Power Spectral Analysis of Networked Control Systems with Data Dropouts

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***Abstract*— This paper derives a closed form expression for a control system's output power spectral density (PSD) as a function of the data dropout probability. We use the PSD to determine a dropout compensator that minimizes the regulator's output power. We show, by example, that the optimal dropout compensator does not always correspond to a filter that minimizes the mean square error between the predicted and dropped feedback measurement.**

***Index Terms*— Networked control system, power spectral density**

I. Introduction

There has recently been great interest in networked control systems (NCS) in which the feedback loop is implemented over a non-deterministic communication network. Non-deterministic networks are unable to deliver data packets within hard real-time deadlines. In such cases, data packets may be excessively delayed due to network congestion. As a result it is often desirable to purposefully drop feedback measurements that are excessively delayed. The rate of data dropouts, therefore, is an important measure of the communication network's service quality (QoS).

Recent work has looked at the impact of data dropouts on control system performance as measured by the $\mathcal{H}_\infty$ system norm [1]. The impact of dropouts on the output signal power of a first-order NCS was studied in [2]. A more general treatment in [3] characterized the PSD for a wide class of NCS under i.i.d. dropouts. Jump-linear system methodologies were used in [4] to determine the output power of an NCS with dropouts governed by a Markov chain. It is also possible to automate this analysis using a software tool known as *Jitterbug* [5]. This paper derives a closed form expression for a control system's output power spectral density (PSD) as a function of the data dropouts (section II). This result is used to synthesize a dropout compensator that minimizes the regulator's output power (section III).

II. Main Result

We consider the networked control system in figure 1. The NCS has two inputs, w and d. w is white noise with zero mean and unit variance. d is an i.i.d. binary process with the distribution of $P(d[n]=1)=\varepsilon, P(d[n]=0)=1-\varepsilon$, where ε is the dropout rate. The loop function $L(z)$ is strictly proper and single-input single-output (SISO). The output signal y drives a data dropout model: when $d[n]=0$, the feedback signal $\overline{y}[n]$ is exactly $y[n]$; when $d[n]=1$, i.e. the output feedback is dropped, $\overline{y}[n]$ will be estimated by $\hat{y}[n]$, the output of the dropout compensator $F(z)$. $F(z)$ is assumed to be strictly proper. The loop function's control signal is $u[n]=w[n]+\overline{y}[n]$, i.e. unity feedback control is used. The output's power, $\mathbf{E}\left[y^2\right]$, is taken as a measure of the control system's performance. So we are studying the attenuation of exogenous signals w and d at the the control system's output y.

The above NCS model will jump in an i.i.d. fashion between two configurations: the open-loop and closed-loop systems. Let $L(z)$ and

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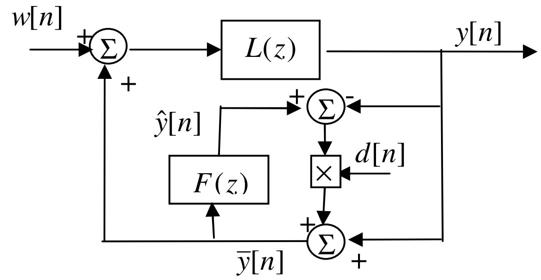

Fig. 1. Networked Control System with Data Dropouts and Dropout Compensation

$F(z)$ have the state space realizations, $L \stackrel{s}{=} \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h & 0 \end{array}\right]$ and $F \stackrel{s}{=} \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & 0 \end{array}\right]$,respectively. The NCS state space equations therefore take the following form:

$$\left\{\begin{array}{rcl} x[k+1] & = & A[k]x[k]+Bw[k] \\ y[k] & = & Cx[k] \end{array}\right. \tag{1}$$

where $P(A[k]=A_0)=1-\varepsilon$, $P(A[k]=A_1)=\varepsilon$. The matrices A_0, A_1, B and C have the form $A_0=\left[\begin{array}{cc} A_h+B_hC_h & 0 \\ B_fC_h & A_f \end{array}\right]$, $B=\left[\begin{array}{c} B_h \\ 0 \end{array}\right]$, $A_1=\left[\begin{array}{cc} A_h & B_hC_f \\ 0 & A_f+B_fC_f \end{array}\right]$, and $C=\left[\begin{array}{cc} C_h & 0 \end{array}\right]$. We can apply results in [6] and [7] to obtain the following stability condition.

Theorem 2.1: The NCS in equation 1 (with $w=0$) is mean square stable if and only if $A_{[2]}=(1-\varepsilon)A_0\otimes A_0+\varepsilon A_1\otimes A_1$ has all eigenvalues within the unit circle, where $\otimes$ is the Kronecker product [8].

Because we're interested in the output signal's power spectral density, we must first verify that the signal is wide sense stationary. This is done by the following theorem.

Theorem 2.2: ([7])All linear outputs of the NCS in equation 1, i.e. the output with the form of $z[k]=Ex[k]+Fw[k]$, are wide sense stationary if $A_{[2]}=(1-\varepsilon)A_0\otimes A_0+\varepsilon A_1\otimes A_1$ has all eigenvalues within the unit circle.

The following theorem states a closed-form expression for the output's power spectral density. The theorem is proven in the appendix (section V).

Theorem 2.3: Consider the NCS in equation 1. Let $\tilde{y}[n]=y[n]-\overline{y}[n]$. If the NCS is mean square stable, the power spectral densities can be computed as

$$S_{yy}(z)=\left|\frac{L(z)}{1-D(z)L(z)}\right|^2 S_{ww}(z)+\left|\frac{D(z)L(z)}{1-D(z)L(z)}\right|^2\frac{\Delta}{1-\varepsilon} \tag{2}$$

$$S_{\tilde{y}\tilde{y}}(z)=\left|\frac{L(z)(D(z)-1)}{1-D(z)L(z)}\right|^2 S_{ww}(z)+\left|\frac{D(z)(1-L(z))}{1-D(z)L(z)}\right|^2\frac{\Delta}{1-\varepsilon} \tag{3}$$

where $|\cdot|$ means magnitude, $D(z)=\frac{1-\varepsilon}{1-\varepsilon F(z)}$, and $\Delta=R_{\tilde{y}\tilde{y}}[0]$ is the variance of the reconstruction error $\tilde{y}$. When $\varepsilon>0$ then Δ is the unique positive solution of the following equation

$$\begin{aligned}\Delta \quad = \quad & \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{L(e^{j\omega})(D(e^{j\omega})-1)}{1-D(e^{j\omega})L(e^{j\omega})}\right|^2 S_{ww}(e^{j\omega})d\omega \\ & +\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{D(e^{j\omega})(1-L(e^{j\omega}))}{1-D(e^{j\omega})L(e^{j\omega})}\right|^2 d\omega\frac{1}{1-\varepsilon}\Delta\end{aligned} \tag{4}$$

III. Optimal Dropout Compensation

The PSD in equation 2 consists of two terms. The first term characterizes the impact of the exogenous disturbance, w, on the output power. The second term characterizes the impact that an equivalent noise term Δ has on the output PSD, $S_{yy}(z)$, where Δ is generated by the dropout process d. These two terms are functions of both the loop function $L(z)$ and dropout compensation filter $F(z)$. The obvious thing to do is choose $F(z)$ to minimize the output power, $\mathbf{E}[y^2]$, for a fixed dropout rate. This section synthesizes such an optimal dropout compensator.

It is important to note that the input disturbance's PSD, S_{ww}, in equation 2 is shaped by a closed loop feedback function of the form $\frac{L}{1-DL}$, whereas the dropout's contribution to the PSD is shaped by the transfer function $\frac{DL}{1-DL}$. The transfer function $D(z) = \frac{1-\varepsilon}{1-\varepsilon F(z)}$ is completely specified by the dropout compensator, F, and the dropout rate. Note that the two transfer functions in equation 2 are complementary so it may be impossible to find a D that minimizes the norms of both transfer functions simultaneously. This is, of course, the classical tradeoff between performance and sensitivity that dominates all feedback controller synthesis problems. In our case, this tradeoff involves balancing how aggressively we try to minimize the effect of the disturbance, w, and dropout noise, Δ, on system performance. This section illustrates that tradeoff by comparing dropout compensators that minimize total output power versus those compensators that minimize the reconstruction error Δ.

It is difficult to design dropout compensators, $F(z)$, directly from our NCS because that system is nonlinear and time-varying. It is possible, however, to identify an equivalent linear time-invariant (LTI) system that generates the same PSD as our original NCS. We then design the optimal dropout compensator for that equivalent system. The equivalent LTI system is shown in figure 2. This is essentially a feedback control system in which $D(z)$ is the feedback controller. In figure 2, we've rewritten $D(z)$ as $(1-\varepsilon)+z^{-1}D_0(z)$ where $D_0(z)$ is proper. Since the dropout compensator $F(z)$ is always strictly proper, we know $D(z)$ must always have this particular form. The control system has two white Gaussian zero-mean noise inputs, w and n whose variances are 1 and $\mathbf{E}[\tilde{y}^2]/(1-\varepsilon)$, respectively. Note that the variance of the noise process n is dependent on the variance of the reconstruction error, $\tilde{y}$. Obviously, the first question we must answer is whether or not this particular LTI system is well-posed. In other words, does there exist an input noise n such that $\mathbf{E}\left[n^2\right] = \mathbf{E}\left[\tilde{y}^2\right]/(1-\varepsilon)$? That question is answered in the affirmative by the following theorem. The theorem is proven in the appendix.

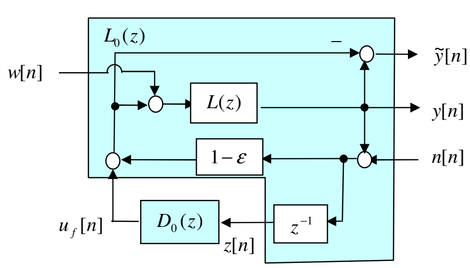

Fig. 2. Equivalent System

Theorem 3.1: Consider the LTI system shown in figure 2 with exogenous inputs w and n where $\mathbf{E}[w^2] = 1$. If the closed-loop system is internally stable and the transfer function $G_{\tilde{y}n}(z)$ satisfies the inequality, $\|G_{\tilde{y}n}\|_2^2 < 1-\varepsilon$, then there exists a noise signal n such that $\mathbf{E}\left[\tilde{y}^2\right] = (1-\varepsilon)\mathbf{E}\left[n^2\right]$

The LTI system whose existence was established in theorem 3.1 is *equivalent* to the original NCS in figure 1 in the sense that both systems generate the same power spectral densities. This assertion is stated in the following theorem whose proof is in the appendix (section V).

Theorem 3.2: If the system in figure 2 is internally stable and its transfer function $G_{\tilde{y}n}$ satisfies $\|G_{\tilde{y}n}\|_2^2 < 1-\varepsilon$ then the NCS in figure 1 is mean square stable and both systems generate the same power spectral densities, $S_{yy}(z)$ and $S_{\tilde{y}\tilde{y}}(z)$.

From theorem 3.2, we know that dropout compensator synthesis for the original NCS can be viewed as a controller synthesis problem for the generalized plant $L_0(z)$ shown in figure 2. In this case, the synthesis problem takes the form

Optimization 3.3: $\min_{D_0(z)} \mathbf{E}\left[y^2\right]$ subject to

$$\mathbf{E}\left[n^2\right] = (1-\varepsilon)\mathbf{E}\left[\tilde{y}^2\right], \quad \mathbf{E}\left[w^2\right] = 1$$

This particular optimization problem is awkward to solve directly because the dropout noise n has a variance that's proportional to the reconstruction error's variance $\mathbf{E}\left[\tilde{y}^2\right]$. The size of the reconstruction error variance, of course, is dependent on our choice of $D_0(z)$. This means that both sides of the equality constraint are dependent on our choice of $D_0(z)$, thereby leading to a problem whose form is inconsistent with many optimization software packages.

In order to solve our synthesis problem, we recast optimization problem 3.3 in a more standard form. Without loss of generality, we take Δ as an additional design parameter that represents a desired reconstruction error variance, $\mathbf{E}\left[\tilde{y}^2\right]$. We also note that the error signal $\tilde{y}$ can be rewritten as $\tilde{y}[n] = \varepsilon y[n] - u_f[n] - (1-\varepsilon)n[n]$. The reconstruction error variance therefore can be written as $\mathbf{E}\left[\tilde{y}^2\right] = \mathbf{E}\left[(\varepsilon y - u_f)^2\right] + (1-\varepsilon)^2\mathbf{E}\left[n^2\right]$. We may rewrite the equality $\mathbf{E}[\tilde{y}^2] = \Delta$ as an inequality constraint without loss of generality. Because $\mathbf{E}\left[n^2\right] = \frac{\Delta}{1-\varepsilon}$, the resulting inequality constraint can be rewritten as $\mathbf{E}\left[(\varepsilon y - u_f)^2\right] \leq \varepsilon\Delta$. If we assume that w and n are multiplied by the same gain $\frac{1}{\sqrt{\Delta}}$, then optimization problem 3.3 is transformed to

Optimization 3.4: $\min_{\Delta} \min_{D_0(z)} \Delta \cdot \mathbf{E}\left[y^2\right]$ subject to

$$\mathbf{E}\left[(\varepsilon y - u_f)^2\right] \leq \varepsilon, \tag{5}$$

$$\mathbf{E}\left[n^2[k]\right] = \frac{1}{1-\varepsilon}, \quad \mathbf{E}\left[w^2[k]\right] = \frac{1}{\Delta} \tag{6}$$

This particular characterization of the synthesis problem is in a more "standard" form that is solved with existing optimization software.

We solved optimization problem 3.4 in two steps. We first note that the inner optimization problem takes the form of a standard linear-quadratic Gaussian (LQG) synthesis. We incorporated the constraint into the performance index as a penalty and solved the unconstrained optimization problem for the augmented performance index $\mathbf{E}\left[y^2 + \lambda(\varepsilon y - u_f)^2\right]$ where λ is a specified positive number. The solution of this optimization problem is a standard LQG controller, denoted as $D_{\Delta,\lambda}(z)$. It can be shown that smaller λ will lead to smaller $\mathbf{E}[y^2]$. This relationship between λ and $\mathbf{E}[y^2]$ stems from the fact that λ plays the role of a weighting function in the LQG performance objective. A small λ, therefore, corresponds to a larger penalty being assigned to $\mathbf{E}[y^2]$. The idea, therefore, is to search for the smallest λ whose corresponding controller $D_{\Delta,\lambda}(z)$ satisfies the constraint in eq. 6. We denote $\mathbf{E}\left[y^2\right]$, under the smallest λ, as $p(\Delta)$. This is exactly the optimal value for the inner part of optimization 3.4. This inner optimization problem was solved for a range of fixed Δ, so that $p(\Delta)$ now becomes a univariate function showing how the optimum performance $\mathbf{E}[y^2]$ varies as a function of the reconstruction error variance Δ. Note that we currently don't know if $p(\Delta)$ is a convex function of Δ. So the point determined by this procedure may only be locally optimal.

As an example, consider a feedback control system with unstable loop function $L(z) = \frac{z+0.8}{z^2+z+1.7}$. We used the above approach to design an optimal dropout compensator for this plant. We refer to this as the LQG dropout compensator. We compared the LQG compensator's performance against 4 popular heuristics. The first heuristic set $F(z) = 0$ and corresponds to zeroing the control signal when a dropout occurs. The second heuristic was $F(z) = z^{-1}$ which is equivalent to reusing the last feedback measurement when a dropout occurs. The third heuristic ("reconstuction compensator") uses an $F(z)$ that minimizes the reconstruction error Δ. The fourth heuristic (Free-ρ) used a Kalman filter in series with a linear quadratic regulator (LQR). The Kalman gain was chosen to minimize the estimation error in the presence of droputs. The LQR's objective was tuned to minimize the output signal power for the given Kalman filter. The output power achieved by all dropout compensators is plotted as a function of the average dropout rate ε in figure 3. The figure shows that the reconstruction estimator and LQG compensation schemes clearly outperform the other three heuristics. The LQG compensator actually does a little better than the reconstruction compensator and surprisingly its minimum value does not occur for $\varepsilon = 0$. This is because the LQG compensator is a better regulator than the default loop function $L(z)$.

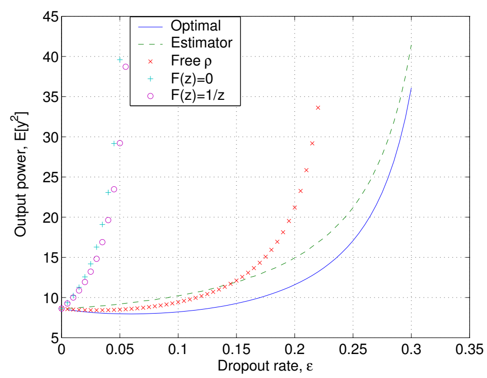

Fig. 3. Performance comparisons under different dropout compensators

As mentioned in the introduction of this section, the two terms in equation 2 suggest that the optimal dropout compensator does not always attempt to minimize the reconstruction error. This fact is illustrated in figure 4. This figure plots the optimum performance level, $p(\Delta)$, achieved for reconstruction errors in the range $0 < \Delta < 0.8$ assuming $\varepsilon = 0.1$. Note that this function is not a monotone increasing function of Δ. It has a definite global minimum that appears to occur for a reconstruction error variance, Δ, of about 0.38. This result confirms our earlier suspicion that optimal dropout compensation should not always try to minimize the reconstruction error.

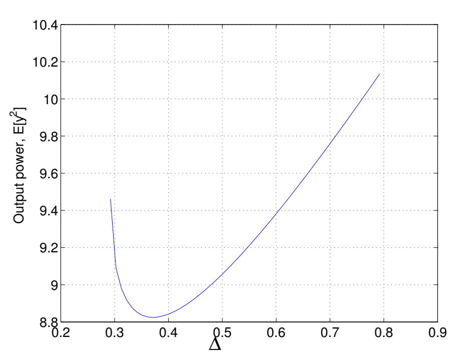

Fig. 4. Value function $p(\Delta)$ for $\varepsilon = 0.1$

IV. Conclusions

This paper derived a closed form expression for the power spectral density of a single-input single-output networked control system with i.i.d. dropouts. The assumptions on independent dropouts may be dropped if one uses jump linear system methods [4]. The expression consists of two terms that can be viewed as closed-loop transfer functions that shape the impact that exogenous input disturbances and dropout noise have on the control system's output signal. We noted that these two terms are complementary in nature, thereby suggesting that optimal dropout compensation schemes should not attempt to perfectly reconstruct the dropped measurements as such reconstructions may increase closed loop system sensitivity to input disturbances. We presented a method for synthesizing the optimal dropout compensator.

V. Appendix: Proof

Proof of Theorem 2.3: Let h and f denote the impulse response functions for $L(z)$ and $F(z)$, respectively. Let w denote the exogenous disturbance signal. The signals, y, $\bar{y}$, and $\hat{y}$ represent the loop function's output signal, the control signal re-injected into the plant, and the dropout compensator's output signal, respectively. These three signals are related through the convolution equations,

$$\left\{ \begin{array}{rcl} y & = & h * (\overline{y} + w) \\ \hat{y} & = & f * \overline{y} \end{array} \right. \tag{7}$$

We first compute the power cross-spectral densities $S_{\overline{y}w}(z)$, $S_{yw}(z)$ and $S_{\hat{y}w}(z)$ relating these output signals to the input w. From equation 7, we get

$$\left\{ \begin{array}{rcl} S_{yw}(z) & = & L(z)\left(S_{\overline{y}w}(z) + S_{ww}(z)\right) \\ S_{\hat{y}w}(z) & = & F(z) S_{\overline{y}w}(z) \end{array} \right. \tag{8}$$

For any m, the correlation $R_{\overline{y}w}$ may be written as

$$\begin{array}{rcl} R_{\overline{y}w}[m] & = & \mathbf{E}\left[\overline{y}[n+m]w[n]\right] \\ & = & \mathbf{E}\left[\overline{y}[n+m]w[n] | d[n+m] = 0\right] P(d[n+m] = 0) \\ & + & \mathbf{E}\left[\overline{y}[n+m]w[n] | d[n+m] = 1\right] P(d[n+m] = 1) \end{array}$$

Because $L(z)$ and $F(z)$ are strictly proper, we know that $y[n]$ and $\hat{y}[n]$ are independent of current and future dropouts, so that the last equation can be rewritten as

$$\begin{array}{rcl} R_{\overline{y}w}[m] & = & \mathbf{E}\left[y[n+m]w[n]\right] P(d[n+m] = 0) \\ & & +\mathbf{E}\left[\hat{y}[n+m]w[n]\right] P(d[n+m] = 1) \\ & = & (1-\varepsilon) R_{yw}[m] + \varepsilon R_{\hat{y}w}[m] \end{array}$$

We then take the double-sided z-transform of the above equation to obtain

$$S_{\overline{y}w}(z) = (1-\varepsilon) S_{yw}(z) + \varepsilon S_{\hat{y}w}(z) \tag{9}$$

Combining equations 8 and 9 generates the following expressions for the cross-spectral densities,

$$\left\{ \begin{array}{rcl} S_{\overline{y}w}(z) & = & \frac{D(z)L(z)}{1-D(z)L(z)} S_{ww}(z) \\ S_{yw}(z) & = & \frac{L(z)}{1-D(z)L(z)} S_{ww}(z) \\ S_{\hat{y}w}(z) & = & \frac{D(z)L(z)F(z)}{1-D(z)L(z)} S_{ww}(z) \end{array} \right. \tag{10}$$

where $D(z) = \frac{1-\varepsilon}{1-\varepsilon F(z)}$.

The convolutions in equation 7 also generate the following equations

$$\begin{aligned} S_{\overline{y}y}(z) &= L(z^{-1})\left(S_{\overline{yy}}(z)+S_{\overline{y}w}(z)\right) && (11)\\ S_{yy}(z) &= L(z)L(z^{-1})\left(S_{\overline{yy}}(z)+S_{\overline{y}w}(z)\right. \\ &\quad \left.+S_{\overline{y}w}(z^{-1})+S_{ww}(z)\right) && (12)\\ S_{\hat{y}\hat{y}}(z) &= F(z)F(z^{-1})S_{\overline{yy}}(z) && (13)\\ S_{\hat{y}y}(z) &= F(z)S_{\overline{y}y}(z) && (14)\\ S_{\overline{y}\hat{y}}(z) &= F(z^{-1})S_{\overline{yy}}(z) && (15) \end{aligned}$$

There are six unknown spectral densities in the above equations. There are the three spectral densities, S_{yy}, $S_{\hat{y}\hat{y}}$, and $S_{\overline{yy}}$. There are also three cross-spectral densities $S_{\overline{y}y}$, $S_{\overline{y}\hat{y}}$, and $S_{\hat{y}y}$. There are, however, only 5 equations given above. Since there are six unknowns and only five equations, we must find another independent equation. The signal $\overline{y}$ is not related to y and $\hat{y}$ through a simple convolution because $\overline{y}$ switches between these two signals.

In order to properly model the correlation of such switching signals, it is convenient to define single-sided power spectral densities.

$$S^{+}_{xy}(z)=\sum_{m=1}^{\infty}R_{xy}[m]z^{-m},\ S^{-}_{xy}(z)=\sum_{m=-\infty}^{-1}R_{xy}[m]z^{-m}$$

The above definitions imply

$$\left\{\begin{aligned} S_{xy}(z) &= S^{+}_{xy}(z)+S^{-}_{xy}(z)+R_{xy}[0]\\ S^{-}_{xy}(z) &= S^{+}_{yx}(z^{-1})\\ S_{\overline{yy}}(z) &= S^{+}_{\overline{yy}}(z)+S^{+}_{\overline{yy}}(z^{-1})+R_{\overline{yy}}[0] \end{aligned}\right. \quad (16)$$

The sixth equation will be obtained by deriving an expression for $S_{\overline{yy}}(z)$. We first note that for $m>0$,

$$\begin{aligned} R_{\overline{yy}}[m] &= \mathbf{E}\left[\overline{y}[n+m]\overline{y}[n]\right]\\ &= \mathbf{E}\left[\overline{y}[n+m]\overline{y}[n]|d[n+m]=0\right]\Pr(d[n+m]=0)\\ &+ \mathbf{E}\left[\overline{y}[n+m]\overline{y}[n]|d[n+m]=1\right]\Pr(d[n+m]=1) \end{aligned}$$

Because $L(z)$ and $F(z)$ are strictly proper, we know that

$$\begin{aligned} R_{\overline{yy}} &= \mathbf{E}\left[y[n+m]\overline{y}[n]\right]Pr(d[n+m]=0)\\ &\quad +\mathbf{E}\left[\hat{y}[n+m]\overline{y}[n]\right]Pr(d[n+m]=1)\\ &= (1-\varepsilon)R_{y\overline{y}}[m]+\varepsilon R_{\hat{y}\overline{y}}[m] \end{aligned}$$

which immediately implies that

$$S^{+}_{\overline{yy}}(z)=(1-\varepsilon)S^{+}_{y\overline{y}}(z)+\varepsilon S^{+}_{\hat{y}\overline{y}}(z)$$

From the PSD identities in equation 16, we know that

$$S^{+}_{y\overline{y}}(z)=S_{y\overline{y}}(z)-S^{-}_{y\overline{y}}(z)-S_{y\overline{y}}[0] \quad (17)$$

A similar technique is used for $m<0$ to obtain

$$S^{-}_{y\overline{y}}(z)=(1-\varepsilon)S^{-}_{yy}(z)+\varepsilon S^{-}_{y\hat{y}}(z) \quad (18)$$

Substituting eq. 18 into eq. 17 yields

$$S^{+}_{y\overline{y}}(z)=S_{y\overline{y}}(z)-(1-\varepsilon)S^{-}_{yy}(z)-\varepsilon S^{-}_{y\hat{y}}(z)-S_{y\overline{y}}[0] \quad (19)$$

Similarly we can obtain

$$S^{+}_{\hat{y}\overline{y}}(z)=S_{\hat{y}\overline{y}}(z)-(1-\varepsilon)S^{-}_{\hat{y}y}(z)-\varepsilon S^{-}_{\hat{y}\hat{y}}(z)-S_{\hat{y}\overline{y}}[0] \quad (20)$$

We now substitute equations 19 and 20 into equation 17 to obtain

$$\begin{aligned} S^{+}_{\overline{yy}}(z) &= (1-\varepsilon)S_{y\overline{y}}(z)+\varepsilon S_{\hat{y}\overline{y}}(z)-(1-\varepsilon)^2S^{-}_{yy}(z)\\ &\quad -\varepsilon^2S^{-}_{\hat{y}\hat{y}}(z)-\varepsilon(1-\varepsilon)S^{-}_{y\hat{y}}(z)\\ &\quad -\varepsilon(1-\varepsilon)S^{-}_{\hat{y}y}(z)-(1-\varepsilon)R_{y\overline{y}}[0]\\ &\quad -\varepsilon R_{\hat{y}\overline{y}}[0] && (21) \end{aligned}$$

Substituting eq. 21 into the third identity in equation 16 yields

$$\begin{aligned} S_{\overline{yy}}(z) &= (1-\varepsilon)\left(S_{y\overline{y}}(z)+S_{y\overline{y}}(z^{-1})\right)\\ &\quad +\varepsilon\left(S_{\hat{y}\overline{y}}(z)+S_{\hat{y}\overline{y}}(z^{-1})\right)\\ &\quad -(1-\varepsilon)^2\left(S^{-}_{yy}(z)+S^{-}_{yy}(z^{-1})\right)\\ &\quad -\varepsilon^2\left(S^{-}_{\hat{y}\hat{y}}(z)+S^{-}_{\hat{y}\hat{y}}(z^{-1})\right)\\ &\quad -\varepsilon(1-\varepsilon)\left(S^{-}_{y\hat{y}}(z)+S^{-}_{\hat{y}y}(z^{-1})\right)\\ &\quad -\varepsilon(1-\varepsilon)\left(S^{-}_{y\hat{y}}(z^{-1})+S^{-}_{\hat{y}y}(z)\right)\\ &\quad -2(1-\varepsilon)R_{y\overline{y}}[0]-2\varepsilon R_{\hat{y}\overline{y}}[0]+R_{\overline{yy}}[0] \end{aligned}$$

We can apply the properties of single sided PSD in eq. 16 to cancel the sum of single sided PSDs in the above equation to obtain our final expression

$$\begin{aligned} S_{\overline{yy}}(z) &= (1-\varepsilon)\left(S_{y\overline{y}}(z)+S_{y\overline{y}}(z^{-1})\right)\\ &\quad +\varepsilon\left(S_{\hat{y}\overline{y}}(z)+S_{\hat{y}\overline{y}}(z^{-1})\right)-(1-\varepsilon)^2S_{yy}(z)\\ &\quad -\varepsilon^2S_{\hat{y}\hat{y}}(z)-\varepsilon(1-\varepsilon)S_{y\hat{y}}(z)\\ &\quad -\varepsilon(1-\varepsilon)S_{\hat{y}y}(z)+(1-\varepsilon)\Delta && (22) \end{aligned}$$

where

$$\begin{aligned} \Delta &= \left(-2\frac{\varepsilon}{1-\varepsilon}R_{\hat{y}\overline{y}}[0]+\frac{1}{1-\varepsilon}R_{\overline{yy}}[0]+\frac{\varepsilon^2}{1-\varepsilon}R_{\hat{y}\hat{y}}[0]\right.\\ &\quad \left.-2R_{y\overline{y}}[0]+(1-\varepsilon)R_{yy}[0]+\varepsilon R_{y\hat{y}}[0]+\varepsilon R_{\hat{y}y}[0]\right). \end{aligned}$$

Equations 11-15 and 22 represent 6 independent equations that we can then solve them for the 6 PSD's. In particular, solving for $S_{yy}(z)$ yields the first PSD in the theorem. Because $\tilde{y}[n]=y[n]-\overline{y}[n]$, we know that

$$S_{\tilde{y}\tilde{y}}(z) = S_{yy}(z)+S_{\overline{yy}}(z)-S_{y\overline{y}}(z)-S_{\overline{y}y}(z)$$

which simplifies to the second PSD in the theorem. A simpler, more meaningful, expression for Δ can be computed. The previously used techniques show that $R_{\overline{y}y}[0]=\varepsilon R_{\hat{y}y}[0]+(1-\varepsilon)R_{yy}[0]$. We then use this relation to simplify our expression for Δ to the form $\Delta=R_{\tilde{y}\tilde{y}}[0]$ where $\tilde{y}=y-\overline{y}$. Because $R_{\tilde{y}\tilde{y}}[0]=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{\tilde{y}\tilde{y}}(e^{j\omega})d\omega$, we can further reduce this expression to that stated in the theorem. When $\varepsilon>0$, then we know that $\Delta=R_{\tilde{y}\tilde{y}}[0]>0$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{L(e^{j\omega})(D(e^{j\omega})-1)}{1-D(e^{j\omega})L(e^{j\omega})}\right|^2S_{ww}(e^{j\omega})d\omega>0$. The existence of a positive solution to equation 4 implies uniqueness. $\diamondsuit$

Proof of Theorem 3.1: Because the system is internally stable, a computation of $\mathbf{E}\left[\tilde{y}^2\right]$ for the LTI system shows that $\mathbf{E}\left[\tilde{y}^2\right]=\|G_{\tilde{y}w}\|_2^2+\|G_{\tilde{y}n}\|_2^2\,\mathbf{E}[n^2]$. Let $\Delta=\mathbf{E}[n^2](1-\varepsilon)=\mathbf{E}[\tilde{y}^2]$, then the preceding equation takes the form, $\Delta=\|G_{\tilde{y}w}\|_2^2+\|G_{\tilde{y}n}\|_2^2\frac{\Delta}{1-\varepsilon}$ which has a non-negative solution when the theorem's inequality constraint is satisfied. $\diamondsuit$

Proof of Theorem 3.2: The state space model of the equivalent system in figure 2 is

$$\left\{\begin{aligned} x_e[k+1] &= A_ex_e[k]+B_ww[k]+B_nn[k]\\ y[k] &= C_yx_e[k]\\ \tilde{y}[k] &= C_{\tilde{y}}x_e[k]-(1-\varepsilon)n[k] \end{aligned}\right. \quad (23)$$

where $B_w=\begin{bmatrix}B_h\\0\end{bmatrix}$, $B_n=(1-\varepsilon)\begin{bmatrix}B_h\\B_f\end{bmatrix}$, $C_y=\begin{bmatrix}C_h & 0\end{bmatrix}$, $C_{\tilde{y}}=\varepsilon\begin{bmatrix}C_h & -C_f\end{bmatrix}$, and $A_e=(1-\varepsilon)A_0+\varepsilon A_1$. The matrices A_0, A_1, B_h, B_f, C_h and C_f are defined in section II.

We first establish that internal stability of the LTI system implies mean square stability of the original NCS. By assumption the LTI system is internally stable, which means that $\|G_{\tilde{y}n}\|_2^2=C_{\tilde{y}}W_nC_{\tilde{y}}^2+(1-\varepsilon)^2$ where W_n satisfies the Lyapunov equation $A_eW_nA_e^T+B_nB_n^T=W_n$. Moreover, because all eigenvalues of A_e lie within the unit circle, we also know there exists a unique $P_0 > 0$ that satisfies the Lyapunov equation $A_e P_0 A_e^T + I = P_0$.

Combining the assumption that $\|G_{\tilde{y}n}\|_2^2 < 1-\varepsilon$ with the Lyapunov equation for W_n yields $C_{\tilde{y}} W_n C_{\tilde{y}}^T < \varepsilon(1-\varepsilon)$. Because this is strict inequality, we know there exists a small positive real number γ such that $C_{\tilde{y}}\left(W_n + \gamma P_0\right) C_{\tilde{y}}^T < \varepsilon(1-\varepsilon)$. We now define the symmetric matrix P by the equation $P = W_n + \gamma P_0$. Based on the matrix definitions given above, we know that for any symmetric matrix $\overline{P}$

$$\begin{aligned} &((1-\varepsilon)A_0\overline{P}A_0^T + \varepsilon A_1\overline{P}A_1^T) - A_e\overline{P}A_e^T \\ &= \frac{1}{\varepsilon(1-\varepsilon)} B_n C_{\tilde{y}} \overline{P} C_{\tilde{y}}^T B_n^T \end{aligned} \tag{24}$$

In particular, we set $\overline{P}$ equal to the matrix P defined in the preceding paragraph. For this particular P we know that $C_{\tilde{y}} P C_{\tilde{y}}^T < \varepsilon(1-\varepsilon)$, so that equation 24 becomes,

$$\begin{aligned} & (1-\varepsilon)A_0PA_0^T + \varepsilon A_1PA_1^T \le A_ePA_e^T + B_nB_n^T \\ = \; & (A_eW_nA_e^T + B_nB_n^T) + \gamma A_eP_0A_e^T \\ = \; & W_n + \gamma(P_0 - I) < W_n + \gamma P_0 = P \end{aligned}$$

Therefore there exists a $P > 0$ such that $(1-\varepsilon)A_0PA_0^T + \varepsilon A_1PA_1^T < P$. We now construct a free jump linear system with the transposed system matrix, $A^T[k]$, of the original NCS in equation 1. We construct a candidate Lyapunov function $V[k] = x^T[k]Px[k]$. Because the switching is i.i.d. in the jump linear system, we use the above equation to show that $\mathbf{E}\left[V[k+1]\right] < \mathbf{E}\left[V[k]\right]$ for all k. This implies that the transposed system is mean square stable and by theorem 2.1 we know that the matrix $A_{[2]}^T = (1-\varepsilon)A_0^T \otimes A_0^T + \varepsilon A_1^T \otimes A_1^T$ has all its eigenvalues within the unit circle. This implies that $A_{[2]}$ is stable and we again use theorem 2.1 to infer the stability of the original NCS. We now show that both systems generate the same power spectral densities. Since the equivalent system is stable, it will generate WSS signals y and $\tilde{y}$. From theorem 3.1, we know that . Computing the PSD's for the equivalent system and using the fact (theorem 3.1) that $\mathbf{E}[n]^2 = \mathbf{E}[\tilde{y}^2]/(1-\varepsilon)$, we can easily show that both systems generate the same PSD's. $\diamondsuit$

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