PERGAMON

Automatica 35 (1999) 491–503

Bounded amplitude performance of switched LPV systems with applications to hybrid systems[1]

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Received 14 October 1997; revised 1 May 1998; received in final form 24 September 1998

Temporal characteristics of a scheduled linear parameter varying system, controlled by switching from a finite collection of setpoint amplitude controllers, are derived and used to construct timed automata models of the switched system.

Abstract

This paper discusses recent results on multiple linear agent control for systems satisfying a bounded amplitude performance constraint. The plant is assumed to be a linear parameter varying (LPV) system scheduled along a nominal parameter trajectory; in this respect, the control problem represents a plant operating between a prespecified set of operating conditions. Linear controllers are designed at setpoints along this scheduling trajectory to satisfy bounded amplitude performance constraints. This paper discusses an approach to analyze the switched system behavior under practical assumptions on the structure of the switching rule. The approach combines the scheduling parameter with LPV system properties to derive bounds on the switching behavior of the system. These estimates are then used to construct a logical model of the switched system behavior in the form of a timed automaton. In this respect, this paper presents a way of extracting logical models of continuous time system behavior.

Keywords: Amplitude response; Automata extraction; Gain scheduling; Lyapunov methods; Parametric variation; Timed automata

1. Introduction

This paper addresses the problem of scheduling a linear parameter varying (LPV) system between a predefined set of operating conditions by *switching* from a set of LTI controllers, called *agents*, so that the controlled system satisfies constraints on signal amplitude. The primary theoretical result of this study is a sufficient condition for bounded amplitude performance expressed as a *dwell time*, a lower bound on the time for which an agent must be kept in feedback with the plant. This dwell time condition is parameterized by solutions to a set of matrix inequalities similar to those defined in Bett and Lemmon (1997) for bounded amplitude control of LPV systems. Analyzing the switched system for the dwell time conditions allows the extraction of a timed logical model of the controlled system. This result has important implications for hybrid systems because it provides a necessary link between continuous state and discrete event representations of the switched system derived from modern linear robust control methodologies.

In the area of scheduled control, this paper provides contributions to the study of switched scheduling under bounded amplitude performance constraints. While the framework established in this paper describes a scheduling problem, the framework is distinct from traditional and modern gain scheduling paradigms (e.g. Apkarian

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[1] This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Guest Editors J. M. Schumacher, A. S. Morse, C. C. Pantelides, and S. Sastry. The authors gratefully acknowledge the partial financial support of the Army Research Office (DAAH04-95-1-0600, DAAH04-96-1-0134, and DAAAG5-98-1-0199).

0005-1098/99/$—see front matter
PII: S0005-1098(98)00180-0

and Gahinet, 1995; Becker and Packard, 1994; Packard 1994; Rugh, 1991; Shamma and Athans, 1990, 1991, 1992), which compute continuous controls even when the design is based on a collection of local controllers. To emphasize the distinction from traditional views, the control technique described in this paper is denoted *switched agent control*. An additional deviation from the gain scheduling literature is the amplitude performance criterion assumed for the paper. Such an objective arises naturally in any application where it is useful to avoid certain regions in the state space, such as robotics or high performance drive positioning. Theoretical and practical results on scheduling control are concerned primarily with $\mathscr{L}_2$ stability and performance; amplitude performance in this context is apparently absent from the literature. The results of this paper indicate that traditional gain scheduling rule demanding slow scheduling is a key concern in switched agent problem when signal amplitude constraints are imposed; this rule manifests itself as the dwell time condition described above.

Because the switched agent controlled system generates a mixture of continuous and discrete-valued signals, it can be seen as a *hybrid system* (Branicky et al., 1998; Stiver et al., 1998). Such systems can be studied as either switched, continuous state systems or as timed, discrete event systems. The continuous state approach offers more insight into the structure of the controlled dynamical system, but analysis and synthesis problems often encounter computational intractability. Examples of this approach can be seen in Johansson, and Rantzer (1998), Morse (1996), Polla and Shamma (1995). The concept of dwell time used in this paper is directly related to that defined in Morse (1996) and alluded to in Polla and Shamma (1995). In the discrete event approach, the controlled system is abstracted into a timed logical model. This abstraction allows one to study distinctly different types of systems in a common domain and may then be used to verify whether or not the supervised system meets timing specifications expressed as temporal logic formulae. In some cases, this approach can offer enormous computational advantages for verifying stability and performance. A drawback is that the abstraction process often ignores the inherent structure of the underlying continuous state dynamical system, often neglecting important questions of stability and performance in the continuous domain and necessitating additional computation and analysis. Examples of this approach are seen in Stiver et al. (1996), where a continuous dynamical system is abstracted into an untimed finite automata, and in Deshpande and Varaiya (1995), where the continuous dynamics are modeled piecewise as simple linear differential inclusions and then abstracted to a timed automaton (Alur and Dill, 1994).

The development of a useful and efficient design methodology for hybrid system design and analysis requires the integration of these distinctly different viewpoints. This paper contributes to such an integration by demonstrating that performance analysis of the switched, continuous state system provides the information required to construct a timed, discrete event representation of the controlled system in the form of a timed automaton. The approach used here is loosely analogous to Deshpande and Varaiya (1995) but the LPV framework used here offers a richer structure and allows the abstraction to be based on guaranteed robustness properties of the system which are computed using modern linear robust control methodologies. Thus, the automata extraction procedure presented here explicitly accounts for continuous state performance constraints.

The remainder of this paper is organized as follows. Section 2 describes the problem setup, including assumptions on the plant and control agents. Section 3 states sufficient conditions for the bounded amplitude performance of a continuous time system whose dynamics switch between two different LPV realizations. These conditions include the aforementioned dwell time condition and provide a basis for the analysis presented in Section 4. Section 4 contains conditions for estimating the switching times of the scheduled system which can then be compared to the dwell time condition. Section 5 demonstrates, by example, how the analysis results can be used to extract timed automata models of a switched dynamical system. Both finite-time and periodic scheduling examples are provided. All proofs are located in Appendix A.

Notation. Throughout the paper, $\|\cdot\|$ denotes the Euclidean vector norm. For a finite constant $T > 0$, the finite-horizon infinity norm of a signal $f: \mathbb{R}^+ \to \mathbb{R}^n$ is defined as $\|f\|_{\infty,[0,T]} := \operatorname{ess\,sup}_{t\in[0,T]}\|f(t)\|$. The linear space $\mathscr{L}^n_\infty[0,T] := \{f: \mathbb{R}^+ \to \mathbb{R}^n \mid \|f\|_{\infty,[0,T]} < \infty\}$ denotes all finite-horizon infinity norm bounded signals; the set of all such signals bounded to the unit sphere is denoted $\mathscr{B}\mathscr{L}^n_\infty[0,T]$. These notions generalize to the infinite horizon case in the obvious way. A *parameter variation set*, $\mathscr{F}_{\tilde{\Theta}}$, consists of all piecewise continuous mappings taking $\mathbb{R}^+$ into a compact $\Theta \subset \mathbb{R}^s$. For a finite $T > 0$, the set $\mathscr{F}_\Theta[0,T]$ denotes the set of all piecewise continuous mappings taking $[0,T]$ into a compact $\Theta \subseteq \tilde{\Theta}$. The notation $\theta \in \mathscr{F}_\Theta$ denotes a function in the parameter variation set; $\theta \in \tilde{\Theta}$ denotes a vector in a compact subset of $\mathbb{R}^s$. A linear parameter varying (LPV) system is a dynamical system whose dynamics evolve as

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B(\theta(t)) \\ C(\theta(t)) & D(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

where $\theta \in \mathscr{F}_\Theta$ and where the matrix mappings are continuous operators on $\mathbb{R}^s$. Finally, a positive definite matrix $P \in \mathbb{R}^{n\times n}$ which satisfies the matrix inequality

$$A'P + PA + \alpha_1 P + \frac{1}{\alpha_2} PB'BP \le 0, \quad \gamma^2 P \ge C'C$$

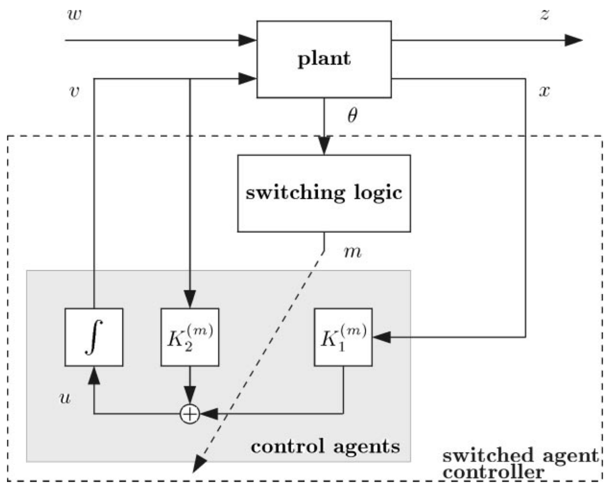


Fig. 1. Control system architecture.

for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_z\times n}$ and scalars α_1, α_2 and γ is denoted $P \in \mathcal{MI}(A, B, C, \alpha_1, \alpha_2, \gamma)$.

2. Problem description

In this section, the control system architecture is described and the performance objective for the controlled system is stated. The components of the system architecture are depicted in Fig. 1.

2.1. Plant dynamics

The plant processes considered in this paper are assumed to take the form of an LPV system

$$\dot{x}(t) = A_1(\theta(t))x(t) + B_1(\theta(t))w(t) + B_2(\theta(t))v(t), \quad (1)$$

$$z(t) = C_1x(t) + D_{12}v(t), \quad (2)$$

with $\theta \in \mathcal{F}_{\tilde{\Theta}}$ where $\tilde{\Theta}$ is some compact subset of $\mathbb{R}^s$. Here, $x \in \mathbb{R}^n$ is the plant state, $v \in \mathbb{R}^{n_v}$ is the control input, $w \in \mathbb{R}^{n_w}$ represents bounded, exogenous disturbances and $z \in \mathbb{R}^{n_z}$ represents plant performance. It is assumed that the disturbance vector $w \in \mathcal{BL}_\infty^{n_w}$.

The parameter vector θ is assumed to be defined by a *parameter mapping*, $\theta(t) = S(x_m(t), x(t), v(t))$; the argument $x_m \in \mathbb{R}^n$ represents an exogenous reference or scheduling variable. The evolution of $\theta(t)$ in $\tilde{\Theta} \subset \mathbb{R}^s$ is called the *parameter trajectory*. A *known* parameter trajectory, called the *nominal parameter trajectory*, is assumed to exist and to represent ideal performance. The nominal parameter trajectory is defined as

$$\theta_{\text{nom}}(t) := S(x_m(t), 0, v_m(t)) \quad (3)$$

The argument $v_m(t)$ represents a nominal control input to the plant. In some cases, such a function may be derived analytically from knowledge of the plant dynamics and the control objectives, e.g. a feedback linearizing control. (Implementation of the feedback linearizing control may not be desirable for robustness reasons or if hardware constraints do not permit.) In other cases, $v_m(t)$ may be found by computing local solutions to linearized problems and interpolating the results.

The parameter mapping, S, is assumed continuous, available for measurement and bounded so that small deviations in $\|z\|$ correspond to small deviations in $\theta(t)$ from $\theta_{\text{nom}}(t)$. Specifically, for points x and v corresponding to small $\|z\| = \|C_1x + D_{12}v\|$,

$$|S_i(x_m, x, v) - S_i(x_m, 0, v_m)| \le k_{zi}\|z\| \quad (4)$$

for known, nonnegative constants k_{zi}, $i = 1, \ldots, s$.

Remark. The framework just described captures the LPV frameworks presented in the literature. When the parameter mapping is purely a function of exogenous parameters, x_m, which are available for measurement, then the system describes a class of LPV gain scheduling problems (see, e.g. Apkarian and Gahinet, 1995; Shamma and Athans, 1991, 1992). Similarly, when θ depends on the system state, the LPV systems which result are sometimes referred to as *quasi-LPV* systems. (Such systems have been studied extensively in, for example, Shamma, and Athans (1990)). LPV (or quasi-LPV) systems arise in nonlinear model reference control problems where ideal performance is measured in terms of a dynamical reference model with state, x_m, and state error, x. A parameter mapping arises from grouping state and control dependent terms in the coefficient matrices of the system.

2.2. Performance objective

Attention in this paper is restricted to *bounded amplitude performance problems*. Two types of amplitude performance problems will be considered. Let T and γ be a fixed positive constants.

2.2.1. Finite-time scheduling

For the LPV system in Eqs. (1) and (2), the finite-time bounded amplitude performance objective is to ensure that, given $\|z(0)\| \le \gamma$,

$$\sup_{w \in \mathcal{BL}_\infty^{n_w}[0,T]} \|z(t)\| \le \gamma. \quad (5)$$

2.2.2. Periodic scheduling

For the LPV system in Eqs. (1) and (2) with a periodic nominal parameter trajectory,

$$\theta_{\text{nom}}(t) = \theta_{\text{nom}}(t + T),$$

the bounded amplitude performance objective in this case is to ensure that, given $\|z(0)\| \le \gamma$,

$$\sup_{w \in \mathcal{BL}_\infty^{n_w}} \|z(t)\| \le \gamma. \quad (6)$$

Both objectives represent important classes of performance problems. The first represents tasks which reach completion in a finite time; the second represents cyclic processes which run continually. The purpose of this paper is to analyze these performance problems for the class of plant processes just described under the control of a class of switched agent controllers described next.

2.3. Switched agent controller structure

There are two primary components to the switched agent controller depicted in Fig. 1: control agents and switching logic. These are now described.

2.3.1. Control agents

Consider a sequence of times, $\{t_i\}$ indexed by $i \in \mathscr{I}_{\mathscr{K}} = \{1, \dots, M\}$. The parameter vectors obtained by sampling the nominal parameter trajectory at times t_i for $i \in \mathscr{I}_{\mathscr{K}}$ form a finite collection of *design points*, $\{\theta_{\text{nom}}(t_i)\}$. The ith design point will be denoted $\theta^{(i)}_{\text{nom}}$. With each $i \in \mathscr{I}_{\mathscr{K}}$, associate a *control agent* designed so that the LPV system in Eqs. (1) and (2) demonstrated amplitude performance in the face of parameter variations about $\theta^{(i)}_{\text{nom}}$. The ith control agent is represented by the system

$$\dot{v} = u; \qquad u = K^{(i)}_1 x + K^{(i)}_2 v$$

where $K^{(i)}_1$ and $K^{(i)}_2$ are constant gain matrices of appropriate dimensions. The collection of control agents, for $i \in \mathscr{I}_{\mathscr{K}}$, is denoted $\mathscr{K}$.

The integrator in the controller is used to eliminate discontinuities in the control input signal to the plant when a switch between control agents occurs. For analysis and synthesis purposes, the integrator will be incorporated into the plant. The modified LPV plant is given by

$$\dot{\hat{x}}(t) = A(\theta(t))\hat{x}(t) + B_w(\theta(t))w(t) + B_u u(t), \tag{7}$$

$$z(t) = C\hat{x}(t), \tag{8}$$

where

$$A(\theta(t)) = \begin{bmatrix} A_1(\theta(t)) & B_2(\theta(t)) \\ 0 & 0 \end{bmatrix},$$

$$B_w(\theta(t)) = \begin{bmatrix} B_1(\theta(t)) \\ 0 \end{bmatrix}, \tag{9}$$

$$B_u = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad C = [C_1 \quad D_{12}]$$

and $\hat{x} = [x' \ \ v']$. The modified plant is now seen to have a control agent $K^{(i)} = [K^{(i)}_1 \ \ K^{(i)}_2]$ so that $u = K^{(i)}\hat{x}$.

2.3.2. Switching logic

The switching logic is the set of rules which define how the control agents are switched into feedback with the plant process. In this paper, the switching logic is assumed to have two components called the *switching sets* and the *nearest neighbor switching rule*.

2.3.3. Switching sets

Associate with each element of $\mathscr{K}$ a compact subset $\Theta_i \subset \tilde{\Theta}$, called a *switching set*, defined by

$$\Theta_i := \left\{ \theta \;\middle|\; \max_j \left| \theta_j - \theta^{(i)}_{\text{nom},j} \right| \le \vartheta_{\text{out}} \right\} \tag{10}$$

where the subscript, j, denotes the jth vector component and ϑ_{out} is a positive constant called the *switching parameter*. The collection of switching sets is denoted $C = \{\Theta_i\}$.

2.3.4. Nearest-neighbor switching rule

Switching between the different control agents in $\mathscr{K}$ is controlled by the parameter vector, θ. Attention in this paper is focused on a *nearest neighbor switching rule*. Suppose that control agent $\mathscr{K}^{(i)}$ is in the feedback loop at time t_0 and assume that $\theta(t_0) \in \Theta_i$. Then the control agent $\mathscr{K}^{(i)}$ will remain in the feedback loop until the earliest time t_s when the parameter trajectory $\theta(t)$ satisfies

$$\max_j |\theta_j(t_s) - \theta^{(i)}_{\text{nom},j}| = \vartheta_{\text{out}}.$$

At time t_s, the control agent $K^{(m)}$ is then switched into the feedback loop where

$$m = \arg\min_{j \in \mathscr{I}_{\mathscr{K}}} \|\theta(t_s) - \theta^{(j)}_{\text{nom}}\|.$$

As described, the switched agent controller switches state feedback controllers into and out of the feedback loop on the basis of the LPV system's current parameter vector. A parameter trajectory $\theta(t)$ will be said to be *legal* if and only if it is continuous (except possibly at switching instants) and $\theta(t) \in \Theta_i$ for all $t \in \{\tau: u(\tau) = K^{(i)}_1 x(\tau) + K^{(i)}_2 v(\tau)\}$. A legal parameter trajectory is denoted $\theta \in \mathscr{F}_{\mathscr{C}}$.

2.3.5. Adequate sampling assumption

From the preceding discussion it is clear a given control agent, $K^{(i)}$, is switched out of the feedback loop when the parameter $\theta(t)$ leaves the switching set Θ_i. The nearest neighbor switching rule requires that the resulting switch will be to the controller, $K^{(j)}$, whose associated design point $\theta^{(j)}_{\text{nom}}$ is closest (with respect to the Euclidean vector norm) to the parameter $\theta(t_s)$ at switching time t_s. To help ensure that $\theta(t_s^+) \in \Theta_j$, it is assumed that for all t there exists $l \in \mathscr{I}_{\mathscr{K}}$ such that

$$|\theta_{\text{nom},j}(t) - \theta^{(l)}_{\text{nom},j}| \le \vartheta_{s,j} \tag{11}$$

where

$$\vartheta_{s,j} \le \vartheta_{\text{out}} - k_{zj}\gamma \tag{12}$$

for $j = 1, \dots, s$. In other words, this is an assumption that $\theta_{\text{nom}}(t)$ has been sampled "adequately" and will be called the *adequate sampling assumption*.

3. Performance of switched LPV systems

Consider a time interval $[0, T]$ with $0 < t_s < T$ and a system that switches between two LPV realizations at time t_s. Associate parameter sets, Θ_1 and Θ_2, with the first and second realizations, respectively. Under appropriate conditions, one may find Bett (1998), Bett and Lemmon (1997) ellipsoidal sets $\mathscr{E}^{(i)}$, $i = 1, 2$, with $\|z\| \le \gamma$ for any $x \in \mathscr{E}^{(i)}$ which are invariant for $\theta \in \mathscr{F}_{\Theta_1}$ and $\theta \in \mathscr{F}_{\Theta_2}$, respectively. These conditions imply that amplitude performance is guaranteed over the interval $[0, T]$ if (1) $x(0) \in \mathscr{E}^{(1)}$, $\theta \in \mathscr{F}_{\Theta_1}[0, t_s]$, (2) $x(t_s) \in \mathscr{E}^{(2)}$ and (3) $\theta \in \mathscr{F}_{\Theta_2}[t_s, T]$.

Assuming that the first condition is satisfied, the second is assured if the state decays to an ellipsoid $r\mathscr{E}^{(1)} \subseteq \mathscr{E}^{(2)}$ for $r \in (0, 1]$. From Theorem 1 of Bett and Lemmon (1997), this decay can be bounded exponentially by solving an appropriately defined matrix inequality. The time, t_d, for the exponential decay to occur represents a lower bound on the switching time, t_s, sufficient for switched system performance and is called a *dwell time* constraint. The dwell time constraint, $t_s > t_d$, guarantees that the state error has had sufficient time to decay so that any transient associated with the switch will not violate performance constraints. This result is formally stated in the following proposition, called the *LPV switching lemma*, which provides the basis for analyzing switched system behavior in this paper.

Proposition 1 (LPV switching lemma). *Consider finite constants $r \in (0, 1]$ and $\gamma > 0$, compact sets $\Theta_1, \Theta_2 \subset \tilde{\Theta}$, continuous matrix mappings $A_i: \mathbb{R}^s \to \mathbb{R}^{n \times n}$, $B_i: \mathbb{R}^s \to \mathbb{R}^{n \times n_u}$ and $C_i: \mathbb{R}^s \to \mathbb{R}^{n_z \times n}$ for $i = 1, 2$. Let $C = \{\Theta_1, \Theta_2\}$. Suppose there exist constants $\alpha > 0$, $\beta > 0$ and $\rho > 0$ and positive definite matrices P_1 and P_2 such that*

$$rP_2 \le P_1 P_1, \tag{13}$$

$$P_1 \in \mathscr{MI}\left(A_1(\theta), B_1(\theta), C_1(\theta), 2\beta + \frac{\alpha}{r}, \alpha, \gamma\right) \quad \forall \theta \in \Theta_1, \tag{14}$$

$$P_2 \in \mathscr{MI}(A_2(\theta), B_2(\theta), C_2(\theta), \rho, \rho, \gamma) \quad \forall \theta \in \Theta_2. \tag{15}$$

Let w, x, and z be the input, state, and output, respectively, of the dynamical system

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} A_1(\theta(t) & B_1(\theta(t)) \\ C_1(\theta(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, & t \in [0, t_s), \\ \begin{bmatrix} A_2(\theta(t)) & B_2(\theta(t)) \\ C_2(\theta(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, & t \in (t_s, T], \end{cases} \tag{16}$$

where $t_s > 0$, $T \in (t_s, \infty)$. If $x'(0) P_1 x(0) \le 1$ and $w \in \mathscr{BL}_\infty^{n_w}$, then for any switching time satisfying

$$t_s > t_d := -\frac{1}{2\beta} \log r \tag{17}$$

with parameter trajectory $\theta(t) \in \mathscr{F}_C$ satisfying $\theta(0) \in \Theta_1$ and $\theta(t_s^+) \in \Theta_2$,

$$\|z\|_{\infty, [0, T]} \le \gamma.$$

Note that because of the integrator in the controller structure, the control input to the plant is continuous. Since the state is also continuous, no discontinuities in the parameter trajectory will occur. Thus, under the nearest neighbor switching rule, the adequate sampling assumption ensures that the parameter variation condition of the lemma is satisfied for the switched systems considered in this paper. The remaining conditions which must be analyzed for the switched system concern the initial condition and dwell time constraints.

Remark. The initial condition and dwell-time constraints of the lemma are directly related to the concept of uniform ultimate boundedness (Corless, 1981). The *LPV switching lemma* possesses the parameter variation condition which is key to the switching behavior considered in this paper. In particular, the conditions of the lemma are special cases of Lemma 5.2 in Khalil (1996). The parameter variation condition may be seen as enforcing the perturbation bound of that result.

4. Switched system performance analysis

The results of this section show that if the parameter variations are "nice" (i.e. if the constants k_{zi} in Eq. (4) are small enough), then $|\theta(t) - \theta_{\text{nom}}(t)|$ is bounded by a known quantity. The result is important because, under the assumption of adequate sampling of $\theta_{\text{nom}}(t)$, it allows one to bound possible times of control agent switches using the nominal parameter trajectory.

4.1. Parameter deviations

The following result bounds the deviation between the nominal parameter trajectory and actual parameter trajectory.

Lemma 2. *Given performance level $\gamma > 0$ and the modified LPV system of Eqs.* (7) *and* (8), *suppose that the control input to the system is given by $u(t) = K\hat{x}(t)$ where K is a constant gain matrix. Let $\bar{\theta}$ be a point on the nominal parameter trajectory and let Θ be any compact subset of $\tilde{\Theta}$ containing $\bar{\theta}$. Suppose that there exists a positive definite matrix P and constants $\alpha > 0$, $\beta \ge 0$,* and $0 < r \le 1$ *such that*

$$P \in \mathscr{MI}\left(A(\theta) + B_u K, B_w(\theta), C, 2\beta + \frac{\alpha}{r}, \alpha, \gamma\right) \quad \forall \theta \in \Theta. \tag{18}$$

For any $T > 0$, *if* $\hat{x}'(0)\ P\hat{x}(0) \le 1$ *and* $w \in \mathscr{BL}_\infty^{n_w}[0, T]$, *then any parameter trajectory* $\theta \in F_\Theta[0, T]$ *satisfies*

$$|\theta_i(t) - \theta_{\mathrm{nom},i}(t)| \le k_{zi}\gamma \max\{\sqrt{r}, \mathrm{e}^{-\beta t}\} \tag{19}$$

for $i = 1, 2, \dots, s$ *and for all* $t \in [0, T]$. *Furthermore, if* $\Theta \subset \tilde{\Theta}$ *is a switching set and*

$$\max_{1 \le i \le s} |\theta_i(T) - \bar{\theta}_i| = \vartheta_{\mathrm{out}}, \tag{20}$$

then the nominal parameter trajectory at time T *satisfies*

$$\vartheta_{\mathrm{out}} - k_{zi}\gamma \max\{\sqrt{r}, \mathrm{e}^{-\beta T}\} \le |\theta_{\mathrm{nom},i}(T) - \bar{\theta}_i| \le \vartheta_{\mathrm{out}} + k_{zi}\gamma \max\{\sqrt{r}, \mathrm{e}^{-\beta T}\} \tag{21}$$

for $i = 1, 2, \dots, s$.

The implication of Lemma 2 is that if the controllers are appropriately designed (so that Eq. (18) is satisfied), $\|z(t)\|$ will be bounded and therefore the parameter trajectory must remain bounded to the nominal parameter trajectory from Eq. (4). This bound tightens while the control agent remains in the feedback loop. This reinforces the intuitive notion that parameter deviation from the nominal parameter trajectory represents modeling error.

Lemma 2 is important because it implies that if the bound on the parameter trajectory is small enough, then $\theta_{\mathrm{nom}}(t)$ represents a reasonable approximation to $\theta(t)$. This further implies that $\theta_{\mathrm{nom}}(t)$ may be used to bound switching times in the multiple agent control system. The time at which $\theta(t)$ intersects the boundary of Θ may be approximated by the times which $\theta_{\mathrm{nom}}(t)$ evolves over points near the boundary of Θ. This is illustrated in Fig. 2; when the conditions of the lemma are satisfied and if $\theta(T)$ lies on the boundary of Θ, then $\theta_{\mathrm{nom}}(T)$ must lie in the shaded region representing the bounds of Eq. (17).

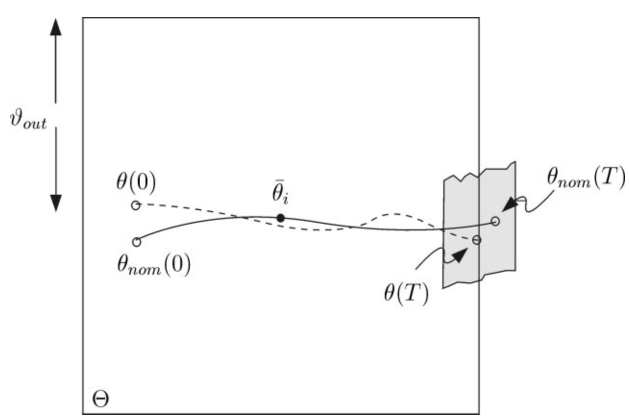


Fig. 2. Estimation of switching times using $\theta_{\mathrm{nom}}(t)$. The shaded region corresponds to bounds from Eq. (21).

Remark. For quasi-LPV systems in any setting, achieved performance objectives must guarantee $\theta \in \mathscr{F}_\Theta$ since θ is state dependant. Failure to satisfy this additional constraint implies existence of a state trajectory which violates $\theta \in \mathscr{F}_\Theta$, hence robust stability/performance cannot be guaranteed.

4.2. Switching time estimation

Lemma 2, combined with the nearest neighbor switching rule, can be used to estimate switching times and the results (destinations) of possible switches. Let

$$k_{\max,i}(t) := k_{zi}\gamma \max\{\sqrt{r}, \mathrm{e}^{-\beta t}\} \tag{22}$$

denote the parameter deviation bound from Lemma 2.

4.2.1. Switching destinations

For $m \in \mathscr{I}_\mathscr{K}$, define the sets

$$\hat{\Theta}_m := \{\theta | \theta \in \hat{\Theta},\ \|\theta - \theta_{\mathrm{nom}}^{(m)}\| \le \|\theta_i - \theta_{\mathrm{nom}}^{(q)}\|,\ q \in \mathscr{I}_\mathscr{K},\ q \ne m\}.$$

The set $\hat{\Theta}_m$ represents the set of all parameter vectors $\theta \in \Theta$ which satisfy

$$m = \arg\min_{j \in \mathscr{I}_\mathscr{K}} \|\theta - \theta_{\mathrm{nom}}^{(j)}\|;$$

if the parameter trajectory at time t_s lies in $\hat{\Theta}_m$, a switch to control agent m will take place according to the nearest neighbor switching rule. The sets $\hat{\Theta}_m$ may be represented by a set of affine inequality constraints on the parameter vector θ. Note that these constraints may be computed in an off-line fashion with knowledge of the design points.

As a consequence of Lemma 2, the times for which $\theta(t) \in \hat{\Theta}_m$ can only be estimated. Define

$$\hat{\Theta}_t := \{\theta \,|\, |\theta_{\mathrm{nom},i}(t) - \theta| \le k_{\max,i}(t) \text{ for all } i = 1, \dots, s\}.$$

For each time t, the set $\hat{\Theta}_t$ is a hyper-rectangle centered at $\theta_{\mathrm{nom}}(t)$ which contains the true parameter trajectory $\theta(t)$. A requirement for a time t to be an estimate of a switching time between control agents $K^{(l)}$ and $K^{(m)}$ is for the intersection $\hat{\Theta}_t \cap \hat{\Theta}_m$ to be nonempty, which is a relatively simple convex feasibility problem.

4.2.2. Switching times

As a consequence of Lemma 2, the times for which the nominal parameter trajectory satisfies

$$\vartheta_{\mathrm{out}} - k_{\max,i}(t) \le |\theta_{\mathrm{nom},i}(t_s) - \theta_{\mathrm{nom},i}^{(l)}| \le \vartheta_{\mathrm{out}} + k_{\max,i}(t)$$

for some $i = 1, \dots, s$, bound the times for which the true parameter trajectory may intersect the boundary of the parameter set Θ_l. This condition represents an additional restriction for a time t to be a possible switching time given that control agent $K^{(l)}$ is in the feedback loop. Note that this requirement may also be expressed with a set of affine inequality constraints on the parameter vector θ which may be computed in an off-line fashion.

Proposition 3. *Given a performance level $\gamma > 0$ and the modified LPV system of Eqs.* (7) *and* (8), *let $\mathscr{K}$ be a set of control agents which forms a control policy with the nearest neighbor switching rule under adequate sampling. Suppose that at time $t = 0$, the control input to the system is given by $u(0) = K^{(l)}\hat{x}(0)$ where $K^{(l)} \in K$ is a constant gain matrix. Let Θ_l be a switching set and suppose there exist positive definite matrix P and constant $\alpha > 0$, $\beta \geq 0$, and $0 < r \leq 1$ such that*

$$P \in \mathscr{MI}\left(A(\theta) + B_u K^{(l)}, B_w(\theta), C, 2\beta + \frac{\alpha}{r}, \alpha, \gamma\right) \quad \forall \theta \in \Theta_l.$$

Define the sets

$$T^{(l,m)} :=$$

$$\left\{ t \,\middle|\, \begin{array}{l} \hat{\Theta}_t \cap \check{\Theta}_m \neq \emptyset, \text{ and for some } 1 \leq i \leq s, \\ \vartheta_{\text{out}} - k_{\max,i}(t) \leq |\theta_{\text{nom},i}(t) - \theta^{(l)}_{\text{nom},i}| \leq \vartheta_{\text{out}} + k_{\max,i}(t) \end{array} \right\}$$

If $\hat{x}'(0)\, P\hat{x}(0) \leq 1$, $w \in \mathscr{BL}_\infty$, and a parameter trajectory θ is generated by the nearest neighbor switching rule under adequate sampling, then the switch time, t_s, between the lth and mth systems satisfies $t_s \in \mathscr{T}^{(l,m)}$.

Given the preceding descriptions, the set $\mathscr{T}^{(l,m)}$ is constructed by finding all times for which the nominal parameter trajectory lies near the switching surface and for which the corresponding set $\hat{\Theta}_t$ has a nonempty intersection with $\hat{\Theta}_m$. Constructing a set $\mathscr{T}^{(l,m)}$ amounts, essentially, to conducting a line search over the nominal parameter trajectory, evaluating a set of convex constraints at each point. Note that the computation is performed off-line; time and computational resources are not a significant issue.

It is apparent from the construction of the switching-time sets, $\mathscr{T}^{(l,m)}$, and from the nearest neighbor switching rule that certain switches will never take place. In fact, given that control agent $\mathscr{K}^{(l)}$ is currently in feedback with the system, under the assumption of adequate sampling, then a switch to control agent $K^{(m)}$ should take place if and only if $m \in \mathscr{I}_l$ where

$$\mathscr{I}_l := \{m \mid m \neq l, \Theta_l \cap \hat{\Theta}_m \neq \emptyset\}. \tag{23}$$

Such a switch from agent $K^{(l)}$ to agent $K^{(m)}$ will be called an *admissible switch.*

4.3. Stability and performance results

In this section, the switching-time sets, $\mathscr{T}^{(l,m)}$, of Proposition 3 are combined with the *LPV switching lemma* to state conditions for bounded amplitude performance of the switched agent controlled system. The conditions are expressed in the following proposition.

Proposition 4. *Given a performance level $\gamma > 0$ and modified LPV system of Eqs.* (7) *and* (8), *let $\mathscr{K}$ be a set of control agents which form a control policy with the nearest neighbor switching rule under the assumption of adequate sampling. Suppose that for each $l \in \mathscr{I}_{\mathscr{K}}$, there exists a positive definite matrix $P^{(l)}$ and constants $\alpha_1^{(l)} \geq \alpha_2^{(l)} > 0$ such that*

$$P^{(l)} \in \mathscr{MI}(A(\theta) + B_u K^{(l)}, B_w(\theta), C, \alpha_1^{(l)}, \alpha_2^{(l)}, \gamma), \quad \forall \theta \in \Theta_l. \tag{24}$$

Denote the agent initially in the feedback loop by $K^{(0)}$. If all possible switches are admissible, $\hat{x}'(0)P^{(0)}\hat{x}(0) \leq 1$ and if, for all admissible switching sequences $k \to l \to m$,

$$\frac{r^{(l,m)}}{\alpha_2^{(l)} - \alpha_1^{(l)} r^{(l,m)}} \log r^{(l,m)} \leq \min T^{(l,m)} - \max T^{(k,l)} \tag{25}$$

where

$$r^{(l,m)} := \max\left\{ r \,|\, rP^{(m)} \leq P^{(l)}, \frac{\alpha_2^{(l)}}{\alpha_1^{(l)}} < r \leq 1 \right\} \tag{26}$$

then for any $T \in [0, \infty)$,

$$\|z\|_{\infty,[0,T]} \leq \gamma \tag{27}$$

and for $T \to \infty$,

$$\|z\|_{\infty,[0,\infty)} \leq \gamma. \tag{28}$$

Proposition 4 is interpreted as follows. Denote

$$R^{(l,m)} := \left\{ r \,|\, rP^{(m)} \leq P^{(l)}, \frac{\alpha_2^{(l)}}{\alpha_1^{(l)}} < r \leq 1. \right\}. \tag{29}$$

The set $\mathscr{R}^{(l,m)}$ represents the set of all possible constants r which yield a positive dwell time, as defined in the LPV switching lemma, for a switch between control agents $K^{(l)}$ and agent $K^{(m)}$. Similarly, the quantity

$$\min \mathscr{T}^{(l,m)} - \max \mathscr{T}^{(k,l)}$$

represents the minimum time that can elapse between switches $k \to l$ and $l \to m$. Eq. (25) therefore represents the dwell time constraint of the LPV switching lemma verified using the switching-time estimates obtained from Proposition 3. Thus, after applying the results of Proposition 3 to obtain the switching sets $\mathscr{T}^{(l,m)}$, Proposition 4 may be applied to analyze bounded amplitude performance in the finite-time and periodic scheduling problems stated in Section 2.2. (Note that Proposition 4 applies to nonperiodic $\theta_{\text{nom}}(t)$ defined for $t \in [0, \infty)$ as well.)

5. Applications to hybrid systems

The switched agent control system is a *hybrid system* because it generates a mixture of discrete event and continuous valued signals. As previously noted, hybrid systems can be studied from two distinct viewpoints; as a supervised collection of real-time computer processes or as switched dynamical systems. While performance of a given switched dynamical system may best be accomplished in the continuous domain, integrating the controlled system with other systems may best be accomplished on the discrete supervisory level. Integration of these two viewpoints is therefore required to develop useful and efficient methodologies for hybrid system design. In this section, the results of the switched system performance analysis (conducted using a continuous time control theoretic approach) are used to construct a timed automaton model of the controlled system. This *automaton extraction* provides part of the link between the the discrete and continuous approaches to analysis and design of hybrid systems. The extraction procedure is demonstrated by example for finite-time and periodic scheduling problems. The remainder of the section begins with a brief description of the timed automata model used in this paper. For more information on automata models and applications, the reader is referred to Alur et al. (1993); Alur and Dill (1994); Burch et al. (1992); Clarke and Emerson (1981) and McMillan (1992).

5.1. Timed automata extraction

A *timed automaton* is defined by the tuple, $(\mathcal{N}, \mathcal{X}, \ell_f, \ell_r, \ell_v)$. $\mathcal{N} = (\mathcal{V}, \mathcal{A}, \bar{\mu}_0)$ is a finite automaton with a finite set of $\mathcal{M}_{\mathcal{V}}$ vertices, $\mathcal{V}$, a set of directed arcs, $\mathcal{A}$, between the vertices and an initial marking vector, $\boldsymbol{\mu}_0$, representing the initial state of the finite automaton. X is a finite set of *local clocks*. The ith clock is characterized by the ordered triple $\mathcal{X}_i = (c_i, x_{i0}, \tau_{i0})$ where $x_{i0} \in \mathbb{R}^n$, $\tau_{i0} \in \mathbb{R}$ and $c_i \in \mathbb{R}^n$. The *local time* of the ith clock, $x_i(\tau)$ $(\tau \geq \tau_{i0})$ generated by clock $\mathcal{X}_i$ is the solution to the initial value problem

$$\dot{x}_i(t) = c_i; \qquad x_i(\tau_{i0}) = x_{i0}. \tag{30}$$

The set of all local times and clock rates at time τ will be called the *clock state* and will be denoted

$$\bar{x}(\tau) = \{(x_i(\tau), c_i)\}_{i=1,2,\ldots,\mathcal{M}_{\mathcal{V}}}. \tag{31}$$

The arcs are labeled with constraints on the local clocks which must be satisfied in order for enabled transitions to occur. When satisfied, the *firing condition*, ℓ_f, means that the arc is free to fire provided that it is already enabled. The *vertex constraint*, ℓ_r, forces the certain local clock states to satisfy an equality constraint. The *reset constraint*, ℓ_v, represents an equality constraint which the clock state is reset to immediately after the firing of a given arc.

The preceding definition of timed automata is essentially the same as that used in Alur and Dill (1994). The description provided above, however, follows notational conventions found in the Petri net literature and appears to be more closely related to the control theoretic approach described earlier. The mappings ℓ_f, ℓ_r and ℓ_v all represent constraints on the clock states which must be satisfied for transitions to occur. In the examples that follow, the vertex constraint is simple because the clocks do not change. In a more general setting, clock constraints can change as the system evolves. The clocks themselves can also be defined as the solution of a more general set of differential equations; the timed automaton is then referred to as a *hybrid automaton* (Alur et al., 1993). In a model reference control problem, one may view the reference model as a clock. The vertex constraint then corresponds to changing the reference model.

Now consider the application of Proposition 4 to the multiple agent system. To apply the proposition, one needs to construct the switch-time sets, $\mathcal{T}^{(l,m)}$, and verify that the dwell time constraints in Eq. (25) are satisfied. If these sets can be constructed, one may next construct a sequential model of the switching behavior with a tree structure. The nodes of the tree correspond to a control agent being switched into feedback with the plant. The nodes can be collected into levels. Each level, $l_s \in \{0, \ldots, N\}$, of the tree contains possible states of the system after l_s switches have taken place, i.e. each level contains the indices of control agents in the loop after l_s switches have taken place. In turn, a state l in level l_s is connected to a state m in level $l_s + 1$ by an arc labeled with the time interval over which the switch could possibly take place, $\mathcal{T}^{(l,m)}$. Because this model is constructed while ensuring that the bounded amplitude performance constraint is satisfied, this model is called a *performance validation tree*. The performance validation tree is now used to construct a timed automaton model for the switched agent system. Suppose that a control agent, $K^{(l)}$, in feedback with the plant is seen as a *state* of the controlled system. Then, assuming that a finite number of switches occurs over a finite interval, $[0, T]$, a finite sequence of states will be reached by the controlled plant during $[0, T]$. The performance validation tree described above represents all possible finite sequences of states which can be assumed by the closed-loop system over $[0, T]$. In other words, the performance validation tree represents all possible trajectories of a timed automaton model of the multiple agent controlled system.

5.2. Finite time scheduling

We now turn to numerical examples illustrating the results presented above. The plant chosen for the purposes of illustration is a second-order nonlinear system given by

$$\dot{x}_{p1} = -x_{p1} + v_1, \tag{32}$$

$$\dot{x}_{p2} = -x_{p2} + (1 + x_{p1}^2)v_2. \tag{33}$$

The finite time scheduling objective considered here is to move the state of the plant from points near $x_p = (2.5, 2)$

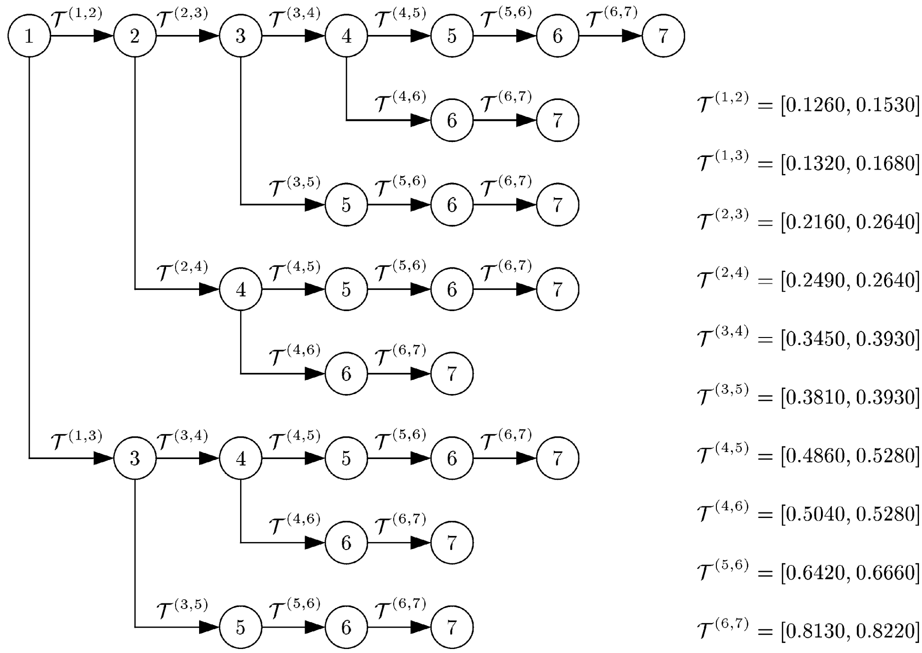


Fig. 3. Performance validation tree for finite time example: $\vartheta_{\text{out}} = 0.4$, $\gamma = 0.068$.

to points near $x_p = (1, 3)$ in one second ($T = 1$) according to the reference model

$$\dot{x}_{m1} = -1.5, \tag{34}$$

$$\dot{x}_{m2} = 1. \tag{35}$$

By defining $x := x_m - x_p$ and choosing a parameter mapping

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{21} x_{p1}^2 - \frac{29}{21} \\ \frac{4}{3} x_{m1} - \frac{7}{3} \\ 2x_{m2} - 5 \end{bmatrix} \tag{36}$$

so that the nominal parameters would all vary between -1 and 1, a quasi-LPV description of the error system is given by

$$\dot{x} = A(\theta)x + B_w(\theta)w + B_v(\theta)v \tag{37}$$

where

$$A(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_w(\theta) = \begin{bmatrix} \frac{3}{4}\theta_2 + \frac{1}{4} \\ \frac{1}{2}\theta_3 + \frac{7}{2} \end{bmatrix},$$
$$\mathrm{B}_v(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{21}{8}\theta_1 - \frac{37}{8} \end{bmatrix} \tag{38}$$

Here, $w = 1$ is introduced as a fictitious disturbance so that the nonlinearities grouped in the $B_w(\theta)$ term are treated as a bounded disturbance. The performance constraint is given by $\|z\|_{\infty,[0,T]} \le \gamma$ where $z = C_1 x + D_{12} v$ with $C_1 = I$ and $D_{12} = 0.01I$. Control agents take the form

$$\dot{v} = u, \tag{39}$$

$$u = K_1^{(l)} x + K_2^{(l)}(v + v^{(l)}), \tag{40}$$

where the index l indicates the design point and $v^{(l)}$ is a constant bias term internal to each agent. The bias term for each design point $\theta^{(l)}$ is chosen as

$$v^{(l)} = -B_v(\theta^{(l)})^{-1} B_w(\theta^{(l)}). \tag{41}$$

Note that the inverse exists in this case for all nominal parameter values.

Control gains were synthesized for the biased systems using the techniques presented in Bett and Lemmon (1997) combined with LMI pole placement constraints (Chilali and Gahinet, 1996). A MATLAB program was written to implement the conditions associated with Propositions 3 and 4. First, the switch time sets $\mathscr{T}^{(l,m)}$ were computed according to the conditions of Proposition 3. The nominal parameter trajectory was searched to determine possible switching times and the resulting switches. The results of the search were used to form a performance validation tree for a fixed performance level γ and switching parameter ϑ_{out}. Fig. 3 depicts one such tree for a multiple agent design with switching parameter $\vartheta_{\text{out}} = 0.4$. The performance validation tree shown in the figure represents a performance level of $\gamma = 0.068$. The performance validation tree is initialized with control agent $K^{(1)}$ in feedback with the plant. A search of $\theta_{\text{nom}}(t)$ indicated a possible switch to agent $K^{(2)}$ for $t \in \mathscr{T}^{(1,2)}$ or to agent $K^{(3)}$ for $t \in \mathscr{T}^{(1,3)}$. This result is indicated by the two branches leaving node 1 and labeled with $\mathscr{T}^{(1,2)}$ and $\mathscr{T}^{(1,3)}$. The second level of the tree is constructed by first assuming that agent $K^{(2)}$ is

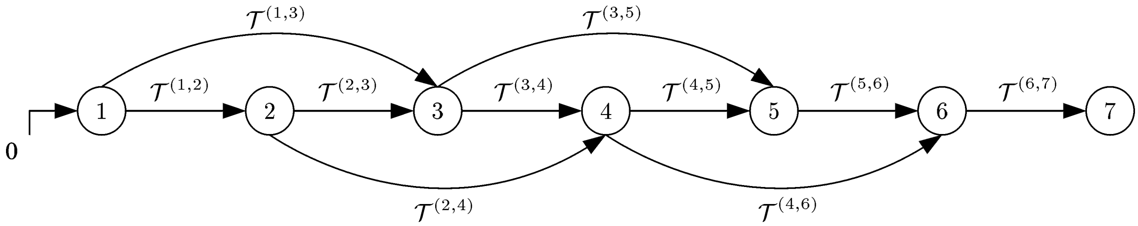
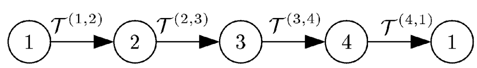


Fig. 4. Timed transition table for finite time example: $\vartheta_{\text{out}} = 0.4$, $\gamma = 0.068$.



$\mathcal{T}^{(1,2)} = [0.1520, 0.1560]$

$\mathcal{T}^{(2,3)} = [0.3600, 0.3960]$

$\mathcal{T}^{(3,4)} = [0.6520, 0.6560]$

$\mathcal{T}^{(4,1)} = [0.9020, 0.9120]$

Fig. 5. Performance validation tree for periodic scheduling example: $\vartheta_{\text{out}} = 0.8$, $\gamma = 0.09$.

switched into feedback at time max $\mathscr{T}^{(1,2)}$; $\theta_{\text{nom}}(t)$ is then searched, resulting in nonempty sets $\mathscr{T}^{(2,3)}$ and $\mathscr{T}^{(2,4)}$ indicating possible switches to agents $K^{(3)}$ or $K^{(4)}$ from agent $K^{(2)}$. The process is repeated until the time interval is exhausted with no further switches.

The performance validation tree of Fig. 3 was then used to construct an automaton model which is represented by the timed transition table shown in Fig. 4. For the underlying finite automaton, $\mathscr{V}$ consists of seven states with connecting directed arcs, $\mathscr{A}$, as indicated in the figure. The initial marking vector satisfies $\mu(1) = 1$, i.e. the automaton is initialized in state 1. There are no reset conditions on the single local clock t. Firing constraints are indicated in the figure, e.g. $t \in T^{(3,4)}$.

The timed automaton represented in Fig. 4 represents an abstraction of the multiple agent controlled system which can be analyzed on the supervisory level. If the switched system described here is one of many similar subsystems, this timed-logical model would be useful in the verification of, for example, desired synchronous behavior among the subsystems.

5.3. Periodic scheduling

The periodic scheduling objective considered here is to cycle the state of the plant described by Eqs. (32) and (33) between points near $x_p = (0.5, 0.5)$ and points near $x_p = (1, 0.5)$ and back with a one second ($T = 1$) period according to the reference model

$$\dot{x}_{m1} = \frac{\pi}{2}\sin 2\pi t; \qquad x_{m1}(0) = 0.5, \tag{42}$$

$$\dot{x}_{m2} = \frac{\pi}{4}\cos 2\pi t; \qquad x_{m2}(0) = 0.5. \tag{43}$$

The control agents used for the periodic scheduling possessed a structure identical to that used for the finite time scheduling. The control synthesis procedure was identical as well.

Because the reference trajectory is periodic and the nominal parameter trajectory evolves over an infinite time horizon, one cannot apply the results of Proposition 3 to compute the sets $\mathscr{T}^{(l,m)}$ in the same manner as for the finite-time case. This difficulty arises because Eq. (22) can no longer be evaluated for all times $t \geq t_0$. However, one can use a more conservative bound by setting $k_{\max,i}(t) = k_{zi}\gamma$ in Eq. (22). Any predicted switch satisfying the dwell time conditions in this more conservative approach must also satisfy the dwell time constraints if the less conservative bound had been used.

Using this more conservative approach, only admissible switches for single period trajectories, initialized with agents consistent with the established switching rules, require evaluation. The switching time estimates represent absolute bounds on when a switch might occur, regardless of the time that the agent was switched into the loop. As an example, for $\vartheta_{\text{out}} = 0.8$, four agents were sufficient to adequately sample the nominal parameter trajectory which was searched to determine possible switching times over a single period of the trajectory using the modified bounds described above. The resulting performance validation tree describing the possible switching behavior for a single period is shown in Fig. 5. The tree indicates that if the controller is initialized with agent $K^{(1)}$ in the feedback loop, by the end of a single reference trajectory period, agent $K^{(1)}$ will have been switched back into the loop. Therefore, in this example, there is only a single switching cycle to be analyzed.

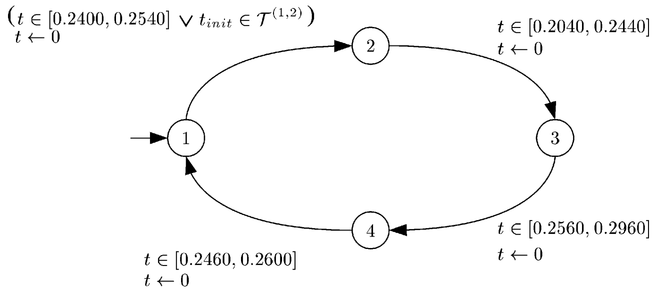


Fig. 6. Timed transition table for periodic scheduling example: $\vartheta_{\text{out}} = 0.8$, $\gamma = 0.09$.

The periodic behavior of the switched system is seen in the timed transition table which can be derived from the performance validation tree. This is shown in Fig. 6. For the timed automaton $\mathscr{V}$ consists of four states with connecting directed arcs, $\mathscr{A}$, as indicated in the figure. The initial marking vector satisfies $\mu(1) = 1$. There are two local clocks, t_{init} and t. The clock t_{init} is never reset and controls only a single switch. The clock t is reset on every firing. Arcs are labeled with the reset constraints and the firing constraints, which are derived from the sets $T^{(l,m)}$ with the reset condition taken into account (as described earlier).

6. Conclusions

This paper has described an approach for analyzing the performance of switched LPV systems required to meet a bounded amplitude performance constraint with respect to a known scheduling trajectory. The LPV systems considered in this paper cover a large class of nonlinear systems which are driven or scheduled along a predetermined path of operating points which may or may not be states of the system.

The central theoretical result of the paper is the *LPV switching lemma* which states sufficient conditions for a system switched between two LPV realizations to satisfy the amplitude performance constraint over a given time interval. Using this result along with knowledge of the scheduling trajectory to compute bounds on the switching times, switching sequences can be checked against the *LPV switching lemma* to establish performance over a sequence of intervals which can be pieced together to establish performance over the entire reference scheduling trajectory. It was shown by example that the performance constraints can be analyzed in this fashion for finite time and periodic scheduling trajectories. Note that the *LPV switching lemma* applied to any system which switches between two LPV realizations. Thus, the results presented here should generalize in a straightforward manner to any control agent structure yielding a controlled system which is an LPV system, e.g. LPV agents.

In addition, this paper has described a method for extracting logical models representing the behavior of a class of scheduled continuous time systems controlled by switching between a finite set of continuous time controllers. The results of this paper show that knowledge of the system scheduling can be combined with robustness properties of LPV systems to derive logical models of the system behavior in the form of timed automata. The results in this paper focus on bounded amplitude performance condition, but there does not appear to be any reason prohibiting the use of these ideas for other performance problems, e.g. $\mathscr{H}_2$, $\mathscr{H}_\infty$. These results are therefore useful for the study of hybrid systems because they provide a link between two distinctly different approaches to hybrid system design and analysis.

Appendix. Proofs

Proof of Proposition 1. By the assumption of the proposition, one can use arguments analogous to those used in the proof of Theorem 3.1 in Bett and Lemmon (1997) (see also Bett, 1998) to define functions $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$V_1(\xi) := \xi' P_1 \xi \quad \text{and} \quad V_2(\xi) := \xi' P_2 \xi$$

so that for any $\theta \in \mathscr{F}_{\mathscr{C}}$ and along any trajectories of the switched system, the time derivatives of V_1 and V_2 must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(x(t)) \leq -2\beta V_1(x(t)) \leq 0 \tag{A.1}$$

for any $t \in [0, t_{\mathrm{s}}]$ and any $x(t)$ and $w(t)$ satisfying $x'(t) P_1 x(t) \geq r$ and $w'(t) w(t) \leq 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2(x(t)) \leq 0 \tag{A.2}$$

for any $t \in (t_{\mathrm{s}}, T]$ and any $x(t)$ and $w(t)$ satisfying $x'(t) P_2 x(t) \geq 1$ and $w'(t) w(t) \leq 1$.

Given that $\beta > 0$, for any $\theta \in \mathscr{F}_{\mathscr{C}}$ and $w \in \mathscr{BL}_\infty^{n_w}$, Eq. (A.1) implies that along system trajectories for $t \in [0, t_{\mathrm{s}}]$, that

$$V_1(x(t)) \leq V_1(x(0)) + \int_0^t -2\beta V_1(x(\tau))\,\mathrm{d}\tau \tag{A.3}$$

so that by the Bellman–Gronwall lemma

$$V_1(x(t)) \leq V_1(x(0)) \exp(-2\beta t). \tag{A.4}$$

Supposing that $V_1(x(0)) \leq 1$, the last equation implies $V_1(x(t)) \leq \exp(-2\beta t) \leq 1$ for all $t \in [0, t_{\mathrm{s}})$. If $t_{\mathrm{s}} > t_{\mathrm{d}}$, then $V_1(x(t)) \leq r$ for all $t > t_{\mathrm{d}}$. Since $z(t) = C_1(\theta(t)) x(t)$ it can be shown that Eq. (14) implies $z'(t) z(t) \leq \gamma^2$ for all $t \in [0, t_{\mathrm{s}}]$.

The state trajectory is continuous at the switch so that $x(t_s^+) = x(t_s)$ which implies that $V_1(t_s^+) \le r$. Combining this fact with Eq. (13) implies that $V_2(t_s^+) \le 1$. Since $z(t) = C_2(\theta(t))x(t)$ for $t > t_s$ and since $\theta(t_s^+) \in \Theta_2$, one can easily show that Eq. (15) implies $z'(t_s^+)z(t_s^+) \le \gamma^2$.

Now suppose that $V_2(x(t)) > 1$ for $t \in (t_s, T]$. This implies that either $V_2(x(t_s^+)) \ge 1$ and $\dot{V}_2(x(t_x^+)) > 0$ with $w'(t_s^+)w(t_s^+) \le 1$ or it implies that there exists a $\tau \in (t_s, T)$ such that $V_2(x(\tau)) \ge 1$ and $\dot{V}_2(x(\tau)) > 0$ with $w'(\tau)w(\tau) \le 1$. Since $\theta(t_s^+) \in \Theta_2$ and since Eq. (A.2) holds for any $\theta \in \mathscr{F}_C$, then $\dot{V}(x(t_s^+)) < 0$ which generates a contradiction. The only conclusion is that $V_2(x(t)) \le 1$ for all $t \in (t_s, T]$. Since $z(t) = C_2(\theta(t))x(t)$ in this time interval, one concludes that $z'(t)z(t) \le \gamma^2$ for all $t \in (t_s, T]$. The result follows immediately.

Proof of Lemma 2. Under the assumptions of the lemma and from the proof of Theorem 3.1 in Bett and Lemmon (1997), for any $\theta \in \mathscr{F}_\Theta[0, T]$

$$\|z(t)\| \le \gamma \max\{\sqrt{r}, e^{-\beta t}\} \tag{A.5}$$

for all t. By assumption,

$$|\theta_i(t) - \theta_{\text{nom},i}(t)| = |S_i(x_m(t), x(t), v(t)) - S_i(x_m(t), 0, v_m(t))| \tag{A.6}$$

for $i = 1, 2, \ldots, s$. The assumptions on the parameter mapping therefore imply that

$$|S_i(x_m(t), x(t), v(t)) - S_i(x_m(t), 0, v_m(t))| \le k_{zi}\gamma \max\{\sqrt{r}, e^{-\beta t}\}. \tag{A.7}$$

Combining these last two equations yields Eq. (19). The remainder of the lemma is proven by combining Eq. (19) with the triangle inequality at $t = T$.

Proof of Proposition 3. Under the assumptions of the proposition, if $\hat{x}'(0)P\hat{x}(0) \le 1$ and $w \in \mathscr{BL}_\infty$, Lemma 2 implies that

$$|\theta_i(t_s) - \theta_{\text{nom},i}(t_s)| \le k_{zi}\gamma \max\{\sqrt{r}, e^{-\beta t_s}\}. \tag{A.8}$$

If the parameter trajectory satisfies the nearest neighbor switching rule, then at t_s,

$$\|\theta(t_s) - \theta_{\text{nom}}^{(m)}\| \le \|\theta(t_s) - \theta_{\text{nom}}^{(q)}\| \tag{A.9}$$

for all $q \in \mathscr{I}_K$ with q not equal to m, or $\theta(t_s) \in \hat{\Theta}_m$. Combining this fact with Eq. (A.8) implies that t_s must satisfy the first condition for $\mathscr{T}^{(l,m)}$. For the second condition, from Lemma 2, $\theta_{\text{nom}}(t_s)$ must satisfy

$$\begin{aligned} \vartheta_{\text{out}} - k_{zi}\gamma \max\{\sqrt{r}, e^{-\beta\tau}\} &\le |\theta_{\text{nom},i}(t_s) - \theta_{\text{nom},i}^{(l)}| \\ &\le \vartheta_{\text{out}} + k_{zi}\gamma \max\{\sqrt{r}, e^{-\beta\tau}\} \end{aligned} \tag{A.10}$$

for some $i = 1, 2, \ldots, s$. One concludes that $t_s \in \mathscr{T}^{(l,m)}$. □

Proof of Proposition 4. Consider an admissible switching sequence $k \to l \to m$ with $\alpha_1^{(i)}$, $\alpha_2^{(i)}$ and $P^{(i)}$ satisfying Eq. (24). Define t_i as the time that control agent $K^{(i)}$ is switched into feedback with the system, $i \in \{k, l, m\}$. Assume, without loss of generality, that $\hat{x}(t_k)'P^{(k)}\hat{x}(t_k) \le 1$. To apply the *LPV switching lemma* to prove performance over the interval $[t_k, t_m]$, it must be established that $\theta(t)$ is legal and that the dwell time constraint is satisfied.

To demonstrate the legality of $\theta(t)$, note that from the integrator controller structure and continuity of the parameter mapping, $\theta(t)$ must be continuous. By the nearest neighbor switching rule and adequate sampling assumption, $\theta(t) \in \mathscr{F}_\mathscr{C}$ (i.e. $\theta(t)$ is legal).

To demonstrate the dwell time constraint, note that from Proposition 3, $\min T^{(l,m)} - \max \mathscr{T}^{(k,l)}$ is a lower bound on $t_l - t_k$. The dwell time requirement is satisfied if there exists an $r \in \mathscr{R}^{(l,m)}$ such that

$$f(r) \le \min \mathscr{T}^{(l,m)} - \max \mathscr{T}^{(k,l)}$$

where

$$f(r) = \frac{r}{\alpha_2^{(l)} - \alpha_1^{(l)} r} \log r$$

and $\mathscr{R}^{(l,m)}$ is as defined in Eq. (29). It is easily shown that $f(r)$ is monotonically decreasing on $\mathscr{R}^{(l,m)}$. Thus, Eq. (25) implies that the dwell time requirement is satisfied for $t_l - t_k$. By the *LPV switching lemma*,

$$\|z(t)\| \le \gamma, \quad t \in [t_k, \tau_l)$$

for any $\tau_l \in [t_l, t_m)$. Furthermore, $\hat{x}(t_l)'P^{(l)}\hat{x}(t_l) \le 1$. As a consequence of the nearest neighbor switching rule, the adequate sampling assumption and the continuity of $\theta(t)$, any finite time interval, $[0, T]$, can contain at most a finite number of switching instants. Assume, without loss of generality, $N + 1$ agents switched in order $0, 1, 2, \ldots, N$. Thus, the interval $[0, T]$ can be written

$$[0, T] = \bigcup_{i=0}^{N-1} [t_i, \tau_{i+1}) + [t_N, T]$$

where $t_0 = 0$ and the interval $[t_i, \tau_{i+1})$ denotes the the admissible switching sequence $i \to i + 1 \to i + 2$ with $\tau_i \in [t_i, t_{i+1}]$ for $i = 1, \ldots, N - 1$ and $\tau_N \in [t_N, T]$. By assumption, $\hat{x}(t_0)'P^{(0)}\hat{x}(t_0) \le 1$ so that with the *LPV switching lemma*, one concludes that

$$\|z(t)\| \le \gamma, \ t \in [t_0, \tau_1) \quad \text{and} \quad \hat{x}(t_1)'P^{(1)}\hat{x}(t_1) \le 1.$$

In similar fashion,

$$\|z(t)\| \le \gamma, \ t \in [t_1, \tau_2) \quad \text{and} \quad \hat{x}(t_2)'P^{(2)}\hat{x}(t_2) \le 1.$$

By induction, one concludes that $\|z(t)\| \le \gamma$ for subintervals $t \in [t_i, \tau_{i+1})$, $i = 1, \ldots, N - 1$, and that $\hat{x}(t_N)'P^{(N)}\hat{x}(t_N) \le 1$. Since Eq. (24) is satisfied for $i = N$ and since $\theta(t) \in \mathscr{F}_\Theta[t_N, T]$ by the fact that there is no other switch according to the nearest neighbor switching rule, $\|z(t)\| \le \gamma$ for $t \in [t_N, T]$ as well. One concludes that performance is satisfied over the finite interval, $\|z\|_{\infty,[0,T]} \leq \gamma$.

For $T \to \infty$, there are two possibilities: switching stops after a finite time, or switching does not stop. In the first case, there are a finite number of switches and performance is proven by the above analysis. In the second case, an infinite number of switches must be considered. Then,

$$[0, \infty) = \bigcup_{i=0}^{\infty} [t_i, \tau_{i+1}).$$

Define

$$\bar{z}_i := \operatorname{ess} \sup_{t \in [t_i, \tau_{i+1})} \|z(t)\|$$

and $Z := \{z \mid \|z\| \leq \gamma\}$ as all points z satisfying performance constraints. Clearly, from the preceding analysis, $\bar{z}_i \in Z$. Furthermore, any limit point of the sequence $\{\bar{z}_i\}$ must lie in the closure of Z. Since Z is closed and bounded, $\bar{z} \in Z$, implying that performance is satisfied in the case of an infinite number of switches. Thus, $\|z\|_{\infty,0,\infty)} \leq \gamma$.

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