Self-Triggered Feedback Systems with State-Independent Disturbances

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***Abstract*— This paper studies self-triggering in sampled-data systems, where the next task release time and finishing time are predicted based on the sampled states. We propose a new self-triggering scheme that ensures finite-gain $\mathcal{L}_2$ stability of the resulting self-triggered feedback systems. This scheme relaxes the assumptions in [1] that the magnitude of the process noise is bounded by a linear function of the norm of the system state. We show that the sample periods generated by this scheme are always greater than a positive constant. We also provide dynamic deadlines for delays and propose a way that may enlarge predicted deadlines without breaking $\mathcal{L}_2$ stability, especially when the predicted deadlines are very short. Simulations show that the sample periods generated by this scheme are longer than those generated by the scheme in [1]. We also show that the predicted deadlines can be extended by our scheme. Moreover, this scheme appears to be robust to the external disturbances.**

I. Introduction

Sampled-data systems are such systems that sample continuous signals and make control decisions based the sampled data. Traditional approaches to implement such systems are based on periodic task models, in which consecutive invocations of a task are released in a periodic manner. Early work [2] is based on Nyquist sampling that ensures perfect reconstruction of the signals. Noticing that perfect reconstruction is much more than we require in a feedback control system, Lyapunov techniques were used to identify the sample period [3]. Further work was done in [4], [5] to bound the inter-sample behavior of nonlinear systems using input-to-state stability techniques. For networked control systems, the maximum admissible time interval (MATI) was introduced by Walsh et al. [6], where a task can be postponed while still maintaining closed-loop system stability. Tighter bounds on MATI were obtained in [7], [8].

As we mentioned above, the preceding approaches are all based on periodic task models. Such models may be undesirable in many situations due to their conservativeness. Under periodic task models, the selection of sample periods is done before the system is deployed. One therefore has to ensure adequate behavior over a wide range of uncertainties. As a result, these selected periods may be shorter than necessary, which results in significant over-provisioning of the real-time system hardware. This over-provisioning may negatively impact the scheduling of other tasks on the same processing system. In these applications it may be better to consider alternatives to periodic task models that can more effectively balance the real-time system's computational cost against the control system's performance.

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In recent years, sporadic task models have been considered for real-time control. A hardware realization of such models is called event-triggering. Under event-triggering the system states are sampled when some error signal exceeds a given threshold [9], [10], [11], [12], [13]. Event-triggering has the ability to dynamically adjust the task periods to variations in the system state. This "on-line" property enables event-triggering to generate longer task periods than periodic task models [1].

One thing worth mentioning is that event-triggering requires a hardware event detector that may be implemented using custom analog integrated circuits (ASIC's) or floating point gate array (FPGA) processors. In some applications, however, it may be unreasonable or impractical to retrofit an existing system with such "event detectors". In these cases, a software approach such as the self-triggered scheme may be more appropriate. Under self-triggering the next task release time and finishing time are predicted by the processing computer based on the sampled data.

A self-triggered task model was introduced by Velasco et al. [14] in which a heuristic rule was used to adjust task periods. Further work was done by Lemmon et al. [15] which chose task periods based on a Lyapunov-based technique. But the authors did not provide analytic bounds for task periods. Most recently, Wang et al. [1] provided the first rigorous examination of what might be required to implement self-triggered feedback control systems for $\mathcal{L}_2$ stability. A scaling law for the execution times of control tasks was derived in [16] for homogeneous systems with asymptotic stability.

A critical assumption in [1] is that the magnitude of the process noise is bounded by a linear function of the norm of the system state. It means that the disturbance should vanish as the state is close to the equilibrium. Such disturbances may arise in uncertain systems when there are unmodeled dynamics caused by fluctuations in plant parameters. In practice, however, the disturbances usually do not depend on the state. With those "independent" disturbances, the self-triggering scheme in [1] cannot theoretically guarantee $\mathcal{L}_2$ stability of the sampled-data system any more. Therefore, it is really important to relax this assumption so that the self-triggering scheme can apply to a wider class of systems.

This paper extends the work in [1]. We present a new self-triggering scheme that ensures finite-gain $\mathcal{L}_2$ stability of the resulting self-triggered feedback systems. This scheme pertains to linear time-invariant systems. The task release time and finishing time are predicted as functions of sampled states. It relaxes the assumptions in [1] that the magnitude of the process noise is bounded by a linear function of the norm of the system state. We show that the sample periods generated by this scheme are always greater than a positive constant. We also provide dynamic deadlines for delays and propose a way that may enlarge predicted deadlines without breaking $\mathcal{L}_2$ stability, especially when the predicted deadlines are very short. Simulations show that the sample periods generated by this scheme are longer than those generated by the scheme in [1]. We also show that our scheme can extend the predicted deadlines. Moreover, this scheme appears to be robust to the external disturbances.

This paper is organized as follows. In section II the problem is formulated. Section III and IV present self-triggering schemes for the sampled-data systems without/with delays, respectively. Simulation results are presented in section V. Finally, conclusions are stated in section VI.

II. System Model

Consider a linear time-invariant system whose state $x : [0,\infty) \to \Re^n$ satisfies the initial value problem,

$$\dot{x}_t = Ax_t + B_1 u_t + B_2 w_t$$

where $u : [0,\infty) \to \Re^m$ is a control input and $w : [0,\infty) \to \Re^l$ is an exogenous disturbance function in $\mathcal{L}_2$ space. In the above equation, $A \in \Re^{n\times n}$, $B_1 \in \Re^{n\times m}$, and $B_2 \in \Re^{n\times l}$ are real matrices of appropriate dimensions.

Assume the unforced system is asymptotically stabilized by the controller

$$u_t = -B_1^T P x_t \tag{1}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix satisfying the $\mathcal{H}_\infty$ algebraic Riccati equation (ARE),

$$0 = PA + A^T P - Q + I + \frac{1}{\gamma^2} P B_2 B_2^T P \tag{2}$$

where

$$Q = P B_1 B_1^T P \tag{3}$$

for some real constant $\gamma > 0$. For notational convenience, let $A_{cl} = A - B_1 B_1^T P$.

This paper considers a sampled-data implementation of the closed-loop system. This means that the plant's control, u, is computed by a computer task. This task is characterized by two monotone increasing sequences of time instants; the release time sequence $\{r_k\}_{k=0}^\infty$ and the finishing time sequence $\{f_k\}_{k=0}^\infty$. The time r_k denotes the time when the kth invocation of a control task (also called a job) is released for execution on the computer's central processing unit (CPU). The time f_k denotes the time when then kth job has finished executing. Each job of the control task computes the control u based on the last sampled state. Upon finishing, the control job outputs this control to the plant. The control signal used by the plant is held constant by a zero-order hold (ZOH) until the next finishing time f_{k+1}. This means that the sampled-data system under study satisfies,

$$\begin{aligned} \dot{x}_t &= Ax_t + B_1 u_t + B_2 w_t \\ u_t &= -B_1^T P x_{r_k} \end{aligned} \tag{4}$$

for $t \in [f_k, f_{k+1})$ and all $k = 0, \dots, \infty$. We define the error $e_t^k : \mathbb{R} \to \mathbb{R}^n$ as $e_t^k = x_t - x_{r_k}$ for all $t \in [r_k, f_{k+1})$.

Definition 2.1: The system (4) is said to be finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than γ if there exist non-negative constants γ and δ such that

$$\left(\int_0^\infty \|x_t\|_2^2 dt\right)^{\frac{1}{2}} \le \gamma \left(\int_0^\infty \|w_t\|_2^2 dt\right)^{\frac{1}{2}} + \delta \tag{5}$$

for any w satisfying $\left(\int_0^\infty \|w_t\|_2^2 dt\right)^{\frac{1}{2}} < \infty$.

In [1], a self-triggering scheme was proposed to ensure finite-gain $\mathcal{L}_2$ stability of the sampled-data system in equation (4) from w to x. But it is not applicable for all w in $\mathcal{L}_2$ space. The scheme is based on the assumption that $\|w_t\|_2 \le W\|x_t\|_2$ holds for some $W > 0$. In practice, however, the disturbances usually do not depend on the state, with which the self-triggering scheme in [1] cannot theoretically guarantee $\mathcal{L}_2$ stability of the sampled-data system any more.

In this paper, we try to find a self-triggering scheme that can relax the assumptions in [1] with the guarantee of finite-gain $\mathcal{L}_2$ stability of the sampled-data system from w to x. In other words, we try to find a self-triggering scheme such that $\mathcal{L}_2$ stability can be preserved for any w in $\mathcal{L}_2$ space. For notation convenience, let $T_k = r_{k+1} - r_k$ denote the kth inter-release time (known as "sample period") and $D_k = f_k - r_k$ denote the time interval between the kth job's release and finishing time (known as "delay").

III. Self-triggered Systems without Delays

In this section, we consider the sample-data systems where task delays are zero ($D_k = 0$). We try to find a self-triggering scheme that ensures finite-gain $\mathcal{L}_2$ stability of such systems. The main idea is that: we first seek some inequality constraint on r_k ($= f_k$) such that $\mathcal{L}_2$ stability can be guaranteed; then we derive the self-triggering scheme that can ensure the satisfaction of this constraint.

Before we show the desired inequality constraint, we need a lemma to help the proof, which provide an upper bound for the derivative of the storage function. To make the paper easy to read, we put all of the proofs in the appendix.

Lemma 3.1: Consider the sampled-data system in equation (4). Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a positive definite function satisfying $V(x) = x^T P x$ with the matrix P given in equation (2). For any real constant $\beta \in (0,1]$, the directional derivative of V satisfies

$$\dot{V} \le -\beta^2 \|x_t\|_2^2 + \gamma^2 \|w_t\|_2^2 + \left(e_t^k\right)^T M e_t^k - x_{r_k}^T N x_{r_k} \tag{6}$$

holds for all $t \in [f_k, f_{k+1})$ and any $k = 0, \dots, \infty$ where M, N satisfy

$$M = (1-\beta^2)I + Q \tag{7}$$
$$N = \frac{1}{2}(1-\beta^2)I + Q, \tag{8}$$

respectively with the matrix Q defined in equation (3).

Remark 3.2: Notice that even for the systems with non-zero delays, lemma 3.1 is applicable. In fact, we will also use this lemma in the proof of theorem 4.1, where the self-triggered systems with non-zero delays are considered.

The inequality constraint on the task release time (it is also task finishing time since we assume the task delay is zero) is presented in the following lemma. For the notation convenience, we define $\rho : \mathbb{R}^n \to \mathbb{R}$ as $\rho(x) = \sqrt{x^T N x}$, $\mu : \mathbb{R}^n \to \mathbb{R}$ as $\mu(x_{r_k}) = \|\sqrt{M} A_{\rm cl} x_{r_k}\|_2$, and $\alpha = \|\sqrt{M} A \sqrt{M}^{-1}\|$.

Lemma 3.3: Consider the sampled-data system in equation (4). Assume $r_0 = 0$ and $r_k = f_k$ for all $k \in \mathbb{Z}^+$. Let β be any positive constant in the interval $(0, 1]$ such that the matrix M defined in equation (7) has full rank. Given a positive constant $\tau \in \mathbb{R}^+$, if

$$r_k \leq r_{k+1} \leq r_k + \tau, \text{and} \tag{9}$$

$$2\int_{f_k}^{f_{k+1}} \frac{\mu(x_{r_k})^2}{\alpha^2} \left(e^{\alpha(t-f_k)} - 1\right)^2 dt \leq \int_{f_k}^{f_{k+1}} \rho(x_{r_k})^2 dt \tag{10}$$

hold for all $k \in \mathbb{Z}^+$ with N defined in equation (8), then the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than η, where

$$\eta = \frac{\sqrt{\gamma^2\alpha^2 + 2\left\|\sqrt{M}B_2\right\|^2 (e^{\alpha\tau} - 1)^2}}{\alpha\beta}. \tag{11}$$

Remark 3.4: The inequality constraint proposed in [1] is $(e_t^k)^T M e_t^k \leq \rho(x_{r_k})^2$ for all $t \in [r_k, f_{k+1})$. The self-triggering scheme in [1] enforces this inequality constraint, thereby assuring the overall system's $\mathcal{L}_2$ stability. This inequality, however, can be relaxed. It is easy to see that the preceding inequality implies the integral inequality constraint

$$\int_{f_k}^{f_{k+1}} (e_t^k)^T M e_t^k dt \leq \int_{f_k}^{f_{k+1}} \rho(x_{r_k})^2 dt \tag{12}$$

and the proof of lemma 3.3 shows that the constraint in equation (12) is sufficient to assure $\mathcal{L}_2$ stability.

Nevertheless, the constraint in equation (12) is still unsuitable for a practical self-triggering scheme. This is because it makes use of e_t^k which also contains the disturbance w_t. Since the exact value of the disturbance is unknown, we cannot use (12) to predict the future time when we expect inequality (12) to be violated.

There are several ways of handling this issue. One approach (that was used in [1]) is to force $\|w_t\|_2 \leq W\|x_t\|_2$, thereby forcing the noise strength to decrease as the system approaches its equilibrium point. This assumption may be justified if the noise term is generated by state-dependent modeling uncertainty, but in general if the disturbance is independent of the process model, this assumption will be overly restrictive.

We were interested in remove the earlier assumption in [1] so that w_t can be any signal in $\mathcal{L}_2$ space. We were able to do this about splitting up the effect that the sampled state x_{r_k} and the noise w_t has on the local error e_t^k. This allows us to isolate those term containing w_t so we can bound $\int_{f_k}^{f_{k+1}} (e_t^k)^T M e_t^k dt$ as a function of w_t plus another term that is only dependent of the sampled state x_{r_k}. The second term leads to the inequality in equation (10) and the term related to w_t contributes to the induced gain η.

Remark 3.5: Lemma 3.3 actually implies an event-triggering scheme for zero-delay systems. The system can use the violation of the inequality

$$2\int_{f_k}^{t} \frac{\mu(x_{r_k})^2}{\alpha^2} \left(e^{\alpha(s-f_k)} - 1\right)^2 ds \leq \int_{f_k}^{t} \rho(x_{r_k})^2 ds \tag{13}$$

to trigger the next task's release with the guarantee of stability of the systems. Notice that equation (13) can be rewritten as

$$\frac{\mu(x_{r_k})^2}{\alpha^2} \left(\frac{e^{2\alpha(t-f_k)}}{\alpha} - \frac{4e^{\alpha(t-f_k)}}{\alpha} + \frac{3}{\alpha} + 2(t - f_k) \right) \leq \rho(x_{r_k})^2 (t - f_k) \tag{14}$$

by taking the integration.

Lemma 3.3 provides a constraint on the task release time. It is easy to see that if we can find some $t \geq f_k$ that makes the equality in equation (14) hold, the next task release time can be predicted. However, it is difficult to obtain such solutions in an explicit form. An alternative way is to get an estimate of the solution that can ensure the satisfaction of equation (10). In this way, $\mathcal{L}_2$ stability can still be maintained. This is formally stated in theorem 3.6, where a self-triggering scheme is presented.

Theorem 3.6: Consider the sampled-data system in equation (4). Assume $r_0 = 0$ and $r_k = f_k$ for all $k \in \mathbb{Z}^+$. Let β be any positive constant in $(0, 1]$ such that the matrix M defined in equation (7) has full rank. Given a positive constant $\tau \in \mathbb{R}^+$, if the next task release time r_{k+1} satisfies

$$r_k \leq r_{k+1} \leq r_k + \min\{\tau, L_1(x_{r_k})\}, \tag{15}$$

for all $k \in \mathbb{Z}^+$, where $L_1 : \mathbb{R}^n \to \mathbb{R}$ is given by

$$L_1(x_{r_k}) = \begin{cases} \frac{1}{\alpha} \ln\left(1 + \frac{\alpha\rho(x_{r_k})}{\sqrt{2}\left\|\sqrt{M}A_{\rm cl}x_{r_k}\right\|_2}\right) & x_{r_k} \neq 0 \\ \infty & x_{r_k} = 0 \end{cases} \tag{16}$$

then there exists a positive constant ξ such that $L_1(x_{r_k}) \geq \xi$ for all $k \in \mathbb{Z}^+$ and the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than η, which η is defined in equation (11).

Remark 3.7: The introduction of τ is the safety requirement of systems. It requires the system updates at least every τ unit-time so that some accidents can be detected. Notice that τ also affects the induced gain.

Remark 3.8: The self-triggering scheme can be $r_{k+1} = r_k + \min\{\tau, L_1(x_{r_k})\}$ for all $k \in \mathbb{Z}$. Then $L_1(x_{r_k}) \geq \xi$ actually implies $T_k \geq \min\{\tau, \xi\} > 0$.

IV. Self-triggered Systems with Delays

This section introduces a self-triggering scheme for the sampled-data systems where the task delays are not zero. In this case, the differential equations associated with two intervals $[r_k, f_k)$ and $[f_k, f_{k+1})$ are

$$\begin{aligned} \dot{x}_t &= Ax_t - B_1 B_1^T P x_{r_{k-1}} + B_2 w_t \quad \text{and} \\ \dot{x}_t &= Ax_t - B_1 B_1^T P x_{r_k} + B_2 w_t, \end{aligned}$$

respectively. We derive bounds on the sample period and task delays to ensure $\mathcal{L}_2$ stability of the systems. Based on these bounds, a self-triggering scheme is proposed. The analysis is similar to that used in theorem 3.6 except that the behaviour of the error, e_t^k, needs to be characterized differently over the intervals $[r_k, f_k)$ and $[f_k, f_{k+1})$. Due to the space limitation, we will not show the bounds on errors over these two intervals. The self-triggering scheme is formally stated in the following theorem.

Theorem 4.1: Consider the sampled-data system in equation (4). Let β be any positive constant in the interval $(0, 1]$ such that the matrix M defined in equation (7) has full rank. Given three positive constant ϵ, $\tau_1, \tau_2 \in \mathbb{R}^+$ and a positive sequence $\{\delta_k\}_{k=0}^{\infty}$ satisfying $\sum_{k=0}^{\infty} \delta_k \leq \infty$, if

- the initial condition is $r_0 = f_0 = 0$,
- the $k+1$th task release time r_{k+1} satisfies

$$r_{k+1} = f_k + \min\{\tau_1, \epsilon L_2(x_{r_k})\}, \tag{17}$$

for all $k \in \mathbb{Z}^+$, where $L_2 : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$L_2(x_{r_k}) = \begin{cases} \frac{1}{\alpha} \ln \left(1 + \frac{\alpha \rho(x_{r_k})}{\sqrt{8} \left\| \sqrt{M} A_{\text{cl}} x_{r_k} \right\|_2} \right) & x_{r_k} \neq 0 \\ \infty & x_{r_k} = 0 \end{cases}$$

- the $k+1$th task finishing time f_{k+1} satisfies

$$\min \left\{ \tau_2, (1-\epsilon) L_2(x_{r_k}), L_3(x_{r_{k+1}}, x_{r_k}; \delta_{k+1}) \right\} \geq f_{k+1} - r_{k+1} \geq 0, \tag{18}$$

where $L_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined by

$$L_3(x_{r_{k+1}}, x_{r_k}; \delta_{k+1}) = \frac{1}{\alpha} \ln \left(1 + \frac{\alpha \sqrt{\rho^2(x_{r_{k+1}}) + 2\delta_{k+1}}}{\sqrt{8} e^{\alpha(\tau_1+\tau_2)} \left\| \sqrt{M} \left(A x_{r_{k+1}} - B_1 B_1^T P x_{r_k} \right) \right\|_2} \right),$$

then the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than a positive constant $\hat{\eta}$.

Remark 4.2: By the self-triggering scheme proposed in theorem 4.1, the $k+1$th task release time is determined when $t = f_k$ and the deadline for the $k+1$th task delay is determined when $t = r_{k+1}$. τ_1 and τ_2 are used to bound the time intervals $[f_k, r_{k+1})$ and $[r_{k+1}, f_{k+1})$, respectively, for the consideration of the system security.

Remark 4.3: By the definition of L_2, it is easy to see that there exists a positive constant $\hat{\xi} \in \mathbb{R}^+$ such that $L_2(x_{r_k}) \geq \hat{\xi} > 0$. This implies the sample periods generated by this self-triggering scheme are always greater than a positive constant.

Remark 4.4: The introduction of δ_k can increase the value of $L_3(x_{r_k}, x_{r_{k-1}}; \delta_k)$. This suggests that by appropriate selecting δ_k, we can to some extent enlarge the deadlines. It may be useful when the predicted deadlines are very short. In that case, some large δ_k is desirable. How to efficiently identify δ_k might be an interesting topic in the future.

V. Simulations

In this section, we used the inverted pendulum problem in [1] to demonstrate the proposed self-triggered scheme. The plant's linearized state equations were

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/\ell & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(M\ell) \end{bmatrix} u \\ &= Ax + Bu \end{aligned}$$

where M was the cart mass, m was the mass of the pendulum bob, ℓ was the length of the pendulum arm, and g was gravitational acceleration. For these simulations, we let $M = 10$, $m = 1$, $\ell = 3$, and $g = 10$. The system state was the vector $x = \begin{bmatrix} y & \dot{y} & \theta & \dot{\theta} \end{bmatrix}^T$ where y was the cart's position and θ was the pendulum bob's angle with respect to the vertical. The system's initial state was the vector $x_0 = \begin{bmatrix} 0.98 & 0 & 0.2 & 0 \end{bmatrix}^T$. The controller is $u = Kx$, where $K = \begin{bmatrix} 2 & 12 & 378 & 210 \end{bmatrix}$. The Lyapunov function for the continuous closed-loop system is $V(x) = \sqrt{x^T P x}$, where P satisfies ARE equation (2) with $\gamma = 200$.

We first used the self-triggered feedback scheme, associated with equation (17) and (18) in theorem 4.1, to trigger the sampling. The parameters are $\tau_1 = 0.15$, $\tau_2 = 0.05$, $\epsilon = 0.8$, and $\delta_k = 0$. In this case, the task release times were generated at time f_k using the equation

$$r_{k+1} = f_k + \min\{\tau_1, \epsilon L_2(x_{r_k})\}$$

and the finishing times were assumed to satisfy

$$\begin{aligned} f_{k+1} &= r_{k+1} + \\ &\quad \min \left\{ \tau_2, (1-\epsilon) L_2(x_{r_k}), L_3(x_{r_{k+1}}, x_{r_k}; \delta_{k+1}) \right\} \end{aligned}$$

which means the delays are equal to the deadlines.

The top plot of figure 1 shows the state trajectories versus time in the resulting self-triggered feedback system. Obviously, the state converges to zero. The bottom plot of figure 1 is the sample periods (cross) and deadlines (dot) generated in the system versus time. We can see a range of variations in periods and deadlines. The average period and deadline are 0.1057 and 0.0056, respectively. It shows that self-triggering can efficiently adjust the sample periods and deadlines in response to changes in the control system.

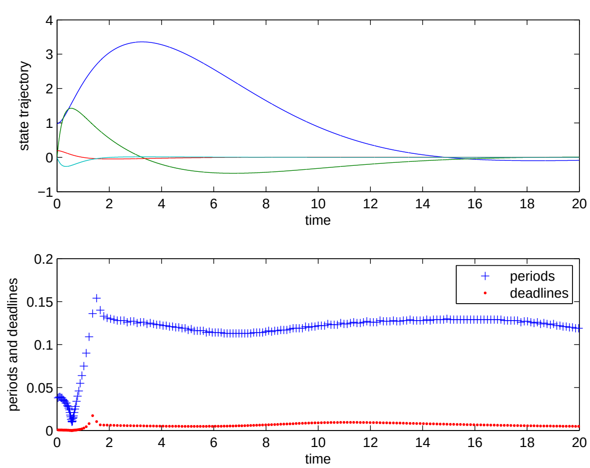

Fig. 1. A self-triggered feedback system

We then set $\delta_k = \frac{10^5}{k^2}$ and re-run the simulation. Notice that $\sum_{k=1}^{\infty} \delta_k \leq \infty$. The resulting self-triggered feedback system is still stable. However, the predicted deadlines in this system are much longer than those in the system with $\delta_k = 0$. This is shown in Figure 2 that plots the deadlines in the systems with $\delta_k = \frac{10^5}{k^2}$ (circle) and $\delta_k = 0$ (dot). It suggests that appropriate selection of δ_k can result in longer deadlines. It provides the possibility of avoiding very short deadlines.

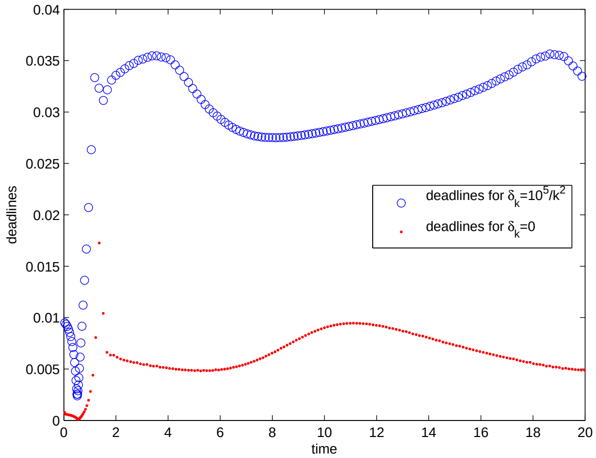

Fig. 2. A comparison between the deadlines generated by the systems with $\delta_k = \frac{10^5}{k^2}$ (circle) and $\delta_k = 0$ (dot)

Then, how to efficiently allocate the resource (selecting δ_k) would be an interesting research topic.

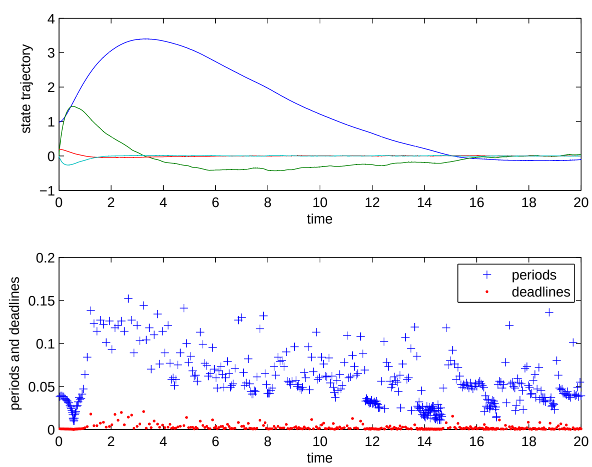

Fig. 3. A self-triggered feedback system with external disturbance $w(t)$

We also examined the robustness of our self-triggered feedback system to the external disturbance with $\tau_1 = 0.15$, $\tau_2 = 0.05$, $\epsilon = 0.8$, and $\delta_k = 0$. The disturbance, w_t, was assumed to be a random variable uniformly distributed over $[-0.2, 0.2]$. The results are plotted in figure 3. The top plot of figure 3 shows the state trajectories of the self-triggered feedback system. The system still converges to a small neighborhood of the equilibrium point. The bottom plot of figure 3 provides the sample periods (cross) and deadlines (dot) generated by this system. Although the periods and deadlines still vary a lot, they are in general much smaller than those in the non-disturbance case. The average period and deadline are 0.0535 and 0.0021, respectively. This verifies the ability of self-triggered feedback systems in adjusting sample periods in response to changes in the control system's external inputs. Based on the results of this simulation, our self-triggering scheme appears to be robust to the external disturbance.

Finally, we compared our self-triggering scheme ($\tau_1 = 0.5$, $\epsilon = 1$) and the self-triggering scheme in [1] with a noise process satisfying $\|w_t\|_2 \leq 0.01\|x_t\|_2$ ($W = 0.01$). In both of the cases, we assume the delays are zero. Recall that the self-triggering scheme in [1] requires $\|w_t\|_2 \leq W\|x_t\|_2$ holds for some $W > 0$ and the $k+1$th task release, r_{k+1}, is triggered in the following way:

$$r_{k+1} = r_k + \frac{1}{\bar{\alpha}} \ln\left(1 + \frac{\bar{\alpha}\|\sqrt{M}x_{r_k}\|_2}{\|\sqrt{M}A_{cl}x_{r_k}\|_2}\right)$$

where $M = (1-\bar{\beta}^2)I + PBB^TP > 0$ with some $\bar{\beta} \in (0,1)$ and $\bar{\alpha} = \|\sqrt{M}A\sqrt{M}^{-1}\| + W\|\sqrt{M}B_2\|\|\sqrt{M}^{-1}\|$. We set $\bar{\beta} = 0.5$ (the value of $\bar{\beta}$ did not significantly affect the simulation results).

The simulation results show the minimal/average/maximal periods generated by our self-triggering scheme and the scheme proposed in [1] are $0.0220/0.1574/0.2290$ and $0.0210/0.0626/0.1030$, respectively. It is obvious that our self-triggered scheme generates much longer sample periods.

VI. Conclusions

This paper proposes a new self-triggering scheme that ensures finite-gain $\mathcal{L}_2$ stability of the resulting self-triggered feedback systems. This scheme relaxes the assumptions in [1] that the magnitude of the process noise is bounded by a linear function of the norm of the system state. We show that the sample periods generated by this scheme are always greater than a positive constant. We also provide dynamic deadlines for delays and propose a way that may enlarge predicted deadlines without breaking $\mathcal{L}_2$ stability, especially when the predicted deadlines are very short. Simulations show that the sample periods generated by this scheme are longer than those generated by the scheme in [1]. We also show that our scheme can extend the predicted deadlines. Moreover, this scheme appears to be robust to the external disturbances.

Appendix

Proof: [Proof of Lemma 3.1] Consider the directional derivative of V for $t \in [f_k, f_{k+1})$:

$$\begin{aligned}
\dot{V} &= \tfrac{\partial V}{\partial x}\left(Ax_t - B_1B_1^TPx_{r_k} + B_2w_t\right) \\
&= -x_t^T(I-Q)x_t - \left\|\gamma w_t - \tfrac{1}{\gamma}B_2^TPx_t\right\|_2^2 \\
&\quad +\gamma^2\|w_t\|_2^2 - 2x_t^TQx_{r_k} \\
&\leq -x_t^T(I-Q)x_t + \gamma^2\|w_t\|_2^2 - 2x_t^TQx_{r_k}.
\end{aligned}$$

Insert $x_t = e_t^k + x_{r_k}$ into the above equation to obtain

$$\begin{aligned}
\dot{V} &\leq -\|x_t\|_2^2 + \left(e_t^k + x_{r_k}\right)^TQ\left(e_t^k + x_{r_k}\right) \\
&\quad -2\left(e_t^k + x_{r_k}\right)^TQx_{r_k} + \gamma^2\|w_t\|_2^2 \\
&= -\beta^2\|x_t\|_2^2 - (1-\beta^2)\|x_t\|_2^2 + \left(e_t^k\right)^TQe_t^k \\
&\quad -x_{r_k}^TQx_{r_k} + \gamma^2\|w_t\|_2^2. \qquad (19)
\end{aligned}$$

Notice that

$$\begin{array}{rcl} \|x_t\|_2^2 & = & \left\|e_t^k + x_{r_k}\right\|_2^2 = \left\|e_t^k\right\|_2^2 + \|x_{r_k}\|_2^2 + 2x_{r_k}e_t^k \\ & = & -\left\|e_t^k\right\|_2^2 + \frac{1}{2}\|x_{r_k}\|_2^2 + \left\|\sqrt{2}e_t^k + \frac{1}{\sqrt{2}}x_{r_k}\right\|_2^2 \\ & \geq & -\left\|e_t^k\right\|_2^2 + \frac{1}{2}\|x_{r_k}\|_2^2 . \end{array} \tag{20}$$

Applying equation (20) into equation (19), we obtain

$$\begin{array}{rcl} \dot{V} & \leq & -\beta^2\|x_t\|_2^2 - (1-\beta^2)\left(-\left\|e_t^k\right\|_2^2 + \frac{1}{2}\|x_{r_k}\|_2^2\right) \\ & & + \left(e_t^k\right)^T Q e_t^k - x_{r_k}^T Q x_{r_k} + \gamma^2\|w_t\|_2^2 \\ & = & -\beta^2\|x_t\|_2^2 + \gamma^2\|w_t\|_2^2 + \left(e_t^k\right)^T M e_t^k - x_{r_k}^T N x_{r_k}, \end{array}$$

where M and N are defined by equation (7) and (8), respectively. ■

Proof: [Proof of Lemma 3.3] By lemma 3.1, we know

$$\dot{V} \leq -\beta^2\|x_t\|_2^2 + \gamma^2\|w_t\|_2^2 + \left(e_t^k\right)^T M e_t^k - x_{r_k}^T N x_{r_k}$$

for all $t \in [f_k, f_{k+1})$. Integrating both sides of the inequality above on t over the interval $[f_k, f_{k+1})$, we obtain:

$$\begin{array}{r} \int_{f_k}^{f_{k+1}} \dot{V}dt \leq -\beta^2\int_{f_k}^{f_{k+1}}\|x_t\|_2^2 dt + \gamma^2\int_{f_k}^{f_{k+1}}\|w_t\|_2^2 dt \\ + \int_{f_k}^{f_{k+1}}\left(e_t^k\right)^T M e_t^k - x_{r_k}^T N x_{r_k} dt. \end{array} \tag{21}$$

Let us consider the term, $\int_{f_k}^{f_{k+1}}\left(e_t^k\right)^T M e_t^k dt$, in the inequality in equation (21). We will show an upper bound on this term. Let $\Phi = \left\{t \in [f_k, f_{k+1}) : \left\|\sqrt{M}e_t^k\right\|_2 = 0\right\}$. The time derivative of $\|\sqrt{M}e_t^k\|_2$ for $t \in [f_k, f_{k+1})\backslash\Phi$ satisfies

$$\begin{array}{rl} & \frac{d}{dt}\left\|\sqrt{M}e_t^k\right\|_2 \leq \left\|\sqrt{M}\dot{e}_t^k\right\|_2 = \left\|\sqrt{M}\dot{x}_t\right\|_2 \\ = & \left\|\sqrt{M}\left(Ax_t - B_1B_1^TPx_{r_k} + B_2w_t\right)\right\|_2 \\ \leq & \left\|\sqrt{M}Ae_t^k\right\|_2 + \left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2 + \left\|\sqrt{M}B_2\right\|\|w_t\|_2 \\ \leq & \alpha\left\|\sqrt{M}e_t^k\right\|_2 + \left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2 + \left\|\sqrt{M}B_2\right\|\|w_t\|_2, \end{array}$$

where the righthand sided derivative is used when $t = f_k$.

Using standard comparison principle on the preceding equation over the interval $t \in [f_k, f_{k+1})$ with the initial condition $\left\|\sqrt{M}e_{f_k}^k\right\|_2 = \left\|\sqrt{M}e_{r_k}^k\right\|_2 = 0$, we have

$$\begin{array}{rcl} \left\|\sqrt{M}e_t^k\right\|_2 & \leq & \frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2}{\alpha}\left(e^{\alpha(t-f_k)} - 1\right) \\ & & + \int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2 ds \end{array} \tag{22}$$

for all $t \in [f_k, f_{k+1})$ because $\left\|\sqrt{M}e_t^k\right\|_2 = 0$ for all $t \in \Phi$.

Notice that equation (22) yields

$$\begin{array}{rcl} \left\|\sqrt{M}e_t^k\right\|_2^2 & \leq & 2\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)} - 1\right)^2 \\ & & +2\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2 ds\right)^2 \end{array} \tag{23}$$

for all $t \in [f_k, f_{k+1})$. Integrating both sides of the inequality in equation (23) on t over the interval $[f_k, f_{k+1})$, we have

$$\begin{array}{l} \int_{f_k}^{f_{k+1}}\left\|\sqrt{M}e_t^k\right\|_2^2 dt \\ \leq 2\int_{f_k}^{f_{k+1}}\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)} - 1\right)^2 d \\ +2\int_{f_k}^{f_{k+1}}\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2 ds\right)^2 dt. \end{array} \tag{24}$$

We now take a look at the second term in the right side of the inequality above. For notational convenience, we define

$$W_k = 2\int_{f_k}^{f_{k+1}}\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2 ds\right)^2 dt. \tag{25}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{array}{l} \left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2 ds\right)^2 \\ \leq \left(\int_{f_k}^t e^{\alpha(t-s)}ds\right)\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 ds\right) \end{array}$$

for all $t \in [f_k, f_{k+1})$. Therefore,

$$\begin{array}{rcl} W_k & \leq & 2\int_{f_k}^{f_{k+1}}\left(\int_{f_k}^t e^{\alpha(t-s)}ds\right) \cdot \\ & & \left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 ds\right)dt \\ & = & 2\int_{f_k}^{f_{k+1}}\frac{1}{\alpha}\left(e^{\alpha(t-f_k)} - 1\right) \cdot \\ & & \left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 ds\right)dt. \end{array} \tag{26}$$

Equation (9) implies $0 \leq r_{k+1} - r_k \leq \tau$. By the assumption that $r_k = f_k$ holds for all $k \in \mathbb{Z}^+$, we have $0 \leq f_{k+1} - f_k \leq \tau$. Therefore, equation (26) can be reduced as

$$\begin{array}{l} W_k \\ \leq \frac{2}{\alpha}\left(e^{\alpha\tau} - 1\right)\int_{f_k}^{f_{k+1}}\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 dsdt \\ = \frac{2(e^{\alpha\tau}-1)}{\alpha}\int_{f_k}^{f_{k+1}}\int_s^{f_{k+1}} e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 dtds \\ = \frac{2(e^{\alpha\tau}-1)}{\alpha^2}\int_{f_k}^{f_{k+1}}\left(e^{\alpha(f_{k+1}-s)} - 1\right)\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2 ds, \end{array} \tag{27}$$

where the first equality is obtained by reversing the order of integration. Applying $f_{k+1} - f_k \leq \tau$ in equation (27) yields

$$W_k \leq \frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau} - 1\right)^2\int_{f_k}^{f_{k+1}}\|w_s\|_2^2 ds. \tag{28}$$

Combining equation (24) and (28), we obtain

$$\begin{array}{rl} & \int_{f_k}^{f_{k+1}}\left\|\sqrt{M}e_t^k\right\|_2^2 dt \\ \leq & 2\int_{f_k}^{f_{k+1}}\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)} - 1\right)^2 dt \\ & +\frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau} - 1\right)^2\int_{f_k}^{f_{k+1}}\|w_s\|_2^2 ds. \end{array} \tag{29}$$

Therefore, equation (21) can be further reduced as

$$\begin{array}{rl} & \int_{f_k}^{f_{k+1}}\dot{V}dt \\ \leq & -\beta^2\int_{f_k}^{f_{k+1}}\|x_t\|_2^2 dt + \gamma^2\int_{f_k}^{f_{k+1}}\|w_t\|_2^2 dt \\ & -\int_{f_k}^{f_{k+1}}x_{r_k}^T N x_{r_k} dt \\ & +2\int_{f_k}^{f_{k+1}}\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)} - 1\right)^2 dt \\ & +\frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau} - 1\right)^2\int_{f_k}^{f_{k+1}}\|w_s\|_2^2 ds \\ = & -\beta^2\int_{f_k}^{f_{k+1}}\|x_t\|_2^2 dt - \int_{f_k}^{f_{k+1}}x_{r_k}^T N x_{r_k} dt \\ & +\left(\gamma^2 + \frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau} - 1\right)^2\right)\int_{f_k}^{f_{k+1}}\|w_t\|_2^2 dt \\ & +2\int_{f_k}^{f_{k+1}}\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)} - 1\right)^2 dt. \end{array} \tag{30}$$

Applying equation (10) in equation (30), we obtain

$$\begin{aligned}&\int_{f_k}^{f_{k+1}} \dot V\, dt \le -\beta^2 \int_{f_k}^{f_{k+1}} \|x_t\|_2^2\, dt \\ &+\left(\gamma^2+\frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau}-1\right)^2\right)\int_{f_k}^{f_{k+1}}\|w_t\|_2^2\, dt.\end{aligned} \tag{31}$$

Summarizing k in both sides of the inequality above from 0 to ∞ , we obtain

$$\begin{aligned}&\int_0^{\infty} \dot V\, dt \le -\beta^2 \int_0^{\infty} \|x_t\|_2^2\, dt \\ &+\left(\gamma^2+\frac{2\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha\tau}-1\right)^2\right)\int_0^{\infty}\|w_t\|_2^2\, dt,\end{aligned}$$

which is sufficient to show that the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than $\frac{\sqrt{\gamma^2\alpha^2+2\left\|\sqrt{M}B_2\right\|^2(e^{\alpha\tau}-1)^2}}{\alpha\beta}$. ■

Proof: [Proof of Theorem 3.6] By the assumption, M defined in equation (7) has full rank. As a result, N defined in equation (8) also has full rank and $M \ge N > 0$. Therefore, by the definition of L_1 in equation (16), we have

$$\begin{aligned} L_1(x_{r_k}) &\ge \frac{1}{\alpha}\ln\left(1+\frac{\alpha\sqrt{\lambda_{\min}(N)}\left\|x_{r_k}\right\|_2}{\sqrt{2\lambda_{\max}\left(A_{\mathrm{cl}}^T M A_{\mathrm{cl}}\right)}\left\|x_{r_k}\right\|_2}\right) \\ &= \frac{1}{\alpha}\ln\left(1+\frac{\alpha\sqrt{\lambda_{\min}(N)}}{\sqrt{2\lambda_{\max}\left(A_{\mathrm{cl}}^T M A_{\mathrm{cl}}\right)}}\right) > 0,\end{aligned}$$

which guarantees the well-poseness of equation (15).

Notice that equation (15) implies

$$\frac{2\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(r_{k+1}-r_k)}-1\right)^2 - x_{r_k}^T N x_{r_k} \le 0. \tag{32}$$

Since $r_k = f_k$ for all $k \in \mathbb{Z}^+$, the equation above can be rewritten as

$$\begin{aligned} 0 &\ge \frac{2\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(f_{k+1}-f_k)}-1\right)^2 - x_{r_k}^T N x_{r_k} \\ &\ge \frac{2\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(s-f_k)}-1\right)^2 - x_{r_k}^T N x_{r_k}\end{aligned} \tag{33}$$

for all $s \in [f_k, f_{k+1})$. Therefore, integrating both sides of this inequality on s over $[f_k, f_{k+1})$ implies that satisfaction of equation (10). Since the hypotheses in lemma 3.3 are satisfied, we can conclude that the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than $\eta = \frac{\sqrt{\gamma^2\alpha^2+2\left\|\sqrt{M}B_2\right\|^2(e^{\alpha\tau}-1)^2}}{\alpha\beta}$. ■

Proof: [Proof of Theorem 4.1] Let $\Phi_1 = \left\{t \in [r_k, f_k) : \left\|\sqrt{M}e_t^k\right\|_2 = 0\right\}$. The time derivative of $\left\|\sqrt{M}e_t^k\right\|_2$ for $t \in [r_k, f_k)\backslash\Phi_1$ satisfies

$$\begin{aligned} &\frac{d}{dt}\left\|\sqrt{M}e_t^k\right\|_2 \le \left\|\sqrt{M}\dot e_t^k\right\|_2 = \left\|\sqrt{M}\dot x_t\right\|_2 \\ =\;& \left\|\sqrt{M}\left(Ax_t - B_1B_1^TPx_{r_{k-1}} + B_2 w_t\right)\right\|_2 \\ \le\;& \alpha\left\|\sqrt{M}e_t^k\right\|_2 + \left\|\sqrt{M}\left(Ax_{r_k} - B_1B_1^TPx_{r_{k-1}}\right)\right\|_2 \\ &+\left\|\sqrt{M}B_2\right\|\|w_t\|_2,\end{aligned}$$

where the righthand sided derivative is used when $t = r_k$.

Using standard comparison principle on the preceding equation over the interval $t \in [r_k, f_k)$ with the initial condition $\left\|\sqrt{M}e_{r_k}^k\right\|_2 = 0$, we have

$$\begin{aligned}\left\|\sqrt{M}e_t^k\right\|_2 \le\;& \frac{\left\|\sqrt{M}\left(Ax_{r_k}-B_1B_1^TPx_{r_{k-1}}\right)\right\|_2}{\alpha}\left(e^{\alpha(t-r_k)}-1\right) \\ &+\int_{r_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds\end{aligned} \tag{34}$$

for all $t \in [r_k, f_k)$ because $\left\|\sqrt{M}e_t^k\right\|_2 = 0$ for all $t \in \Phi_1$.

Let $\Phi_2 = \left\{t \in [f_k, f_{k+1}) : \left\|\sqrt{M}e_t^k\right\|_2 = 0\right\}$. Following the similar analysis in the proof lemma 3.3, the time derivative of $\left\|\sqrt{M}e_t^k\right\|_2$ for $t \in [f_k, f_{k+1})\backslash\Phi_2$ satisfies

$$\begin{aligned}\frac{d}{dt}\left\|\sqrt{M}e_t^k\right\|_2 \;\le\;& \alpha\left\|\sqrt{M}e_t^k\right\|_2 + \left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2 \\ &+\left\|\sqrt{M}B_2\right\|\|w_t\|_2,\end{aligned}$$

where the righthand sided derivative is used when $t = f_k$.

Using standard comparison principle on the preceding equation over the interval $t \in [f_k, f_{k+1})$ with the initial condition

$$\begin{aligned}\left\|\sqrt{M}e_{f_k}^k\right\|_2 \le\;& \frac{\left\|\sqrt{M}\left(Ax_{r_k}-B_1B_1^TPx_{r_{k-1}}\right)\right\|_2}{\alpha}\left(e^{\alpha D_k}-1\right) \\ &+\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds\end{aligned}$$

obtained from equation (34), we have

$$\begin{aligned}&\left\|\sqrt{M}e_t^k\right\|_2 \\ \le\;& e^{\alpha(t-f_k)}\frac{\left\|\sqrt{M}\left(Ax_{r_k}-B_1B_1^TPx_{r_{k-1}}\right)\right\|_2}{\alpha}\left(e^{\alpha D_k}-1\right) \\ &+e^{\alpha(t-f_k)}\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds \\ &+\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2}{\alpha}\left(e^{\alpha(t-f_k)}-1\right) \\ &+\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds\end{aligned} \tag{35}$$

holds for all $t \in [f_k, f_{k+1})$ since $\left\|\sqrt{M}e_t^k\right\|_2 = 0$ for all $t \in \Phi_2$. By squaring both sides of the inequality in equation (35), we obtain

$$\begin{aligned}&\left\|\sqrt{M}e_t^k\right\|_2^2 \\ \le\;& 4e^{2\alpha(t-f_k)}\frac{\left\|\sqrt{M}\left(Ax_{r_k}-B_1B_1^TPx_{r_{k-1}}\right)\right\|_2^2}{\alpha^2}\left(e^{\alpha D_k}-1\right)^2 \\ &+4e^{2\alpha(t-f_k)}\left(\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds\right)^2 \\ &+4\frac{\left\|\sqrt{M}A_{\mathrm{cl}}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)}-1\right)^2 \\ &+4\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\, ds\right)^2\end{aligned} \tag{36}$$

holds for all $t \in [f_k, f_{k+1})$. By equation (17) and (18), we have $f_{k+1} - f_k \le \tau_1 + \tau_2$. Therefore, equation (18) implies that

$$\begin{aligned}&4e^{2\alpha(t-f_k)}\frac{\left\|\sqrt{M}\left(Ax_{r_k}-B_1B_1^TPx_{r_{k-1}}\right)\right\|_2^2}{\alpha^2}\left(e^{\alpha D_k}-1\right)^2 \\ &\qquad\le \frac{1}{2}x_{r_k}^T N x_{r_k} + \delta_k\end{aligned} \tag{37}$$

holds for all $t \in [f_k, f_{k+1})$. Again, by equation (17) and (18), we have $f_{k+1} - f_k \le L_2(x_{r_k})$, which implies that

$$4\frac{\left\|\sqrt{M}A_{\rm cl}x_{r_k}\right\|_2^2}{\alpha^2}\left(e^{\alpha(t-f_k)}-1\right)^2 \le \frac{1}{2}x_{r_k}^T N x_{r_k} \qquad (38)$$

holds for all $t \in [f_k, f_{k+1})$. Applying equation (37) and (38) into equation (36) yields

$$\begin{aligned} & \left\|\sqrt{M}e_t^k\right\|_2^2 \\ \le\; & x_{r_k}^T N x_{r_k} + \delta_k \\ & +4\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 \\ & +4e^{2\alpha(t-f_k)}\left(\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 \end{aligned} \qquad (39)$$

for all $t \in [f_k, f_{k+1})$. By lemma 3.1, we know

$$\dot V \le -\beta^2\|x_t\|_2^2 + \gamma^2\|w_t\|_2^2 + \left(e_t^k\right)^T M e_t^k - x_{r_k}^T N x_{r_k} \qquad (40)$$

holds for all $t \in [f_k, f_{k+1})$ with $V(x) = x^T P x$. Applying equation (39) into the preceding inequality implies that

$$\begin{aligned} \dot V \;\le\; & -\beta^2\|x_t\|_2^2 + \gamma^2\|w_t\|_2^2 + \delta_k \\ & +4e^{2\alpha(t-f_k)}\left(\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 \\ & +4\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 \end{aligned}$$

holds for all $t \in [f_k, f_{k+1})$. Integrating both sides of the preceding inequality on t over $[f_k, f_{k+1})$ yields

$$\begin{aligned} & \int_{f_k}^{f_{k+1}} \dot V dt \\ & \le -\beta^2\int_{f_k}^{f_{k+1}}\|x_t\|_2^2\,dt + \gamma^2\int_{f_k}^{f_{k+1}}\|w_t\|_2^2\,dt + \int_{f_k}^{f_{k+1}}\delta_k dt \\ & + \int_{f_k}^{f_{k+1}} 4e^{2\alpha(t-f_k)}\left(\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 dt \\ & + \int_{f_k}^{f_{k+1}} 4\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 dt. \end{aligned} \qquad (41)$$

Let us now take a look at the fourth item in the right side of the inequality in equation (41). By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \int_{f_k}^{f_{k+1}} 4e^{2\alpha(t-f_k)}\left(\int_{r_k}^{f_k} e^{\alpha(f_k-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 dt \\ & \le \int_{f_k}^{f_{k+1}} 4e^{2\alpha(t-f_k)}dt \cdot \\ & \quad \int_{r_k}^{f_k} e^{2\alpha(f_k-s)}ds \int_{r_k}^{f_k}\left\|\sqrt{M}B_2\right\|^2\|w_s\|_2^2\,ds \\ & = \tfrac{2}{\alpha}\left(e^{2\alpha(f_{k+1}-f_k)}-1\right)\tfrac{1}{2\alpha}\left(e^{2\alpha(f_k-r_k)}-1\right)\cdot \\ & \quad \left\|\sqrt{M}B_2\right\|^2\int_{r_k}^{f_k}\|w_s\|_2^2\,ds \\ & \le \tfrac{2}{\alpha}\left(e^{2\alpha(\tau_1+\tau_2)}-1\right)\tfrac{1}{2\alpha}\left(e^{2\alpha\tau_2}-1\right)\left\|\sqrt{M}B_2\right\|^2\cdot \\ & \quad \int_{f_{k-1}}^{f_k}\|w_s\|_2^2\,ds. \end{aligned} \qquad (42)$$

Following the similar analysis used in lemma 3.3, we obtain an upper bound on the fifth item in the right side of the inequality in equation (41):

$$\begin{aligned} & \int_{f_k}^{f_{k+1}} 4\left(\int_{f_k}^t e^{\alpha(t-s)}\left\|\sqrt{M}B_2\right\|\|w_s\|_2\,ds\right)^2 dt \\ \le\; & \tfrac{4\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha(\tau_1+\tau_2)}-1\right)^2\int_{f_k}^{f_{k+1}}\|w_s\|_2^2\,ds. \end{aligned} \qquad (43)$$

Applying equation (42) and (43) into (41), we obtain

$$\begin{aligned} & \int_{f_k}^{f_{k+1}} \dot V dt \\ & \le -\beta^2\int_{f_k}^{f_{k+1}}\|x_t\|_2^2\,dt + \gamma^2\int_{f_k}^{f_{k+1}}\|w_t\|_2^2\,dt + \int_{f_k}^{f_{k+1}}\delta_k dt \\ & + \tfrac{\left(e^{2\alpha(\tau_1+\tau_2)}-1\right)\left(e^{2\alpha\tau_2}-1\right)\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\int_{f_{k-1}}^{f_k}\|w_s\|_2^2\,ds \\ & + \tfrac{4\left\|\sqrt{M}B_2\right\|^2}{\alpha^2}\left(e^{\alpha(\tau_1+\tau_2)}-1\right)^2\int_{f_k}^{f_{k+1}}\|w_s\|_2^2\,ds. \end{aligned}$$

Summarizing k in the inequality above from 0 to ∞ yields

$$\begin{aligned} & \int_0^\infty \dot V dt \\ & \le -\beta^2\int_0^\infty\|x_t\|_2^2\,dt + (\tau_1+\tau_2)\sum_{k=0}^\infty\delta_k \\ & \quad + \int_0^\infty\|w_s\|_2^2\,ds\cdot \\ & \quad \frac{\gamma^2\alpha^2 + \left\|\sqrt{M}B_2\right\|^2\left(\left(e^{2\alpha(\tau_1+\tau_2)}-1\right)\left(e^{2\alpha\tau_2}-1\right)+4\left(e^{\alpha(\tau_1+\tau_2)}-1\right)^2\right)}{\alpha^2}. \end{aligned} \qquad (44)$$

Since $\sum_{k=0}^\infty \delta_k \le \infty$, the inequality above is sufficient to show the sampled-data system is finite-gain $\mathcal{L}_2$ stable from w to x with an induced gain less than $\hat\eta = \frac{\sqrt{\gamma^2\alpha^2+\left\|\sqrt{M}B_2\right\|^2\left(\left(e^{2\alpha(\tau_1+\tau_2)}-1\right)(e^{2\alpha\tau_2}-1)+4\left(e^{\alpha(\tau_1+\tau_2)}-1\right)^2\right)}}{\alpha\beta}$. ■

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