SOFT REAL-TIME SCHEDULING OF NETWORKED CONTROL SYSTEMS WITH DROPOUTS GOVERNED BY A MARKOV CHAIN

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Abstract. This paper derives an equation for a networked control system's (NCS) performance as a function of the network's dropout process. We assume that the NCS is modelled as a discrete-time system whose dropout process is governed by a Markov chain. The equation computes the system output's power as a function of the Markov chain's probability transition matrix, Q. This equation is used to pose an optimization problem whose solution yields the transition matrix Q that optimizes closed loop output power for fixed average dropout rates. The "optimized" Markov chain is interpreted as a specification on the quality of service (QoS) delivered by real-time schedulers in the communication network. This paper, therefore, provides a method for obtaining specifications on real-time schedulers that assure overall feedback system performance. For a specific example, we compare the performance of this "optimal" scheduler to a soft version of schedulers implementing the popular (m, k)-firm guarantee model. The comparison demonstrates that schedulers enforcing the optimal Markovian policy may outperform (m, k)-firm guarantee schedulers.

1. Introduction

In recent years there has been considerable interest in networked control systems (NCS) [1] [2] [3]. The feedback loops in an NCS are implemented over a communication network. The measurements are sent to the controller over the network, the controller computes the control output, and then uses the network to send the control to the actuators. This paper focuses on NCS implemented by non-deterministic networks. In a non-deterministic network, the measurement and control packets are not reliably delivered to their destination. Data may be dropped in the feedback path for a variety of reasons. An interesting question is: how much does the overall control system's performance degrade in the presence of such data dropouts? This paper provides an answer to that question under the assumption that the dropout process is governed by a Markov chain.

There is little prior work investigating the effect of dropouts on NCS performance. In [3] [4], networked control systems with dropouts are modelled as asynchronous switched systems. The approach replaces the true switched system with an "averaged system" and then provides some sufficient stability conditions on the system. Because only average dropout rates are used, the achieved results may be very conservative. Another important contribution is found in [5]. In this work, the dropouts are modelled by a Markov chain with two states and are treated as vacant sampling. This work proposes two approaches for handling data dropouts: using past control signals or estimating the lost data and computing new control signals. The stability of an optimal LQ controller under the two approaches is analyzed. This work, unfortunately, does not provide a rigorous analysis of the dropout model and only demonstrates the results through examples. A recent work [6] derived a closed form expression for the power spectral density of a single-input single-output discrete-time networked control system whose dropout process is modelled as an i.i.d. process. The scope of [6] is limited by its assumption on the i.i.d. nature of the dropout process.

The work in this paper extends [6] to dropout processes governed by Markov chains. We assume that the feedback measurements are randomly dropped with a distribution selected from an underlying Markov chain. The main result of this paper is an equation that expresses the power in the networked control system's output signal as a function of the Markov chain's probability transition matrix, Q. This result directly relates a measure of system performance (output signal power) that is commonly used by control system engineers to a measure of the network's quality of service (QoS) (dropout rate) that is commonly used by real-time system engineers. This result provides, in our opinion, a method by which real-time engineers can identify QoS measures that directly address NCS performance issues of interest to control system engineers.

The main result of this paper can be used to obtain an "optimal" specification on the network's quality of service (QoS). In particular, we use our equation for the output power to pose an optimization problem whose solution is the Markov chain that minimizes the output signal power for a fixed average dropout rate. The resulting Markov chain is interpreted as a specification on the network's optimal quality of service. This QoS specification is optimal in the sense that if our real-time system has a scheduler that can enforce the specified dropout behavior, then we can guarantee that no other soft real-time scheduler will result in an NCS whose output power is smaller. This paper demonstrates this observation by comparing the NCS's performance obtained with schedulers implementing the "optimal" Markov specification against the performance of schedulers enforcing soft (m, k) deadlines [7]. The (m, k)-firm guarantee model requires that at least k out of m consecutive packets be delivered by the network. Prior work [8] has empirically demonstrated that the (m, k)-model can greatly improve control system performance in overloaded networks. The results in this paper show, however, that an (m, k) policy for dropouts is not necessarily the optimal policy from the control system's point of view. This paper identifies a scheduling policy that apparently outperforms the (m, k)-policy for a specific example system.

The remainder of this paper is organized as follows. Section 2 goes through the mathematical preliminaries. Section 3 presents the assumed NCS model. Section 4 presents results on the asymptotic stability and wide-sense stationarity of switched discrete-time systems whose switching is controlled by Markov chains. The main result of the paper is an equation for the NCS output power. This equation is found in section 5. Section 6 uses the main result to pose an optimization problem whose solution specifies the "optimal" dropout process. This section then compares the performance of that optimal dropout policy against a soft (m, k) dropout policy. Concluding remarks will be found in section 7. The proofs of all theorems and technical lemmas have been moved to the appendix (section 8).

2. Mathematical Preliminaries

Let x be a random vector and let $\mathbf{E}[x]$ denote the expectation of x. A real-valued discrete-time stochastic process $x = \{x[n]\}$ is convergent in the mean square sense if there exists a random vector $\overline{x}$ such that

$$\lim_{n\to\infty} \mathbf{E}\left[(x[n] - \overline{x})^T (x[n] - \overline{x})\right] = 0.$$

It can be shown [9] that a random process $x = \{x[n]\}$ is *convergent in the mean square sense* if and only if

$$\lim_{n\to\infty} \sup_{m\geq n} \mathbf{E}\left[(x[m] - x[n])^T (x[m] - x[n])\right] = 0.$$

A *jump linear system* is a linear dynamical system whose system matrices are random processes. It has the form

$$\left\{ \begin{array}{lcl} x[n+1] & = & A[n]x[n] + B[n]w[n] \\ y[n] & = & C[n]x[n] + D[n]w[n] \end{array} \right. , \tag{2.1}$$

where $x = \{x[n]\}$ is the system's state process; $w = \{w[n]\}$ is a random process representing an exogenous input; $y = \{y[n]\}$ is the system's output process; and $\{A[n]\}$, $\{B[n]\}$, $\{C[n]\}$, and $\{D[n]\}$ are matrix valued random processes. This paper confines its attention to strictly proper jump linear systems, i.e. $D[n] = 0$. The state $x[n]$, therefore, depends on only the past parameters $\{(A[n-1], B[n-1]), (A[n-2], B[n-2]), (A[n-3], B[n-3]), \cdots\}$ and the past inputs $\{w[n-1], w[n-2], w[n-3], \cdots\}$.

If the input process $w[n] = 0$, then we say that system 2.1 is a free jump linear system. A free jump linear system is said to be *asymptotically stable in the mean square sense* [10] whenever

$$\lim_{n\to\infty} \mathbf{E}\left[x[n]^T x[n] \mid x_0\right] = 0$$

for any initial states $x[0] = x_0$. If the corresponding free jump linear system is asymptotically stable in the mean square sense, then the process $x = \{x[n]\}$ in equation 2.1 will eventually forget its dependence on the initial condition.

A random process $x = \{x[n]\}$ is said to be *wide sense stationary* (WSS) if its mean is constant and its covariance is shift invariant. In other words, $\{x[n]\}$ is WSS if and only if $\mathbf{E}\left[x[n]\right] = \text{constant} = \mu_x$ and $\mathbf{E}\left[(x[k]-\mu_x)(x[l]-\mu_x)^T\right] = \mathbf{E}\left[(x[k+n]-\mu_x)(x[l+n]-\mu_x)^T\right]$ for arbitrary n. Obviously if $\{x[n]\}$ is WSS, $\mathbf{E}\left[x[k]x^T[l]\right] = \mathbf{E}\left[x[k+n]x^T[l+n]\right]$ for arbitrary n. The mean of the WSS process $x = \{x[n]\}$ is denoted as μ_x and the correlation matrix of this process is denoted as $R_{xx}[m] = \mathbf{E}\left[x[n+m]x[n]^T\right]$, where n can be arbitrarily chosen because of the wide sense stationarity of x.

The *power semi-norm* of a WSS process $x = \{x[n]\}$ is

$$\|x\|_{\mathcal{P}} = \sqrt{\text{Trace}(R_{xx}[0])}. \tag{2.2}$$

Power semi-norm is taken as a measure of control system performance.

A random process $q = \{q[n]\}$ is called a *Markov chain* if its state space S is discrete,i.e. $S = \{1, 2, \cdots\}$, and for any $n \geq 2, t_1 < \cdots < t_n$ and any $i_1, \cdots, i_n \in S$

$$\Pr(q[t_n] = i_n \mid q[t_{n-1}] = i_{n-1}, \cdots, q[t_1] = i_1) = \Pr(q[t_n] = i_n \mid q[t_{n-1}] = i_{n-1}), \tag{2.3}$$

which is called Markov property.

A Markov chain is *time-homogeneous* if its probability distribution is time-invariant, i.e. for any states $i_1, \cdots, i_n$, any time instants $t_1, \cdots, t_n$ and any time shift L,

$$\Pr(q[t_n = i_n, \cdots, q[t_1] = i_1) = \Pr(q[t_n + L] = i_n, \cdots, q[t_1 + L] = i_1). \tag{2.4}$$

In this paper, all Markov chains are limited to time-homogeneous ones.

For a time-homogeneous Markov chain, the n step transition probability from state i to state j is defined as $q_{ij}[n] = \Pr(x[n+L] = j \mid x[L] = i)$,where L can be chosen arbitrarily. The one step transition probability $q_{ij}[1]$ is denoted as q_{ij} for simplicity. The *probability transition matrix* Q is defined as $Q = (q_{ij})$, i.e. the element of Q at the i-th row j-th column is q_{ij}.

A pair of states, i and j, *communicate* if there exist n and m such that $q_{ij}[n] > 0$ and $q_{ji}[m] > 0$. If any pair of states of a Markov chain communicate, we say the chain is *irreducible*.

If the greatest common divisor of $\{n \mid q_{ii}[n] > 0, i = 1, 2, \cdots\}$ is 1, the Markov chain is *aperiodic*.

For a time-homogeneous irreducible aperiodic Markov chain with $N (N < \infty)$ states, the following limit exists

$$\lim_{n\to\infty} q_{ij}[n] = \pi_j (i, j = 1, 2, \cdots, N). \tag{2.5}$$

The *limiting distribution* $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \end{bmatrix}$ can be computed from the following equation

$$\left\{ \begin{array}{rcl} \pi_j & = & \sum_{i=1}^N q_{ij}\pi_i \\ 1 & = & \sum_{i=1}^N \pi_i \\ \pi_j & > & 0 \end{array} \right. . \tag{2.6}$$

The limit, $\lim_{n\to\infty} \Pr(q[n] = j) = \pi_j)$, is called the *steady state* of the Markov chain. In this paper, the dropout Markov chain is time-homogeneous, irreducible and aperiodic. So the above limiting results are applicable. It is usually assumed that the dropout Markov chain always stays in the steady state.

The mean of x, $\mu_x = \mathbf{E}[x[n]]$, is the ensemble average of $x[n]$ (because x is WSS, $\mathbf{E}[x[n]]$ is constant for any n.). The *time average* of x for n steps can be defined as $\hat{\mathbf{E}}_n[x] = \frac{1}{n}\sum_{i=1}^n x[i]$. x is *ergodic* if

$$\lim_{n\to\infty} \hat{\mathbf{E}}_n[x] = \mu_x \ \text{ in the mean square sense.} \tag{2.7}$$

It can be shown [11] that a WSS process $x = \{x[n]\}$ is ergodic if

$$\lim_{m\to\infty} \mathbf{E}\left[(x[n+m] - \mu_x)(x[n] - \mu_x)^T\right] = 0.$$

When $\mu_x = 0$, the above condition is equivalent to $\lim_{m\to\infty} R_{xx}[m] = 0$.

Some of the technical proofs in section 8 make use of the *Kronecker product*, $\otimes$. The Kronecker product of two matrices $A = (a_{ij})_{M\times N}$, $B = (b_{pq})_{P\times Q}$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}B & a_{M2}B & \cdots & a_{MN}B \end{bmatrix}.$$

For simplicity, $A \otimes A$ is denoted as $A^{[2]}$ and $A \otimes A^{[n]}$ is denoted as $A^{[n+1]}$ $(n \geq 2)$.

For two vectors x and y, $x \otimes y$ simply rearranges the columns of xy^T into a vector. So for two WSS processes $\{x[n]\}$ and $\{y[n]\}$, $\lim_{n\to\infty} \mathbf{E}[x[n] \otimes y[n]] = 0$ if and only if $\lim_{n\to\infty} \mathbf{E}\left[x[n]y^T[n]\right] = 0$. It then follows that a zero-mean WSS process $x = \{x[n]\}$ is ergodic if

$$\lim_{m\to\infty} \mathbf{E}[x[n+m] \otimes x[n]] = 0.$$

The following property of Kronecker products will be frequently used in the technical proofs,

$$(A_1\ A_2 \cdots A_n) \otimes (B_1\ B_2 \cdots B_n) = (A_1 \otimes B_1)(A_2 \otimes B_2) \cdots (A_n \otimes B_n), \tag{2.8}$$

where $A_i, B_i (i = 1, 2, \cdots, n)$ are all matrices with appropriate dimensions.

In the computations on Kronecker product, two operators, **vec** and **devec**, are frequently used. They are defined as follows.

Definition 2.1. **vec** is applied to a matrix $A = (a_{ij})_{M\times N}$. $vec(A) = [a_{11}\ a_{21} \cdots a_{M1}\ a_{12} \cdots a_{M2} \cdots a_{1N} \cdots a_{MN}]^T$.

Definition 2.2. devec is the inverse of **vec** for a square matrix. When A is square, $devec\,(vec(A)) = A$.

It is straightforward to see that

$$\mathbf{E}\left[x\, y^T\right] = devec\left(\mathbf{E}\left[y \otimes x\right]\right),$$

for two random vectors with the same dimensions, x and y.

3. Data Dropout Model

A block diagram of a discrete-time networked control system is shown in figure 1. The system shown in this figure consists of a discrete-time loop function with strictly proper frequency response function $H(e^{j\omega})$. This loop function generates the output signal y. y drives a model of the data dropout process which generates the feedback signal $\overline{y}$. The plant $H(e^{j\omega})$ is driven by the input signal $u = \overline{y} + w$ which is simply the sum of an exogenous disturbance signal w and the feedback signal $\overline{y}$.

A state space representation of $H(e^{j\omega})$ is

$$\left\{ \begin{array}{rcl} x_P[n+1] & = & A_P x_P[n] + B_P u[n] \\ y[n] & = & C_P x_P[n] \end{array} \right. , \tag{3.1}$$

where $x_P[n] \in \mathcal{R}^{n_P}$, $u[n] \in \mathcal{R}^p$, $y[n] \in \mathcal{R}^m$, and the matrices A_P, B_P and C_P have the appropriate dimensions.

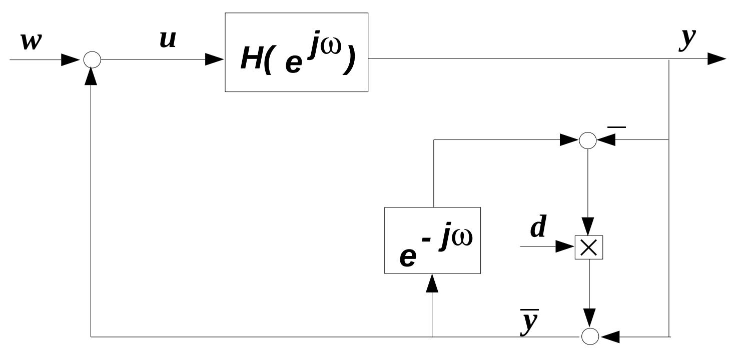

FIGURE 1. Networked Control System with Data Dropouts

The system shown in figure 1 has two inputs. The exogenous input disturbance process w is assumed to be white with zero mean. The other input is the dropout process $d = \{d[n]\}$. When $d[n] = 0$, $\overline{y}[n] = y[n]$, i.e. the measurement is sent out successfully; when $d[n] = 1$, $\overline{y}[n] = \overline{y}[n-1]$, i.e. the measurement is dropped and the past measurement is reused. d is random with the distribution selected from an underlying Markov chain. The Markov chain is time-homogeneous, irreducible and aperiodic with N $(N < \infty)$ states, $\{q_1, q_2, \cdots, q_N\}$, and the probability transition matrix, $Q = (q_{ij})_{N\times N}$. The state of the Markov chain at time step n is denoted as $q[n]$. We define a function $f(\cdot)$ such that $d[n] = f(q[n])$. The function f, therefore, maps the state of the Markov chain into the actual dropouts.

A state space representation of the NCS in figure 1 is

$$\left\{ \begin{array}{rcl} x[n+1] & = & A[n]x[n] + Bw[n] \\ y[n] & = & Cx[n] \end{array} \right. , \tag{3.2}$$

where $x[n] \in \mathcal{R}^{n_0} (n_0 = n_P + m)$, $w[n] \in \mathcal{R}^p$, $y[n] \in \mathcal{R}^m$. $\{A[n]\}$ is switching matrix-valued random process. When $d[n] = 0$, $A[n] = A^0$; when $d[n] = 1$, $A[n] = A^1$. $A^0 = \begin{bmatrix} A_P + B_P C_P & 0_{n_P \times m} \\ C_P & 0_{m \times m} \end{bmatrix}$, $A^1 = \begin{bmatrix} A_P & B_p \\ 0_{m \times n_P} & I_m \end{bmatrix}$, $B = \begin{bmatrix} B_P \\ 0_{m \times p} \end{bmatrix}$,$C = \begin{bmatrix} C_P & 0_{m \times m} \end{bmatrix}$, where $0_{n_P \times m}$ denotes a zero matrix with n_P rows m columns and the other matrices have the similiar meanings, I_m denotes an identity matrix with the dimension of m.

For convenience, $A[n]$ is denoted as A_i when $q[n] = q_i$. Obviously $A_i = A^0$ when $d[n] = f(q_i) = 0$; $A_i = A^1$ when $d[n] = f(q_i) = 1$. A very important matrix is constructed with A_i and Q:

$$A = \begin{bmatrix} q_{11} A_1 \otimes A_1 & q_{21} A_1 \otimes A_1 & \cdots & q_{N1} A_1 \otimes A_1 \\ q_{12} A_2 \otimes A_2 & q_{22} A_2 \otimes A_2 & \cdots & q_{N2} A_2 \otimes A_2 \\ \vdots & \vdots & \vdots & \vdots \\ q_{1N} A_N \otimes A_N & q_{2N} A_N \otimes A_N & \cdots & q_{NN} A_N \otimes A_N \end{bmatrix} \tag{3.3}$$

Throughout the paper, the following four assumptions are taken:

(1) w is a zero-mean white input noise with finite variance $R_{ww}[0]$
(2) The dropout Markov chain is time-homogeneous, irreducible and aperiodic with N $(N < \infty)$ states. The probability transition matrix is $Q = (q_{ij})_{N \times N}$. The steady state of the Markov chain is $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \end{bmatrix}$, which is computed from equation 2.6.
(3) The disturbance process, w, is independent from the dropouts.
(4) The matrix, A defined in equation 3.3, is asymptotically stable, i.e. $\lambda_{max}(A) = \lambda_0 < 1$, where $\lambda_{max}(\cdot)$ denotes the maximum magnitude of the eigenvalues of a matrix.

Under assumption 2, we know that the Markov chain will eventually reach the steady state shown in equations 2.5 and 2.6. Throughout this paper, we assume the Markov chain stays in its steady state.

4. Stability and Wide Sense Stationarity

In order to study control system performance (as measured by the output's power semi-norm), we first need to show that the NCS is stable and the output process is wide sense stationary. Theorem 4.1 provides a necessary and sufficient condition for the NCS to be asymptotically stable in the mean square sense. Based on this stability result, a sufficient condition for the wide sense stationarity of the state and the output are presented in theorem 4.2 and corollary 4.3, respectively. These results are based in large part upon results cited or proven in [12] and [13].

Theorem 4.1. *Consider the free jump linear system ($w = 0$) given in equation 3.2 under assumption 2. The free jump linear system is asymptotically stable in the mean square sense if and only if assumption 4 is valid.*

Remark: Theorem 4.1 shows that the stability of the NCS depends on not only the system matrices A^0 and A^1, but also the probability transition matrix Q which describes the dropout rule in a random manner.

Remark: The proof of the sufficient part of theorem 4.1 is repeated from [13]. [13] claims that necessity is obvious, but does not formally prove the result. We provide a complete formal proof of the theorem's necessity in the appendix. To our best knowledge the necessity proof is new.

The disturbance input signal, w, is assumed to be WSS. The following theorem shows that the state of the NCS is also WSS under certain conditions.

Theorem 4.2. *Consider the NCS in equation 3.2. Under assumptions 1—4, the state process* $x = \{x[n]\}$ *is wide sense stationary.*

Remark: It is not surprising that x is WSS, because the input w is WSS, the dropouts d are time-homogeneous and d is independent from w. The proof is somewhat standard. We've included it in the appendix to make the paper self-contained. The following corollary shows that all linear outputs of the NCS are WSS.

Corollary 4.3. *Consider the NCS in equation 3.2. Under assumptions 1—4, any linear output* $\{\underline{y}_n = \underline{C}[n]x[n] + \underline{D}[n]w[n]\}$ *is WSS, where* $\underline{C}[n] = \underline{C}^0$, $\underline{D}[n] = \underline{D}^0$ *when* $d[n] = 0$; $\underline{C}[n] = \underline{C}^1$, $\underline{D}[n] = \underline{D}^1$ *when* $d[n] = 1$.

Remark: The proof of corollary 4.3 is similar as the one of theorem 4.3. So the proof is omitted. By corollary 4.3, it follows that any two linear outputs, $\{\underline{y}_1[n]\}$ and $\{\underline{y}_2[n]\}$ are mutually WSS by considering wide sense stationarity of the combined linear output $\{\underline{y}[n] = [(\underline{y}_1[n])^T \ (\underline{y}_2[n])^T]^T\}$. The state x can also be treated as a linear output. So x and all linear outputs are mutually WSS. Similarly w and all linear outputs are also mutually WSS.

5. Main Result

For a system with noise as the input, the output signal's power semi-norm may be taken as a measure of the control system's performance. In this section, an equation for the output's power semi-norm is computed. The computed power semi-norm is a function of the dropout Markov chain's probability transition matrix. This equation is presented in theorem 5.1.

Experimental results are presented to support the correctness of the result. In the experiment, the power semi-norm for an NCS is predicted using the result of theorem 5.1 for various average dropout rates. A simulation of the NCS was used to compute the time average of the output signal's power semi-norm. Theorem 5.2 states and proves sufficient conditions for the NCS to be ergodic. Using this theorem, we identify an interval of dropout rates over which the example system is known to be ergodic. The predicted and time-averaged power semi-norms of the output agree closely for those dropout rates over which ergodicity can be guaranteed. These experimental results therefore appear to support the correctness of paper's main result.

If the NCS is asymptotically stable in the mean square sense, then the initial state will be eventually forgotten. So if we set the initial time as $-\infty$, the initial state may be taken as zero without loss of generality. In a similar way, we can assume that the dropout Markov chain is at steady state. In the following theorems, these assumptions on the initial time, initial state and Markov chain's steady state will be used without explicit explanations.

Theorem 5.1. *Consider the NCS in equation 3.2 under assumptions 1—4. Let the initial time be* $-\infty$*. Denote the conditional correlations:* $P_i = \pi_i \mathbf{E}\left[x_n \otimes x_n \mid q[n-1] = q_i\right]$ *for* $i = 1, 2, \cdots, N$*. Then the power semi-norm of* y *can be computed through*

$$\|y\|_{\mathcal{P}} = \sqrt{\text{Trace}\left(devec\left(C^{[2]}\sum_{i=1}^{N} P_i\right)\right)}. \tag{5.1}$$

where P_i *satisfies the equation*

$$P_i = A_i^{[2]}\sum_{k=1}^{N} q_{ki}P_k + \pi_i B^{[2]}\mu_{w_2},$$

$A_i = A^0$ *when* $f(q_i) = 0$*,* $A_i = A^1$ *when* $f(q_i) = 1$*, and* $\mu_{w_2} = vec(R_{ww}[0])$*.*

Remark: The power semi-norm in equation 5.1 is a function of the probability transition matrix Q as well as the closed-loop parameters A^0 and the open-loop parameters A^1. So the power semi-norm can be predicted with the system parameters and Q, i.e. the dropout rule.

We applied theorem 5.1 on a simple NCS. The assumed plant was an unstable system with transfer function of $H(e^{j\omega}) = \frac{e^{j\omega}+2}{e^{j2\omega}+e^{j\omega}+2}$. Assuming unity gain feedback, the closed loop transfer function becomes $e^{-j\omega} + 2e^{-j2\omega}$, which is stable. The dropout process therefore switches our system between a stable and unstable configuration. The input noise, w, is white zero-mean Gaussian. The dropout Markov chain has $N = 3$ states; its probability transition matrix is $Q = \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, where ε is a parameter between 0 and 1. This particular Markov chain specifies a soft (m, k) scheduling model with $m = 3$, $k = 2$ [14].

We used theorems 4.1 and theorem 4.2 to determine that the system is asymptotically stable in the mean square sense and wide sense stationary when $\varepsilon \leq 24.9\%$. We then used theorem 5.1 to numerically evaluate the power semi-norm for dropout rates, ε, between 0 and 24%. The predicted power levels are plotted in figure 2 as a function of ε. The plots show that as the dropout rate increases, the output's power semi-norm approaches infinity, which is expected since we are switching between a stable closed loop and unstable open-loop system. The question is whether or not the theorem accurately predicts how quickly the performance degrades as we increase ε?

To answer this question, we created a `Matlab simulink` model of the example system. The simulation was used to experimentally generate the system's output signal under various ε. We used these output traces to compute a time average of the output signal's power. This time average was taken as an estimate of the system's true expected output power, $\mathbf{E}\left[y^{[2]}\right]$. For a sample path of length $2L$ time steps, the time average is computed as

$$\hat{\mathbf{E}}_L\left[y^{[2]}\right] = \frac{1}{L}\left(y^{[2]}[L+1] + y^{[2]}[L+2] + \cdots + y^{[2]}[L+L]\right). \tag{5.2}$$

The following theorem provides the conditions for $\hat{\mathbf{E}}_L\left[y^{[2]}\right]$ to efficiently approach $\mathbf{E}\left[y^{[2]}\right]$.

Theorem 5.2. *Under the assumptions of theorem 5.1, if the following two additional assumptions are satisfied,*

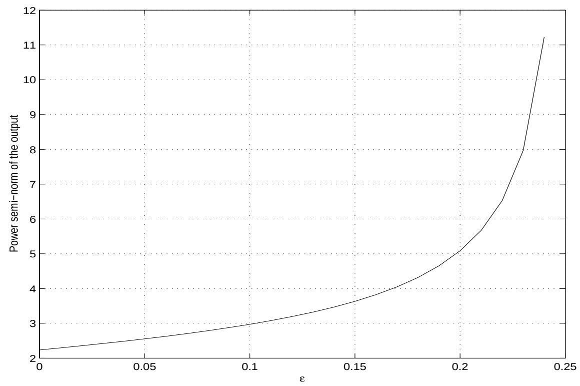

FIGURE 2. Theoretical Predition of Output's Power Semi-Norm ($0 \leq \varepsilon \leq 24\%$)

(1) *All the fourth order moments of w exist,* $\mathbf{E}\left[w[n+k_1] \otimes w[n+k_2] \otimes w[n+k_3] \otimes w[n+k_4]\right]$ *is shift-invariant with respect to n, and* $\mathbf{E}\left[w[k_1] \otimes w[k_2] \otimes w[k_3] \otimes w[k_4]\right] = 0$ *except for the following 3 cases:*
   - $k_1 = k_2$ *and* $k_3 = k_4$*;*
   - $k_1 = k_3$ *and* $k_2 = k_4$*;*
   - $k_1 = k_4$ *and* $k_2 = k_3$*.*

   Moreover, $\mathbf{E}\left[(w[n])^{[4]}\right] = \mu_{w_4} < \infty$, $\mathbf{E}\left[(w[n])^{[2]}\right] = \mu_{w_2}$, *and* $\mathbf{E}\left[(w[k])^{[2]} \otimes (w[l])^{[2]}\right] = (\mu_{w_2})^{[2]}$ *when* $k \neq l$*.*

(2) *All eigenvalues of* $A_{(4)} = diag(A_i^{[4]})(Q^T \otimes I_{n_0^4})$ *lie inside the unit circle, where n_0 is the dimension of the NCS.*

Then

$$\lim_{L \to \infty} \hat{\mathbf{E}}_L\left[y^{[2]}\right] = \mathbf{E}\left[y^{[2]}\right] \text{ in the mean square sense,} \tag{5.3}$$

where $\hat{\mathbf{E}}_L\left[y^{[2]}\right]$*, the time average of the process* $\{y^{[2]}[n]\}$*, is computed through equation 5.2.*

Remark: This theorem may be used to determine an upper bound, $\bar{\varepsilon}$ such that for $\varepsilon < \bar{\varepsilon}$ the time-averaged power (equation 5.2) is guaranteed to converge to the true ensemble average predicted in theorem 5.1. So for this range of ε, we can verify theorem 5.1 by simply comparing $\hat{\mathbf{E}}_L\left[y^{[2]}\right]$ and $\mathbf{E}\left[y^{[2]}\right]$. For $\varepsilon > \bar{\varepsilon}$, no conclusions can be drawn about the correctness of the theorem in this manner, because we cannot guarantee the ergodicity of the NCS.

Remark: If w is generated by passing white noise w_0 through a stable linear filter and w_0 satisfies all the requirements in assumption (1) of theorem 5.2, then theorem 5.2 is still valid. The proof is straightforward and is not included in this paper.

For the example in this paper, theorem 5.2 guarantees the convergence of the estimate in equation 5.3 if ε is below 6.1%. This upper bound 6.1% is smaller than the upper bound (24.9%) by theorem 5.1.

The networked control system was simulated with various dropout rates between 0 and 6%. For each value of ε, we ran 5 different simulations for $200,000$ time steps and then estimated the power semi-norm by the time average in equation 5.2. The simulation results are shown in figure 3. The figure shows close agreement between the predicted and the experimentally estimated power semi-norms of the output. So we have high confidence in the correctness of the results stated in theorems 5.1 and 5.2.

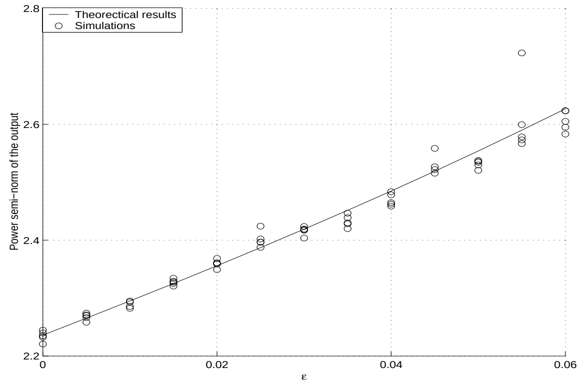

FIGURE 3. Output's Power Semi-Norm ($0 \leq \varepsilon \leq 6\%$)

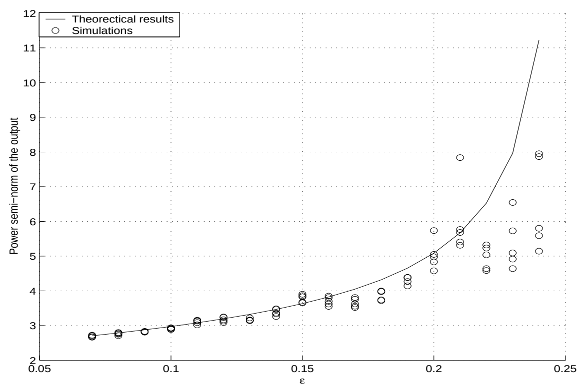

FIGURE 4. Output's Power Semi-Norm ($7\% \leq \varepsilon \leq 24\%$)

For completeness, we also present results for the networked control system with dropout rates between 7% and 24%. Figure 4 shows that the predicted and the experimentally estimated power semi-norms of the outputs disagree under large dropout rates. As discussed early, the disagreement comes from the violation of the ergodicity conditions in theorem 5.2. Figure 4 still shows close agreement for dropout rates less than 10%. This thereby shows that theorem 5.2 is somewhat conservative in its estimate of the interval over which the system is ergodic.

Remark: To our best knowledge, the results in theorem 5.1 and 5.2 are new. It is useful to compare the results in this paper to those obtained in [6]. The overall approach followed in this section is similar to our earlier work for i.i.d. dropouts [6]. There are, however, some important differences. The first obvious difference is that this paper's results pertain to a more general class of dropout processes than those assumed in [6]. The general approach used in both papers is different. In [6], we focused on an input-output characterization of the NCS, so that [6]'s main result actually obtained a closed form expression for the output signal's power spectral density. In this paper, we adopted a state-based approach. The state-based approach only yielded an equation for computing the power semi-norm, so that the analytical characterization of the spectral density is missing. In return for this, however, it should be apparent that the results in this paper apply to multi-input multi-output systems ([6] restricted its attention to SISO systems).

6. Optimal Dropout Policies and Scheduling

Theorem 5.1 presents an equation in which the output's power semi-norm, $\|y\|_{\mathcal{P}}$, is a function of the Markov chain's probability transition matrix Q. An obvious thing to do now is to minimize $\|y\|_{\mathcal{P}}$ with respect to Q. This section poses this optimization problem for the example system used in section 5, solves for the optimal Q and compares the performance of this Markov dropout process against the soft $(3, 2)$-dropout rule that was analyzed in section 5. The results in this section are purely empirical as we don't attempt to establish conditions under which the optimization is well-posed. The purpose of this section is to merely demonstrate the feasibility of determining an optimal dropout process and then use the example to suggest a possible use for this result.

For mathematical tractability, let's consider a special class of Markov chain. The state of the chain at time instant n is determined by the last two consecutive values of the dropout process. We assume there are four states, q_1, q_2, q_3 and q_4 which are defined as follows

$$\begin{aligned} q_1[n] &: d[n-1]=0, d[n]=0,\\ q_2[n] &: d[n-1]=0, d[n]=1,\\ q_3[n] &: d[n-1]=1, d[n]=0,\\ q_4[n] &: d[n-1]=1, d[n]=1. \end{aligned}$$

When $q[n] = q_i$ $(i = 1, 2, 3, 4)$, the next packet is dropped with the probability ε_i. In other words, the probability of delivering the packet is $\Pr(d[n+1] = 0 \mid q[n] = q_i) = 1-\varepsilon_i$. With these notational conventions, the probability transition matrix for this Markov chain is

$$Q = \begin{bmatrix} 1-\varepsilon_1 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & 1-\varepsilon_2 & \varepsilon_2 \\ 1-\varepsilon_3 & \varepsilon_3 & 0 & 0 \\ 0 & 0 & 1-\varepsilon_4 & \varepsilon_4 \end{bmatrix}.$$

This dropout process satisfies assumption 2, therefore it has a steady state distribution $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}$ which is a function of ε_i. We define the *average dropout rate* as $\overline{\varepsilon} = \sum_{i=1}^{4} \pi_i \varepsilon_i$. With these definitions, we can formally state the optimization problem as follows:

$$\begin{array}{ll} \text{minimize:} & \sqrt{\text{Trace}\left(devec\left(C^{[2]}\sum_{i=1}^{N} P_i\right)\right)}, \\ \text{with respect to:} & \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \\ \text{subject to:} & \overline{\varepsilon} \geq \varepsilon_0, \\ & P_i = A_i^{[2]}\sum_{k=1}^{N} q_{ki}P_k + \pi_i B^{[2]}\mu_{w_2}, \\ & \pi = Q\pi, \\ & 1 = \sum_{i=1}^{N} \pi_i, \\ & \lambda_{\max}(A) < 1. \end{array} \tag{6.1}$$

The constraints in this optimization problem are taken directly from theorem 5.1. The last constraint ensures that the solution is stable in the mean square sense.

We used `Matlab`'s optimization toolbox to numerically solve this optimization problem. This problem, essentially, has two constraints: the stability constraint $\lambda_{\max}(A) < 1$; and the average dropout rate constraint, $\overline{\varepsilon} \geq \varepsilon_0$. The optimization problem was solved by using a standard gradient-descent algorithm. In particular, we used the `Matlab` function `fmincon()` after a suitable initial condition was identified. The initial condition for our problem had to be stable and this condition was identified by solving the preliminary optimization problem given below:

$$\begin{array}{ll} \text{minimize:} & (\overline{\varepsilon} - \varepsilon_0)^2 \\ \text{subject to:} & \lambda_{\max}(A) < 1 \end{array}$$

Using the initial condition $\varepsilon_i = 0$ (for $i = 1, \ldots, 4$), we used `fmincon()` to identify a stable solution and then used this as the initial condition for the full optimization problem (equation 6.1). For this stable initial condition, the solution to the full optimization problem was obtained using `fmincon()`. The optimal ε_i are plotted on the left hand side of figure 5 for various average dropout rates, ε_0 between 0 and 0.5. Directed graphs representing the optimal dropout policy and the $(3, 2)$ policy used in section 5 are shown on the right hand side of figure 5

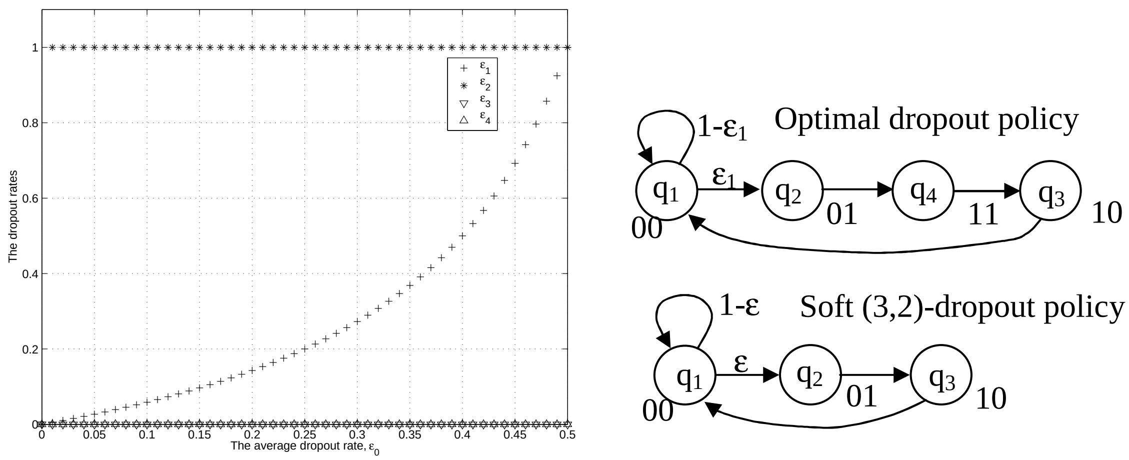

FIGURE 5. Optimal ε for $0 \leq \varepsilon_0 \leq 0.5$ (left) and the Dropout Policies (right) Used in This Comparison.

It is interesting to note that the optimal dropout policy only allows two types of dropout sequences. Either there are no dropouts or there are two consecutive dropouts following by two successes. This optimal rule runs counter to the heuristic inherent in the $(3, 2)$-rule. A direct comparison of the two policies will be found in figure 6. This figure shows the estimated power semi-norm for the NCS under the optimal dropout policy (plus), an i.i.d. dropout policy (circle), and the $(3, 2)$-rule (asterisk). The estimated power semi-norm is plotted as a function of the average dropout rate ε_0. What is apparent here is that the optimal dropout policy achieves lower power levels over a wide range average dropout rates. What is surprising, however, is that the $(3, 2)$-heuristic appears to actually be worse than an i.i.d. dropout process. This observation seems to imply that the heuristic inherent in the (m, k)-firm guarantee is not necessarily meaningful for all feedback control systems.

We found the optimization exercise in this section interesting for a number of reasons. To understand these reasons, we first digress to talk about the way in which real-time system engineers specify the quality of service on feedback streams. Real-time system engineers must build the computer

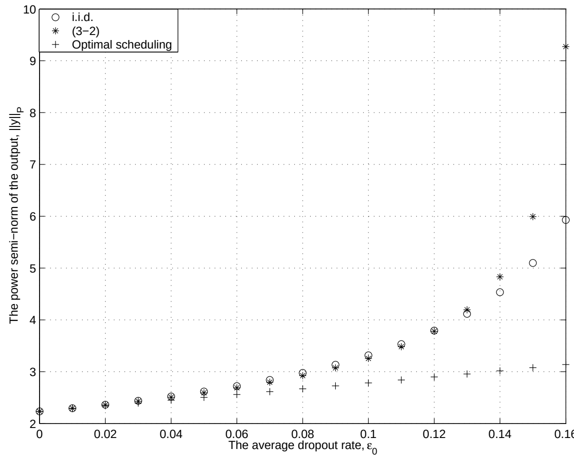

FIGURE 6. Comparisons of the three dropout models

networks that guarantee (as much as possible) the delivery of packets across the network by a specified deadline. This is particularly difficult in networks where congestion, scheduling conflicts, or processor faults may result in network overloading. In an overloaded network, the system is no longer capable of delivering their packets by the specified deadline. A common strategy used to handle this problem is to simply drop packets that are excessively delayed. This is done in soft real-time control systems as well as multi-media streaming applications. The problem is that there appears to be no clear way of relating a chosen dropout policy to the performance of the overall system.

An example of this is seen with the (m, k)-firm guarantee model. The (m, k)-firm guarantee model seeks to guarantee the delivery of at least k out of m consecutive packets. In [8] it was shown that such a policy can greatly improve the performance of overloaded network control systems. This rule, however, is based on a simple (though reasonable) heuristic that a fixed percentage of the packets should be delivered at a regular rate. Other than simulation and empirical studies, however, there is no reason to believe that such heuristics are reasonable for feedback control systems. The underlying model presented in section 3 is a switched linear system, and it is well known from the hybrid system's community [15] that certain switching sequences can have a dramatic impact on overall system stability. The (m, k)-heuristic does not take this possibility into account and this fact suggests that there may be many situations in which the heuristic may actually do more harm than good. The question is whether we can identify such "bad" switching sequences without having to resort to exhaustive simulation?

The results in theorem 5.1 and the optimization exercise in this section seem to shed some light on this question. This result seems to provide a direct relationship between the control system performance and the dropout process' probability transition matrix. Moreover, as demonstrated in this section, it may be possible to find "optimal" dropout process' that optimize the control system performance. To some extent, we can see the optimization as "searching" for those "bad" switching sequences and excising them from the dropout rule. In this regard, the optimization presented in this paper seems to provide a systematic way of designing dropout policies that are based on an analytical understanding of the closed loop system. This may be a significant improvement over current practice in which heuristics are proposed and then evaluated primarily through simulation analysis.

7. Conclusions

This paper studied networked control systems with data dropouts that were generated by an underlying Markov chain. The main result of this paper is an equation that computes the output's power semi-norm as a function of the system state realization and the probability transition matrix of the dropout Markov chain. We also established sufficient conditions under which we can expect the NCS to be ergodic so that time-averages of the system's signals can be used to reliably evaluate NCS performance. The results in this paper extend earlier results in [6] to multi-input multi-output systems as well as to a more general class of dropout processes.

Probably one of the most significant aspects of these results is that they can be used to place specifications on the quality of service provided by networks in support of real-time networked control systems. The equation in theorem 5.1 may be used to pose an optimization problem whose solution is the Markov chain that optimizes the NCS's performance as measured by its power semi-norm. The importance of this "optimal" Markov chain is that it specifies the "optimal" dropout process in a way that includes the soft (m, k)-policy, a heuristic policy that is widely used in soft real-time applications. The results in this paper appear to provide a practical way by which control system engineers can provide real-time system engineers with QoS specifications that truly assure the closed loop system's overall performance. Our future work is studying whether nor not these results can provide the basis for a systematic method for the "design" of optimal scheduling policies, a design approach that takes analysis, rather than simulation, as the primary means for verifying a scheduling policy's impact on closed loop system performance.

8. Appendix

To enhance the readability of the manuscript we moved all of the technical lemmas and proofs to this section. This section proves theorems 4.1, 4.2, 5.1 and 5.2. Throughout this section, the following notational conventions will be followed. The system transition matrix for the NCS in equation 3.2 is

$$\Phi(n;m) = \begin{cases} \prod_{l=m}^{n-1} A[l], & \text{if } m < n \\ I_{n_0}, & \text{if } m \geq n \end{cases},$$

where I_{n_0} is an identity matrix with the dimension of n_0. With this matrix, the system's state at time instant n can be expressed as $x[n] = \Phi(n;0)x[0]$. A $(Nn_0) \times n_0$ matrix C_I is defined as

$$C_I = \begin{bmatrix} I_{n_0} & I_{n_0} & \cdots & I_{n_0} \end{bmatrix}.$$

Proof of Theorem 4.1: These proofs are similar to proofs in [12] and [13]. The chief novelty here appears to lie in our proof for the necessary part of the theorem.

First we prove the NCS is asymptotically stable in the mean square sense when assumption 4 is valid. For initial time 0, let the initial state of the NCS in equation 3.2 be $x[0] = x_0$ and the initial distribution of $q[0]$ be $p = \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}$ $(\Pr(q[0] = q_i) = p_i)$. Then $\mathbf{E}\left[(x^{[2]}[n]) \mid x_0\right] =$

$\mathbf{E}\left[(\Phi(n;0))^{[2]}\right] x_0^{[2]}$. Define $\Phi_i[n] = \Pr(q[n-1] = q_i)\mathbf{E}\left[(\Phi(n;0))^{[2]} \mid q[n-1] = q_i\right](i = 1, 2, \cdots, N)$. Then

$$\mathbf{E}\left[(\Phi(n;0))^{[2]}\right] = \sum_{i=1}^{N} \Phi_i[n]. \tag{8.1}$$

$\Phi_i[n]$ can be recursively computed as follows.

$$\begin{aligned}
\Phi_i[n] &= \Pr(q[n-1] = q_i)\mathbf{E}\left[(A[n-1]\Phi(n-1;0))^{[2]} \mid q[n-1] = q_i\right] \\
&= \Pr(q[n-1] = q_i)\mathbf{E}\left[(A[n-1])^{[2]}(\Phi(n-1;0))^{[2]} \mid q[n-1] = q_i\right] \\
&= \Pr(q[n-1] = q_i)A_i^{[2]}\mathbf{E}\left[(\Phi(n-1;0))^{[2]} \mid q[n-1] = q_i\right] \\
&= \Pr(q[n-1] = q_i)A_i^{[2]}\sum_{k=1}^{N}\mathbf{E}\left[(\Phi(n-1;0))^{[2]} \mid q[n-1] = q_i, q[n-2] = q_k\right] \\
&\quad \Pr(q[n-2] = q_k \mid q[n-1] = q_i) \\
&= \Pr(q[n-1] = q_i)A_i^{[2]}\sum_{k=1}^{N}\mathbf{E}\left[(\Phi(n-1;0))^{[2]} \mid q[n-2] = q_k\right]\frac{q_{ki}\Pr(q[n-2] = q_k)}{\Pr(q[n-1] = q_i)} \\
&= \sum_{k=1}^{N} q_{ki}A_i^{[2]}\Phi_k[n-1],
\end{aligned}$$

where the second equality comes from the property of Kronecker product in equation 2.8; the fifth equality comes from the Markov property of the dropout Markov chain in equation 2.3.

Combine all $\Phi_i[n](i = 1, 2, \cdots, N)$ into a bigger matrix

$$V_\Phi[n] = \begin{bmatrix} \Phi_1^T[n] & \Phi_2^T[n] & \cdots & \Phi_N^T[n] \end{bmatrix}^T. \tag{8.2}$$

Then

$$V_\Phi[n] = A \;\; V_\Phi[n-1]. \tag{8.3}$$

The solution of the above equation is

$$V_\Phi[n] = A^n V_\Phi[0], \tag{8.4}$$

where $V_\Phi[0] = p^T \otimes I_{n_0^2}$.

Obviously when assumption 4 is valid, i.e. $\lambda_{max}(A) < 1$, $\lim_{n\to\infty} V_\Phi[n] = 0$. Because $\mathbf{E}\left[(\Phi(n;0))^{[2]}\right] = C_I V_\Phi[n]$, $\lim_{n\to\infty}\mathbf{E}\left[(\Phi(n;0))^{[2]}\right] = 0$. Then we get $\lim_{n\to\infty}\mathbf{E}\left[x^{[2]}[n] \mid x_0\right] = 0$ by equation 8.1 for any initial state x_0 and any initial distribution of $q[0]$, i.e.the NCS is asymptotically stable in the mean square sense.

Second we prove that assumption 4 is valid for an asymptotically stable NCS. When the free jump linear systm is asymptotically stable in the mean square sense, i.e. $\lim_{n\to\infty}\mathbf{E}\left[x^{[2]}[n] \mid x_0\right] = 0$ for any initial state x_0 and any initial distribution of $q[0]$, we get $\lim_{n\to\infty}\mathbf{E}\left[(\Phi[n,0])^{[2]}\right] = 0$. In the following, we first prove that $\lim_{n\to\infty}\Phi_i[n] = 0$ for all $i \in \{1, 2, \cdots, N\}$.

Choose any $z_0, w_0 \in \mathcal{R}_0^n$, then

$$\mathbf{E}\left[(z_0^{[2]})^T(\Phi(n;0))^{[2]}w_0^{[2]}\right] = \mathbf{E}\left[\left(z_0^T\Phi(n;0)w_0\right)^2\right].$$

By equation 8.1, we get

$$\mathbf{E}\left[\left(z_0^T\Phi[n,0]w_0\right)^2\right] = \sum_{i=1}^{N}\Pr(q[n-1]=q_i)\mathbf{E}\left[\left(z_0^T\Phi[n,0]w_0\right)^2 \mid q[n-1]=q_i\right]. \tag{8.5}$$

Because $\lim_{n\to\infty}\mathbf{E}\left[\left(z_0^T\Phi(n;0)w_0\right)^2\right]=0$ and $\lim_{n\to\infty}\Pr(q[n-1]=q_i)=\pi_i>0$ by equation 2.6, it follows that

$$\lim_{n\to\infty}\mathbf{E}\left[\left(z_0^T\Phi(n;0)w_0\right)^2 \mid q[n-1]=q_i\right]=0,$$

for any z_0 and w_0, i.e. $\lim_{n\to\infty}(z_0^{[2]})^T\Phi_i[n]w_0^{[2]}=0$ for any z_0 and w_0. Then we know

$$\lim_{n\to\infty}\Phi_i[n]=0. \tag{8.6}$$

By the above equation and the definition of $V_\Phi[n]$, we know $\lim_{n\to\infty}V_\Phi[n]=0$. Then equation 8.4 yields that

$$\lim_{n\to\infty}A^n(p^T\otimes I_{n_0}) \tag{8.7}$$

for any initial distribution p. By taking p as $[1,0,0,\cdots]$, $[0,1,0,\cdots]$, $\cdots$, we can easily get

$$\lim_{n\to\infty}A^n=0. \tag{8.8}$$

So $\lambda_{max}(A)<1$, i.e. assumption 4 is valid. The proof is completed. $\diamondsuit$

Based on the computations on $\Phi(n;0)$ in the proof of theorem 4.1, we can put a bound on $\mathbf{E}\left[\Phi(n;0)\Phi^T(n;0)\right]$ in the follwing lemma.

Lemma 8.1. *Consider the NCS in equation 3.2 under assumptions 1—4. There exists a nonnegative matrix Φ_0 such that*

$$\mathbf{E}\left[\Phi(n;0)\Phi^T(n;0)\right]\le\sigma_0^n\Phi_0.$$

Proof: From the proof of theorem 4.1, we can get $V_\Phi[n]=A^nV_\Phi[0]$, $\lambda_{max}(A)=\sigma_0$. Because $\mathbf{E}\left[\Phi(n;0)\Phi^T(n;0)\right]=devec\left(C_IV_\Phi[n]\right)$, the lemma can be proven easily. $\diamondsuit$

The shift-invariance property of the dropout Markov chain yields that
$\mathbf{E}\left[\Phi(n;k)\Phi^T(n;k)\right]=\mathbf{E}\left[\Phi(n-k;0)\Phi^T(n-k;0)\right]$. So the following upper bound can be put

$$\mathbf{E}\left[\Phi(n;k)\Phi^T(n;k)\right]\le\sigma_0^{(n-k)}\Phi_0. \tag{8.9}$$

When the NCS in equation 3.2 is asymptotically stable in the mean square sense, we can ignore the initial state by taking the initial time at $-\infty$. It is very important to derive the expression of $x[n]$. The following lemma gives an expression.

Lemma 8.2. *Consider the NCS in equation 3.2 with initial time of $-\infty$. Under assumptions 1—4, the state $x[n]$ can be expressed with the following infinite series.*

$$x[n]=\sum_{k=0}^{\infty}\Phi(n;n-k)Bw[n-k-1]. \tag{8.10}$$

Furthermore,

$$\mathbf{E}\left[x[n]x^T[n]\right]<\infty.$$

Proof: If the infinite series in equation 8.10 makes sense, $x[n]$ can obviously be computed as the equation. So we just need to prove that the infinite series in equation 8.10 is convergent in the mean square sense.

Let $e_i \in \mathcal{R}^{n_0}$, whose ith element is 1 and all other elements are zero. The ith component of equation 8.10 can be expressed as

$$e_i^T x[n] = \sum_{k=0}^{\infty} e_i^T \Phi(n; n-k) B w[n-k-1]. \tag{8.11}$$

Let $S(p,q)$ denote the partial summation

$$S(p,q) = \sum_{k=p}^{p+q} e_i^T \Phi(n; n-k) B w[n-k-1], \tag{8.12}$$

where $p \geq 0$, $q \geq 1$.

It can be shown that

$$\sqrt{\mathbf{E}\left[S(p,q)^2\right]} \leq \sum_{k=p}^{p+q} \sqrt{\mathbf{E}\left[e_i^T \Phi(n; n-k) B w[n-k-1]\right]^2}. \tag{8.13}$$

Now consider a single term of the summation in equation 8.13.

$$\begin{aligned} \mathbf{E}\left[e_i^T \Phi(n; n-k) B w[n-k-1]\right]^2 &= \mathbf{E}\left[e_i^T \Phi(n; n-k) B w[n-k-1] w^T[n-k-1] B^T \Phi^T(n; n-k) e_i\right] \\ &= \mathbf{E}\left[e_i^T \Phi(n; n-k) B R_{ww}[0] B^T \Phi^T(n; n-k) e_i\right]. \end{aligned}$$

Let $\sigma_B = \lambda_{max}(B R_{ww}[0] B^T)$, then

$$\mathbf{E}\left[e_i^T \Phi(n; n-k) B R_{ww}[0] B^T \Phi^T(n; n-k) e_i\right] \leq \sigma_B e_i^T \mathbf{E}\left[\Phi(n; n-k) \Phi^T(n; n-k)\right] e_i. \tag{8.14}$$

By equation 8.9, we can put an upper bound:

$$\mathbf{E}\left[e_i^T \Phi(n; n-k) B w_{n-k-1}\right]^2 \leq \sigma_B \sigma_0^k e_i^T \Phi_0 e_i. \tag{8.15}$$

With the preceding relation, we get

$$\begin{aligned} \sqrt{\mathbf{E}\left[S(p,q)^2\right]} &\leq \sqrt{\sigma_B e_i^T \Phi_0 e_i} \sum_{k=p}^{p+q} \sigma_0^k \\ &\leq M \sigma_0^p, \end{aligned}$$

where $M = \frac{\sqrt{\sigma_B e_i^T \Phi_0 e_i}}{1-\sigma_0}$.

Because $\sigma_0 < 1$, $\lim_{p\to\infty} \sup_{q\geq 1} \mathbf{E}\left[S^2(p,q)\right] = 0$. So we know that the summation in equation 8.11 is convergent in the mean square sense.

If we set $p = 0$ and $q = \infty$, $S(p,q) = e_i^T x[n]$. So

$$\begin{aligned} \mathbf{E}\left[(e_i^T x[n])^2\right] &\leq M \\ &< \infty. \end{aligned}$$

Because i is choosen arbitrarily, we can get the convergence of the infinite series in equation 8.10 and the finiteness of $\mathbf{E}\left[x[n]x^T[n]\right]$. $\diamondsuit$

Proof of theorem 4.2: As shown by lemma 8.2, $x[n]$ exists in the mean square sense and has finite correlation. So we just need to prove the mean of $x[n]$ is constant and the correlation $\mathbf{E}\left[x[n+m]x^T[n]\right]$ is shift-invariant with respect to n.

Rewrite the expression of $x[n]$ from equation 8.10

$$x[n] = \sum_{k=0}^{\infty} \Phi(n; n-k) B w[n-k-1]. \tag{8.16}$$

So the mean of $x[n]$ can be computed as

$$\begin{aligned}\mathbf{E}[x[n]] &= \sum_{k=0}^{\infty} \mathbf{E}[\Phi(n; n-k)B]\mathbf{E}\left[w[n-k-1]\right] \\ &= 0,\end{aligned}$$

where the first equality follows from the independence between dropouts and w; the second equality follows from the fact w is zero-mean.

The correlation $\mathbf{E}\left[x[n+m]x^T[n]\right]$ can be expressed as

$$\begin{aligned}\mathbf{E}\left[x[n+m]x^T[n]\right] &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathbf{E}\left[\Phi(n+m; n+m-k_1)Bw[n+m-k_1-1]w^T[n-k_2-1]B^T\Phi^T(n; n-k_2)\right] \\ &= \sum_{k=0}^{\infty} \mathbf{E}\left[\Phi(n+m; n-k)BR_{ww}[0]B^T\Phi^T(n; n-k)\right].\end{aligned}$$

Because the dropout Markov chain is time-homogeneous and the initial time is set to $-\infty$, the dropout Markov chain stays in the steady state. Then $\mathbf{E}\left[\Phi(n+m; n-k)BR_{ww}[0]B^T\Phi^T(n; n-k)\right]$ is shift-invariant with respect to n. So $\{x[n]\}$ is WSS. $\diamondsuit$

Proof of theorem 5.1: Corollary 4.3 guarantees that $\{y[n]\}$ is WSS. Then the power semi-norm of y can be computed as

$$\begin{aligned}\|y\|_{\mathcal{P}} &= \sqrt{Trace\left(\mathbf{E}\left[y[n]y^T[n]\right]\right)} \\ &= \sqrt{Trace\left(devec\left(\mathbf{E}\left[y^{[2]}[n]\right]\right)\right)} \\ &= \sqrt{Trace\left(devec\left(C^{[2]}\mathbf{E}\left[x^{[2]}[n]\right]\right)\right)}.\end{aligned}$$

The above equation shows $\mathbf{E}\left[x^{[2]}[n]\right]$ has to be computed in order to get $\|y\|_{\mathcal{P}}$. Because the initial time is set to $-\infty$, the dropout Markov chain can be assumed to stay at the steady state, i.e. $\Pr(q[n] = q_i) = \pi_i$ $(i = 1, 2, \cdots, N)$, where π_i is defined in equations 2.5 and 2.6. Let $P_i[n] = \pi_i \mathbf{E}\left[x^{[2]}[n] \mid q[n-1] = q_i\right]$. Then

$$\mathbf{E}\left[x^{[2]}[n]\right] = \sum_{i=1}^{N} P_i[n].$$

$P_i[n+1]$ can be recursively computed as follows.

$$\begin{aligned}
P_i[n+1] &= \pi_i \mathbf{E}\left[(A[n]x[n]+Bw[n])^{[2]} \mid q[n]=q_i\right] \\
&= \pi_i A_i^{[2]} \mathbf{E}\left[x^{[2]}[n] \mid q[n]=q_i\right] + \pi_i B^{[2]}\mu_{w_2} \\
&= \pi_i A_i^{[2]} \sum_{k=1}^{N} \mathbf{E}\left[x^{[2]}[n] \mid q[n]=q_i, q[n-1]=q_k\right] \Pr(q[n-1]=q_k \mid q[n]=q_i) + \pi_i B^{[2]}\mu_{w_2} \\
&= \pi_i A_i^{[2]} \sum_{k=1}^{N} \mathbf{E}\left[x^{[2]}[n] \mid q[n-1]=q_k\right] \Pr(q[n-1]=q_k \mid q[n]=q_i) + \pi_i B^{[2]}\mu_{w_2} \\
&= A_i^{[2]} \sum_{k=1}^{N} q_{ki} P_k[n] + \pi_i B^{[2]}\mu_{w_2},
\end{aligned}$$

where the second equality follows from $\mathbf{E}\left[x[n]\otimes w[n]\right]=0$ (because $x[n]$ depends on only the past noise inputs $\{w[n-1], w[n-2], \cdots\}$); the fourth equality follows from the Markov property of the dropouts.

Let $V_P[n] = \begin{bmatrix} P_1^T[n] & P_2^T[n] & \cdots & P_N^T[n] \end{bmatrix}^T$. Then the recursive computations on $P_i[n]$ yield

$$V_P[n+1] = AV_P[n] + \pi^T \otimes (B^{[2]}\mu_{w_2}). \tag{8.17}$$

Because the initial time is set to $-\infty$, the solution of the above equation is

$$\begin{aligned}
V_P[n] &= \sum_{l=-\infty}^{n} A^{n-l}\left(\pi^T \otimes (B^{[2]}\mu_{w_2})\right) \\
&= \sum_{k=0}^{\infty} A^{k}\left(\pi^T \otimes (B^{[2]}\mu_{w_2})\right) \\
&= \mathit{constant},
\end{aligned}$$

where the second equality comes from the substitution of the variable , $k=n-l$.

Because $V_P[n] = \mathit{constant}$, $P_i[n]$ is also constant with respect to n. Then $P_i[n]$ can be simplified as P_i. The proof is completed. $\diamondsuit$

Proof of theorem 5.2: In this proof, the dropout Markov chain is assumed to stay at the steady state π, which is defined in equations 2.5 and 2.6. The notations in the proof of theorem 5.1, P_i and $V_P[n]$, are reused.

At the begining, the wide sense stationarity of $\{y^{[2]}[n]\}$ is proved.

$$y^{[2]}[n] = C^{[2]}x^{[2]}[n]. \tag{8.18}$$

So $\{y^{[2]}[n]\}$ is WSS if $\{x^{[2]}[n]\}$ is WSS.

Let $P_{4,i}[n] = \pi_i \mathbf{E}\left[x^{[4]}[n] \mid q[n-1]=q_i\right]$, $V_{P_4}[n] = \begin{bmatrix} P_{4,1}^T[n] & P_{4,2}^T[n] & \cdots & P_{4,N}^T[n] \end{bmatrix}^T$. Then $\mathbf{E}\left[x^{[4]}[n]\right] = (C_I \otimes I_{n_0^2})V_{P_4}[n]$. $P_{4,i}[n+1]$ can be recuresively computed as

$$\begin{aligned}
P_{4,i}[n+1] &= \pi_i \mathbf{E}\left[(A[n]x[n]+Bw[n])^{[4]} \mid q[n]=q_i\right] \\
&= \pi_i A_i^{[4]} \mathbf{E}\left[x^{[4]}[n] \mid q[n]=q_i\right] + \pi_i B^{[4]}\mu_{w_4} + \mu_i[n].
\end{aligned}$$

where

$$\begin{aligned}
\mu_i[n] &= \mu_{i,1100}[n] + \mu_{i,1010}[n] + \mu_{i,1001}[n] + \mu_{i,0011}[n] + \mu_{i,0101}[n] + \mu_{i,0110}[n], \\
\mu_{i,1100}[n] &= \pi_i \mathbf{E}\left[(Bw[n]) \otimes (Bw[n]) \otimes (A[n]x[n]) \otimes (A[n]x[n]) \mid q[n] = q_i\right], \\
\mu_{i,1010}[n] &= \pi_i \mathbf{E}\left[(Bw[n]) \otimes (A[n]x[n]) \otimes (Bw[n]) \otimes (A[n]x[n]) \mid q[n] = q_i\right], \\
&\vdots \\
\mu_{i,0110}[n] &= \pi_i \mathbf{E}\left[(A[n]x[n]) \otimes (Bw[n]) \otimes (Bw[n]) \otimes (A[n]x[n]) \mid q[n] = q_i\right].
\end{aligned}$$

It can be shown that

$$\mu_{i,1100}[n] = (B^{[2]}\mu_{w_2}) \otimes \left(A_i^{[2]} \sum_{k=1}^{N} q_{ki} P_k \right).$$

So $\mu_{i,1100}[n]$ is constant with respect to n. Similarly $\mu_{i,1010}[n], \cdots, \mu_{i,0110}[n]$ can also be shown to be constant with respect to n. Then $\mu_i[n]$ is constant with respect to n, which is denoted as μ_i. Let $\mu = \left[\begin{array}{cccc} \mu_1^T & \mu_2^T & \cdots & \mu_N^T \end{array}\right]^T$. So $P_{4,i}[n+1]$ can be expressed as

$$P_{4,i}[n+1] = A_i^{[4]} \sum_{k=1}^{N} q_{ki} P_{4,k}[n] + \pi_i B^{[4]} \mu_{w_4} + \mu_i.$$

Then

$$V_{P_4}[n+1] = A_{(4)} V_{P_4}[n] + \pi^T \otimes (B^{[4]}\mu_{w_4}) + \mu.$$

Because the initial time is $-\infty$ and $A_{(4)}$ is asymptotical stable,

$$V_{P_4}[n] = \sum_{l=0}^{\infty} A_{(4)}^l \left(\pi^T \otimes (B^{[4]}\mu_{w_4}) + \mu \right).$$

So $V_{P_4}[n]$ is constant with respect to n.

Let $F_{i,n}[m] = \pi_i \mathbf{E}\left[x^{[2]}[n+m] \otimes x^{[2]}[n] \mid q[n+m-1] = q_i\right]$, $V_{F_n}[m] = \left[\begin{array}{cccc} F_{1,n}^T[m] & F_{2,n}^T[m] & \cdots & F_{N,n}^T[m] \end{array}\right]^T$. Then

$$\mathbf{E}\left[x^{[2]}[n+m] \otimes x^{[2]}[n]\right] = (C_I \otimes I_{n_0^2}) V_{F_n}.$$

When $m \geq 1$, $F_{i,n}[m]$ can be recursively computed as

$$\begin{aligned}
F_{i,n}[m] &= \pi_i \mathbf{E}\left[(A[n+m-1]x[n+m-1] + Bw[n+m-1])^{[2]} \otimes x_n^{[2]} \mid q[n+m-1] = q_i\right] \\
&= \pi_i (A_i^{[2]} \otimes I_{n_0^2}) \mathbf{E}\left[x^{[2]}[n+m-1] \otimes x^{[2]}[n] \mid q[n+m-1] = q_i\right] + (\pi_i B^{[2]}\mu_{w_2}) \otimes \mathbf{E}\left[x^{[2]}[n] \mid q[n+m-1] = q_i\right] \\
&= (A_i^{[2]} \otimes I_{n_0^2}) \sum_{k=1}^{N} q_{ki} F_{k,n}[m-1] + (B^{[2]}\mu_{w_2}) \otimes \sum_{k=1}^{N} q_{ki}(m) P_k.
\end{aligned}$$

So

$$V_{F_n}[m] = (A \otimes I_{n_0^2}) V_{F_n}[m-1] + (B^{[2]}\mu_{w_2}) \otimes (((Q^m)^T \otimes I_{n_0^2}) V_P). \tag{8.19}$$

Because $V_{F_n}[0] = V_{P_4}[n]$ is constant with respect to n and $(A \otimes I_{n_0^2})$ is asymptotically stable, the above equation yields that $V_{F_n}[m]$ is constant with respect to n. So $\{x_n^{[2]}\}$ is WSS.

When $m \to \infty$, $Q^m \to Q_0$, where Q_0 is a $N \times N$ matrix, whose elements in the i-th column are all π_i. So when $m \to \infty$,

$$V_{F_n}[m] \to \sum_{l=0}^{\infty}(A \otimes I_{n_0^2})^l (B^{[2]}\mu_{w_2}) \otimes ((Q_0^T \otimes I_{n_0^2})V_P).$$

By the similiar arguments as the computations for $\mathbf{E}\left[x^{[4]}[n]\right]$, we can get

$$\begin{aligned}\mathbf{E}\left[x^{[2]}[n]\right] &= C_I \sum_{l=0}^{\infty} A^l \left(\pi^T \otimes (B^{[2]}\mu_{w_2})\right) \\ &= constant\end{aligned}$$

Denote $\mathbf{E}\left[x^{[2]}[n]\right]$ as μ_{x_2}. Because $\mathbf{E}\left[x^{[2]}[n+m] \otimes x^{[2]}[n]\right] = C_I V_{F_n}[m]$, we can get

$$\mathbf{E}\left[x^{[2]}[n+m] \otimes x^{[2]}[n]\right] - \mu_{x_2} \otimes \mu_{x_2} \to 0,$$

when $m \to \infty$. So $\{x^{[2]}[n]\}$ is ergodic. Then we can get the ergodicity of $\{y^{[2]}[n]\}$. The proof is completed. $\diamondsuit$

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