*} have to be greater. Being responsible for the between-cluster covariances of (x_1, x_2, x_3) and (x_4, x_5, x_6) with (x_7, x_8, x_9), ϕ_{13*} and ϕ_{23*} will be affected due to the changes of factor loadings within (x_1, x_2, x_3) and (x_4, x_5, x_6), but we are unable to determine the direction of their changes. The parameters λ_7 to λ_9 explain the within-cluster associations among x_7 to x_9, and their values will not affect the between-cluster covariances of (x_1, x_2, x_3) and (x_4, x_5, x_6). The effect of the misspecification is accounted for by ϕ_{13*} and ϕ_{23*} before reaching (x_7, x_8, x_9), thus, λ_{7*} to λ_{9*} and ψ_{77*} to ψ_{99*} will equal their population values corresponding to the correctly specified model.

TABLE 1
The Directions of Change in Parameters When a Positive Error Covariance or Extra Factor Loading Exists in the Population and the Model is Represented by Figure 4

θ_*	$\sigma_{e_1 e_2}$	$\sigma_{e_1 e_4}$	λ_{320}
λ_{1*}	+	+	−
λ_{2*}	+	−	−
λ_{3*}	−	−	+
λ_{4*}	0	+	0
λ_{5*}	0	−	0
λ_{6*}	0	−	0
λ_{7*}	0	0	0
λ_{8*}	0	0	0
λ_{9*}	0	0	0
ϕ_{12*}	−	+	+
ϕ_{13*}	−	?	?
ϕ_{23*}	0	?	0
ψ_{11*}	−	−	+
ψ_{22*}	−	+	+
ψ_{33*}	+	+	?
ψ_{44*}	0	−	0
ψ_{55*}	0	+	0
ψ_{66*}	0	+	0
ψ_{77*}	0	0	0
ψ_{88*}	0	0	0
ψ_{99*}	0	0	0

Notice that x_3 does not load on f_2 in Figure 4. When the correct model needs x_3 to load on f_2 with a positive loading[5] λ_{320}, all the covariances between x_3 and the other variables (σ_{3i}s) are positively perturbed. The relation of x_3 with (x_4, x_5, x_6) and (x_7, x_8, x_9) needs to pass $f_1 \leftrightarrow f_2$ and $f_1 \leftrightarrow f_3$, which will be accounted for by ϕ_{12} and ϕ_{13}, respectively. The between-cluster covariances of (x_4, x_5, x_6) with (x_7, x_8, x_9) are intact. To explain the intact covariances, we have $\lambda_{4*} = \lambda_{40}, \ldots, \lambda_{9*} = \lambda_{90}$, $\phi_{23*} = \phi_{230}$, and $\psi_{44*} = \psi_{440}, \ldots, \psi_{99*} = \psi_{990}$. The model needs to mainly explain the perturbed σ_{3i} by $x_3 \leftarrow f_1 \leftrightarrow f_2$ with the model-implied covariance $m_{x_3 f_2} = \lambda_3 \phi_{12}$. So there must exist $\lambda_{3*}\phi_{12*} > \lambda_{30}\phi_{120}$. Notice that it is x_3 that has a greater covariance with f_2, not x_1 or x_2, so λ_{3*} must be greater. When ϕ_{120} and ϕ_{130} are comparable, $\sigma_{x_3 f_2} = \lambda_3 \phi_{12}$ is positively perturbed more by the additional $\lambda_{320} > 0$ than $\sigma_{x_3 f_3} = \lambda_3 \phi_{13}$, ϕ_{12*} also needs to be greater to explain the greater perturbation. With a greater ϕ_{12*}, λ_{1*} and λ_{2*} need to be smaller to explain the intact covariances between (x_1, x_2) and (x_4, x_5, x_6); consequently, ψ_{11*} and ψ_{22*} will be greater. Because $\lambda_{320} > 0$ also makes σ_{33} greater, we cannot determine the direction of change in ψ_{33*}. The parameter ϕ_{13} is responsible for the covariance between the clusters (x_1, x_2, x_3) and (x_7, x_8, x_9). Due to the change in λ_{1*} to λ_{3*}, and the greater σ_{3i}, $i = 7$ to 9, we cannot determine the direction of change in ϕ_{13*}. The results of the analysis for Figure 4 are summarized in Table 1.

STRUCTURAL EQUATION MODELS

We consider structural equation models with two and three factors. Similarities and differences between the two models allow us to understand the functions of similar parameters under different constraints. Compared to the previous section, the analysis for structural equation models is more complex. However, we employ the same technique as was used for the confirmatory factor models. That is, when excluding a path with a nonzero loading between two variables, the existing paths indirectly connecting these two variables in the model need to compensate. Loading estimates of the paths may have positive or negative biases. The technique of analyzing the path allows us to determine the directions of bias in many of these estimates.

A Model With Two Factors

Figure 5 is a path diagram of a simple structural equation model with six indicators and two factors, where $m_{f_1 f_1} = \phi_{11}$, $m_{d_2 d_2} = \varphi_{22}$, and $m_{e_i e_i} = \psi_{ii}$.

[5]When λ has double nonzero subscripts, the first indicates the order of the variable and the second indicates the order of the factor.

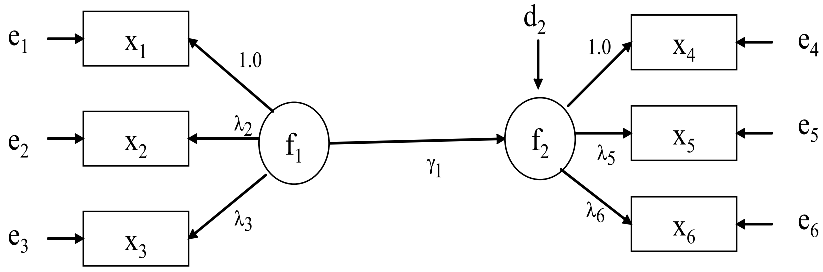

FIGURE 5 The path diagram for a structural equation model with two factors.

The model does not contain any error covariances and all the indicators are unidimensional. We consider the existence of positive covariances $\sigma_{e_1e_2}$, $\sigma_{e_2e_3}$, $\sigma_{e_4e_5}$, $\sigma_{e_5e_6}$, $\sigma_{e_1e_4}$, $\sigma_{e_1e_5}$, $\sigma_{e_2e_4}$, and $\sigma_{e_2e_5}$ and double loadings represented by $x_1 \leftarrow f_2$, $x_2 \leftarrow f_2$, $x_4 \leftarrow f_1$, $x_5 \leftarrow f_1$ in the population. The loading parameters will be denoted by λ_{12}, λ_{22}, λ_{41}, and λ_{51}, respectively. For the same reason as with the three-factor model in the previous section, when a pair of errors within a cluster are correlated and the correlation is excluded from the model, the relation within the other cluster is still correctly evaluated. Similarly, when $\lambda_{i20} > 0$, $i = 1, 2, 3$, the relation within the cluster (x_4, x_5, x_6) is not affected; when $\lambda_{i10} > 0$, $i = 4, 5, 6$, the relation within the cluster (x_1, x_2, x_3) is not affected either. The results are summarized in Table 2. Now readers may analyze Figure 5 and compare their results with those in Table 2. To avoid repetition, we only

TABLE 2
The Directions of Change in Parameters When a Positive Error Covariance or Extra Factor Loading Exists in the Population and the Model is Represented by Figure 5

θ_*	$\sigma_{e_1e_2}$	$\sigma_{e_2e_3}$	$\sigma_{e_4e_5}$	$\sigma_{e_5e_6}$	$\sigma_{e_1e_4}$	$\sigma_{e_1e_5}$	$\sigma_{e_2e_4}$	$\sigma_{e_2e_5}$	λ_{120}	λ_{220}	λ_{410}	λ_{510}
λ_{2*}	−	+	0	0	−	−	+	+	−	+	0	0
λ_{3*}	−	+	0	0	−	−	−	−	−	?	0	0
λ_{5*}	0	0	−	+	−	+	−	+	0	0	−	+
λ_{6*}	0	0	−	+	−	−	−	−	0	0	−	−
ϕ_{11*}	+	−	0	0	+	+	−	−	+	−	0	0
γ_{1*}	−	−	+	−	?	?	+	+	−	+	+	+
φ_{22*}	?	+	+	?	−	−	−	−	−	−	?	?
ψ_{11*}	−	+	0	0	−	−	+	+	?	+	0	0
ψ_{22*}	?	?	0	0	?	?	−	−	?	?	0	0
ψ_{33*}	?	?	0	0	?	?	?	?	?	?	0	0
ψ_{44*}	0	0	−	+	−	?	?	?	0	0	?	?
ψ_{55*}	0	0	?	?	?	−	?	−	0	0	?	?
ψ_{66*}	0	0	?	?	?	?	?	?	0	0	?	?

discuss parameters that are affected and provide the outline for the analyses that lead to the results.

When $\sigma_{e_1e_2} > 0$ in Figure 5, $m_{12*} = \lambda_{2*}\phi_{11*}$ has to be greater to account for the extra association between x_1 and x_2. Because $m_{23} = \lambda_3 m_{12}$, λ_{3*} has to be smaller to account for the greater m_{12*}. A smaller λ_{3*} needs a greater ϕ_{11*} to explain the intact σ_{31}. Greater ϕ_{11*} and $(m_{x_1f_1*} + m_{x_2f_1*} + m_{x_3f_1*})/3 = (1 + \lambda_{2*} + \lambda_{3*})\phi_{11*}/3$ need a smaller γ_{1*} for the intact between-cluster covariances. It might seem that λ_{2*} can be smaller or greater. However, due to a greater ϕ_{11*}, similar to λ_{3*}, λ_{2*} should be smaller to explain the intact between-cluster covariances. A greater ϕ_{11*} leads to a smaller ψ_{11*}. We are unable to predict the directions of change in φ_{22*}, ψ_{22*}, and ψ_{33*}. When a parameter θ_1 is closely related to two other parameters θ_2 and θ_3, if θ_{2*} needs a smaller θ_{1*} and θ_{3*} needs a greater θ_{1*}, we will not be able to predict the direction of change in θ_{1*} in general.

When $\sigma_{e_2e_3} > 0$, the path $x_2 \leftarrow f_1 \rightarrow x_3$ is responsible for explaining the greater σ_{23} with $m_{23} = \lambda_2\lambda_3\phi_{11}$. Due to the parallel positions of λ_2 and λ_3 in the model, both of them have to be either greater or smaller. Notice that $\lambda_2 = m_{23}/m_{13}$, $\lambda_3 = m_{23}/m_{12}$, and σ_{13} and σ_{12} are intact, λ_{2*} and λ_{3*} have to be greater. Due to greater λ_{2*} and λ_{3*}, ϕ_{11*} has to be smaller, which leads to a greater ψ_{11*}. Similarly, the $(m_{x_1f_1*} + m_{x_2f_1*} + m_{x_3f_1*})/3$ is greater due to the stronger within-cluster associations, which leads to a smaller γ_{1*} for the intact between-cluster associations. A smaller γ_{1*} together with a smaller ϕ_{11*} needs a greater φ_{22*} for the intact $m_{f_2f_2}$. We are unable to predict the directions of change in ψ_{22*} and ψ_{33*}.

When $\sigma_{e_4e_5} > 0$, $m_{45*} = \lambda_{5*}m_{f_2f_2*}$ has to be greater. Because $m_{56} = \lambda_6 m_{45}$, λ_{6*} has to be smaller. Parallel to λ_{6*}, λ_{5*} needs to be smaller for the intact between-cluster covariances. Smaller λ_{5*} and λ_{6*} need a greater $m_{f_2f_2*}$. The only function of φ_{22*} is to adjust for $m_{f_2f_2*}$; a greater $m_{f_2f_2*}$ needs a greater φ_{22*}. Notice that only ϕ_{11} and γ_1, not φ_{22*}, contribute to the between-cluster covariances, hence smaller λ_{5*} and λ_{6*} need a greater γ_{1*}. A greater $m_{f_2f_2*}$ leads to a smaller ψ_{44*}. We are unable to predict the directions of change in ψ_{55*} and ψ_{66*}.

When $\sigma_{e_5e_6} > 0$, $m_{56*} = \lambda_{5*}\lambda_{6*}m_{f_2f_2*}$ has to be greater. Due to their parallel positions, both λ_{5*} and λ_{6*} have to be greater. Greater λ_{5*} and λ_{6*} need a smaller γ_{1*} to explain the intact between-cluster covariances and a smaller $m_{f_2f_2*}$ to explain σ_{45} and σ_{46}. A smaller $m_{f_2f_2*}$ leads to a greater ψ_{44*}. We are unable to predict the directions of change in φ_{22*}, ψ_{55*}, and ψ_{66*}.

When $\sigma_{e_1e_4} > 0$, $m_{14*} = m_{f_1f_2*} = \phi_{11*}\gamma_{1*}$ has to be greater. There are three possibilities: (a) γ_{1*} is much greater and ϕ_{11*} is slightly smaller or greater; (b) ϕ_{11*} is much greater and γ_{1*} is slightly smaller or greater; and (c) both ϕ_{11*} and γ_{1*} are greater. Because the only function of φ_{22*} is to adjust for $m_{f_2f_2*}$, the values of ϕ_{11*} and γ_{1*} will not affect the associations within the cluster

(x_4, x_5, x_6). Under (a), λ_{2*} and λ_{3*} need to be much smaller to explain the between-cluster covariances and they need to remain approximately the same to explain the covariances within (x_1, x_2, x_3), so (a) is unlikely to happen. Under (c), λ_{2*} and λ_{3*} only need to adjust for the effect of a greater ϕ_{11*} for the intact covariances within (x_1, x_2, x_3) and they need to adjust for both the effect of a greater ϕ_{11*} and a greater γ_{1*} in modeling the intact σ_{24}, σ_{25}, σ_{26}, and σ_{34}, σ_{35}, σ_{36}, so (c) is unlikely to happen either. Notice that all the model-implied covariances contain ϕ_{11}. Except for m_{14}, all also contain at least a λ_i. The most likely scenario is therefore (b). Thus, ϕ_{11*} should be greater, λ_{2*}, λ_{3*}, λ_{5*} and λ_{6*} will be smaller; ψ_{11*} will be smaller. Smaller λ_{5*} and λ_{6*} lead to a greater $m_{f_2 f_2 *}$ and thus, a smaller ψ_{44*}. A greater $\phi_{11*}\gamma_{1*}$ also leads to a smaller φ_{22*}. We are unable to determine the directions of change in γ_{1*}, ψ_{22*}, ψ_{33*}, ψ_{55*}, and ψ_{66*}.

When $\sigma_{e_1 e_5} > 0$, $m_{15*} = \lambda_{5*}\phi_{11*}\gamma_{1*}$ has to be greater. Because $\phi_{11*}\gamma_{1*}$ is shared by all the between-cluster covariances, λ_{5*} should be much greater. A smaller $\phi_{11*}\gamma_{1*}$ is not supported by any of the between-cluster covariances either. A greater m_{15*} needs smaller λ_{2*} and λ_{3*} for the intact between-cluster covariances of (x_2, x_3) with (x_4, x_5, x_6). Smaller λ_{2*} and λ_{3*} lead to a greater ϕ_{11*}, which further leads to a smaller ψ_{11*}. Smaller λ_{2*} and λ_{3*} also support a greater $\phi_{11*}\gamma_{1*}$. A smaller λ_{6*} is necessary to explain the intact σ_{16} and σ_{56}. A greater λ_{5*} most likely leads to a smaller ψ_{55*}. A greater $\phi_{11*}\gamma_{1*}$ also needs a smaller φ_{22*} for the intact covariances within (x_4, x_5, x_6). We are unable to determine the directions of change in γ_{1*}, ψ_{22*}, ψ_{33*}, ψ_{44*}, and ψ_{66*}.

When $\sigma_{e_2 e_4} > 0$, parallel to $\sigma_{e_1 e_5} > 0$, λ_{2*} is much greater and $\phi_{11*}\gamma_{1*}$ is also greater; λ_{3*}, λ_{5*}, and λ_{6*} are smaller. A much greater λ_{2*} needs a smaller ϕ_{11*} for the intact covariances within (x_1, x_2, x_3). A smaller ϕ_{11*} needs a greater γ_{1*} and a greater ψ_{11*}. A much greater λ_{2*} also most likely leads to a smaller ψ_{22*}. A greater $\phi_{11*}\gamma_{1*}$ needs a smaller φ_{22*} for the intact covariances within (x_4, x_5, x_6). We are unable to determine the directions of change in ψ_{33*}, ψ_{44*}, to ψ_{66*}.

When $\sigma_{e_2 e_5} > 0$, $m_{25*} = \lambda_{2*}\lambda_{5*}\phi_{11*}\gamma_{1*}$ has to be greater. Because $\phi_{11*}\gamma_{1*}$ is responsible for all the between-cluster covariances, λ_{2*} and λ_{5*} have to be greater. There is no reason for $\phi_{11*}\gamma_{1*}$ to be smaller either. A greater λ_{2*} needs a smaller ϕ_{11*} for the intact covariances within (x_1, x_2, x_3). Smaller λ_{3*} and λ_{6*} are also necessary for explaining the between-cluster associations. Greater λ_{2*} and λ_{5*} also most likely lead to smaller ψ_{22*} and ψ_{55*}. A smaller ϕ_{11*} needs a greater γ_{1*} and a greater ψ_{11*}. We are unable to determine the directions of change in ψ_{33*}, ψ_{44*}, and ψ_{66*}.

In Figure 5, x_1 does not load on f_2. If the correct model needs an extra path $x_1 \leftarrow f_2$ with loading $\lambda_{120} > 0$, the stronger association between x_1 and f_2 has to be explained by $m_{x_1 f_2} = \gamma_1\phi_{11}$, thus $\gamma_{1*}\phi_{11*}$ will be greater. Notice that $m_{x_2 f_2} = \lambda_2 m_{x_1 f_2}$ and $m_{x_3 f_2} = \lambda_3 m_{x_1 f_2}$, hence λ_{2*} and λ_{3*} have to be smaller to explain the associations of x_2 and x_3 with the cluster (x_4, x_5, x_6). Due to smaller λ_{2*} and λ_{3*}, ϕ_{11*} has to be much greater to explain the intact σ_{23} and the extra associations of x_1 with x_2 and x_3. Notice that $m_{f_2 f_2 *}$ is not affected by the misspecification, and γ_1 is the path coefficient of $f_1 \to f_2$. A much greater ϕ_{11*} needs a smaller γ_{1*} to explain the intact covariances of (x_2, x_3) with (x_4, x_5, x_6). A greater $\phi_{11*}\gamma_{1*}$ also leads to a smaller φ_{22*} for the intact $m_{f_2 f_2 *}$. Notice that σ_{11} is positively perturbed by the additional $\lambda_{120} > 0$. We are unable to predict the directions of change in ψ_{11*} to ψ_{33*}.

When x_2 needs to load on f_2 with $\lambda_{220} > 0$, $m_{x_2 f_2 *} = \lambda_{2*}\phi_{11*}\gamma_{1*}$ has to be greater. Because $m_{f_1 f_2 *} = \phi_{11*}\gamma_{1*}$ is shared by all the between-cluster covariances, λ_{2*} has to be greater. There is no reason for $\phi_{11*}\gamma_{1*}$ to become smaller. A greater λ_{2*} needs a smaller ϕ_{11*} to explain the covariances within (x_1, x_2, x_3). A smaller ϕ_{11*} leads to a greater ψ_{11*}. A smaller ϕ_{11*} also needs a greater γ_{1*} for not a smaller $\phi_{11*}\gamma_{1*}$. Because $\phi_{11*}\gamma_{1*}$ cannot be smaller, φ_{22*} is most likely smaller. We are unable to predict the directions of change in λ_{3*}, ψ_{22*}, and ψ_{33*}.

When x_4 needs to load on f_1 with $\lambda_{410} > 0$, $m_{x_4 f_1 *} = \phi_{110}\gamma_{1*}$ has to be greater, thus γ_{1*} has to be greater. A greater γ_{1*} needs smaller λ_{5*} and λ_{6*} to explain the intact $\sigma_{x_5 f_1}$ and $\sigma_{x_6 f_1}$. Smaller λ_{5*} and λ_{6*} need a greater $m_{f_2 f_2 *}$ to interpret the intact σ_{56}. Because σ_{44}, σ_{45}, and σ_{46} are positively perturbed by $\lambda_{410} > 0$, we are unable to predict the directions of change in φ_{22*} and ψ_{44*} to ψ_{66*}.

When x_5 needs to load on f_1 with $\lambda_{510} > 0$, $m_{x_5 f_1 *} = \lambda_{5*}\phi_{11*}\gamma_{1*}$ has to be greater. Because $m_{f_1 f_2 *} = \phi_{11*}\gamma_{1*}$ is shared by all the between-cluster covariances, λ_{5*} has to be much greater. There is no reason for $\phi_{11*}\gamma_{1*}$ to become smaller and γ_{1*} also needs to be greater. A greater $\phi_{11*}\gamma_{1*}$ needs a smaller λ_{6*} for the intact between-cluster covariances σ_{16}, σ_{26}, and σ_{36}. We are unable to predict the directions of change in φ_{22*} and ψ_{44*} to ψ_{66*}.

A Model With Three Factors

Figure 6 is a path diagram for a structural equation model with nine indicators and three factors, where $m_{f_1 f_1} = \phi_{11}$, $m_{d_j d_j} = \varphi_{jj}$, $j = 2, 3$; $m_{e_i e_i} = \psi_{ii}$, $i = 1$ to 9. In this model, λ_2, λ_3, and ϕ_{11} are responsible for explaining the covariances within the cluster (x_1, x_2, x_3). They are also involved in the between-cluster covariances among the indicators. Similarly, λ_5, λ_6, and $m_{f_2 f_2}$ are mainly responsible for explaining the covariances within (x_4, x_5, x_6); λ_8, λ_9, and $m_{f_3 f_3}$ are mainly responsible for explaining the covariances within (x_7, x_8, x_9). The parameters γ_1 to γ_3 need to explain the relation between the clusters. The ψ_{ii}s are mainly responsible for explaining the marginal variances of the observed variables. Due to the three predicted relations among the latent factors in Figure 6, analyzing the path is relatively more complicated than that for the one predicted relation in Figure 5. We need to use covariance algebra to facilitate the analysis. Let $\mathbf{f} = (f_1, f_2, f_3)'$. The Appendix provides the detail

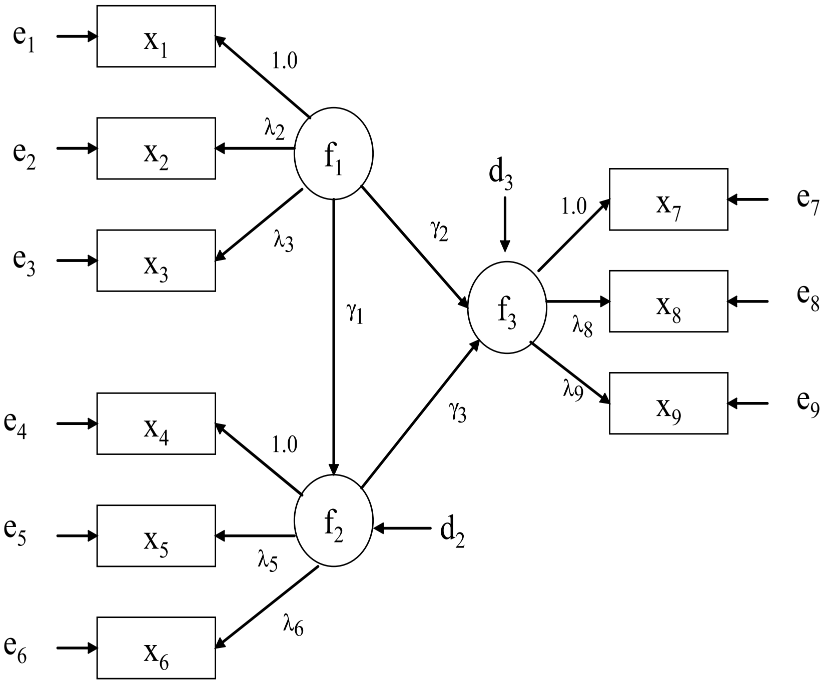

FIGURE 6 The path diagram for a structural equation model with three factors.

leading to the model-implied covariance matrix

$$\mathrm{Cov}(\mathbf{f}) = \begin{pmatrix} \phi_{11} & \gamma_1\phi_{11} & \phi_{11}(\gamma_2+\gamma_1\gamma_3) \\ \gamma_1\phi_{11} & \gamma_1^2\phi_{11}+\varphi_{22} & \gamma_1\phi_{11}(\gamma_2+\gamma_1\gamma_3)+\gamma_3\varphi_{22} \\ \phi_{11}(\gamma_2+\gamma_1\gamma_3) & \gamma_1\phi_{11}(\gamma_2+\gamma_1\gamma_3)+\gamma_3\varphi_{22} & \phi_{11}(\gamma_2+\gamma_1\gamma_3)^2+\gamma_3^2\varphi_{22}+\varphi_{33} \end{pmatrix}. \quad (2)$$

The model-implied covariance between any two indicators is the two factor loadings times the corresponding element in this matrix. Due to $f_1 \to f_3 \leftarrow f_2$, some of the parameters in Figure 6 need to play different roles from the parallel ones in Figure 5. For example, the only duty of φ_{22*} in Figure 5 is to adjust for $m_{f_2f_2*}$; any changes in ϕ_{11*} and γ_{1*} do not need to consider $m_{f_2f_2*}$. In

Figure 6, φ_{22} is also involved in $m_{f_2 f_3}$ and $m_{f_3 f_3}$. When a parameter is involved in more model-implied variances and covariances, it will be subjected to more restraints, but the principle of the analysis will remain the same. With a greater population covariance between two variables, the direct or shortest path between the two will mainly respond to the misspecification. Other paths will also be affected, but to a lesser degree. We first discuss the directions of change for the model parameters when errors are correlated, followed by the effect of double factor loadings in the population but not in the model, and then that of excluding the γs. When considering double factor loadings represented by paths $x_1 \leftarrow f_2$, $x_1 \leftarrow f_3$, $x_2 \leftarrow f_2$, $x_2 \leftarrow f_3$, $x_4 \leftarrow f_1$, $x_4 \leftarrow f_3$, $x_5 \leftarrow f_1$, $x_5 \leftarrow f_3$, $x_7 \leftarrow f_1$, $x_7 \leftarrow f_2$, $x_8 \leftarrow f_1$, $x_8 \leftarrow f_2$ we will use, respectively, λ_{12}, λ_{13}, λ_{22}, λ_{23}, λ_{41}, λ_{43}, λ_{51}, λ_{53}, λ_{71}, λ_{72}, λ_{81}, and λ_{82} to denote the loading parameters.

Within the cluster (x_1, x_2, x_3). Suppose a pair of errors within the cluster (x_1, x_2, x_3) are positively correlated in Figure 6. Similar to our analysis for the confirmatory three-factor model in the previous section and the two-factor structural equation model in this section, the within-cluster covariances of (x_4, x_5, x_6) and (x_7, x_8, x_9) are not affected, nor are the between-cluster associations of (x_4, x_5, x_6) with (x_7, x_8, x_9). Parameters λ_5, λ_6, λ_8, λ_9, and ψ_{44} to ψ_{99} will be correctly evaluated. The model-implied variances and covariance $m_{f_2 f_2}$, $m_{f_2 f_3}$, and $m_{f_3 f_3}$ will also equal their population counterparts corresponding to the correctly specified model. The remaining analysis for excluding within-cluster error covariances is also similar to that in obtaining Tables 1 and 2. The results are summarized in Table 3. Readers are strongly encouraged to analyze Figure 6 themselves and compare their results with those in Table 3. Those who cannot get the same results may consult the following paragraphs corresponding to the exclusion of each error covariance.

When $\sigma_{e_1 e_2} > 0$, the greater σ_{12} has to be explained through the path $x_1 \leftarrow f_1 \rightarrow x_2$ by the model-implied covariance $m_{12} = \lambda_2 \phi_{11}$. Thus, $\lambda_{2*}\phi_{11*} > \lambda_{20}\phi_{110}$. Because $m_{23} = \lambda_3 m_{12}$, there exists $\lambda_{3*} < \lambda_{30}$. Notice that the positions of λ_2 and λ_3 are parallel in Figure 6, similar to that in Figure 5, a greater ϕ_{11*} alone takes care of the extra association between x_1 and x_2. Actually, if λ_{2*} becomes greater and λ_{3*} becomes smaller, the model cannot explain the associations of x_2 and x_3 with the other two clusters. Due to a greater ϕ_{11*}, ψ_{11*} becomes smaller. To explain the extra covariance in σ_{12}, the average association of x_1, x_2, and x_3 with f_1, given by $(m_{x_1 f_1 *} + m_{x_2 f_1 *} + m_{x_3 f_1 *}) = \phi_{11*}(1 + \lambda_{2*} + \lambda_{3*})/3$, is stronger. The parameters γ_1 and γ_2 need to explain the between-cluster covariances of (x_1, x_2, x_3) through f_1 with (x_4, x_5, x_6) and (x_7, x_8, x_9). They have to be smaller to explain the intact between-cluster covariances in the population. We cannot determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, ψ_{22*}. and ψ_{33*}.

TABLE 3
The Directions of Change in Parameters When a Within-Cluster Positive Error Covariance Exists in the Population and the Model is Represented by Figure 6

θ_*	$\sigma_{e_1e_2}$	$\sigma_{e_2e_3}$	$\sigma_{e_4e_5}$	$\sigma_{e_5e_6}$	$\sigma_{e_7e_8}$	$\sigma_{e_8e_9}$
λ_{2*}	−	+	0	0	0	0
λ_{3*}	−	+	0	0	0	0
λ_{5*}	0	0	−	+	0	0
λ_{6*}	0	0	−	+	0	0
λ_{8*}	0	0	0	0	−	+
λ_{9*}	0	0	0	0	−	+
ϕ_{11*}	+	−	0	0	0	0
γ_{1*}	−	−	+	−	0	0
γ_{2*}	−	−	?	?	+	−
γ_{3*}	?	?	?	?	+	−
φ_{22*}	?	+	+	?	0	0
φ_{33*}	?	?	?	?	+	?
ψ_{11*}	−	+	0	0	0	0
ψ_{22*}	?	?	0	0	0	0
ψ_{33*}	?	?	0	0	0	0
ψ_{44*}	0	0	−	+	0	0
ψ_{55*}	0	0	?	?	0	0
ψ_{66*}	0	0	?	?	0	0
ψ_{77*}	0	0	0	0	−	+
ψ_{88*}	0	0	0	0	?	?
ψ_{99*}	0	0	0	0	?	?

When $\sigma_{e_2e_3} > 0$, $m_{23*} = \lambda_{2*}\lambda_{3*}\phi_{11*}$ has to be greater. It is impossible for λ_{2*} or λ_{3*} to be smaller. Due to greater λ_{2*} and λ_{3*}, ϕ_{11*} has to be smaller. For the same reason as when $\sigma_{e_1e_2} > 0$, γ_{1*} and γ_{2*} have to be smaller. A smaller ϕ_{11*} together with a smaller γ_{1*} leads to a greater φ_{22*}. A smaller ϕ_{11*} leads to a greater ψ_{11*}. We are unable to determine the directions of change in γ_{3*}, φ_{33*}, ψ_{22*}, and ψ_{33*}.

Within the cluster (x_4, x_5, x_6). As summarized in Table 3, the misspecification of ignoring error covariances within (x_4, x_5, x_6) does not affect the evaluation of the within-cluster parameters of (x_1, x_2, x_3) and (x_7, x_8, x_9). The variances and covariance $m_{f_1f_1*} = \phi_{11*}$, $m_{f_1f_3*} = \phi_{11*}(\gamma_{2*} + \gamma_{1*}\gamma_{3*})$, and $m_{f_3f_3*} = \phi_{11*}(\gamma_{2*} + \gamma_{1*}\gamma_{3*})^2 + \gamma_{3*}^2\varphi_{22*} + \varphi_{33*}$ will also equal their population counterparts corresponding to the correctly specified model.

When $\sigma_{e_4e_5} > 0$, σ_{45} will be greater, which has to be explained by $m_{45} = \lambda_5 m_{f_2f_2}$. For the same reason as in Figure 5, $m_{f_2f_2*}$ will be greater, λ_{5*} and λ_{6*} will be smaller, and ψ_{44*} will be smaller. Due to smaller λ_{5*} and λ_{6*}, and

$m_{f_1f_2*} = \gamma_{1*}\phi_{110}$, γ_{1*} has to be greater to explain the intact between-cluster covariances of (x_1, x_2, x_3) with (x_4, x_5, x_6). Notice that $m_{f_2f_2} = \gamma_1^2\phi_{11} + \varphi_{22}$. Because γ_1 is mainly responsible for the between-cluster associations, a greater γ_{1*} alone is not enough for the greater σ_{45}, and φ_{22*} should be greater for the greater $m_{f_2f_2*}$. We are unable to determine the directions of change in γ_{2*}, γ_{3*}, φ_{33*}, ψ_{55*}, and ψ_{66*}.

When $\sigma_{e_5e_6} > 0$, similar to when $\sigma_{e_2e_3} > 0$, λ_{5*} and λ_{6*} will be greater, $m_{f_2f_2*}$ has to be smaller. Notice that $m_{f_1f_2*} = \gamma_{1*}\phi_{110}$. Due to greater λ_{5*} and λ_{6*}, γ_{1*} has to be smaller to explain the between-cluster covariances of (x_1, x_2, x_3) with (x_4, x_5, x_6). A smaller $m_{f_2f_2*}$ leads to a greater ψ_{44*}. Due to the stronger associations within (x_4, x_5, x_6), γ_{3*} will change to adjust for the between-cluster covariances. Because too many parameters affect $m_{f_2f_3}$, we are unable to determine the directions of change in γ_{3*}. We are unable to determine the directions of change in γ_{2*}, φ_{22*}, φ_{33*}, ψ_{55*}, and ψ_{66*} either.

Within the cluster (x_7, x_8, x_9). Using the same logic as in the previous sections, ignoring possible covariances among e_7, e_8, and e_9 does not affect the parameters λ_2, λ_3, λ_5, λ_6, and ψ_{11} to ψ_{66}. Because $m_{f_1f_1}$, $m_{f_1f_2}$, and $m_{f_2f_2}$ are also correctly evaluated, ϕ_{11}, γ_1, and φ_{22} will not be affected either.

When $\sigma_{e_7e_8} > 0$, $m_{f_3f_3*}$ is greater, λ_{8*} and λ_{9*} are smaller, and ψ_{77*} is smaller. Because $m_{f_3f_3} = \phi_{11}(\gamma_2 + \gamma_1\gamma_3)^2 + \gamma_3^2\varphi_{22} + \varphi_{33}$ and γ_2 and γ_3 are mainly responsible for the between-cluster associations, the only function of φ_{33} is to adjust for $m_{f_3f_3}$; thus, φ_{33*} should be greater. Note that φ_{33*} does not contribute to the between-cluster associations. Due to smaller λ_{8*} and λ_{9*}, γ_{2*} and γ_{3*} need to be greater to explain the intact between-cluster covariances of (x_7, x_8, x_9) with (x_1, x_2, x_3) and (x_4, x_5, x_6). We are unable to determine the directions of change in ψ_{88*} and ψ_{99*}.

When $\sigma_{e_8e_9} > 0$, both λ_{8*} and λ_{9*} are greater, $m_{f_3f_3*}$ is smaller, and ψ_{77*} is greater. Due to the greater average path $(m_{x_7f_3*} + m_{x_8f_3*} + m_{x_9f_3*})/3$, γ_{2*}, γ_{3*} will be smaller to explain the intact between-cluster covariances. We are unable to predict the directions of change in φ_{33*}, ψ_{88*}, and ψ_{99*}.

Between clusters (x_1, x_2, x_3) and (x_4, x_5, x_6). First, the misspecification of ignoring a covariance between (e_1, e_2, e_3) and (e_4, e_5, e_6) does not affect the estimation of parameters within cluster (x_7, x_8, x_9). Thus, λ_{8*}, λ_{9*}, ψ_{77*}, ψ_{88*}, and ψ_{99*} will equal their counterparts corresponding to the correctly specified model; $\sigma_{f_3f_3}$ will also be correctly predicted by $m_{f_3f_3*}$. The results for excluding between-cluster error covariances are summarized in Table 4. We encourage readers to analyze Figure 6 to obtain their own version of the table.

When $\sigma_{e_1e_4} > 0$ in Figure 6, the path $f_1 \rightarrow f_2$ is responsible for explaining the greater σ_{14} with $m_{14} = m_{f_1f_2} = \gamma_1\phi_{11}$. Different from the model in Figure 5,

TABLE 4
The Directions of Change in Parameters When a Between-Cluster Positive Error Covariance Exists in the Population and the Model is Represented by Figure 6

θ	$\sigma_{e_1e_4}$	$\sigma_{e_1e_5}$	$\sigma_{e_2e_4}$	$\sigma_{e_2e_5}$	$\sigma_{e_1e_7}$	$\sigma_{e_1e_8}$	$\sigma_{e_2e_7}$	$\sigma_{e_2e_8}$	$\sigma_{e_4e_7}$	$\sigma_{e_4e_8}$	$\sigma_{e_5e_7}$	$\sigma_{e_5e_8}$
λ_{2*}	?	?	?	?	−	−	+	+	0	0	0	0
λ_{3*}	?	?	?	?	−	−	−	−	0	0	0	0
λ_{5*}	?	?	?	?	0	0	0	0	−	−	+	+
λ_{6*}	?	?	?	?	0	0	0	0	−	−	−	−
λ_{8*}	0	0	0	0	−	+	−	+	−	+	−	+
λ_{9*}	0	0	0	0	−	−	−	−	−	−	−	−
ϕ_{11*}	?	?	?	?	+	+	−	−	0	0	0	0
γ_{1*}	+	+	+	+	−	−	+	+	+	+	?	?
γ_{2*}	?	?	?	?	+	+	+	+	−	−	?	?
γ_{3*}	?	?	?	?	?	?	?	?	+	+	+	+
φ_{22*}	−	−	−	−	?	?	?	?	+	+	?	?
φ_{33*}	?	?	?	?	?	?	?	?	?	?	?	?
ψ_{11*}	?	?	?	?	−	−	+	+	0	0	0	0
ψ_{22*}	?	?	?	?	?	?	?	?	0	0	0	0
ψ_{33*}	?	?	?	?	?	?	?	?	0	0	0	0
ψ_{44*}	?	?	?	?	0	0	0	0	−	−	?	?
ψ_{55*}	?	?	?	?	0	0	0	0	?	?	?	?
ψ_{66*}	?	?	?	?	0	0	0	0	?	?	?	?
ψ_{77*}	0	0	0	0	−	?	?	?	−	?	−	?
ψ_{88*}	0	0	0	0	?	?	?	?	?	?	?	?
ψ_{99*}	0	0	0	0	?	?	?	?	?	?	?	?

f_1 is also at the origin of the prediction relation $f_1 \to f_3$, the parameter ϕ_{11} is subject to the constraints of the intact covariances between (x_1, x_2, x_3) and (x_7, x_8, x_9). Because $m_{f_3f_3}$ is not affected by $\sigma_{e_1e_4} > 0$, there is little flexibility in ϕ_{11}. Although γ_1 also contributes to $m_{f_3f_3}$, the contribution is an indirect effect, so there is more flexibility in γ_1 than in ϕ_{11}. Actually, ϕ_{11} has two contributions to $m_{f_3f_3}$; one is the direct effect through $f_1 \to f_3$, and the other is the indirect effect through $f_1 \to f_2 \to f_3$. In the indirect effect, ϕ_{11} and γ_1 are comparable in their contributions. Thus, a greater $\gamma_{1*}\phi_{11*}$ needs a greater γ_{1*}. With a greater γ_{1*}, ϕ_{11*} can be greater or smaller; a greater ϕ_{11*} corresponds to smaller λ_{2*} and λ_{3*} and vice versa. However, it is hard to predict the direction of change in ϕ_{11*}. Similarly, λ_{2*}, λ_{3*}, λ_{5*} and λ_{6*} need to explain both the within- and between-cluster associations of (x_1, x_2, x_3) and (x_4, x_5, x_6). For example, σ_{24} and σ_{34} need smaller λ_{2*} and λ_{3*} whereas σ_{12}, σ_{13}, and σ_{23} may need them to be greater. Consequently, it is hard to predict their directions of change. The parameter φ_{22} mainly reflects the prediction error in $f_1 \to f_2 \leftarrow d_2$, a greater γ_{1*} will lead to a smaller φ_{22*}. We are also unable to determine the directions of change in γ_{2*}, γ_{3*}, φ_{33*}, and ψ_{11*} to ψ_{66*}.

Similar to $\sigma_{e_1e_4} > 0$, when a positive covariance between one of (e_1, e_2, e_3) and one of (e_4, e_5, e_6) exists, γ_{1*} will be greater and φ_{22*} will be smaller. We are unable to determine the directions of change in λ_{2*}, λ_{3*}, λ_{5*}, λ_{6*}, γ_{2*}, γ_{3*}, and φ_{33*}.

Between clusters (x_1, x_2, x_3) and (x_7, x_8, x_9). When ignoring a covariance between (e_1, e_2, e_3) and (e_7, e_8, e_9), the parameters within the cluster (x_4, x_5, x_6) are not affected, as indicated in Table 4; $m_{f_2f_2*}$ will be correctly evaluated. The shortest path from x_i, $i = 1, 2, 3$ to x_j, $j = 7, 8, 9$ is through $f_1 \to f_3$. Although the indirect path $f_1 \to f_2 \to f_3$ may partially explain the covariances between (x_1, x_2, x_3) and (x_7, x_8, x_9), because the model parameters within the cluster (x_4, x_5, x_6) are not affected, any extra association between (x_1, x_2, x_3) and (x_7, x_8, x_9) passed through the indirect path is very limited. A greater ϕ_{11*} needs to pass both the direct and indirect paths to achieve a greater $m_{f_1f_3*}$, so γ_{2*} is mainly responsible for a greater σ_{ij}, $i = 1, 2, 3$ and $j = 7, 8, 9$. Notice that, unlike φ_{22*}, the only function of φ_{33*} is to adjust for $m_{f_3f_3*}$; the covariances within the cluster (x_7, x_8, x_9) have little to do with the values of γ_{2*} and ϕ_{11*}.

When $\sigma_{e_1e_7} > 0$, γ_{2*} should be greater and so is $m_{17*} = m_{f_1f_3*}$. Notice $m_{27} = \lambda_2 m_{f_1f_3}$. With a greater $m_{f_1f_3*}$, λ_{2*} has to be smaller to explain σ_{27}. Similarly, λ_{3*}, λ_{8*}, and λ_{9*} have to be smaller. Due to smaller λ_{2*} and λ_{3*}, ϕ_{11*} has to be greater to explain the within-cluster covariances of (x_1, x_2, x_3). A greater ϕ_{11*} implies smaller γ_{1*} and ψ_{11*}. Similarly, $m_{f_3f_3*}$ has to be greater and ψ_{77*} has to be smaller. Our analysis cannot determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, ψ_{22*}, ψ_{33*}, ψ_{88*}, and ψ_{99*}.

When $\sigma_{e_1e_8} > 0$, γ_{2*} should be greater and so are $m_{f_1f_3*}$ and m_{18*}. Because $m_{18} = \lambda_8 m_{f_1f_3}$ and $m_{f_1f_3}$ is shared by all the paths between (x_1, x_2, x_3) and (x_7, x_8, x_9), λ_{8*} has to be greater. Notice $m_{28} = \lambda_2 m_{18}$, $m_{38} = \lambda_3 m_{18}$, and $m_{19} = \lambda_9 m_{f_1f_3}$; hence λ_{2*}, λ_{3*}, and λ_{9*} have to be smaller. Due to smaller λ_{2*} and λ_{3*}, ϕ_{11*} has to be greater to explain the within-cluster associations of (x_1, x_2, x_3). A greater ϕ_{11*} leads to smaller γ_{1*} and ψ_{11*}. We are unable to determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, ψ_{22*}, ψ_{33*}, and ψ_{77*} to ψ_{99*}.

When $\sigma_{e_2e_7} > 0$, the direct path loading γ_{2*} from f_1 to f_3 as well as $m_{f_1f_3*}$, m_{27*} and λ_{2*} should be greater. Notice $m_{28} = \lambda_8 m_{27}$ and $m_{29} = \lambda_9 m_{27}$, $m_{37} = \lambda_3 m_{f_1f_3}$; hence λ_{3*}, λ_{8*}, and λ_{9*} have to be smaller for the corresponding intact between-cluster covariances. Due to greater λ_{2*}, ϕ_{11*} should be smaller to explain σ_{12}. A smaller ϕ_{11*} leads to greater γ_{1*} and ψ_{11*}. Our analysis cannot determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, ψ_{22*}, ψ_{33*}, and ψ_{77*} to ψ_{99*}.

When $\sigma_{e_2e_8} > 0$, both λ_{2*} and λ_{8*} should be greater. Due to a greater λ_{2*} and λ_{8*}, ϕ_{11*}, λ_{3*}, and λ_{9*} have to be smaller to explain the intact within-cluster covariances. A smaller ϕ_{11*} needs a greater γ_{2*}; a smaller ϕ_{11*} also leads to a greater γ_{1*} and ψ_{11*}. Our analysis cannot determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, ψ_{22*}, ψ_{33*}, and ψ_{77*} to ψ_{99*}.

Between clusters (x_4, x_5, x_6) and (x_7, x_8, x_9). With possible covariances between (e_4, e_5, e_6) and (e_7, e_8, e_9) in Figure 6, the parameters within the cluster (x_1, x_2, x_3) are not affected, nor is $\phi_{11} = m_{f_1 f_1}$. The path $f_2 \to f_3$ is mainly responsible for explaining the covariances between the indicators in (x_4, x_5, x_6) and (x_7, x_8, x_9). Although $m_{f_2 f_3}$ involves γ_1 and γ_2, due to $\phi_{11*} = \phi_{110}$ and the intact between-cluster covariances of (x_1, x_2, x_3) with (x_4, x_5, x_6) and (x_7, x_8, x_9), the strength of γ_{1*} and γ_{2*} to affect $m_{f_2 f_3}$ is limited. Actually, γ_1 is mainly responsible for the between-cluster associations of (x_1, x_2, x_3) and (x_4, x_5, x_6); γ_2 is mainly responsible for the between-cluster associations of (x_1, x_2, x_3) and (x_7, x_8, x_9).

When $\sigma_{e_4 e_7} > 0$, both γ_{3*} and $m_{47} = m_{f_2 f_3 *}$ have to be greater. Notice $m_{57} = \lambda_5 m_{47}$, $m_{67} = \lambda_6 m_{47}$, $m_{48} = \lambda_8 m_{47}$, and $m_{49} = \lambda_9 m_{47}$. Due to a greater m_{47*}, λ_{5*}, λ_{6*}, λ_{8*}, and λ_{9*} have to be smaller. Due to smaller λ_{5*} and λ_{6*}, $m_{f_2 f_2 *}$ has to be greater to explain the within-cluster covariances of (x_4, x_5, x_6); ψ_{44*} has to be smaller. The parameter φ_{22*} should be greater because it is mainly responsible for adjusting the variance of f_2. Due to the smaller average loadings within the cluster (x_4, x_5, x_6), γ_{1*} has to be greater to explain the between-cluster associations of (x_1, x_2, x_3) with (x_4, x_5, x_6). Notice $m_{f_1 f_3} = \phi_{11}(\gamma_1\gamma_3 + \gamma_2)$. Due to greater γ_{1*} and γ_{3*}, γ_{2*} has to be smaller. Due to smaller λ_{8*} and λ_{9*}, $m_{f_3 f_3 *}$ is greater, and ψ_{77*} is smaller. We cannot determine the directions of change in φ_{33*}, ψ_{55*}, ψ_{66*}, ψ_{88*}, and ψ_{99*}.

When $\sigma_{e_4 e_8} > 0$, γ_{3*} as well as m_{48*} and $m_{f_2 f_3 *}$ will be greater. Due to the unique position of x_8, λ_{8*} should be greater. Notice that $m_{49} = \lambda_9 m_{f_2 f_3}$, $m_{58} = \lambda_5 m_{48}$, and $m_{68} = \lambda_6 m_{48}$, hence λ_{5*}, λ_{6*}, and λ_{9*} have to be smaller. Due to smaller λ_{5*} and λ_{6*}, $m_{f_2 f_2 *}$ has to be greater to explain the intact within-cluster covariances of (x_4, x_5, x_6). Consequently, φ_{22*}, whose main function is to adjust for $m_{f_2 f_2 *}$, has to be greater; ψ_{44*} has to be smaller. Because γ_1 is mainly for the between-cluster associations of (x_1, x_2, x_3) with (x_4, x_5, x_6) and the average factor loading within the cluster (x_4, x_5, x_6) is smaller, γ_{1*} has to be greater. Notice $m_{f_1 f_3} = \phi_{11}(\gamma_2 + \gamma_1\gamma_3)$. Due to greater γ_{1*} and γ_{3*}, γ_{2*} has to be smaller. We cannot determine the directions of change in φ_{33*}, ψ_{55*} to ψ_{99*}.

When $\sigma_{e_5 e_7} > 0$, λ_{5*}, $m_{f_2 f_3 *}$, m_{57*} will be greater. Notice $m_{x_6 f_2} = \lambda_6 m_{f_2 f_3 *}$, $m_{58} = \lambda_8 m_{57}$, and $m_{59} = \lambda_9 m_{57}$; hence λ_{6*}, λ_{8*}, and λ_{9*} need to be smaller. Due to smaller λ_{8*} and λ_{9*}, $m_{f_3 f_3 *}$ will be greater, which leads to a smaller ψ_{77*}. Our analysis cannot determine the directions of change in γ_{1*}, γ_{2*}, φ_{22*}, φ_{33*}, ψ_{44*} to ψ_{66*}, ψ_{88*}, and ψ_{99*}.

When $\sigma_{e_5 e_8} > 0$, λ_{5*}, λ_{8*}, $m_{f_2 f_3 *}$, m_{58*} will be greater; λ_{6*} and λ_{9*} will be smaller. We are unable to determine the directions of change in γ_{1*}, γ_{2*}, φ_{22*}, φ_{33*}, ψ_{44*} to ψ_{99*}.

Extra factor loadings for x_1 or x_2. When x_1 or x_2 needs to load on f_2 or f_3, the model represented by Figure 6 is misspecified. Because the within- and between-cluster associations of (x_4, x_5, x_6) and (x_7, x_8, x_9) are not affected by the misspecification, λ_{5*}, λ_{6*}, λ_{8*}, λ_{9*}, and ψ_{44*} to ψ_{99*} will equal their counterparts in the correctly specified model. The variances and covariance $m_{f_2 f_2 *}$, $m_{f_2 f_3 *}$, and $m_{f_3 f_3 *}$ will also be correctly evaluated. In the following we only discuss the remaining parameters when an extra loading exists in the population for the model represented by Figure 6. The results corresponding to excluding all the factor loadings are summarized in Table 5. Readers should analyze Figure 6 to get their own version of the table.

When x_1 needs to load on f_2 with $\lambda_{120} > 0$, $m_{x_1 f_2 *} = \phi_{11*}\gamma_{1*}$ has to be greater. A greater $\phi_{11*}\gamma_{1*}$ needs smaller λ_{2*} and λ_{3*} to explain the intact $\sigma_{x_2 f_2}$ and $\sigma_{x_3 f_2}$. Smaller λ_{2*} and λ_{3*} need a much greater ϕ_{11*} to interpret the intact σ_{23} and greater σ_{12} and σ_{13}. A much greater ϕ_{11*} needs a smaller γ_{1*} to explain the intact covariances of (x_2, x_3) with (x_4, x_5, x_6). For a similar reason, γ_{2*} needs to be smaller. A much greater ϕ_{11*} or greater $\phi_{11*}\gamma_{1*}$ also leads to a smaller φ_{22*} to have $m_{f_2 f_2 *} = m_{f_2 f_2 0}$. We are unable to determine the directions

TABLE 5
The Directions of Change in Parameters When an Extra Positive Factor Loading Exists in the Population and the Model is Represented by Figure 6

θ	λ_{120}	λ_{130}	λ_{220}	λ_{230}	λ_{410}	λ_{430}	λ_{510}	λ_{530}	λ_{710}	λ_{720}	λ_{810}	λ_{820}
λ_{2*}	−	−	+	+	0	0	0	0	0	0	0	0
λ_{3*}	−	−	−	−	0	0	0	0	0	0	0	0
λ_{5*}	0	0	0	0	−	−	+	+	0	0	0	0
λ_{6*}	0	0	0	0	−	−	−	−	0	0	0	0
λ_{8*}	0	0	0	0	0	0	0	0	−	−	+	+
λ_{9*}	0	0	0	0	0	0	0	0	−	−	−	−
ϕ_{11*}	+	+	−	−	0	0	0	0	0	0	0	0
γ_{1*}	−	−	+	+	+	+	+	?	0	0	0	0
γ_{2*}	−	−	+	+	−	−	?	?	+	+	+	−
γ_{3*}	?	?	?	?	−	−	?	+	?	+	?	+
φ_{22*}	−	−	?	?	?	?	−	?	0	0	0	0
φ_{33*}	?	?	?	?	?	?	?	?	?	?	?	?
ψ_{11*}	?	?	+	+	0	0	0	0	0	0	0	0
ψ_{22*}	?	?	?	?	0	0	0	0	0	0	0	0
ψ_{33*}	?	?	+	+	0	0	0	0	0	0	0	0
ψ_{44*}	0	0	0	0	?	?	+	+	0	0	0	0
ψ_{55*}	0	0	0	0	?	?	?	?	0	0	0	0
ψ_{66*}	0	0	0	0	?	?	+	+	0	0	0	0
ψ_{77*}	0	0	0	0	0	0	0	0	?	?	+	+
ψ_{88*}	0	0	0	0	0	0	0	0	?	?	?	?
ψ_{99*}	0	0	0	0	0	0	0	0	?	?	+	+

of change in γ_{3*} and φ_{33*}. Due to $\lambda_{120} > 0$ positively perturbing σ_{11}, we are unable to determine the directions of change in ψ_{11*} to ψ_{33*} either.

When x_1 needs to load on f_3 with $\lambda_{130} > 0$, the stronger association between x_1 and f_3 has to be explained by $m_{x_1 f_3}$. By similar logic as with $\lambda_{120} > 0$, ϕ_{11*} has to be much greater; λ_{2*} and λ_{3*} have to be smaller; and γ_{1*}, γ_{2*}, and φ_{22*} have to be smaller. We are unable to determine the directions of change in γ_{3*}, φ_{33*}, and ψ_{11*} to ψ_{33*}.

When x_2 needs to load on f_2 with $\lambda_{220} > 0$, the extra association between x_2 and f_2 has to be explained by $m_{x_2 f_2} = \lambda_2\gamma_1\phi_{11}$; thus $\lambda_{2*}\gamma_{1*}\phi_{11*}$ will be greater. Notice that $m_{x_1 f_2} = \gamma_1\phi_{11}$ and $m_{x_3 f_2} = \lambda_3\gamma_1\phi_{11}$, $\gamma_{1*}\phi_{11*}$ cannot be too much greater, so λ_{2*} has to be much greater. A much greater λ_{2*} needs smaller ϕ_{11*} and λ_{3*} to explain the associations within (x_1, x_2, x_3). A smaller ϕ_{11*} leads to greater γ_{1*}, γ_{2*}, ψ_{11*}, and ψ_{33*}. We are unable to determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, and ψ_{22*}.

When x_2 needs to load on f_3 with $\lambda_{230} > 0$, the extra association between x_2 and f_3 has to be explained by $m_{x_2 f_3}$. By similar logic as when $\lambda_{220} > 0$, λ_{2*} has to be much greater; ϕ_{11*} and λ_{3*} have to be smaller; and γ_{1*}, γ_{2*}, ψ_{11*}, and ψ_{33*} have to be greater. We are unable to determine the directions of change in γ_{3*}, φ_{22*}, φ_{33*}, and ψ_{22*}.

Extra factor loadings for x_4 or x_5. When x_4 or x_5 needs to load on f_1 or f_3, the model represented by Figure 6 is also misspecified. The within- and between-cluster covariances of (x_1, x_2, x_3) and (x_7, x_8, x_9) are not affected. Thus, λ_{2*}, λ_{3*}, λ_{8*}, λ_{9*}, ψ_{11*} to ψ_{33*} and ψ_{77*} to ψ_{99*} still equal those corresponding to the correctly specified model, as indicated in Table 5. The variances and covariance ϕ_{11*}, $m_{f_1 f_3 *}$, and $m_{f_3 f_3 *}$ will also be correctly evaluated.

When x_4 needs to load on f_1 with $\lambda_{410} > 0$, the stronger association between x_4 and f_1 has to be explained by $m_{x_4 f_1} = \gamma_1\phi_{11}$, thus γ_{1*} has to be much greater. A greater γ_{1*} needs smaller λ_{5*} and λ_{6*} to explain the intact covariances of (x_5, x_6) with (x_1, x_2, x_3). A much greater γ_{1*} most likely needs to have smaller γ_{2*} and γ_{3*} to keep $m_{f_1 f_3} = \phi_{110}(\gamma_{2*} + \gamma_{1*}\gamma_{3*})$ the same. We are unable to determine the directions of change in φ_{22*}, φ_{33*}, and ψ_{44*} to ψ_{66*}.

When x_4 needs to load on f_3 with $\lambda_{430} > 0$, the extra association between x_4 and f_3 has to be explained by $m_{x_4 f_3 *}$. Greater $m_{x_4 f_3 *}$ leads to smaller λ_{5*} and λ_{6*} to explain the intact covariances of (x_5, x_6) with (x_7, x_8, x_9). Smaller λ_{5*} and λ_{6*} need a greater γ_{1*} to explain the between-cluster associations of (x_4, x_5, x_6) with (x_1, x_2, x_3). A greater γ_{1*} needs to have smaller γ_{2*} or γ_{3*} or both to keep $m_{f_1 f_3 *}$ unchanged. It is difficult for us to determine which case. We are unable to determine the directions of change in φ_{22*}, φ_{33*}, and ψ_{44*} to ψ_{66*}.

When x_5 needs to load on f_1 with $\lambda_{510} > 0$, the extra association between x_5 and f_1 has to be explained by $m_{x_5 f_1} = \lambda_5\gamma_1\phi_{11}$, thus $\lambda_{5*}\gamma_{1*}\phi_{110}$ will be greater.

Because $f_1 \to f_2$ is the direct path for the extra association, γ_{1*} will be greater but cannot be too much greater, because $m_{x_4 f_1} = \gamma_1 \phi_{11}$ and $m_{x_6 f_1} = \lambda_6 \gamma_1 \phi_{11}$, so λ_{5*} has to be much greater. A much greater λ_{5*} needs smaller $m_{f_2 f_2 *} = \gamma_{1*}^2 \phi_{110} + \varphi_{22*}$ and λ_{6*} to explain the associations within (x_4, x_5, x_6). Because γ_{1*} is mainly responsible for the between-cluster associations of (x_1, x_2, x_3) with (x_4, x_5, x_6), φ_{22*} has to be smaller. A smaller $m_{f_2 f_2 *}$ leads to greater ψ_{44*} and ψ_{66*}. We are unable to determine the directions of change in γ_{2*}, γ_{3*}, φ_{33*}, and ψ_{55*}.

When x_5 needs to load on f_3 with $\lambda_{530} > 0$, the extra association between x_5 and f_3 has to be explained by $m_{x_5 f_3} = \lambda_5 m_{f_2 f_3}$. By an analysis similar to when $\lambda_{510} > 0$, λ_{5*} has to be much greater. A much greater λ_{5*} needs smaller $m_{f_2 f_2 *} = \gamma_{1*}^2 \phi_{11*} + \varphi_{22*}$ and λ_{6*} to explain the associations within (x_4, x_5, x_6). A smaller $m_{f_2 f_2 *}$ needs greater γ_{3*}, ψ_{44*}, and ψ_{66*}. We are unable to determine the directions of change in γ_{1*}, γ_{2*}, φ_{22*}, φ_{33*}, and ψ_{55*}.

Extra factor loadings for x_7 or x_8. When x_7 or x_8 needs to load on f_1 or f_2, the model represented by Figure 6 is again misspecified. The within- and between-cluster associations of (x_1, x_2, x_3) and (x_4, x_5, x_6) will not be affected. Thus, λ_{2*}, λ_{3*}, λ_{5*}, λ_{6*}, and ψ_{11*} to ψ_{66*} still equal the population values corresponding to the correctly specified model, as indicated in Table 5. The variances and covariance $m_{f_1 f_1 *}$, $m_{f_1 f_2 *}$, and $m_{f_2 f_2 *}$ will also be correctly evaluated with $\phi_{11*} = \phi_{110}$, $\gamma_{1*} = \gamma_{10}$, and $\varphi_{22*} = \varphi_{220}$.

When x_7 needs to load on f_1 with $\lambda_{710} > 0$, the extra association has to be explained by $m_{x_7 f_1} = m_{f_3 f_1} = \phi_{11}(\gamma_2 + \gamma_1 \gamma_3)$. Being mainly responsible for the association between f_1 and f_3, γ_{2*} has to be greater. Smaller λ_{8*} and λ_{9*} are needed to explain the associations of (x_8, x_9) with (x_1, x_2, x_3). We are unable to determine the directions of change in γ_{3*}, φ_{33*}, and ψ_{77*} to ψ_{99*}.

When x_7 needs to load on f_2 with $\lambda_{720} > 0$, the extra association has to be explained by $m_{x_7 f_2} = m_{f_3 f_2}$. Being mainly responsible for the path $f_2 \to f_3$, γ_{3*} should be greater. A greater $m_{f_3 f_2 *}$ needs smaller λ_{8*} and λ_{9*} to properly explain the between-cluster associations of (x_7, x_8, x_9) with (x_4, x_5, x_6). Smaller λ_{8*} and λ_{9*} need greater γ_{2*} for the between-cluster associations of (x_1, x_2, x_3) and (x_7, x_8, x_9). We are unable to determine the directions of change in φ_{33*} and ψ_{77*} to ψ_{99*}.

When x_8 needs to load on f_1 with $\lambda_{810} > 0$, the extra association has to be explained by $m_{x_8 f_1} = \lambda_8 m_{f_3 f_1}$. Being mainly responsible for the association between f_1 and f_3, γ_{2*} has to be greater. Because $m_{f_3 f_1}$ is shared by all the variables in (x_1, x_2, x_3) and (x_7, x_8, x_9) for between-cluster covariances, λ_{8*} has to be much greater. A much greater λ_{8*} needs smaller λ_{9*} and $m_{f_3 f_3 *}$ to explain the within-cluster associations of (x_1, x_2, x_3), which further lead to greater ψ_{77*} and ψ_{99*}. We are unable to determine the directions of change in γ_{3*}, φ_{33*}, and ψ_{88*}.

When x_8 needs to load on f_2 with $\lambda_{820} > 0$, the extra association has to be explained by $m_{x_8 f_2} = \lambda_8 m_{f_3 f_2}$. Being mainly responsible for the association between f_3 and f_2, γ_{3*} has to be greater. Because $m_{f_3 f_2}$ is shared by (x_4, x_5, x_6) and (x_7, x_8, x_9) for their between-cluster associations, λ_{8*} has to be much greater. A greater λ_{8*} needs smaller λ_{9*} and $m_{f_3 f_3 *}$ to explain the within-cluster associations of (x_7, x_8, x_9), which further leads to greater ψ_{77*} and ψ_{99*}. A much greater λ_{8*} most likely needs a smaller γ_{2*} for the between-cluster associations of (x_1, x_2, x_3) and (x_7, x_8, x_9). We are unable to determine the directions of change in φ_{33*} and ψ_{88*}.

Excluding the γ_j. The remaining analysis for Figure 6 is when one of the paths among the latent factors f_1, f_2, and f_3 are excluded. Similar to the previous analyses, we consider cases when a positive γ_j exists in the population but not in the model. We discuss the effect in the order of excluding γ_1, γ_2, and γ_3. The results of the analysis are summarized in Table 6, where $\times$ means the parameter is fixed at zero or the path represented by that parameter is excluded from the model.

TABLE 6
The Directions of Change in Parameters
When $\gamma_j > 0$ Exists in the Population
But Is Excluded From the Model in Figure 6

θ_*	γ_1	γ_2	γ_3
λ_{2*}	0	0	?
λ_{3*}	0	0	?
λ_{5*}	0	?	0
λ_{6*}	0	?	0
λ_{8*}	0	0	0
λ_{9*}	0	0	0
ϕ_{11*}	0	0	?
γ_{1*}	$\times$	+	+
γ_{2*}	+	$\times$	+
γ_{3*}	+	+	$\times$
φ_{22*}	+	−	−
φ_{33*}	?	?	?
ψ_{11*}	0	0	?
ψ_{22*}	0	0	?
ψ_{33*}	0	0	?
ψ_{44*}	0	?	0
ψ_{55*}	0	?	0
ψ_{66*}	0	?	0
ψ_{77*}	0	0	0
ψ_{88*}	0	0	0
ψ_{99*}	0	0	0

When $\gamma_{10} > 0$ in Figure 6 and the path $f_1 \to f_2$ is removed from the model, let $m_{f_2 f_2} = \phi_{22} = \varphi_{22}$. The model-implied covariance matrix of (f_1, f_2, f_3) can be obtained from the matrix in Equation 2 by letting $\gamma_1 = 0$. It follows from either the model-implied covariance matrix or the path diagram that the values of ϕ_{11} and γ_2 are not identified in $m_{f_1 f_3} = \phi_{11}\gamma_2$. Similarly, the values of ϕ_{22} and γ_3 are confounded in $m_{f_2 f_3} = \phi_{22}\gamma_3$. Because $m_{f_1 f_2} = 0$ whatever values the parameters are, ϕ_{11*} will adjust for the covariances within (x_1, x_2, x_3) and γ_{2*} will take whatever ϕ_{11*} is and adjust for the covariance between (x_1, x_2, x_3) and (x_7, x_8, x_9). Thus, $\phi_{11*} = \phi_{110}$, $\lambda_{2*} = \lambda_{20}$, and $\lambda_{3*} = \lambda_{30}$. Similarly, $\lambda_{5*} = \lambda_{50}$, $\lambda_{6*} = \lambda_{60}$, and $\varphi_{22*} = \phi_{22*} = \sigma_{f_2 f_2} = \gamma_{10}^2\phi_{110} + \varphi_{220}$. The only function of φ_{33} is to adjust for the variance of f_3, as being predicted by f_1 and f_2, the between-cluster covariances can and have to be explained by γ_{2*} and γ_{3*}, so $\lambda_{8*} = \lambda_{80}$ and $\lambda_{9*} = \lambda_{90}$. The parameters ψ_{ii*}s also equal their counterparts corresponding to the correctly specified model. Notice that in the population there exists $\sigma_{f_1 f_3} > \phi_{110}\gamma_{20}$, hence γ_{2*} has to be greater to properly explain the covariances between (x_1, x_2, x_3) and (x_7, x_8, x_9). Similarly, $\sigma_{f_2 f_3} > \phi_{22*}\gamma_{30}$, γ_{3*} has to be greater to explain the covariances between (x_4, x_5, x_6) and (x_7, x_8, x_9). We are unable to predict the direction of change in φ_{33*}.

When $\gamma_{20} > 0$ in Figure 6 and the path $f_1 \to f_3$ is removed from the model, the model-implied covariance matrix of (f_1, f_2, f_3) can be obtained from the matrix in Equation 2 by letting $\gamma_2 = 0$. Similar to when $f_1 \to f_2$ is removed, γ_1 and ϕ_{11} are confounded in explaining the relation among the factors, ϕ_{11*} will equal ϕ_{110} to explain the covariances within (x_1, x_2, x_3), and γ_{1*} will take a value to best represent the between-cluster associations. Similarly, φ_{33*} will adjust for $m_{f_3 f_3 *}$ to fully explain the covariance within the cluster (x_7, x_8, x_9). Thus, λ_{2*}, λ_{3*}, λ_{8*}, λ_{9*}, ψ_{11*} to ψ_{33*}, and ψ_{77*} to ψ_{99*} will equal their counterparts corresponding to the correctly specified model. When $f_1 \to f_3$ existed, the association between f_1 and f_3 was mostly through the direct path; now, it has to be through the indirect path. Although the path $f_2 \to f_3$ still exists, due to missing $f_1 \to f_3$, now $m_{f_2 f_3} = \gamma_1^2\gamma_3\phi_{11} + \gamma_3\varphi_{22}$. It follows from the matrix in Equation 2 that, in the population, $\sigma_{f_2 f_3} > \gamma_{10}^2\gamma_{30}\phi_{110} + \gamma_{30}\varphi_{220}$. With $\phi_{11*} = \phi_{110}$, both γ_{1*} and γ_{3*} have to be greater to balance the effect of the excluded path. Notice that the main and direct effect of φ_{22} is to adjust for $m_{f_2 f_2}$ although it also contributes to $m_{f_2 f_3}$. Due to a greater γ_{1*}, φ_{22*} should become smaller. We are unable to determine the directions of change in λ_{5*}, λ_{6*}, ψ_{44*} to ψ_{66*}, and φ_{33*}.

When $\gamma_{30} > 0$ in Figure 6 and the path $f_2 \to f_3$ is removed from the model, the model-implied covariance matrix of (f_1, f_2, f_3) can be obtained from Equation 2 by letting $\gamma_3 = 0$. Now, the only function of φ_{22} is to adjust for $m_{f_2 f_2}$, so the parameters within the cluster (x_4, x_5, x_6) can still be evaluated correctly. For the same reason, the parameters within the cluster (x_7, x_8, x_9) will be evaluated correctly. Without the path $f_2 \to f_3$, the covariance between f_2 and

f_3 is explained by their sharing the same predictor f_1, with the model-implied covariance $m_{f_2 f_3} = \phi_{11}\gamma_1\gamma_2$. In Figure 6, there are two paths for f_1 and f_3 to correlate, one is the direct path $f_2 \to f_3$, and the other is the indirect path $f_1 \to f_2 \to f_3$. Without $f_2 \to f_3$, the direct path will pick up the association in the indirect path. Actually, in the covariance matrix in Equation 2, γ_2 always appears in the form $\gamma_2 + \gamma_1\gamma_3$. Thus, without γ_3, γ_{2*} will at least increase up to $\gamma_{20} + \gamma_{10}\gamma_{30}$. However, an increased γ_{2*} alone is not enough for modeling $\sigma_{f_2 f_3} = \gamma_{10}\phi_{110}(\gamma_{20} + \gamma_{10}\gamma_{30}) + \gamma_{30}\varphi_{220}$. Notice that γ_1 and γ_2 are parallel in modeling $\sigma_{f_2 f_3}$. Because ϕ_{11*} needs to model the covariances within (x_1, x_2, x_3), both γ_{1*} and γ_{2*} need to be greater. Due to the parallel positions of γ_1 and γ_2, a greater $m_{f_2 f_3 *} = \gamma_{1*}\gamma_{2*}\phi_{11*}$ most likely leads to a greater $\gamma_{1*}^2\phi_{11*}$, and thus a smaller φ_{22*}. A greater γ_{1*} may need a smaller ϕ_{11*} or λ_{2*} or λ_{3*} to explain the covariances between (x_1, x_2, x_3) and (x_4, x_5, x_6). We are unable to determine the directions of change in ϕ_{11*}, λ_{2*}, λ_{3*}, φ_{33*}, and ψ_{11*} to ψ_{33*}.

EXAMPLES

The previous sections illustrated the technique of analyzing the path to identify parameter changes due to model misspecification. This section applies the technique to three models. The first one is a continuation of Example 1, in which we show that the predicted change in γ_{3*} by analyzing the path agrees with the change in $\hat{\gamma}_3$ when the model is changed. The second one is based on a data set from Chatterjee, Handcock, and Simonoff (1995); the model is a one-factor model. We verify whether the parameter change as predicted by analyzing the path concurs with that in real data. The last one is based on a data set from Holzinger and Swineford (1939). That model is a confirmatory three-factor model, the same as the one analyzed earlier. Because we do not know the correct model in any of the examples, our analysis is to compare parameter estimates between different models. These examples allow us to see how the technique of analyzing the path performs with real data, especially whether it can still predict the change of parameter estimates when the model with more parameters may only represent a closer fit.

Example 1 Continued

Applying the same analysis as for the three-factor structural equation model in the previous section to Figure 1, γ_{3*} should increase when (a) $e_3 \leftrightarrow e_5$ is removed; (b) $e_4 \leftrightarrow e_6$ is removed; (c) both $e_3 \leftrightarrow e_5$ and $e_4 \leftrightarrow e_6$ are removed; and (d) $e_3 \leftrightarrow e_5$ and $e_4 \leftrightarrow e_6$ are removed, as well as when f_1 together with education and occupational status index are out of the model. With the real data, the maximum likelihood estimates $\hat{\gamma}_3$s are

model change	(a)	(b)	(c)	(d)	(e)
$\hat{\gamma}_3$	0.655	0.617	0.705	0.789	0.607
p	0.000	0.275	0.000	0.000	0.316

where (e) is for the full model in Figure 1. Our analysis therefore applies not only to the population parameter values, but also to their estimates. The second row of numbers are the p values when referring the normal distribution-based likelihood ratio statistics, with $N = 932$, to the corresponding chi-square distributions. Both the full model and the model without the path $e_4 \leftrightarrow e_6$ are statistically acceptable. However, they generate different $\hat{\gamma}_3$. If one is an unbiased estimate, the other might be a biased estimate. A complication with real data is that the difference between the different $\hat{\gamma}_3$s might be just due to sampling error. Yuan et al. (2003) provided a procedure for testing the significance of the difference.

Example 2

Chatterjee et al. (1995, p. 299) contains a data set of descriptive statistics for 105 guards of the National Basketball Association (NBA) for the 1992–1993 season. Among the variables are total minutes played, points scored per game, assists per game, and rebounds per game. Yuan and Bentler (1998) proposed a one-factor model for these four variables. To make the four variables have comparable standard deviations we divide the first three variables by 880, 6, and 2, respectively. Such a change of scale makes the convergence of the iterative procedure for obtaining the maximum likelihood estimates much faster. It has no effect on the substantive aspects of the analysis but makes the factor loadings and error variances of similar magnitude.

Fitting the sample covariance matrix to a one-factor model, as represented in Figure 2 (without the correlated error), leads to the first column of numbers in Table 7. However, the LM test in EQS indicates that the model will fit the data significantly better if allowing e_2 and e_4 to correlate. Adding the parameter ψ_{24} to the model results in the second column of numbers under $\hat{\boldsymbol{\theta}}_0$ in Table 7. Due to the model change, the likelihood ratio statistic changes from 15.281 with 2 *df* to 3.557 with 1 *df*; the LM test implies that no other model modification is needed at the .05 level. If we regard the modified model as the correct model, then $\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_0$ contains both the sampling error and the biases. The fourth column, under D, is the signs of bias obtained by analyzing the path, which agree with the actual change on all the estimates.

Yuan and Bentler (1998) noted that cases numbered 2, 4, and 6 in the NBA data are influential and should be downweighted or removed for proper analysis. After the three cases are removed, the parallel results are shown on the right

TABLE 7
Maximum Likelihood Estimates for Two Different Confirmatory Factor Models Using NBA Data and by Removing Three Cases

	The Original Sample				*Cases 2, 4, 6 Removed*			
θ	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\theta}}_0$	$\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_0$	*D*	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\theta}}_0$	$\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_0$	*D*
λ_1	0.897	0.977	−0.080	−	0.911	0.959	−0.048	−
λ_2	0.895	0.818	0.078	+	0.805	0.759	0.046	+
λ_3	0.895	0.913	−0.018	−	0.937	0.945	−0.008	−
λ_4	1.086	0.959	0.127	+	1.004	0.937	0.067	+
ψ_{11}	0.208	0.058	0.150	+	0.160	0.070	0.089	+
ψ_{22}	0.218	0.351	−0.133	−	0.197	0.268	−0.071	−
ψ_{33}	0.749	0.716	0.033	+	0.704	0.690	0.014	+
ψ_{44}	0.478	0.737	−0.260	−	0.355	0.485	−0.130	−
ψ_{24}	×	0.238			×	0.131		

side of Table 7. Again, the predicted change and the actual change in parameter estimates agree. Without Cases 2, 4, and 6, the LM test also implies that adding the parameter ψ_{24} can improve the model fit significantly; the likelihood ratio statistic changes from 7.500 to 1.201 due to the model change.

Example 3

Holzinger and Swineford (1939) contains test scores on the following subtests or variables: visual perception, cubes, lozenges, paragraph comprehension, sentence completion, word meaning, addition, counting dots, and straight-curved capitals. The first three variables were designed to measure spatial ability, the next three variables were designed to measure verbal ability, and the last three variables were administered with a limited time and were designed to measure a speed factor in performing the tasks. Thus, Holzinger and Swineford's design can be represented by the confirmatory factor model in Figure 4. However, all nine available statistics for testing $\boldsymbol{\Sigma} = \mathbf{M}(\boldsymbol{\theta})$ indicate that the model does not fit the sample (see Yuan & Bentler, 2007a), the largest p value is .015. The LM test indicates that the model would fit the data much better when adding either $e_7 \leftrightarrow e_8$ or $x_9 \leftarrow f_1$ to the model represented by Figure 4. Assuming the model with a free ψ_{78} is correct, parameter estimates $\hat{\lambda}_7$, $\hat{\lambda}_8$, and $\hat{\psi}_{99}$ will contain positive biases according to our analysis; $\hat{\lambda}_9$, $\hat{\phi}_{13}$, $\hat{\phi}_{23}$, $\hat{\psi}_{77}$, and $\hat{\psi}_{88}$ will contain negative biases; the rest of the parameter estimates do not contain biases. Assuming the model with a free λ_{91} is correct, then $\hat{\lambda}_9$, $\hat{\phi}_{13}$, $\hat{\psi}_{77}$, and $\hat{\psi}_{88}$ will contain positive biases; and $\hat{\lambda}_7$ and $\hat{\lambda}_8$ will contain negative biases; and $\hat{\lambda}_1$

to $\hat{\lambda}_6$ and $\hat{\psi}_{11}$ to $\hat{\psi}_{66}$ do not contain any biases. These results will be contrasted with numerical values of the estimates next.

Because standard deviations of the nine variables differ a lot, we divide them, respectively, by 6, 4, 8, 3, 4, 7, 23, 20, 36 to keep each marginal standard deviation between 1 and 2. This change of scale makes the factor loadings and error variances of a similar magnitude; thus, it facilitates our comparison of parameter change. Table 8 contains the maximum likelihood estimates $\hat{\boldsymbol{\theta}}$, corresponding to the model represented in Figure 4; and $\hat{\boldsymbol{\theta}}_{01}$, corresponding to the model with $e_7 \leftrightarrow e_8$ being added to Figure 4 and $\hat{\boldsymbol{\theta}}_{02}$, corresponding to the model with $x_9 \leftarrow f_1$ being added to Figure 4. For easy comparison, the differences $\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{01}$ and $\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{02}$ as well as the directions of change in θ_*, under D, are also included in Table 8. For all the θ_*s that are going to change according to the analysis, their estimates move in the predicted direction. For those $\theta_* = \theta_0$ according to the analysis, their estimates also become smaller or larger, most likely due to sampling error. For those the analysis cannot predict, their change might be due to sampling error as well as biases.

TABLE 8
Maximum Likelihood Estimates for Three Different Confirmatory Factor Models Using Data of Holzinger and Swineford (1939)

θ	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\theta}}_{01}$	$\hat{\boldsymbol{\theta}}_{02}$	$\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{01}$	D	$\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{02}$	D
λ_1	0.779	0.821	0.819	−0.041	0	−0.040	0
λ_2	0.574	0.548	0.543	0.026	0	0.031	0
λ_3	0.721	0.693	0.688	0.028	0	0.033	0
λ_4	0.974	0.975	0.975	−0.001	0	−0.001	0
λ_5	0.964	0.965	0.964	−0.001	0	0.000	0
λ_6	0.938	0.936	0.937	0.002	0	0.001	0
λ_7	0.682	0.440	0.708	0.242	+	−0.026	−
λ_8	0.837	0.568	0.900	0.269	+	−0.063	−
λ_9	0.720	1.007	0.453	−0.287	−	0.267	+
ϕ_{12}	0.541	0.542	0.554	−0.001	0	−0.013	0
ϕ_{13}	0.522	0.624	0.392	−0.102	−	0.130	+
ϕ_{23}	0.335	0.381	0.240	−0.046	−	0.096	?
ψ_{11}	0.721	0.654	0.657	0.066	0	0.064	0
ψ_{22}	0.905	0.934	0.940	−0.029	0	−0.035	0
ψ_{33}	0.560	0.601	0.607	−0.040	0	−0.046	0
ψ_{44}	0.317	0.315	0.315	0.002	0	0.002	0
ψ_{55}	0.422	0.421	0.422	0.001	0	−0.000	0
ψ_{66}	0.409	0.413	0.411	−0.004	0	−0.002	0
ψ_{77}	0.604	0.876	0.568	−0.272	−	0.036	+
ψ_{88}	0.402	0.780	0.293	−0.378	−	0.109	+
ψ_{99}	0.540	0.045	0.479	0.495	+	0.062	?
ψ_{78}	×	0.388	×				
λ_{91}	×	×	0.460				

We might have observed that, in Table 8, a parameter or its estimate may change to different directions when the model is modified differently. If the model with λ_{91} being included represents the correct model, then $\hat{\lambda}_7$ and $\hat{\lambda}_8$ contain negative biases, whereas $\hat{\lambda}_9$, $\hat{\phi}_{13}$, $\hat{\psi}_{77}$, and $\hat{\psi}_{88}$ contain positive biases. If the model with ψ_{78} being included represents the correct model, then these estimates contain biases in the opposite directions. This is because the two modified models are not mathematically equivalent. Actually, all the reliable statistics indicate both the modified models fit the data well. Again, the challenge with practical data is that we do not know which model is the correct one.

DISCUSSION

SEM, and covariance structural analysis in particular, has been widely used in the social and behavior sciences. Due to the effort of many methodologists, the capacity of SEM in modeling more complex data structures is increasing. When simultaneously modeling the relation among manifest and latent variables as well as measurement errors, misspecification is inevitable. Diagnostic tools that facilitate the understanding of the relation among different parts of the model and how misspecification affects model parameters are needed. Advanced mathematical and statistical procedures have been developed to study the effect of misspecification (e.g., Yuan et al., 2003), but they are not designed for model diagnosis. The technique of analyzing the path is not only intuitive but also facilitates a good understanding of the relation of various parts of the model and provides a clear picture of reactions of parameters to a misspecification. Thus, it is a valuable tool for model diagnosis. However, like any other method, analyzing the path needs conditions to work well. We discuss these conditions and related results later.

We intend to introduce the technique of analyzing the path for model diagnosis. The mathematics behind the analysis is Equation 1, which holds only when $\mathbf{\Sigma}$ is in a neighborhood of $\mathbf{\Sigma}_0$. The validity of analyzing the path is also restricted to when $\mathbf{\Sigma}$ is close to $\mathbf{\Sigma}_0$. Notice that the coefficients a_{ij}s in Equation 1 depend on the discrepancy function used in estimating the model, so analyzing the path also depends on the discrepancy function employed. Actually, the function $\mathbf{g}(\mathbf{\Sigma})$ introduced earlier is closely related to the weight matrix in the estimating equation (Yuan et al., 2003). Among all the discrepancy functions, only the normal distribution-based likelihood function corresponds to a weight matrix decided solely by the structural model $\mathbf{M}(\boldsymbol{\theta})$. When the model is misspecified, different discrepancy functions are not equivalent (Yuan & Chan, 2005). Because analyzing the path only uses information from the model and not from the weight matrix, the results obtained only correspond to the most widely used ML method. Of course, the function $\boldsymbol{\theta} = \mathbf{g}(\mathbf{\Sigma})$ also depends on $\mathbf{M}(\boldsymbol{\theta})$. Thus, the a_{ij} in Equation 1 is model dependent, which implies that analyzing the path is also model specific. There is no general conclusion that a parameter has to become smaller or greater when either $\mathbf{\Sigma}$ or $\mathbf{M}$ changes arbitrarily. When $\mathbf{M}$ is given, $\Delta\theta = \Delta\theta(\mathbf{\Sigma})$ is just a function of $\mathbf{\Sigma}$. When σ_{ij} is perturbed, the bias $\Delta\theta$ is characterized by Equation 1 and is predictable by analyzing the path. The models discussed in this article aim to be simple and to mimic those typically encountered in textbooks. The same technique and logic can be equally applied to more complex situations. For example, with more indicators in a cluster, the corresponding θ_* will also change according to the analysis but the actual values of the change may be smaller due to their subjection to more model and population constraints. Equality constrains are often used in SEM. When parameters θ_1 and θ_2 are set equal but their population values θ_{10} and θ_{20} in the unrestricted model are not equal, then their common estimated value in the restricted model will be somewhere between θ_{10} and θ_{20}.

We have only discussed biases in unstandardized parameter estimates due to model misspecification. In the applied literature, standardized estimates are reported more often than unstandardized estimates. Because standardized estimates are functions of unstandardized estimates, biases in unstandardized parameter estimates imply biases in standardized estimates. Let $\hat{\lambda}_{si}$ be the standardized estimate of the factor loading λ_i and $\hat{\psi}_{si}$ be the standardized estimate of the standard deviation of e_i in the model represented by either Figure 2 or 4. Then

$$\hat{\lambda}_{si} = \hat{\lambda}_i / s_i, \quad \hat{\psi}_{si} = \hat{\psi}_i / s_i, \tag{3}$$

where $\hat{\psi}_i = \hat{\psi}_{ii}^{1/2}$ and s_i is the sample standard deviation of x_i. Because s_i is consistent for σ_i, $\hat{\lambda}_{si}$ and $\hat{\psi}_{si}$ are consistent estimates of $\lambda_{si*} = \lambda_{i*}/\sigma_i$ and $\psi_{si*} = \psi_{i*}/\sigma_i$, respectively. Thus, the biases in $\hat{\lambda}_{si}$ and $\hat{\psi}_{si}$ will be in the same directions as those in $\hat{\lambda}_i$ and $\hat{\psi}_{ii}$, respectively. The simple relation in Equation 3 is due to $\phi_{kk} = 1$ in Figures 2 and 4. When $\phi_{kk} \neq 1$ as in Figures 1, 3, 5, and 6,

$$\hat{\lambda}_{si} = \hat{\lambda}_i \hat{\phi}_k / s_i, \quad \hat{\psi}_{si} = \hat{\psi}_i / s_i,$$

where $\hat{\phi}_k = \hat{\phi}_{kk}^{1/2}$ is the estimate of the standard deviation of the kth factor. Then the direction of the bias in $\hat{\lambda}_{si}$ depends on the magnitude of the biases in $\hat{\lambda}_i$ and $\hat{\phi}_{kk}$. When $\hat{\lambda}_i$ contains a positive bias and $\hat{\phi}_{kk}$ contains a negative bias, the bias in $\hat{\lambda}_{si}$ can be either negative or positive. So the directions of biases in standardized estimates are more difficult to determine by analyzing the path.

Analyzing the path is not a mathematical proof; the directions of change in parameters or their estimates will most likely happen according to the analysis.

They may not always happen as predicted under all conditions. First, our analysis is based on a model containing only a single misspecification. When a model contains multiple misspecifications or misses multiple parameters, a parameter may need to become larger by one misspecification and to become smaller by another. The resulting value of the parameter will be the accumulated changes, depending on the specific value of each change. Then analyzing the path cannot predict the direction of the accumulated parameter change. If dealing with one misspecification at a time, even when neither of the models is correct, the result of analyzing the path will still be valuable, as illustrated by the examples in the previous section, where we do not have the true model. Second, outliers may create problems when applying the technique to data. One outlier in a single sample may move a parameter estimate to an arbitrary place, and analyzing the path cannot predict such kinds of biases. When data contain outliers or heavy tails, one may need to remove the outliers, use a robust procedure, or transform the data before applying the ML procedure (see Yuan, Chan, & Bentler, 2000). Third, our analysis is based on all the model parameter values being comparable. When one parameter has a dominated value, a change in this parameter may have a dominant effect that is different from our analysis. In practice, one may rescale the variables to have comparable standard deviations, as was done to the NBA data and data of Holzinger and Swineford (1939) in the previous section. Fourth, analyzing the path cannot identify the directions of change on all model parameters. This typically happens when θ is associated with several other parameters in explaining a variance or covariance, some of which need θ_* ($\hat{\theta}$) to be greater and some of which need θ_* ($\hat{\theta}$) to be smaller. Fifth, analyzing the path cannot identify the directions of change on parameters when latent variables are omitted in the model. A structure equation or confirmatory factor analysis model is at least partially justified by theory or previous exploratory analysis, so missing latent variables in the model are possible but unlikely. Most of these limitations are not unique to analyzing the path. For example, a single outlier can move the commonly used likelihood ratio statistic to an arbitrary value (Yuan & Bentler, 2001). Similarly, a statistical test may not be able to tell whether the population value of a parameter is changed either due to effect size, sampling errors, or crude estimation of standard errors, especially with misspecified models (see Yuan & Hayashi, 2006). Actually, none of the differences of the estimates in Table 8 is zero. Analyzing the path tells us that some of these nonzero numbers are just due to sampling errors.

Analyzing the path is parallel to the functional relation in Equation 1 but provides an intuitive tool for model diagnosis. Different from the functional relation is the statistical relation that characterizes the reaction (fluctuation) of parameter estimates with sampling errors occurring to all the variables. The fluctuation and correlation among parameter estimates, caused by sampling errors, can be described by the asymptotic covariance matrix of the estimates.

For normally distributed data, the asymptotic covariance matrix is obtained by inverting the related information matrix. With a particular change in the model, the biases in parameter estimates have little to do with their correlations. For example, with positive population factor loadings in Figure 2 and normally distributed data, their estimates are positively correlated. When a pair of errors in Figure 2 are correlated and ignored, two loading estimates contain positive biases and two contain negative biases. Actually, asymptotic or population correlations or covariances among parameter estimates depend on the distribution of the data. For given σs, the correlations among $\hat{\theta}$s change when the distribution of the sample changes; the values of θ_* and θ_0 remain fixed when an estimation procedure is chosen, regardless of how the population distribution changes. Similarly, the predicted value change of a parameter estimate by analyzing the path does not imply the corresponding standard error will change proportionally.

The focus of this article is the biases in parameter estimates due to model misspecification. For certain parameters in statistical modeling, estimates with small biases may have much smaller variances when compared with those of the unbiased estimates. In the context of ML estimation for covariance structure models, adding variables or reducing the number of parameters usually leads to smaller variances (Kano, Bentler, & Mooijaart, 1993; Yung & Bentler, 1999). Biased estimates due to model misspecification do not necessarily have smaller variances (Yuan & Hayashi, 2006).

In summary, analyzing the path is an intuitive tool for model diagnosis for the most widely used ML procedure. It not only provides information on biases of parameter estimates when a model is misspecified but also facilitates a good understanding of the relation of various parts of the model in a path diagram. Like any other method of statistical modeling, it needs conditions to work effectively.

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APPENDIX

This appendix provides the detail leading to Equation 2. The relation of f_1, f_2, and f_3 in Figure 6 are expressed in equations as

$$f_2 = \gamma_1 f_1 + d_2,$$

$$\begin{aligned} f_3 &= \gamma_2 f_1 + \gamma_3 f_2 + d_3 \\ &= \gamma_2 f_1 + \gamma_3(\gamma_1 f_1 + d_2) + d_3 \\ &= (\gamma_2 + \gamma_1\gamma_3) f_1 + \gamma_3 d_2 + d_3, \end{aligned}$$

where $E(f_1) = 0$, $E(f_2) = 0$, $E(f_3) = 0$, $E(d_2) = 0$, $E(d_3) = 0$, $E(f_1 d_2) = 0$, $E(f_1 d_3) = 0$, $E(f_2 d_3) = 0$, $E(d_2 d_3) = 0$, and $\mathrm{Var}(f_1) = \phi_{11}$ according to the model assumption. Thus,

$$\begin{aligned} \mathrm{Var}(f_2) &= E[(\gamma_1 f_1 + d_2)^2] \\ &= E(\gamma_1^2 f_1^2 + 2\gamma_1 f_1 d_2 + d_2^2) \\ &= \gamma_2^2\phi_{11} + \varphi_{22}, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(f_3) &= E\{[(\gamma_2 + \gamma_1\gamma_3) f_1 + \gamma_3 d_2 + d_3]^2\} \\ &= E[(\gamma_2 + \gamma_1\gamma_3)^2 f_1^2 + \gamma_3^2 d_2^2 + d_3^2 + 2(\gamma_2 + \gamma_1\gamma_3)\gamma_3 f_1 d_2 \\ &\quad + 2(\gamma_2 + \gamma_1\gamma_3) f_1 d_3 + 2\gamma_3 d_2 d_3] \\ &= (\gamma_2 + \gamma_1\gamma_3)^2\phi_{11} + \gamma_3^2\varphi_{22} + \varphi_{33}, \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(f_1, f_2) &= E[f_1(\gamma_1 f_1 + d_2)] \\ &= \gamma_1\phi_{11}, \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(f_1, f_3) &= E\{f_1[(\gamma_2 + \gamma_1\gamma_3) f_1 + \gamma_3 d_2 + d_3]\} \\ &= (\gamma_2 + \gamma_1\gamma_3)\phi_{11}, \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(f_2, f_3) &= E\{(\gamma_1 f_1 + d_2)[(\gamma_2 + \gamma_1\gamma_3) f_1 + \gamma_3 d_2 + d_3]\} \\ &= E[\gamma_1(\gamma_2 + \gamma_1\gamma_3) f_1^2 + \gamma_1\gamma_3 f_1 d_2 + \gamma_1 f_1 d_3 \\ &\quad + (\gamma_2 + \gamma_1\gamma_3) f_1 d_2 + \gamma_3 d_2^2 + d_2 d_3] \\ &= \gamma_1(\gamma_2 + \gamma_1\gamma_3)\phi_{11} + \gamma_3\varphi_{33}. \end{aligned}$$

Putting these elements into a matrix yields Equation 2.