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Direct Adaptive Stabilization of Linear Systems using Query-based Protocols

M.D. Lemmon *, C.J. Bett
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556 USA

Abstract

Query-based inference is a machine learning paradigm which has been used for learning Boolean functions from examples. This paper shows how such a protocol can be used for direct adaptive control of linear systems. The proposed procedure employs the central-cut ellipsoid method to iteratively search for a set of control gains which are feasible solutions to a system of linear inequalities. The value of using this approach is that such inference protocols can be shown to converge after a finite number of updates. This convergence time scales in a polynomial manner, $O(n^2 \ln n)$, with the number, n, of control gains to be determined. The convergence time is also bounded below by a function of the uncontrolled system's eigenvalues. These results thereby suggest that inductive inference protocols may represent a feasible method for direct adaptive control which can be practical for large scale linear systems.

1 Introduction

This paper confines its attention to single-input multi-output control systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{1}$$

where $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{x} \in \Re^n$, $\mathbf{b} \in \Re^n$, and $u \in \Re$. The control law is assumed to be

$$u = \mathbf{k}'\mathbf{x} \tag{2}$$

and $\mathbf{k} \in \Re^n$. The vector $\mathbf{x}$ is called the state vector and u is called the control signal.

The system of equations 1 and 2 will be said to be quadratically stabilizable if and only if there exists a positive definite symmetric matrix $\mathbf{P}$ such that the Lyapunov inequality

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{k}\mathbf{b}'\mathbf{P} + \mathbf{P}\mathbf{b}\mathbf{k}' < 0 \tag{3}$$

holds. In this paper, two matrices $\mathbf{P}$ and $\mathbf{Q}$ stand in relation $\mathbf{P} > \mathbf{Q}$ if and only if $\mathbf{P} - \mathbf{Q}$ is positive definite. Traditionally, the problem of quadratic stabilization has focused on finding a $\mathbf{P}$ and $\mathbf{k}$ given $\mathbf{A}$ and $\mathbf{b}$ such that inequality 3 holds. Recent work [Bernussou 1989] [Geromel 1991] suggested using linear programming techniques to find the stabilizing $\mathbf{P}$ and $\mathbf{k}$ assuming that $\mathbf{A}$ and $\mathbf{b}$ lie in a known convex set. The advantage of this approach is that it allows the determination of robust controls given known bounds on the system matrices.

These robust control gains, however, often trade away system performance for stability. In many applications, such a tradeoff may not be acceptable and this requires that system uncertainty be reduced in order to improve system performance. One way this can be done is by adapting the control law on the basis of observed system performance. For linear systems, this so-called "adaptive" control has traditionally relied on tools from parameter estimation theory in order to identify plant models [Kosut 1992]. The appropriate control gains are then computed from the identified plant. This approach to adaptive control is often called "indirect" adaptation. In this paper, a <u>direct</u> adaptation technique is presented. The proposed approach uses inductive

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inference or so-called "query-based" learning protocols. It extends prior work in which inductive inference procedures were used to identify "supervisable" events in hybrid control systems [Lemmon 1993b] .

In that the proposed algorithm is a direct adaptation method based on inductive inference protocols, it represents a significant departure from conventional indirect methods. The justificiation for this departure is based on the following results: 1) provable convergence after a finite number of updates, 2) upper bounds on the convergence time which scale in a polynomial manner with the number of control gains to be determined, 3) lower bounds on the convergence time which can be directly related back to the uncontrolled system's modes, 4) and experimental results which demonstrate fast convergence for high order systems. These results imply that the proposed direct adaptation strategy converges in a finite time that scales in a polynomial manner with problem complexity. Thus, such adaptation methods may provide practical methods for the control of large scale dynamical systems.

2 Stabilization of Linear Systems using Inductive Protocols

This section derives the components for a query-based algorithm which stabilizes the linear system given by equations 1 and 2.

The algorithm is based on the Lyapunov inequality (eq. 3). By appropriately transforming the state variables, it can be assumed without loss of generality that $\mathbf{P} = \mathbf{I}_n$ where $\mathbf{I}_n$ is an n by n identity matrix. As a consequence of this simplification, the Lyapunov inequality therefore reduces to

$$\tilde{\mathbf{A}} + \mathbf{E} < 0 \qquad (4)$$

where

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{A}' + \mathbf{A} && (5) \\ \mathbf{E} &= \mathbf{k}\mathbf{b}' + \mathbf{b}\mathbf{k}' && (6) \end{aligned}$$

The preceding inequality will hold if and only if for all $\mathbf{x} \in \Re^n$

$$\mathbf{x}' \left(\tilde{\mathbf{A}} + \mathbf{k}\mathbf{b}' + \mathbf{b}\mathbf{k}' \right) \mathbf{x} < 0 \qquad (7)$$

Note that this inequality is linear in the control gains $\mathbf{k}$, so it can be rewritten as a system of linear inequalities. Using the fact that the <u>uncontrolled</u> system's state velocity is $\dot{\mathbf{x}}_u = \mathbf{A}\mathbf{x}$, inequality 7 can be rewritten as

$$\mathbf{x}'\dot{\mathbf{x}}_u + (\mathbf{x}'\mathbf{k})(\mathbf{x}'\mathbf{b}) < 0 \qquad (8)$$

The preceding inequalities determine the set of control gains $\mathbf{k}$ which stabilize the uncontrolled system. A gain $\mathbf{k}$ which satsifies inequality 8 will be called a feasible gain and the set of all feasible gains will be denoted by the set K.

Equation 8 provides a set of inequalities which the uncontrolled system's state and state velocity have to satisfy if the assumed gains stablize the system. These inequalities form the basis of a Boolean functional called the "stabilization" oracle. This Boolean functional has the form,

$$O_s(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{k}) = \begin{cases} 0 & \text{if } \mathbf{x}'\dot{\mathbf{x}}_u + (\mathbf{x}'\mathbf{k})(\mathbf{x}'\mathbf{b}) < 0 \\ 1 & \text{otherwise} \end{cases} \qquad (9)$$

The stabilization oracle is "queried" by the system to determine whether or not the current set of gains are consistent with the hypothesis that the controlled system is stable. Therefore the output of the oracle algorithm can be interpreted as a declaration on the validity of the stability hypothesis.

The fact that the set of stabilizing control gains form a feasible point of a system of linear inequalities can be used to develop an algorithm which searches for the stabilizing gains. There are various numerical procedures for finding feasible points of such linear inequality systems. One such procedure is the so-called ellipsoid method [Bland 1981] [Groetschel 1988]. This method generates a sequence of ellipsoids which contain the convex body formed by a system of linear inequalities. Under appropriate assumptions, the algorithm can be shown to converge after a finite number of iterations. This fact was used quite effectively in earlier work which developed event identification algorithms for hybrid control systems [Lemmon 1993b]. A brief description of the algorithm as it pertains to query-based adaptation is provided below.

Assume that the convex body, K, formed by the above system of linear inequalities (Eq. 8) is contained within an n-dimensional ellipsoid,

$$E(\mathbf{Q},\mathbf{k}) = \left\{\mathbf{x} \in \Re^n : (\mathbf{x}-\mathbf{k})'\mathbf{Q}^{-1}(\mathbf{x}-\mathbf{k}) \leq 1\right\} \tag{10}$$

where $\mathbf{Q} > 0$. Note that this assumption is valid if it is assumed that the set of "feasible" gains have a bounded magnitude,

$$\mathbf{k}'\mathbf{k} < k_{\max}^2 \tag{11}$$

Inequality 11 is therefore added to the inequalities in equation 8 so that the set, K, of feasible gains clearly forms a bounded convex body contained with the ellipsoid $E(k_{\max}^2\mathbf{I}_n, \mathbf{0})$.

Let $\mathbf{k}$ be the current control gain to be tested by the oracle and assume that $\mathbf{x}'\mathbf{b} > 0$. Let data, $\mathbf{x}$ and $\dot{\mathbf{x}}$, collected by the experiment for the uncontrolled system (i.e., $u = 0$) be declared inconsistent by the oracle. This implies that inequality 8 does not hold, so that any control gain, $\mathbf{l}$, such that

$$\mathbf{x}'\mathbf{l} > \mathbf{x}'\mathbf{k} + \frac{\mathbf{x}'\dot{\mathbf{x}}}{\mathbf{x}'\mathbf{b}} \tag{12}$$

cannot possibly be a feasible gain. The above inequality can be simplified to

$$\mathbf{x}'\mathbf{l} > \mathbf{x}'\mathbf{k} \tag{13}$$

by recognizing that due to the failed query, $\mathbf{k}$ cannot possibly be in the feasible set, K.

We can now use the inequality of equation 13 to find a smaller ellipsoid containing K. Let H be the set

$$H = \{\mathbf{l} \in \Re^n : \mathbf{x}'\mathbf{l} > \mathbf{x}'\mathbf{k}\} \tag{14}$$

As noted above it can be assumed that the convex body of feasible gains, K, is contained within an ellipsoid, $E(\mathbf{Q},\mathbf{k})$, centered at $\mathbf{k}$. In view of the preceding arguments it can now be inferred that

$$K \subset K_1 = H^c \cap E(\mathbf{Q},\mathbf{k}) \tag{15}$$

Clearly K_1 is also a bounded convex body. The central-cut ellipsoid method [Bland 1981] [Groetschel 1988] provides a numerical procedure for computing an ellipsoid of minimal volume containing K_1.

If $E(\mathbf{Q}_i,\mathbf{k}_i)$ is the ellipsoid bounding the set, K, of feasible gains prior to the ith consecutive failed oracle query, then the minimal-volume ellipsoid generated by the ellipsoid algorithm is given by

$$\mathbf{d} = sgn(\mathbf{x}'\mathbf{b})\frac{\mathbf{Q}_i\mathbf{x}}{\sqrt{\mathbf{x}\mathbf{Q}_i\mathbf{x}}} \tag{16}$$

$$\mathbf{k}_{i+1} = \mathbf{k}_i - \frac{1}{n+1}\mathbf{d} \tag{17}$$

$$\mathbf{Q}_{i+1} = \frac{n^2}{n^2-1}\left(\mathbf{Q}_i - \frac{2}{n+1}\mathbf{d}\mathbf{d}^t\right) \tag{18}$$

where $\mathbf{x}$ and $\dot{\mathbf{x}}$ are the state and state velocities gathered by the experiment for the $(i+1)$st failed oracle query. See [Groetschel 1988] for a detailed derivation of these equations.

On the basis of the preceding remarks, the proposed adaptation algorithm can be formally stated.

- **Initial Hypothesis:** Let $i = 0$ and let $\mathbf{Q}_i = k_{\max}^2\mathbf{I}_n$ and $\mathbf{k}_i = 0$. This matrix and vector describe an initial ellipsoid, $E(\mathbf{Q}_0,\mathbf{k}_0)$ which is known to contain all feasible control gains.

- **Experiment:** Measure the uncontrolled system's (i.e. $u = 0$) state vector, $\mathbf{x}$, and its time rate of change $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. This forms the data collection to be input to the oracle.

- **Oracle Query:** Compute the oracle's response using equation 9.

- **Update Algorithm:** If $O_s(\mathbf{x},\dot{\mathbf{x}},\mathbf{k}_i) = 1$ (i.e. a failed oracle query) then update the control gain using equations 16, 17, and 18.

  Loop: Go to **Experiment**.

Note that the inductive inference protocol described above requires little or no previous knowledge of the system dynamics to stabilize the system. The algorithm 'learns' from mistakes, modifying the hypotheses based only upon state and velocity measurements. Also, no attempt is made to identify the system dynamics because the knowledge is unnecessary for system stabilization. This is a marked departure from traditional adaptive control techniques. In addition, as will be proven in the following section, the system can be stabilized in a known and finite time.

3 Convergence and Complexity

The algorithm outlined above represents a significant departure from conventional approaches to adaptive linear control. The justification for this departure lies in finite time convergence results for the central-cut ellipsoid method. In this section, these results are stated and some bounds are derived relating the convergence time to system parameters.

Theorem 1 *Let K denote the set of "feasible" control gains and let v denote the volume of an n-dimensional ellipsoid contained within K. The proposed algorithm will determine a feasible control gain after no more than $2n\ln(V/v)$ failed oracle queries where V is the volume of an n-dimensional sphere of radius $k_{\max}$.*

Proof: The proof of the theorem follows from standard applications of the central cut ellipsoid method [Groetschel 1988].

The preceding theorem shows that the proposed algorithm converges after a finite number of updates. This important convergence result has significant consequences which will be noted in the following section. The result, however, is only valuable if we can establish some results indicating the size of this bound and how it scales as a function of the system being controlled. The following theorem shows that the maximum number of failed queries L derived in the preceding theorem will grow in a polynomial manner with the state space dimension.

Theorem 2 *Let K be the set of feasible gains and assume that K encloses an ellipsoid with a covariance matrix $\mathbf{Q}$ whose eigenvalues are all less than δ where $1 > \delta > 0$. Under the assumptions of the preceding theorem, a feasible gain will be found after no more than $2n^2\ln n + n^2\ln\delta^{-1} + 2n\ln V$ failed oracle queries.*

Proof: Because of the constraints on $\mathbf{Q}$, the volume of the minimal ellipse is no more than $\delta^{n/2}V_n$ where V_n is the volume of an n-dimensional ellipsoid with unit radius. Note, however, that $n^{-n} < V_n < 2^{-n}$ which implies that the ellipsoid contained within K will have a volume no larger than $n^{-n}\delta^{n/2}$. Using this for v in theorem 1 yields the bound. **QED**.

The preceding theorem provides an upper bound on the number of failed queries. Assume that δ represents the intrinsic precision with which the eigenvalues of the ellipsoid can be specified. The preceding theorem therefore provides an upper bound on the failed queries as a function of the system's intrinsic numerical precision. The result, however, also provides specific results concerning the growth of the bound as a function of the system's state space dimension. This growth rate appears to be bounded on the order of $O(n^2\ln n) \approx O(n^{2.5})$ which suggests that the procedure's convergence time grows at a modest rate with the system's complexity (i.e. state space dimension). It would also be valuable if this growth rate could be related back to the uncontrolled system. The following theorem establishes such a relationship.

Theorem 3 *Let K be the set of feasible gains. Let $\lambda_i(\tilde{\mathbf{A}})$ be the ith eigenvalue for the matrix $\tilde{\mathbf{A}} = \mathbf{A}' + \mathbf{A}$. Let*

$$\beta = -\frac{1}{2}\frac{\sum_i \lambda_i}{|k_{\max}||\mathbf{b}|} \quad (19)$$

where the summation is over all eigenvalues λ_i of the symmetric matrix $\mathbf{A} + \mathbf{A}'$. Then the number of failed oracle queries, L, must satisfy

$$L > (1 - n\beta)^2 \quad (20)$$

Proof: Let μ_i and $\mathbf{d}_i$ be the ith eigenvalue and eigenvector, respectively, of the matrix $\mathbf{E} = \mathbf{k}\mathbf{b}' + \mathbf{b}\mathbf{k}'$. Note that $\tilde{\mathbf{A}}$ and $\mathbf{E}$ are symmetric matrices so that

$$\tilde{\mathbf{A}} + \mathbf{E} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i \mathbf{a}_i^t + \mu_1 \mathbf{d}_1 \mathbf{d}_1^t + \mu_n \mathbf{d}_n \mathbf{d}_n^t \quad (21)$$

where $\mathbf{a}_i$ are orthonormal eigenvectors of $\tilde{\mathbf{A}}$, λ_i are eigenvalues of $\tilde{\mathbf{A}}$, d_1 and d_n are the eigenvectors of $\mathbf{E}$ with largest and smallest eigenvalues, μ_1 and μ_n, respectively. Well known results from matrix perturbation theory imply that the eigenvalues of $\tilde{\mathbf{A}} + \mathbf{E}$ can be rewritten as

$$\lambda_i(\tilde{\mathbf{A}} + \mathbf{E}) = \lambda_i(\tilde{\mathbf{A}}) + m_{1i}\mu_1 + m_{ni}\mu_n \quad (22)$$

where $\sum_i m_{1i} = 1$ and $\sum_i m_{ni} = 1$. For $\mathbf{k}$ to be a feasible gain, then $\tilde{\mathbf{A}} + \mathbf{E}$ must be negative definite which implies that all its eigenvalues are negative. Adding up all of the inequalities implies that

$$\sum_i \lambda_i(\tilde{\mathbf{A}}) + \mu_1 + \mu_n < 0 \qquad (23)$$

The eigenvalues of $\mathbf{E}$ are functions of the gain vector $\mathbf{k}$. It can be easily shown that all eigenvalues of $\mathbf{E}$ will be

$$\mu_i = \begin{cases} \mathbf{b}'\mathbf{k} + |\mathbf{b}||\mathbf{k}| & \text{if } i = 1 \\ 0 & \text{if } 1 < i < n \\ \mathbf{b}'\mathbf{k} - |\mathbf{b}||\mathbf{k}| & \text{if } i = n \end{cases} \qquad (24)$$

Inserting these values back into the original inequality implies

$$\sum_i \lambda_i + 2\mathbf{b}'\mathbf{k} < 0 \qquad (25)$$

Recall from our original constraints on $\mathbf{k}$, that its magnitude is less than $k_{\max}$. This constraint therefore requires that feasible $\mathbf{k}$ be enclosed within the ellipsoid, $E(k_{\max}^2 \mathbf{I}_n, \mathbf{0})$. This ellipsoid has a volume $k_{\max}^n V_n$ where V_n is the volume of an n-dimensional sphere with unit radius. The inequality implied above requires that feasible gains also be in the halfspace defined by

$$\mathbf{b}'\mathbf{k} < -\frac{1}{2}\sum_i \lambda_i(\tilde{\mathbf{A}}) \qquad (26)$$

. Once again we have a single cut of an ellipsoid. The cut, however, for $\beta < 0$, is somewhat deeper than the central cuts used by the original algorithm.

The depth of the cut is parameterized in the following inequality by β

$$\mathbf{b}'\mathbf{k} < \beta |k_{\max}||\mathbf{b}| \qquad (27)$$

where

$$\beta = -\frac{1}{2}\frac{\sum_i \lambda_i}{|k_{\max}||\mathbf{b}|} \qquad (28)$$

The cut yields a convex body which can be shown to be contained within a minimal volume ellipsoid whose volume is [Groetschel 1988] $V_n k_{\max}^n e^{-(1-n\beta)^2/2n}$. **QED**

Theorem 3 tells us that as the unstable eigenvalues of the system become increasingly positive with respect to the stable eigenvalues, the minimum number of updates required to stabilize the system will increase. This agrees with intuition which suggests that such systems will require larger gain vectors for stabilization.

Initial experiments were performed where the dimension of an unstable system was varied from $n = 2$ to $n = 20$. Random initial conditions were provided and the algorithm was simulated to verify that the system was indeed stabilized. The results are summarized in Figure 1. The results indicate that the algorithm remains effective for large dimension systems.

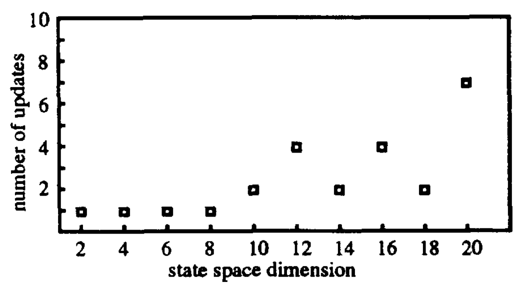

Figure 1: Updates vs. dimension of state space

4 Advantages and Limitations

The theorems of the preceding section are significant for several reasons. First they show that the control gains can be located after a finite number of failed queries. Therefore, the theorem asserts that the system only need perceive itself as "unstable" a finite number of times before system stability can be guaranteed. The second important aspect of this result is that the theorem's bound implies that the convergence time will have polynomial time complexity. This means that as the system becomes more and more complex (i.e. larger state spaces), the time required to learn the system control gains will grow at a modest rate. In other words, query-based adaptation represents a practical method for adaptation and identification of potentially high-dimensional systems. Experimentation showed that the theoretical bounds on the algorithm convergence can be extremely conservative. This is likely due to the way in which the algorithm updates the gain hypothesis. The ellipsoidal search technique is well-known to be an inefficient implementation of the "method of centers". It is to be expected that more efficient interior point algorithms based on logarithmic barrier functions will provide more efficient methods of searching for stabilizing solutions.

It should be noted, however, that the covergence bounds are not with respect to system time, but rather with respect to failed oracle time. This is an important distinction for it is quite possible that there may be a long period of time between consecutive oracle declarations of failure. Consequently, convergence of the proposed algorithm can be extremely long in "system" time and may, in fact, never converge at all. At first glance, this observation may seem to cast doubt upon the value of theorem 1. Upon closer consideration, however, it provides further insight into the method. Recall that the oracle will always declare failures if the Lyapunov inequality is not satisfied. In other words, if the system is exhibiting "unstable" behaviour, the gains will be modified. For the times between failures, the system appears to be stable and there is, therefore, no reason to change the gains. From this viewpoint, it can be seen that the bound of theorem 1 is very meaningful since it is measured with respect to the only quantity of physical interest to the system; the number of times the system "stumbles".

Finally, it must be observed that the preceding algorithm assumes a perfect oracle that never makes an incorrect declaration. In practice, oracles will not be perfect as they will be affected by such things as measurement noise. Such oracles which are not deterministic are called stochastic oracles and theories exist on how to deal with the problems of stochastic oracles in other environments[Nemirovsky 1983]. Methods of dealing with stochastic oracles may simply involve shallower cuts by the update algorithm, possibly defined by the noise variance. This is clearly an area for future research.

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