Event-Triggering in Distributed Networked Systems with Data Dropouts and Delays

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Abstract. This paper studies distributed networked systems with data dropouts and transmission delays. We propose a decentralized event-triggering scheme, where a subsystem broadcasts its state information to its neighbors only when the subsystem's local state error exceeds a specified threshold. The novelty of this scheme is its complete decentralization, which means that a subsystem's broadcast decisions are made using its local sampled data, the maximal allowable transmission delay of a subsystem's broadcast is predicted based on the local information, a subsystem locally identifies the maximal allowable number of its successive data dropouts, and the designer's selection of the threshold only requires information about an individual subsystem and its immediate neighbors. With the assumption that the number of each subsystem's successive data dropouts is less than the bound identified by that subsystem, if the bandwidth of the network is limited so that the transmission delays are always greater than a positive constant, the resulting system is globally uniformly ultimately bounded using our scheme; otherwise, the resulting system is asymptotically stable.

1 Introduction

A networked control system (NCS) is a system wherein numerous physically coupled subsystems are geographically distributed throughout the system. Control and feedback signals are exchanged through a real-time network among the system's components. Specific examples of NCS include electrical power grids and transportation networks. The networking of control effort can be advantageous in terms of lower system costs due to streamlined installation and maintenance costs. The introduction of real-time network infrastructure, however, raises new challenges regarding the impact that communication reliability has on the control system's performance. In real-time networks, information is transmitted in discrete time rather than continuous-time. Moreover, all real networks have bandwidth limitation that can cause delays in message delivery that may have a major impact on overall system stability [1].

For this reason, some researchers begun investigating the timing issue in NCS. One packet transmission problem was considered in [2], [3], where a supervisor

* The authors gratefully acknowledge the partial financial support of the National Science Foundation (NSF CNS-0720457)

summarizes all subsystem data into this single packet. As a result such schemes may be impractical for large-scale systems. Asynchronous transmission was considered in [4], [5], [6], where derived bounds on the maximum admissible transfer interval (MATI) that a message can be delayed while still maintaining closed-loop system stability. All of this prior work confined its attention to control area network (CAN) buses where centralized computers are used to coordinate the information transmission.

One thing worth mentioning is that these schemes mentioned above require extremely detailed models of subsystem interactions and the execution of communication protocols must be done in a highly centralized manner. Both of these requirements can greatly limit the scalability of centralized approaches to NCS. On the other hand, because the MATI is computed before the system is deployed, it must ensure adequate behavior over a wide range of possible input disturbances. As a result, it may be conservative.

To overcome these issues, decentralized event-triggering feedback schemes were proposed in [7] and [8] for linear and nonlinear systems, respectively, where a subsystem broadcasts its state information to its neighbors only when "needed". In this case, "needed" means that some measure of the subsystem's local state error exceeds a specified threshold [9], [10]. In this way, event-triggering makes it possible to reduce the frequency with which subsystems communicate and therefore use network bandwidth in a extremely frugal manner. Most recently, an implementation of event-triggering in sensor-network was introduced in [11]. An important assumption in all preceding work is that no data dropouts and delays occur in such systems. In real-time network, however, especially wireless network, data dropouts and delays always exist. Therefore, it suggests a more comprehensive consideration of such systems.

This paper studies the distributed NCS with data dropouts and transmission delays. Unlike the prior work that modelled data dropouts as stochastic processes using a centralized approach [12], [13], we propose a decentralized event-triggering scheme that enables a subsystem to locally identify the maximal allowable number of its successive data dropouts. The novelty of this scheme is its "complete" decentralization. By "complete", it means that (1) a subsystem's broadcast decisions are made using its local sampled data, (2) the maximal allowable transmission delay (also called "deadline") of a subsystem's broadcast can be predicted based on the local information, (3) a subsystem locally identifies the maximal allowable number of its successive data dropouts, and (4) the designer's selection of the threshold only requires information about an individual subsystem and its immediate neighbors.

Our analysis applies to nonlinear continuous systems. With the assumption that the number of each subsystem's successive data dropouts is less than the bound identified by that subsystem, if the bandwidth of the network is limited so that the transmission delays are always greater than a positive constant, the resulting NCS is globally uniformly ultimately bounded using our scheme; otherwise, the resulting NCS is asymptotically stable. We use an example to illustrate the design procedure.

The paper is organized as follows: section 2 formulates the problem; the decentralized approach to design the local triggering event is introduced in section 3; Transmission delays and data dropouts are considered in section 4 and 5, respectively; Simulation results are presented in section 6; In section 7, the conclusions are drawn.

2 Problem Formulation

Consider a distributed NCS containing N subsystems, denoted as $\mathbf{P}_i$. Let $\mathcal{N} = \{1, 2, \cdots, N\}$. $Z_i \in \mathcal{N}$ denotes the set of subsystems that $\mathbf{P}_i$ can get information from; $D_i \subset \mathcal{N}$ denotes the set of subsystems that directly drive $\mathbf{P}_i$'s dynamics; $U_i \in \mathcal{N}$ denotes the set of subsystems that can receive $\mathbf{P}_i$'s broadcasted information; $S_i \in \mathcal{N}$ denotes the set of subsystems who are directly driven by $\mathbf{P}_i$.

The state equation of the ith subsystem is

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_{D_i}(t), u_i) \\ u_i &= \gamma_i(x_{Z_i}(t)) \\ x_i(0) &= x_{i0}. \end{aligned} \tag{1}$$

To outline the analysis, we assume $x_i \in \mathbb{R}^n$ for all i. The results can be easily extended to the case where the states have different dimensions. In equation (1), $x_{D_i} = \{x_j\}_{j \in D_i}$, $x_{Z_i} = \{x_j\}_{j \in Z_i}$, $\gamma_i : \mathbb{R}^{n|Z_i|} \to \mathbb{R}^{m_i}$ is the given feedback strategy of agent i satisfying $\gamma_i(0) = 0$, and $f_i : \mathbb{R}^{n|D_i|} \times \mathbb{R}^{m_i} \to \mathbb{R}^n$ is a given continuous function satisfying $f_i(0, 0) = 0$.

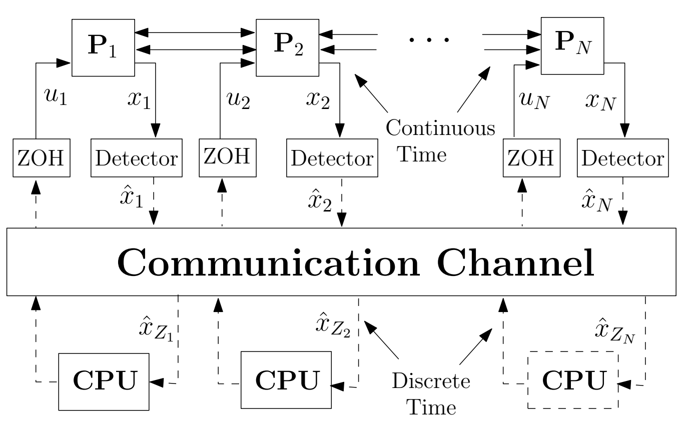

Fig. 1. The infrastructure of the real-time NCS

This paper considers a real-time implementation of this distributed NCS. The infrastructure of such an implementation is plotted in figure 1. In such a system,

$\mathbf{P}_i$ can only detect its own state, x_i. If the local "error" signal exceeds some given threshold, which can be detected by hardware detectors, $\mathbf{P}_i$ will sample and broadcast its state information to its neighbors through a real-time network. Therefore, $\mathbf{P}_i$'s control, u_i, at time t is computed based on its neighbors' latest broadcast states (also called "measured states") at time t, denoted as $\hat{x}_{Z_i}(t)$. We assume that the time spent in computing the control and sending the control back to the plant is zero. The control signal used by $\mathbf{P}_i$ is held constant by a zero-order hold (ZOH) unless one of its neighbors makes another broadcast. This means that $\mathbf{P}_i$ has the following state equation,

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_{D_i}(t), u_i) \\ u_i &= \gamma_i(\hat{x}_{Z_i}(t)) \\ x_i(0) &= x_{i0}. \end{aligned} \tag{2}$$

Subsystem i's broadcast can characterized by two monotone increasing sequences of time instants: the broadcast release time sequence $\{r_k^i\}_{k=1}^{\infty}$ and the broadcast finishing time $\{f_k^i\}_{k=1}^{\infty}$, where $r_k^i \leq f_k^i \leq r_{k+1}^i$ holds for all $k = 1, 2, \cdots, \infty$. The time r_k^i denotes the time when the kth broadcast is released by $\mathbf{P}_i$ for transmission through the channel. At this time, we assume there is no delay between sampling and broadcast release. The time f_k^i denotes the time when the kth broadcast information by $\mathbf{P}_i$ is received by its neighbors.

3 Decentralized Broadcast-triggering Events Design

In this section, we study a decentralized approach to characterize the broadcast time sequence. Inequality constraints on each subsystem's broadcast release and finishing time are provided to ensure asymptotic stability of the overall system. These constraints can be locally determined by individual subsystems. To obtain the decentralized method, we first introduce a theorem in [8] that provide a centralized approach to derive the time constraints on r_k^i and f_k^i. For notation convenience, we define $e_k^i : [r_k^i, f_{k+1}^i) \to \mathbb{R}^n$ as $e_k^i(t) = x_i(t) - x_i(r_k^i)$ for $\forall t \in [r_k^i, f_{k+1}^i)$. Notice that $\hat{x}_i(t) = x_i(r_k^i)$ for all $t \in [f_k^i, f_{k+1}^i)$.

Theorem 1 ([8]). *Consider the NCS in equation* (2)*. Assume that there exist a smooth, proper, positive-definite function $V : \mathbb{R}^{nN} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\phi_i, \psi_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \cdots, N$ such that the inequality*

$$\sum_{i \in \mathcal{N}} \frac{\partial V(x)}{\partial x_i} f_i\left(x_{D_i}, \gamma_i(y_{Z_i})\right) \leq \sum_{i \in \mathcal{N}} -\phi(_i\|x_i\|_2) + \sum_{i \in \mathcal{N}} \psi_i(\|x_i - y_i\|_2) \tag{3}$$

holds for all $x, y \in \mathbb{R}^{nN}$. If for any $i \in \mathcal{N}$, there exists a constant $\rho_i \in (0,1)$ such that subsystem i's broadcast release time sequence, $\{r_k^i\}_{k=1}^{\infty}$, and finishing time sequence, $\{f_k^i\}_{k=1}^{\infty}$, satisfy

$$-\rho_i \phi_i(\|x_i(t)\|_2) + \psi_i(\|e_k^i(t)\|_2) \leq 0 \tag{4}$$

for all $t \in [f_k^i, f_{k+1}^i)$ and all $k \in \mathbb{N}$, then the NCS is asymptotically stable.

Theorem 1 shows that the satisfaction of equation (4) guarantees asymptotic stability of the NCS. Based on this theorem, deriving local time constraints is equivalent to constructing class $\mathcal{K}$ functions ϕ_i and ψ_i. The following theorem provides a decentralized approach to design such class $\mathcal{K}$ functions.

Theorem 2. *Consider the NCS defined in equation* (2)*. Assume that, for any subsystem $i \in \mathcal{N}$, there exist a continuous, positive-definite functions $V_i : \mathbb{R}^n \to \mathbb{R}$, positive constants $\alpha_i, \beta_i, \kappa_i \in \mathbb{R}$, and control law $\gamma_i : \mathbb{R}^{n|Z_i|} \to \mathbb{R}^{m_i}$ satisfying the following two conditions*

$$\frac{\partial V_i(x_i)}{\partial x_i} f_i(x_{D_i}, \gamma_i(y_{Z_i})) \leq -\alpha_i \|x_i\|_2 + \sum_{j \in D_i \cup Z_i} \beta_j \|x_j\|_2 + \sum_{j \in Z_i} \kappa_j \|x_j - y_j\|_2 \quad (5)$$

$$\alpha_i - |S_i \cup U_i| \beta_i > 0 \quad (6)$$

Then $\phi_i, \psi_i : \mathbb{R} \to \mathbb{R}$, defined by $\phi_i(s) = a_i s$ and $\psi_i(s) = b_i s$, satisfy equation (4) *in theorem 1, where $a_i = \alpha_i - |S_i \cup U_i| \beta_i$ and $b_i = |U_i| \kappa_i$.*

Proof. It is easy to see that

$$\begin{aligned}
&\sum_{i \in \mathcal{N}} \frac{\partial V(x_i)}{\partial x_i} f_i(x_{D_i}, \gamma_i(y_{Z_i})) \\
\leq &\sum_{i \in \mathcal{N}} -\alpha_i \|x_i\|_2 + \sum_{j \in D_i \cup Z_i} \beta_j \|x_j\|_2 + \sum_{j \in Z_i} \kappa_j \|x_j - y_j\|_2 \\
= &\sum_{i \in \mathcal{N}} \left(-\alpha_i + |S_i \cup U_i| \beta_i\right) \|x_i\|_2 + \sum_{i \in \mathcal{N}} |U_i| \kappa_i \|x_i - y_i\|_2,
\end{aligned}$$

where the equality is obtained by resorting the items according to index i. □

Remark 1. Equation (5) and (6) may have a more general form, where $\alpha_i \|x_i\|_2$, $\beta_j \|x_j\|_2$, and $\kappa_j \|x_j - y_j\|_2$ are replaced by some class $\mathcal{K}$ functions. Using the more general form, however, will require additional assumptions on those class $\mathcal{K}$ functions in the later discussion, such as Lipschitz condition. It will make the paper hard to read. To outline the main idea of this paper, we just use equation (5) and (6) as a sufficient condition to construct ϕ_i and ψ_i in theorem 1.

Remark 2. Equation (5) suggests that subsystem i is finite-gain $\mathcal{L}_2$ stable from $(\{x_j\}_{j \in D_i \cup Z_i}, \{x_j - y_j\}_{j \in Z_i})$ to x_i.

We will find that it is convenient in the later work to use a slightly weaker sufficient condition for asymptotic stability where the state error $e_k^i(t)$ is bounded by a function of the sampled data $x_i(r_k^i)$ as stated in the following corollary.

Corollary 1. *Consider the NCS in equation* (2)*. Assume that equation* (3) *holds. If for any $i \in \mathcal{N}$, subsystem i's broadcast release time sequence, $\{r_k^i\}_{k=1}^{\infty}$, and finishing time sequence, $\{f_k^i\}_{k=1}^{\infty}$, satisfy*

$$\|e_k^i(t)\|_2 \leq c_i \|x_i(r_k^i)\|_2 \quad (7)$$

for all $t \in [f_k^i, f_{k+1}^i)$ and all $k \in \mathbb{N}$, where $c_i :\in \mathbb{R}$ is defined by

$$c_i = \frac{\rho_i a_i}{\rho_i a_i + b_i}, \tag{8}$$

for some $\rho_i \in (0,1)$, then the NCS is asymptotically stable.

Proof. By the definitions of c_i equation (8), equation (7) is equivalent to

$$b_i \|e_k^i(t)\|_2 + \rho_i a_i \|e_k^i(t)\|_2 \leq \rho_i a_i \|x_i(r_k^i)\|_2. \tag{9}$$

for all $t \in [f_k^i, f_{k+1}^i)$ and all $k \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} b_i \|e_k^i(t)\|_2 &\leq \rho_i a_i \|x_i(r_k^i)\|_2 - \rho_i a_i \|e_k^i(t)\|_2 \\ &\leq \rho_i a_i \|x_i(r_k^i) + e_k^i(t)\|_2 = \rho_i a_i \|x_i(t)\|_2 \end{aligned}$$

for all $t \in [f_k^i, f_{k+1}^i)$ and all $k \in \mathbb{N}$. Since the hypotheses of theorem 1 are satisfied, we can conclude that the NCS is asymptotically stable. □

Remark 3. The inequalities in equations (4) or (7) can both be used as the basis for a decentralized event-triggered feedback control system. Note that both inequalities are trivially satisfied at $t = r_k^i$. If we let the delay be zero for each broadcast ($r_k^i = f_k^i$) and assume there are not any data dropouts, then by triggering the release times $\{r_k\}_{k=0}^{\infty}$ anytime before the inequalities in equations (4) or (7) are violated, we will ensure the sampled-data system's stability.

Theorem 2 and corollary 1 provide ways to identify the broadcast release time, r_k^i. However, we still do not know how to predict maximal allowable delays for each broadcast. In other words, we do not have an explicit constraint on f_k^i yet. In the following section, we will consider the bounds on f_k^i.

4 Event-Triggering with Delays

In this section, we quantify maximal allowable delays for each subsystems that will not break the stability of the NCS. An upper bound on the kth broadcast finishing time is derived in a decentralized manner as a function of the previously sampled local states.

Before introducing the results, we would like to point out that since we consider asymptotic stability or uniformly ultimately boundness in this paper, it is easy to see that the state trajectory of the NCS will fall into some compact set $S \subset \mathbb{R}^n$. Therefore, there always exists $p_i > 0$ such that

$$f_i\left(x_{D_i}(t), \gamma_i\left(\hat{x}_{Z_i}(t)\right)\right) \leq p_i \tag{10}$$

holds for any $t \geq 0$ and $i \in \mathcal{N}$.

To obtain the upper bound on the delays, we need a lemma to identify the behaviour of $e_{k-1}^i(t)$ and $e_k^i(t)$ over the time interval $[r_k^i, f_k^i)$. Ideally, we hope that $\|e_{k-1}^i(f_k^i)\|_2 \leq c_i \|x_i(r_{k-1}^i)\|_2$ holds. In that case, the constraint $\|e_{k-1}^i(t)\|_2 \leq$

$c_i\|x_i(r^i_{k-1})\|_2$ will not be violated over $[f^i_{k-1}, f^i_k)$. At the same time, we require $\|e^i_k(f^i_k)\|_2 \leq \delta_i c_i \|x_i(r^i_k)\|_2$ holds for some $\delta_i \in (0,1)$. This is to ensure $r^i_{k+1} \geq f^i_k$ when we use the violation of $\|e^i_k(t)\|_2 \leq \delta_i c_i \|x_i(r^i_k)\|_2$ to trigger r^i_{k+1}. The lemma is stated as follows.

Lemma 1. *Consider subsystem i in equation* (2)*. Assume that equation* (10) *holds with some $p_i \in \mathbb{R}^+$. For any $k \in \mathbb{N}$, if*

$$\|e^i_{k-1}(r^i_k)\|_2 \leq \delta_i c_i \|x_i(r^i_{k-1})\|_2 \tag{11}$$

$$f^i_k - r^i_k \leq \min\left\{\frac{(1-\delta_i)c_i}{p_i}\|x_i(r^i_{k-1})\|_2, \frac{\delta_i c_i}{p_i}\|x_i(r^i_k)\|_2\right\} \tag{12}$$

hold with any $\delta_i \in (0,1)$, then

$$\|e^i_{k-1}(t)\|_2 \leq c_i \|x_i(r^i_{k-1})\|_2 \tag{13}$$

$$\|e^i_k(t)\|_2 \leq \delta_i c_i \|x_i(r^i_k)\|_2 \tag{14}$$

hold for all $t \in [r^i_k, f^i_k)$.

Proof. Consider the derivative of $\left\|e^i_{k-1}(t)\right\|_2$ over the time interval $[r^i_k, f^i_k)$.

$$\frac{d}{dt}\left\|e^i_{k-1}(t)\right\|_2 \leq \left\|\dot{e}^i_{k-1}(t)\right\|_2 = \|\dot{x}_i(t)\|_2 = \|f_i\left(x_{D_i}, \gamma_i\left(\hat{x}_{Z_i}\right)\right)\|_2 \leq p_i$$

holds for all $t \in [r^i_k, f^i_k)$. Solving the preceding inequality with initial condition $\left\|e^i_{k-1}(r^i_k)\right\|_2$ implies

$$\left\|e^i_{k-1}(t)\right\|_2 \leq p_i(t - r^i_k) + \left\|e^i_{k-1}(r^i_k)\right\|_2 \leq p_i(f^i_k - r^i_k) + \left\|e^i_{k-1}(r^i_k)\right\|_2 \tag{15}$$

holds for all $t \in [r^i_k, f^i_k)$. By equation (12), we know

$$p_i(f^i_k - r^i_k) \leq (1-\delta_i)c_i\|x_i(r^i_{k-1})\|_2 \tag{16}$$

Applying equation (11) and (16) into (15), we know equation (13) holds. With a similar analysis, we can show the satisfaction of equation (14). □

Remark 4. We can actually substitute the assumption in equation (10) by the following inequality

$$f_i\left(x_{D_i}, \gamma_i\left(\hat{x}_{Z_i}\right)\right) \leq \sum_{j \in D_i} L^1_{ij}\|x_j\|_2 + \sum_{j \in Z_i} L^2_{ij}\|\hat{x}_j\|_2$$

with $L^1_{ij}, L^2_{ij} \geq 0$. With the preceding inequality, if $D_i \subseteq Z_i$, we may obtain a tighter bound on delays other than the one given by equation (12) using comparison principle; if $D_i \not\subseteq Z_i$, with an addition assumption that $\|x_i\|_2 \leq q_i$ for any $i \in \mathcal{N}$ and some $q_i \geq 0$, we can still get a tighter bound on delays using similar analysis. However, the discussion of the topology of communication graph and the complexity of these new bounds may keep the readers away from the main purpose of this paper that is how to decentralize the event-triggering scheme. Since using new bounds will lead to the same conclusion, we just use the assumption in equation (10) to make the analysis easy to understand.

With lemma 1, we can present the following theorem where the upper bounds on delays of subsystems' broadcasts are given to guarantee asymptotic stability of the event-triggered NCS.

Theorem 3. *Consider the NCS in equation* (2)*. Assume that equation* (3) *holds and* $f_i\left(x_{D_i}, \gamma_i\left(\hat{x}_{Z_i}\right)\right) \leq p_i$ *with some* $p_i \in \mathbb{R}^+$*. If, for any* $i \in \mathcal{N}$*, the broadcast release time* r^i_{k+1} *is triggered by the violation of the inequality*

$$\|e^i_k(t)\|_2 \leq \delta_i c_i \|x_i(r^i_k)\|_2 \tag{17}$$

for some $\delta_i \in (0,1)$ *and the broadcast finishing time,* f^i_{k+1}*, satisfies*

$$f^i_{k+1} - r^i_{k+1} \leq \min\left\{\frac{(1-\delta_i)c_i}{p_i}\|x_i(r^i_k)\|_2, \frac{\delta_i c_i}{p_i}\|x_i(r^i_{k+1})\|_2\right\}, \tag{18}$$

then the NCS is asymptotically stable.

Proof. Since the hypotheses in lemma 1 hold, we have

$$\|e^i_k(t)\|_2 \leq c_i \|x_i(r^i_k)\|_2 \tag{19}$$

hold for all $t \in [r^i_{k+1}, f^i_{k+1})$ and all $k \in \mathbb{N}$.

We also know by equation (17) that

$$\|e^i_k(t)\|_2 \leq \delta_i c_i \|x_i(r^i_k)\|_2 \tag{20}$$

holds for all $t \in [r^i_k, r^i_{k+1})$ and all $k \in \mathbb{N}$.

Combining equation (19), (20) yields

$$\|e^i_k(t)\|_2 \leq c_i \|x_i(r^i_k)\|_2 \tag{21}$$

for all $t \in [r^i_k, f^i_{k+1})$ and all $k \in \mathbb{N}$. Therefore, by corollary 1, the NCS is asymptotically stable. □

Remark 5. Notice that the maximal allowable delay of subsystem i's $k+1$st broadcast only depend on local information. In other words, subsystem i can predict the deadline by itself. The cost of such decentralization is that the deadlines will go to zero as the state converges to the equilibrium.

In theorem 3, subsystem i predicts the deadline for its $k+1$st broadcast delay at time r^i_{k+1} when the state $x_i(r^i_{k+1})$ is sampled. It is more reasonable to have subsystem i predict the deadline for its $k+1$st delay ahead of time, such as at time r^i_k. Corollary 2 provides such a deadline as a function of $x_i(r^i_k)$.

Corollary 2. *Assume that all hypotheses in theorem 3 are satisfied except that equation* (18) *is replaced by*

$$f^i_{k+1} - r^i_{k+1} \leq \min\left\{\frac{(1-\delta_i)c_i}{p_i}\|x_i(r^i_k)\|_2, \frac{\delta_i c_i(1-\delta_i c_i)}{p_i}\|x_i(r^i_k)\|_2\right\}, \tag{22}$$

then the NCS is asymptotically stable.

Proof. By the definition of c_i in equation (8), we know $c_i < 1$ and therefore $1-\delta_i c_i > 0$ with $\delta_i \in (0,1)$. Based on equation (17), we know $(1-\delta_i c_i)\|x_i(r_k^i)\|_2 \leq \|x_i(r_{k+1}^i)\|_2$. So equation (22) implies the satisfaction of equation (18). □

As we can see from equation (18) in theorem 3, the predicted deadlines for subsystem i's broadcast delays go to zero as the state converges to the equilibrium point. If the channel capacity is not taken into account, this result is acceptable. However, if the bandwidth of the network is limited, the broadcast delays have to be greater than a positive constant. In that case, the overall system will not be asymptotically stable. Instead, the state will eventually stay in a small neighborhood of the equilibrium, which means that the system is globally uniformly ultimately bounded. The size of the neighborhood depends on the length of the maximal delay. The results are formally stated as follows.

Corollary 3. *Assume that all hypotheses in theorem 3 are satisfied except that equation* (18) *is replaced by*

$$f_{k+1}^i - r_{k+1}^i \leq \min\left\{\frac{(1-\delta_i)c_i\epsilon}{p_i}, \frac{\delta_i c_i \epsilon}{p_i}\right\} \tag{23}$$

for some positive constant $\epsilon \in \mathbb{R}^+$, then the NCS is globally uniformly ultimately bounded.

Proof. Following a similar analysis to the proof of theorem 3, we know that $\dot{V} \leq \sum_{i\in\mathcal{N}} -(1-\rho_i)a_i\|x_i(t)\|_2$ for $\|x_i(t)\|_2 \geq \epsilon$ with some $\rho_i \in (0,1)$, which means that the NCS is globally uniformly ultimately bounded. □

5 Event-Triggering with Data Dropouts

In the previous sections, we did not consider the occurrence of data dropouts. In other words, whenever a broadcast release is triggered, the local state of the related subsystem will be sampled and transmitted to its neighbors successfully. In this section, we take data dropouts into account, which frequently happen in NCS. We assume that data dropouts only happen when the sampled states are sent to the controllers through the network.

Let us take a look at what happen in the system when a data package is lost. We first consider the case where the network uses Transmission Control Protocol (TCP). By TCP, the subsystem will be notified if transmission fails. So when data dropouts happens, the subsystem just needs to keep sending the newly sampled state unless it is transmitted successfully. Also, the local triggering event will not be updated until transmission succeeds. So it is a trivial case.

A more interesting thing happens with the network using User Datagram Protocol (UDP). By UDP, the subsystem will not be notified when transmission fails. In that case, when the hardware detector located at subsystem i detects the occurrence of the local event, the local state will be sampled and ready to be transmitted to its neighbors through the channel. At the same time, the event will be updated from k to $k+1$ with the newly sampled state. Once the transmission fails (in other words, the sampled state is lost), the controllers will not receive the sampled state. So the control inputs will not be updated. Notice that in this case, the local event will be updated, but the control inputs will not.

In the following discussion, we intend to address the number of data dropouts in such NCS (UDP) with the guarantee of stability. In fact, we provide a decentralized approach that enables each subsystem to locally identify the largest number of its successive data dropouts that the subsystem can tolerant. The idea is to have events happen earlier than the violation of the inequality in equation (17) so that even if some data is lost, equation (17) can still be satisfied.

Before we introduce the results, we need to define two different types of release: the triggered release, $\hat{r}^i_j$, and the successful release, r^i_k. $\hat{r}^i_j$ is the time when the jth broadcast of subsystem i is released (but not necessarily transmitted successfully). r^i_k is the time when the kth successful broadcast of subsystem i is released. Obviously, $\{r^i_k\}_{k=1}^{\infty}$ is a subsequence of $\{\hat{r}^i_j\}_{j=1}^{\infty}$. For notation convenience, we define $\hat{e}^i_j : \mathbb{R} \to \mathbb{R}^n$ as $\hat{e}^i_j(t) = x_i(t) - x_i(\hat{r}^i_j)$.

Theorem 4. *Consider the NCS in equation* (2)*. Assume that equation* (3) *holds and* $f_i\left(x_{D_i}, \gamma_i\left(\hat{x}_{Z_i}\right)\right) \leq p_i$ *with some* $p_i \in \mathbb{R}^+$*. If, for any* $i \in \mathcal{N}$ *and some* $\delta_i \in (0,1)$*, the next broadcast release time is triggered by the violation of*

$$\|\hat{e}^i_j(t)\|_2 \leq \hat{\delta}_i c_i \|x_i(\hat{r}^i_j)\|_2 \tag{24}$$

for some $\hat{\delta}_i \in (0, \delta_i)$*, the* k*th successful broadcast finishing time,* f^i_k*, satisfies*

$$f^i_k - r^i_k \leq \min\left\{\frac{(1-\delta_i)c_i}{p_i}\|x_i(r^i_{k-1})\|_2, \frac{\hat{\delta}_i c_i(1-\delta_i c_i)}{p_i}\|x_i(r^i_{k-1})\|_2\right\}, \tag{25}$$

and the largest number of successive data dropouts, $n_i \in \mathbb{Z}$*, satisfies*

$$n_i \leq log_{(1+\hat{\delta}_i c_i)}(1+\delta_i c_i) - 1 \tag{26}$$

then the NCS is still asymptotically stable.

Proof. Consider subsystem i over the time interval $[r^i_k, f^i_{k+1})$. For notation convenience, we assume $r^i_k = \hat{r}^i_0 < \hat{r}^i_1 < \cdots < \hat{r}^i_{n_i} < \hat{r}^i_{n_i+1} = r^i_{k+1}$. Since the hypotheses in lemma 1 hold, we have $\|e^i_k(t)\|_2 \leq \hat{\delta}_i c_i \|x_i(r^i_k)\|_2$ for all $t \in [r^i_k, f^i_k)$ and all $k \in \mathbb{N}$. Since $\|e^i_k(\hat{r}^i_1)\|_2 = \hat{\delta}_i c_i \|x_i(r^i_k)\|_2$, we have $f^i_k \leq \hat{r}^i_1$, namely that subsystem i does not release broadcasts during $[r^i_k, f^i_k)$.

Consider $\|e^i_k(t)\|_2$ for any $t \in [\hat{r}^i_j, \hat{r}^i_{j+1})$. We have

$$\|e^i_k(t)\|_2 = \|x_i(t) - x_i(r^i_k)\|_2 \leq \sum_{l=0}^{j-1} \|x_i(\hat{r}^i_{l+1}) - x_i(\hat{r}^i_l)\|_2 + \|x_i(t) - x_i(\hat{r}^i_j)\|_2$$

for $\forall t \in [\hat{r}^i_j, \hat{r}^i_{j+1})$. Applying equation (24) into the preceding equation yields

$$\|e^i_k(t)\|_2 \leq \sum_{l=0}^{j} \hat{\delta}_i c_i \|x_i(\hat{r}^i_j)\|_2 \tag{27}$$

for all $t \in [\hat{r}^i_j, \hat{r}^i_{j+1})$. Therefore,

$$\|e^i_k(t)\|_2 \le \sum_{l=0}^{n_i} \hat{\delta}_i c_i \|x_i(\hat{r}^i_l)\|_2 \tag{28}$$

holds for all $t \in [r^i_k, r^i_{k+1})$.

Because $\|\hat{e}^i_j(\hat{r}^i_{j+1})\|_2 = \|x_i(\hat{r}^i_{j+1}) - x_i(\hat{r}^i_j)\|_2 = \hat{\delta}_i c_i \|x_i(\hat{r}^i_j)\|_2$, we have

$$\|x_i(\hat{r}^i_{j+1})\|_2 \le (1 + \hat{\delta}_i c_i)\|x_i(\hat{r}^i_j)\|_2$$

and therefore

$$\|x_i(\hat{r}^i_{j+1})\|_2 \le (1 + \hat{\delta}_i c_i)^{j+1}\|x_i(\hat{r}^i_0)\|_2 = (1 + \hat{\delta}_i c_i)^{j+1}\|x_i(r^i_k)\|_2 \tag{29}$$

for $j = 0, 1, 2, \cdots, n_i$.

Applying equation(29) into (28) yields

$$\|e^i_k(t)\|_2 \le \sum_{l=0}^{n_i} \hat{\delta}_i c_i (1 + \hat{\delta}_i c_i)^l \|x_i(r^i_k)\|_2 = \left((1 + \hat{\delta}_i c_i)^{n_i+1} - 1\right)\|x_i(r^i_k)\|_2 \tag{30}$$

for all $t \in [r^i_k, r^i_{k+1})$.

By equation (26), we know $(1 + \hat{\delta}_i c_i)^{n_i+1} - 1 \le \delta_i c_i$. Therefore, equation (30) implies $\|e^i_k(t)\|_2 \le \delta_i c_i \|x_i(r^i_k)\|_2$ for all $t \in [r^i_k, r^i_{k+1})$. Since the hypotheses in corollary 2 are satisfied, we conclude that the NCS is asymptotically stable. □

Remark 6. By equation (26), we know the maximal allowable number of each subsystem's successive data dropouts can be identified locally, depending on the selection of c_i, δ_i, and $\hat{\delta}_i$. If subsystem i wants the maximal allowable number of data dropouts to be large, $\hat{\delta}_i$ must be small enough. In general, however, small $\hat{\delta}_i$ will result in short broadcast periods. Therefore, there is a tradeoff between the maximal allowable number of data dropouts and the broadcast periods.

Similar to corollary 3, we have the following result for the case with fixed transmission deadlines.

Corollary 4. *Assume that all hypotheses in theorem 4 are satisfied except that equation* (25) *is replaced by*

$$f^i_k - r^i_k \le \min\left\{\frac{(1-\delta_i)c_i\epsilon}{p_i}, \frac{\hat{\delta}_i c_i \epsilon}{p_i}\right\} \tag{31}$$

for some small positive constant $\epsilon > 0$, *then the NCS is still globally uniformly ultimately bounded.*

Proof. Following a similar analysis to the proof in theorem 4, we have $\|e^i_k(t)\|_2 \le \delta_i c_i \|x_i(r^i_k)\|_2$ for all $t \in [r^i_k, r^i_{k+1})$. Since the hypotheses in corollary 3 are satisfied, we conclude that the NCS is globally uniformly ultimately bounded. □

Based on the preceding results, we are able to present the decentralized event-triggering scheme.

Decentralized Event-Triggering Scheme
1. Select positive constants $\beta_i, \kappa_i \in \mathbb{R}^+$ for $i = 1, \cdots, N$;
2. For subsystem i,
   (1) Find $V_i : \mathbb{R}^n \to \mathbb{R}$, $\alpha_i \in \mathbb{R}^+$, and $\gamma_i : \mathbb{R}^{n|Z_i|} \to \mathbb{R}^{m_i}$ satisfying equation (5), (6);
   (2) Compute p_i satisfying equation (10);
   (3) Select $\rho_i \in (0, 1)$ and compute c_i based on equation (8);
   (4) Select $\delta_i \in (0, 1)$, $\hat{\delta}_i \in (0, \delta_i)$ and use the violation of the inequality in equation (24) to trigger the broadcast release;
   (5) Predict the deadline for the delay in the kth successful broadcast of subsystem i $(f_k^i - r_k^i)$ at r_{k-1}^i by equation (25) or equation (31);
   (6) Identify the maximal allowable number of successive data dropouts by equation (26).

6 An Illustrative Example

This section presents simulation results demonstrating the decentralized event-triggering scheme. The system under study is a collection of coupled carts (figure 2), which are coupled together by springs. The ith subsystem state is the vector $x_i = \begin{bmatrix} y_i \ \dot{y}_i \end{bmatrix}^T$ where y_i is the ith cart's position. We assume that at the equilibrium of the system, all springs are unstretched.

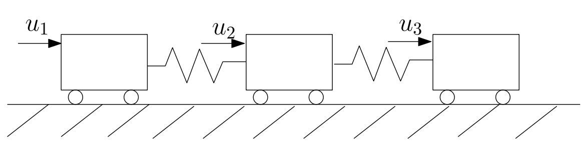

Fig. 2. Three carts coupled by springs

The state equation for the ith cart is $\dot{x}_i = A_i x_i + B_i u_i + H_{i,i-1} x_{i-1} + H_{i,i+1} x_{i+1}$ where $A_i = \begin{bmatrix} 0 & 1 \\ -\mu_i k & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_{ij} = \begin{bmatrix} 0 & 0 \\ \nu_{ij} & 0 \end{bmatrix}$.

In the preceding equation, we have $k = 5$ is the spring constant, $\mu_1 = \mu_N = 1$ and $\mu_i = 2$ for $i = 2, \cdots, N-1$. Also $\nu_{ij} = 1$ for $i \notin \{1, N\}$ and $j \in \{i-1, i+1\}$ and $\nu_{12} = \nu_{N,N-1} = 1$. Otherwise, $\nu_{ij} = 0$.

The control input of subsystem i is

$$u_i = K_i \hat{x}_i + L_{i,i-1} \hat{x}_{i-1} + L_{i,i+1} \hat{x}_{i+1}, \tag{32}$$

where $K_1 = K_N = \begin{bmatrix} -4 \ -6 \end{bmatrix}$, $K_i = \begin{bmatrix} 1 \ -6 \end{bmatrix}$ for $i = 2, \cdots, N-1$, and $L_{i,i-1} = L_{i,i+1} = \begin{bmatrix} -1 \ 0 \end{bmatrix}$ except that $L_{10} = L_{N,N+1} = 0$.

We first considered the case with $N = 3$. According to the decentralized event-triggering scheme, we obtained $c_2 = 0.3622$ and $c_1 = c_3 = 0.4451$. The initial state x_{i0} of subsystem i was randomly generated satisfying $\|x_{i0}\|_2 \leq 1$. We set $p_i = 20$, $\epsilon = 0.1$, $\hat{\delta}_i = 0.9$, and $\hat{\delta}_i = 0.2$. We ran the event-triggered system for 8 seconds with the assumption that the number of successive data dropouts are the same as the maximal allowable number and the delay is equal to the deadline. The simulation results show that the system is asymptotically stable, but not globally uniformly ultimately bounded as we stated in corollary 4. This is because the decentralization leads to the conservativeness of the theoretical results. The data of this simulation is listed in table 1.

Table 1. Results on Running a Decentralized Event-Triggered Networked System

	Subsystem 1	Subsystem 2	Subsystem 3
Maximal Allowable Number of Successive Data Dropouts	2	3	2
Predicted Deadline	2.226×10^{-4}	1.811×10^{-4}	2.226×10^{-4}
Number of Broadcasts Release	153	229	155
Number of Successful Broadcasts	50	56	50
Average Period of Broadcasts	0.0523	0.0349	0.0516
Average Period of Successful Broadcasts	0.1600	0.1429	0.1600

We then examine the relationship between the maximal allowable number of successive data dropouts, n_i, and the predicted deadline. In particular, we studied subsystem 1. We changed δ_1 from 0 to 1 and $\hat{\delta}_i$ from 0 to δ_1. The other parameters remain the same. The resulting changes in n_i and the deadline are shown in figure 3, where each pair of n_i and the deadline is associated with a pair of $(\delta_i, \hat{\delta}_i)$. We may see from the plot that as n_i increases, the predicted deadline decreases. That is because large n_i suggests tiny $\hat{\delta}_i$ and large δ_i, which results in short deadline according to equation (31).

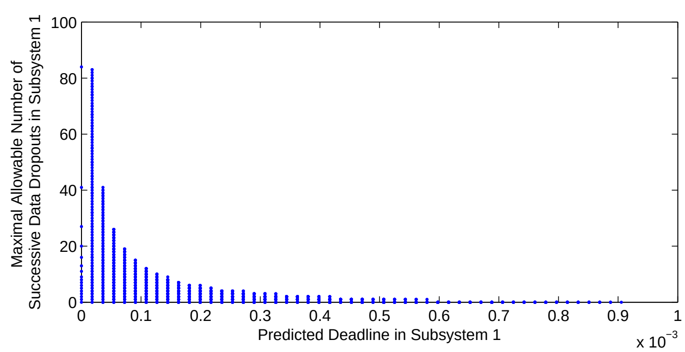

Fig. 3. Maximal allowable number of successive data dropouts versus the deadline

7 Conclusions

This paper studies distributed NCS with data dropouts and transmission delays. We propose a decentralized event-triggering scheme for such systems. The novelty of this scheme is its complete decentralization, which means that a subsystem's broadcast decisions are made using its local sampled data, the maximal allowable transmission delay of a subsystem's broadcast is predicted based on the local information, a subsystem locally identifies the maximal allowable number of its successive data dropouts, and the designer's selection of the threshold only requires information about an individual subsystem and its immediate neighbors. Our analysis applies to nonlinear continuous systems. With the assumption that the number of each subsystem's successive data dropouts is less than the bound identified by that subsystem, if the bandwidth of the network is limited so that the transmission delays are always greater than a positive constant, the resulting system is globally uniformly ultimately bounded using our scheme; otherwise, the resulting system is asymptotically stable.

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