Robust Performance of Soft Real-time Networked Control Systems with Data Dropouts [1]

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Abstract

This paper derives the power spectral density(PSD) of the output generated by a discrete-time feedback control system in which feedback measurements are dropped with a known probability, ε . This class of systems is a model for soft real-time control systems in which the feedback path is implemented on a non-deterministic computer network. The PSD computed in this paper is a function of the dropout probability. The dropout probability is taken as a measure of the network quality of service (QoS). Based on the derived PSD, the power semi-norm of the output is predicted. So a direct way of linking control system performance(as measured by the power semi-norm of the output) to the network's QoS (as measured by the dropout probability) is provided.

1 Introduction

In recent years there has been considerable interest in implementing sampled data control systems over non-deterministic communication networks [4] [5]. A non-deterministic network is one in which data packets cannot be delivered within hard deadlines. In these cases, the networked control system becomes a soft real-time system. In soft real-time networks, data packets may be excessively delayed due to network congestion. For control purposes, however, it often turns out that delayed feedback data is more harmful than no data at all. As a result it is often desirable to purposefully drop packets that are greatly delayed. Industrial practice appears to support this ad hoc approach to soft real-time networks [1] [2]. The rate of data dropouts therefore becomes a measure of the soft real-time system's quality of service (QoS). The challenge faced by control engineers is finding a way of relating this particular QoS measure to actual control system performance.

There has, however, been relatively little work studying the impact of data dropouts on control system performance. Most of the prior work has only focused on the impact of dropouts on overall system stability. In [9] [10], networked control systems with dropouts are modelled as asynchronous switched systems. The approach replaces the true switched system with an "averaged system" and then studies the stability of the system. In [8], the dropouts are governed by a Markov chain and are treated as vacant sampling. The work proposes two approaches for handling data dropouts: using past control signals or estimating the lost data and computing new control signals. The stability of an optimal LQ controller under the two approaches is analyzed. This work unfortunately does not provide a rigorous analysis and demonstrates the results only through examples.

This paper presents a formal analysis that directly relates control system performance to the data dropout rate in a soft real-time system. Let y be the control system's output and y be wide sense stationary(WSS). Control system performance is characterized by the power semi-norm, $\|y\|_{\mathcal{P}} = \sqrt{\text{Trace}(R_{yy}[0])}$, where $R_{yy}[0]$ is the autocorrelation of y. This paper derives the PSD of y, $S_{yy}(e^{j\omega})$, for a single-input single-output discrete-time control system with data dropouts. The resulting PSD is a function of the dropout rate (dropout probability) ε so we can directly relate system performance $\|y\|_{\mathcal{P}}$ to the data dropout rate through $R_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(e^{j\omega})d\omega$. This relationship, therefore, provides control engineers a way of finding bounds on the dropout rates that assures a specified level of control system performance.

The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries in this paper. Section 3 presents the modelling framework used in this paper. Section 4 states sufficient conditions for the stability and the wide sense stationarity of the networked control system. Section 5 states the power spectral density of the output and presents experimental data supporting the result. Final remarks will be found in section 6. The proofs of all theorems have been moved to the appendix (section 7) to improve the paper's readability.

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2 Mathematical Preliminaries

Let $x = \{x[n]\}$ be a random vector process. When x is wide sense stationary, denote the mean as $\mu_x = \mathbf{E}[x[n]]$, the correlation as $R_{xx}[m] = \mathbf{E}\left[x[n+m]x^T[n]\right]$, and the power spectral density as $S_{xx}\left(e^{j\omega}\right)$ ($e^{j\omega}$ is often dropped to improve the readability, i.e. $S_{xx}\left(e^{j\omega}\right)$ will be simply denoted as S_{xx}.) Refer to [11] for the detail of the above definitions. Because the power spectral density is the Fourier transform of the correlation [1],

the power semi-norm of y can be computed through

$$\|y\|_{\mathcal{P}} = \sqrt{\mathrm{Trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(e^{j\omega})d\omega\right)}$$

In section 7, the positive and negative single-sided power spectral densities of a WSS process x are used, which are defined by the equations

$$\begin{aligned} S_{xx}^{+}(e^{j\omega}) &= \sum_{m=1}^{\infty} R_{xx}[m]e^{-jm\omega} \\ S_{xx}^{-}(e^{j\omega}) &= \sum_{m=-\infty}^{-1} R_{xx}[m]e^{-jm\omega} \end{aligned}$$

By the above definition, it follows that

$$S_{xx} = S_{xx}^{+} + S_{xx}^{-} + R_{xx}[0]$$

Convergence in mean square sense is used in this paper. Refer to [11] for its definition. It can be shown [6] that a random vector process $x = \{x[n]\}$ is convergent in mean square if and only if

$$\lim_{n\to\infty}\sup_{m\geq n}\mathbf{E}\left[(x[m]-x[n])^T(x[m]-x[n])\right] = 0$$

A jump linear system is a linear dynamical system with random system matrices as follows.

$$\left\{\begin{array}{rcl} x[n+1] &=& A[n]x[n] + B[n]w[n] \\ y[n] &=& C[n]x[n] + D[n]w[n] \end{array}\right. \tag{2.1}$$

where $\{A[n]\}$, $\{B[n]\}$, $\{C[n]\}$, and $\{D[n]\}$ are matrix valued random processes. If the input process $w[n] = 0$, then we say that the system is a free jump linear system. A free jump linear system is said to be *stochastically asymptotically stable* in mean square sense [7] if $\lim_{n\to\infty}\mathbf{E}\left[x[n]^Tx[n] \mid x_0\right] = 0$ for any initial state $x[0] = x_0$.

[1]Most of the WSS processes in this paper are zero mean, so their covariances (auto-covariances) and correlations (autocorrelations) are equal. Therefore we interchangeably use these terms throughout the paper.

3 Data Dropout Model

A block diagram of the discrete-time networked control system is shown in figure 1.

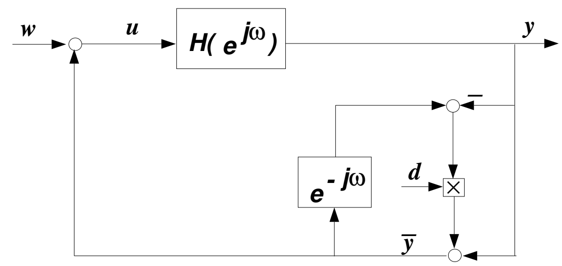

Figure 1: The Control System with Data Dropouts

In the system shown in figure 1, the loop function $H(e^{j\omega})$ is single-input single-out and strictly proper. $w = \{w[n]\}$ is the exogenous input, which is wide sense stationary with zero mean. $d = d[n]$ is the dropout process, which is independently identically distributed (i.i.d.) with probability distribution of $\Pr(d[n]=1) = \varepsilon$ and $\Pr(d[n]=0) = 1-\varepsilon$. When $d[n]=0$, $\overline{y}[n] = y[n]$, i.e. the measurement is sent sucessfully; when $d[n]=1$, $\overline{y}[n] = \overline{y}[n-1]$, i.e. the measurement is dropped and the last measurement is reused. ε is the dropout rate. The dropout process d and the input process w are assumed to be independent.

A state space representation of the system is

$$\left\{\begin{array}{rcl} x[n+1] &=& A[n]x[n] + Bw[n] \\ y[n] &=& Cx[n] \end{array}\right. \tag{3.1}$$

where x is the state, $\{A[n]\}$ is a random switching process. When $d[n]=0$, $A[n]=A_0$; when $d[n]=1$, $A[n]=A_1$. Obviously $\{A[n]\}$ is an i.i.d. matrix valued process with the probability distribution of $\Pr(A[n]=A_1) = \varepsilon$ and $\Pr(A[n]=A_0) = 1-\varepsilon$.

4 Wide Sense Stationarity

In order to determine the power spectral density of the output process $y = \{y[n]\}$, we first need to show that the overall system in equation 3.1 is stable and y is wide sense stationary. This section states sufficient conditions for the stability of the overall system and the wide sense stationarity of y by the following theorems and corollaries. Their formal proofs will be found in section 7.

Theorem 4.1 *Consider the (free) jump linear system given in equation 3.1 with $w=0$. Let $A = \{A[n]\}$ be a matrix valued i.i.d. process such that $\Pr(A[n]=A_1) = \varepsilon$ and $\Pr(A[n]=A_0) = 1-\varepsilon$. Let the matrix X equal*

$(1-\varepsilon)A_0^T A_0 + \varepsilon A_1^T A_1$. *The free jump linear system is stochastically asymptotically stable in the mean square sense if there exists a constant* $\sigma_X \in \Re^+$ *such that*

$$\sqrt{\lambda_{\max}(X)} = \sigma_X < 1$$

where $\lambda_{\max}(\cdot)$ *denotes the the maximal modulus of the eigenvalues of a matrix.*

Under the specified conditions, the above theorem guarantees that the state processes of the free jump linear system converge to 0 independently of the initial state x_0. We may, therefore, without loss of generality take the initial condition to be zero ($x_0 = 0$) in equation 3.1 and then study the behavior of the resulting jump linear system. The theorem is conservative, because it depends on the exact state space realization, i.e. A_0 and A_1. The following corollary reduces this conservativity.

Corollary 4.2 *Consider the (free) jump linear system given in equation 3.1 with* $w = 0$. *Let* $A = \{A[n]\}$ *be an i.i.d. matrix valued process such that* $\Pr(A[n] = A_1) = \varepsilon$ *and* $\Pr(A[n] = A_0) = 1 - \varepsilon$. *The free jump linear system is asymptotically stable in the mean square sense if there exists a nonsingular matrix* P *and a constant* $\sigma_{X_P} \in \Re^+$ *such that*

$$\sqrt{\lambda_{\max}(X_P)} = \sigma_{X_P} < 1$$

where $X_P = (1-\varepsilon)(P^{-1}A_0P)^T(P^{-1}A_0P) + \varepsilon(P^{-1}A_1P)^T(P^{-1}A_1P)$.

The proof of corollary 4.2 is similar to the proof of theorem 4.1, the details are omitted here. Corollary 4.2 and theorem 4.1 take the same assumptions, so theorem 4.1 can always be freely replaced by corollary 4.2. Based on the above results, the wide-sense stationarity of the state process $x = \{x[n]\}$ is characterized by the following theorem.

Theorem 4.3 *Under the assumptions of theorem 4.1, the state process* x *is wide sense stationary.*

The following corollary gives the result for the wide sense stationarity of the outputs. The proof is similar to the one of theorem 4.3 and is omitted. The measurement $\overline{y}$ can also be taken as output.

Corollary 4.4 *When all conditions in theorem 4.3 are satisfied, all output processes are wide sense stationary.*

5 Main Result and Example

This section states the main result of this paper in theorem 5.1, which is an expression for the power spectral density $S_{yy}(e^{j\omega})$. Experimental data are then provided to support the correctness of this result. The formal derivation is presented in section 7.

Theorem 5.1 *Consider the system in equation 3.1.* w *is wide sense stationary with zero mean, whose power spectral density is* S_{ww}. *The dropout process* d *is i.i.d. with the probability distribution of* $\Pr(d[n] = 1) = \varepsilon$ *and* $\Pr(d[n] = 0) = 1 - \varepsilon$. d *is independent of* w. *If there exists a nonsingular matrix* P *and a constant* $\sigma_{X_P} \in \Re^+$ *such that* $\sqrt{\lambda_{\max}(X_P)} = \sigma_{X_P} < 1$ *with* $X_P = (1-\varepsilon)(P^{-1}A_0P)^T(P^{-1}A_0P) + \varepsilon(P^{-1}A_1P)^T(P^{-1}A_1P)$, *the overall system is stochastically asymptotically stable in mean square sense, the output process* $y = \{y[n]\}$ *is wide sense stationary and the power spectral density of* y *can be computed as follows.*

$$S_{yy} = \left|\frac{H}{1-DH}\right|^2 S_{ww} + \left|\frac{DH}{1-DH}\right|^2 \Delta \qquad (5.1)$$

where $D(e^{j\omega}) = \frac{1-\varepsilon}{1-\varepsilon e^{-j\omega}}$, *and* Δ *can be computed through the following equation*

$$\begin{aligned}\Delta &= \frac{1}{\pi(1-\varepsilon)}\int_{-\pi}^{\pi} D^*(D-1)\left|\frac{H}{1-DH}\right|^2 S_{ww}dw \\ &\quad + \Delta\frac{1}{\pi(1-\varepsilon)}\int_{-\pi}^{\pi}(1-H)\left|\frac{D}{1-DH}\right|^2 dw\end{aligned}$$

The power spectral density in equation 5.1 consists of two terms. The first term is the usual term we would expect to see if a WSS process w were driving a unity gain feedback system. For this first term, data dropouts introduce an additional transfer function D in series with the plant, H. The second term in equation 5.1 is more interesting. This term models the explicit effect that the dropout process d has on the system's output. It is the inclusion of this second term that greatly complicates the derivations in section 7.

In order to experimentally verify the correctness of the theorem, we used the theorem to predict the power semi-norm of the output of a simple feedback control system. This system is then simulated to experimentally estimate the same power semi-norm. The assumed plant transfer function was an unstable system with transfer function $H(e^{j\omega}) = \frac{e^{j\omega}+2}{e^{j2\omega}+e^{j\omega}+2}$. w is a Gaussian wide sense stationary process with the power spectral density of $S_{ww} = \frac{1}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})}$. By theorem 5.1, when the dropout rate $\varepsilon \leq 18.9\%$, the overall system is stochastically asymptotically stable and the power spectral density of y can be predicted by equation 5.1. The power spectral densities under this range of dropout rates are shown in figure 2. The theoretically predicted and experimentally estimated power semi-norms of y are shown in figure 3. A `Matlab simulink` model was created for the simulation. For each value of ε, 5 simulations were run for 200,000 time steps. The expectation of the output's power, $R_{yy}[0]$, was estimated by the time average of the output's power over the last half of the output samples, $\hat{R}_{yy}[0]$. The power semi-norm of the output was estimated by $\sqrt{\hat{R}_{yy}[0]}$. This estimation approach is very practical. It will, however, put more contraints on the acceptable dropout rate. In this example, the upper bound of the dropout rate to guarantee the efficiency of the estimation approach is 5.6%, which is less than the bound 18.9%. This difference results from that the output process cannot be guaranteed to be ergodic $\varepsilon > 5.6\%$. The problem on ergodicity will be explained in detail in our future papers.

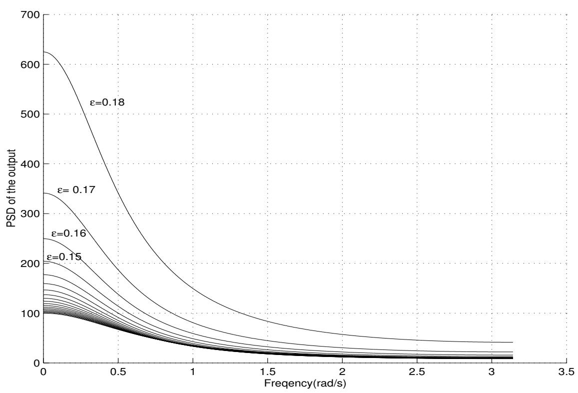

Figure 2: Output's Power Spectral Density($0 \leq \varepsilon \leq 18\%$)

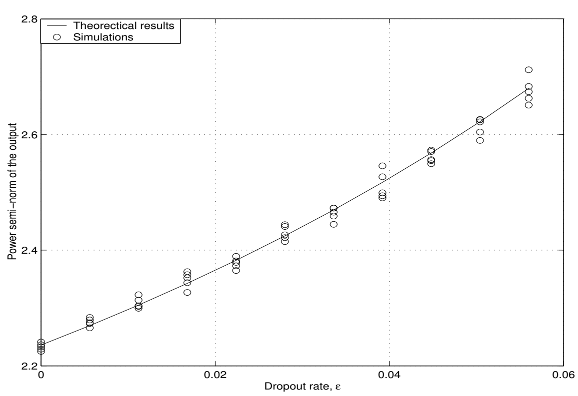

Figure 3: Output's Power Semi-Norm ($0 \leq \varepsilon \leq 5.6\%$)

In figure 3, the plot shows that as the dropout rate increases, the power semi-norm of the output increases. This trend is expected since the system is switching between a stable closed loop and unstable open-loop system. Figure 3 shows excellent agreement between the experimental data and theoretical results, so we have high confidence in the correctness of the results stated in theorem 5.1.

6 Conclusions

This paper studied the networked control system with dropouts. It first identified sufficient conditions that can be used to identify an interval of dropout rates over which the networked control system is stochastically asymptotically stable and wide sense stationary. It then formally derived the PSD of the output which is a closed-form function of the dropout rate, this PSD is used to compute the power semi-norm of the system's output and experiments validate the result through simulations.

The results are significant because they provide a formal analysis relating control system performance (measured by the power semi-norm of the output) to QoS of the network (measured by the dropout rate). It therefore allows control engineers to specify bounds on the network's QoS (dropout rate) that enforce control system performance. Embedded system engineers would then use this dropout rate bound as a specification for the networked implementation of the control system.

The results in this paper showed that the output power of the system could be accurately predicted through analysis. Figure 4 shows the output trace under a dropout rate of 0.04. Although the system is stable and the control system performance measured by the power semi-norm of the output is not bad, there still exist occasional high amplitude bursts. These bursts are of short duration, but the existence of such bursts is very troubling for it indicates that even though output power is acceptable, the system may still exhibit large signal amplitudes. For practical implementations of networked control systems in the presence of dropouts, we must find some way of bounding the size of these bursts.

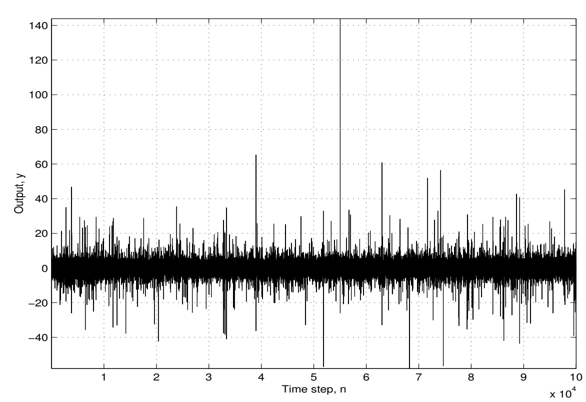

Figure 4: Output signal trace with burst

This observation suggests one avenue of future work. This paper assumed that the dropout process was i.i.d. We believe that it is possible to control the burst size by considering a more restricted set of dropout processes. This restricted dropout process is realized by a judicious choice for the real-time scheduler. One such candidate is a scheduler implementing an (m,k)-firm guarantee model [3]. This approach represents a natural way of constraining the dropout process to bound signal burstiness. But it is only one approach and there may certainly be other constraints that constrain burstiness and are still easily implemented by real-time schedulers. Our future work intends to explore this relationship between burstiness, real-time scheduling, and dropouts in greater detail.

7 Appendix: Proof

Proof of Theorem 4.1: Let the initial state of the overall system be $x[0]=x_0$. Then by $x^TXx \le \sigma_X^2 x^T x$ for any x, we can get

$$\begin{aligned}
& \mathbf{E}\left[x^T[n+1]x[n+1] \mid x_0\right] \\
= \; & \mathbf{E}\left[x^T[n]A^T[n]A[n]x[n] \mid x_0\right] \\
= \; & \mathbf{E}\left[x^T[n]\mathbf{E}\left[A^T[n]A[n]\right]x[n] \mid x_0\right] \\
= \; & \mathbf{E}\left[x^T[n]Xx[n] \mid x_0\right] \\
\le \; & \sigma_X^2 \mathbf{E}\left[x^T[n]x[n] \mid x_0\right]
\end{aligned}$$

Recursively applying the above inequality allows us to conclude that

$$\mathbf{E}\left[x^T[n+1]x[n+1] \mid \right] \le \sigma_X^{2(n+1)} x_0^T x_0$$

which is sufficient to ensure that $\lim_{n\to\infty}\mathbf{E}[x^T[n+1]x[n+1] \,|\, x_0] = 0$ for all x_0. $\diamondsuit$

Proof of theorem 4.3: The initial time of the system is assumed to be $-\infty$. By theorem 4.1, the initial state can be assumed to be zero without loss of generality. Then the state can be expressed as follows.

$$x[n+1] = \sum_{k=0}^{\infty} \Phi(n+1; n+1-k)Bw[n-k]$$

where

$$\Phi(n;k) = \begin{cases} A[n-1]A[n-2]\cdots A[k], & n>k \\ I, & otherwise \end{cases}$$

Based on the above expression, it can be shown by simply computations that $\mathbf{E}[x[n+1]] = 0$ and $\mathbf{E}\left[x[n+m+1]x^T[n+1]\right]$ is shift invariant with respect to n. So $x = \{x[n]\}$ is wide sense stationary with zero mean. $\diamondsuit$

Proof of Theorem 5.1: Let the impulse response of $H(e^{j\omega})$ be $h = \{h[n]\}$. Because $H(e^{j\omega})$ is strictly proper, $h[n]=0$ when $n \le 0$. So the output y can be expressed as

$$y = h * (\overline{y} + w) \tag{7.1}$$

where $*$ denotes convolution. The following identities can easily be obtained from equation 7.1.

$$\begin{aligned}
S_{yw} &= H(S_{\overline{y}w} + S_{ww}) && (7.2) \\
S_{y\overline{y}} &= H\left(S_{\overline{yy}} + S_{\overline{y}w}^*\right) && (7.3) \\
S_{yy} &= HH^*\left(S_{ww} + S_{\overline{y}w} + S_{\overline{y}w}^* + S_{\overline{yy}}\right) && (7.4)
\end{aligned}$$

First $R_{\overline{y}w}[m]$ is computed directly.

$$\begin{aligned}
& R_{\overline{y}w}[m] \\
= \; & \mathbf{E}\left[\overline{y}[n+m]w[n]\right] \\
= \; & \Pr(d[n+m]=0)\mathbf{E}\left[\overline{y}[n+m]w[n] | d[n+m]=0\right] \\
& + \Pr(d[n+m]=1)\mathbf{E}\left[\overline{y}[n+m]w[n] | d[n+m]=1\right] \\
= \; & (1-\varepsilon)R_{yw}[m] + \varepsilon R_{\overline{y}w}[m-1]
\end{aligned}$$

Taking the Fourier transform for the above equation yields

$$S_{\overline{y}w} = (1-\varepsilon)S_{yw} + \varepsilon e^{-j\omega}S_{\overline{y}w} \tag{7.5}$$

Combining equations 7.2 and 7.5, we can get

$$S_{\overline{y}w} = \frac{DH}{1-DH}S_{ww} \tag{7.6}$$

Second $R_{y\overline{y}}[m]$ is computed similarly for $m<0$.

$$R_{y\overline{y}}[m] = (1-\varepsilon)R_{yy}[m] + \varepsilon R_{y\overline{y}}[m+1]$$

We now take the negative single-sided Fourier transform for the above equation and obtain $S_{y\overline{y}}^-(e^{j\omega}) = (1-\varepsilon)S_{yy}^-(e^{j\omega}) + \varepsilon e^{j\omega}S_{y\overline{y}}^-(e^{j\omega}) + \varepsilon e^{j\omega}R_{y\overline{y}}[0]$. Then it follows that

$$S_{y\overline{y}}^- = D^* S_{yy}^- + \frac{\varepsilon e^{j\omega}}{1-\varepsilon e^{j\omega}}R_{y\overline{y}}[0] \tag{7.7}$$

The similar procedure can be repeated for $R_{\overline{yy}}[m]$ for $m>0$ and yields

$$S_{\overline{yy}}^+ = (1-\varepsilon)S_{y\overline{y}}^+ + \varepsilon e^{-j\omega}S_{\overline{yy}}^+ + \varepsilon e^{-j\omega}R_{\overline{yy}}[0]$$

Then $S_{y\overline{y}}^+$ can be expressed as

$$S_{y\overline{y}}^+ = D^{-1}S_{\overline{yy}}^+ - \frac{\varepsilon e^{-j\omega}}{1-\varepsilon}R_{\overline{yy}}[0] \tag{7.8}$$

$S_{y\overline{y}}$ is given by the equation

$$S_{y\overline{y}} = S_{y\overline{y}}^- + S_{y\overline{y}}^+ + R_{y\overline{y}}[0] \tag{7.9}$$

Inserting the results of equations 7.7 and 7.8 into the above equation yields

$$\begin{aligned}
S_{y\overline{y}} = \; & D^*S_{yy}^- + D^{-1}S_{\overline{yy}}^+ + \frac{1}{1-\varepsilon e^{j\omega}}R_{y\overline{y}}[0] \\
& - \frac{\varepsilon e^{-j\omega}}{1-\varepsilon}R_{\overline{yy}}[0]
\end{aligned} \tag{7.10}$$

Combining equations 7.3 and 7.10 yields

$$\begin{aligned} S_{yy}^{-} &= \left[D^{-1}\right]^{*} \Big(H(S_{\overline{yy}} + S_{\overline{y}w}^{*}) - D^{-1}S_{\overline{yy}}^{+} - \\ &\quad \frac{1}{1-\varepsilon e^{j\omega}} R_{y\overline{y}}[0] + \frac{\varepsilon e^{-j\omega}}{1-\varepsilon} R_{\overline{yy}}[0] \Big) \end{aligned} \quad (7.11)$$

By the definition of the negative single-sided power spectral density, we know $S_{yy} = S_{yy}^{-} + \left[S_{yy}^{-}\right]^{*} + R_{yy}[0]$. Inserting equation 7.11 into it yields

$$\begin{aligned} S_{yy} &= \left[D^{-1}\right]^{*} H(S_{\overline{yy}} + S_{\overline{y}w}^{*}) + \left[D^{-1}\right] H^{*}(S_{\overline{yy}} + S_{\overline{y}w}) \\ &\quad - \left[D^{-1}\right]^{*} \left[D^{-1}\right] S_{\overline{yy}} + \Delta \end{aligned} \quad (7.12)$$

where $\Delta = \frac{1-\varepsilon^2}{(1-\varepsilon)^2} R_{\overline{yy}}[0] - \frac{2}{1-\varepsilon} R_{y\overline{y}}[0] + R_{yy}[0]$.

Combining equations 7.4, 7.6 and 7.12, $S_{\overline{yy}}$ yields

$$\begin{aligned} S_{\overline{yy}} &= \left[\frac{DH}{1-DH}\right]\left[\frac{DH}{1-DH}\right]^{*} S_{ww} \\ &\quad + \left[\frac{D}{1-DH}\right]\left[\frac{D}{1-DH}\right]^{*} \Delta \end{aligned}$$

We then use the above result together with equations 7.4 and 7.6 to obtain the final expression of S_{yy}.

$$S_{yy} = \left|\frac{H}{1-DH}\right|^2 S_{ww} + \left|\frac{DH}{1-DH}\right|^2 \Delta$$

The last part deduces the algorithm for computing Δ. The direct computation of $R_{\overline{yy}}[0]$ shows that $R_{\overline{yy}}[0] = R_{yy}[0]$. So $\Delta = \frac{2}{1-\varepsilon}(R_{\overline{yy}}[0] - R_{y\overline{y}}[0])$. By the definition of PSD, we get

$$\Delta = \frac{2}{1-\varepsilon} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(S_{\overline{yy}}(e^{j\omega}) - S_{y\overline{y}}(e^{j\omega})\right) d\omega$$

Substitute the expressions of $S_{\overline{yy}}(e^{j\omega})$ and $S_{y\overline{y}}(e^{j\omega})$ into the above equation and we will get the final result of Δ.

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