Control System Performance under Dynamic Quantization: the scalar case

Michael Lemmon and Qiang Ling

***Abstract*— This paper derives an upper bound on the quantization error generated by a scalar quantized feedback control system. We assume a dynamic quantization policy in which feedback data is randomly dropped in accordance with an (m,k)-firm guarantee rule. Our main result identifies the minimum quantization level required to assure a specified signal-to-quantization ratio (SQR). We also show that these performance bounds scale in an exponential manner with $k-m$, thereby suggesting that real-time systems enforcing an (m,k)-firm guarantee rule should seek to keep k as small as possible.**

I. Introduction

This paper studies the performance of a scalar quantized feedback control system shown in figure 1. This paper extends the work in [1] to bound the steady state quantization error achieved by the feedback system. The encoder/decoder use a dynamic quantization policy and the encoded data is dropped by the communication channel in accordance with the (m,k)-firm guarantee rule. The (m,k)-firm guarantee rule is a task model in which at least m out of k consecutive jobs meet their deadlines. This firm real-time constraint has been used for overload management in real-time control systems [2].

Most of the previous work in this area has identified fundamental upper bounds on the minimum number of quantization levels assuring closed loop stability [3] [4] [5] [6] [7] [8] [9] [1] [10] [11]. There has been relatively little work examining system performance [1] [12]. In [1], the stability of the quantized control system shown in figure 1 was ensured by determining a bound on the quantization noise. This bound, however, was too loose to be a practical performance measure. More recently, [12] examined the performance of scalar statically quantized feedback control systems with delays.

In this paper we also confine our attention to scalar quantized feedback systems. For such systems it is relatively easy to obtain tight bounds on the quantization error. The main result of this paper is an upper bound on the quantization error and a study of how this bound scales with the number of quantization levels, Q, the rate at which data is dropped, and the open loop system's pole location. From this study, we identify the minimum quantization level required to assure a specified *signal-to-quantization* ratio (SQR).

The remainder of this paper is organized as follows. Section II introduces the quantized feedback control system under study. Section III derives an upper bound on the quantization error of the system in the presence of dropped feedback measurements. Section IV uses this bound to determine the minimum quantization level required to enforce a specified performance level under an (m,k)-firm guarantee rule. Section V examines the dependence of this bound on the (m,k)-rule itself. Section VI summarizes the paper's conclusions.

II. Problem Statement

Figure 1 is a block diagram of the control system under study. The plant's output at time n is a real scalar denoted as x_n. The state process $\{x_n\}$ satisfies the following difference equation,

$$x_{n+1} = a x_n + u_n + w_n$$

where a is a real number. If $|a|<1$, then the plant is said to be *stable*, otherwise it is unstable. The signal $\{w_n\}$ is a bounded exogenous input such that $|w_n| < M$ for all n. The signal $\{u_n\}$ is the control signal generated by the controller from the feedback signal $\{\hat{x}_n\}$. In this paper we assume that $u_n = k\hat{x}_n$, where k is some scalar control gain. Throughout this paper we assume that $a>1$ and $|a+k|<1$.

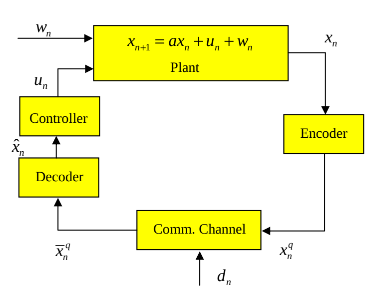

Fig. 1. Quantized Feedback Control System

The control system shown in figure 1 is a *quantized* control system. In such systems, the plant's output is quantized by an *encoder*, the quantized signal is transmitted over an unreliable feedback channel, and a *decoder* at the other end of the channel reconstructs the feedback measurement. The encoder maps the plant output, x_n, onto one of Q discrete symbols. The resulting quantized signal is denoted as x_n^q.

The authors are with Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556; *Email:* lemmon,qling@nd.edu; *Phone:* 574-631-8309; *Fax:* 574-631-4393. The authors gratefully acknowledge the partial financial support of the Army Research Office (DAAG19-01-0743), the National Science Foundation (NSF-CCR02-8537, NSF-ECS02-25265)

The signal received by the decoder is denoted as $\overline{x}_n^q$. Note that the feedback channel is assumed to be unreliable, which means that the channel has a finite chance of dropping the quantized feedback signal. In particular, let $\{d_n\}$ be a binary random process that we call the *dropout process*. The signal received at the decoder is denoted as $\overline{x}_n^q$ and it satisfies the equation

$$\overline{x}_n^q = \begin{cases} \emptyset & \text{if } d_n = 1 \\ x_n^q & \text{if } d_n = 0 \end{cases}$$

In other words, if $d_n = 1$ (a dropout occurs), then the symbol received at the decoder is the "empty" symbol. If $d_n = 0$ (no dropout occurs), then the symbol received at the decoder is simply the quantized measurement, x_n^q. The received symbol is used by the decoder to reconstruct the plant's output. This reconstructed estimate is denoted as $\hat{x}_n$ and it is used by the controller to generate the control signal u_n.

The dropout process, $\{d_n\}$, is assumed to satisfy the (m,k)-firm guarantee rule. This means that there are at least m successful transmissions over the feedback path in k consecutive attempts. So if m and k are given, then d_n satisfies the (m,k)-firm guarantee rule if and only if

$$k - m \geq \sum_{i=1}^{k} d_{n+i}$$

for all n. For a $\{d_n\}$ that satisfies the (m,k)-firm guarantee rule, it will be convenient to define the maximum dropout rate as

$$\varepsilon = 1 - \frac{m}{k} \tag{1}$$

The encoder and decoder use the dynamic quantization policy described in [1]. We focus on this particular quantizer because it is well-known that if $w_n = 0$, then the closed loop system without dropouts is asymptotically stable if and only if the number of quantization levels satisfies the following inequality $Q > \max(1, |a|)$.

Under the dynamic quantization policy, the "meaning" of the quantization symbol is time varying. In particular, the quantization policy assumes that the true plant output, x_n, at time n lies within the closed interval $I_n = [x_n^\ell, x_n^u]$ where x_n^ℓ and x_n^u denote the two endpoints of the interval. We assume that I_n is partitioned into Q subintervals of equal length. The ith subinterval is $[x_{n,i}^\ell, x_{n,i}^u]$ where $x_{n,i}^u = x_{n,i+1}^\ell$ for $i = 1, \ldots, Q-1$. We denote this subinterval as $I_{n,i}$. If the plant output x_n lies within the ith sub-interval $I_{n,i}$, then the measurement is encoded with the index i and that index (symbol) is then transmitted to the decoder. If the transmitted index is successfully received by the decoder (i.e., $d_n = 0$), then the feedback measurement, $\hat{x}_n$ and endpoints of the interval $I_{n+1} = [x_{n+1}^\ell, x_{n+1}^u]$ are updated according to the formulae

$$\begin{aligned} \hat{x}_n &= \frac{x_{n,i}^\ell + x_{n,i}^u}{2} \\ x_{n+1}^\ell &= a x_{n,i}^\ell - M + u_n \\ x_{n+1}^u &= a x_{n,i}^u + M + u_n \end{aligned}$$

If the transmitted symbol is dropped (i.e., $d_n = 1$) then the estimate, $\hat{x}_n$, and endpoints of I_{n+1} are updated according to the following formulae,

$$\begin{aligned} \hat{x}_n &= \frac{x_n^\ell + x_n^u}{2} \\ x_{n+1}^\ell &= a x_n^\ell - M + u_n \\ x_{n+1}^u &= a x_n^u + M + u_n \end{aligned}$$

This quantization rule is *dynamic* because each quantization symbol is associated with an interval that changes over time according to the update rules given above.

Remark: It is important to note that this dynamic quantization policy requires that the encoder and decoder be synchronized at time 0. Specifically, this means that they agree upon the same x^ℓ and x^u and the system model (i.e., the system parameter a) prior to starting. Moreover, there is the implicit assumption here that both encoder and decoder know when a symbol has been dropped. In practical terms, this means that the decoder has to acknowledge (ACK) the receipt of the symbol and it means that the ACK must be reliably received by the encoder. These are strong, though somewhat common, assumptions and the practical implementation of such schemes must be concerned with developing fault-tolerant methods for enforcing these assumptions.

III. Main Result

Prior work has established that if $Q > \max(1, |a|)$ then the unforced ($w_n = 0$) system without dropouts will be asymptotically stable. A straightforward extension of this prior work [11] shows that the unforced system is asymptotically stable if and only if

$$Q \geq \left\lceil (\max(1, |a|))^{\frac{1}{1-\varepsilon}} \right\rceil \tag{2}$$

where ε is the average dropout rate given in equation 1. In all of this prior work, however, the underlying assumption is that there is no input disturbance. This section uses the techniques in [1] to compute an upper bound on the quantization error when there is bounded input disturbance (i.e. $|w_n| < M$ for all n). For the scalar system considered in this paper, the bound is tight in the sense that there exists a disturbance process $\{w_n\}$ that actually achieves the bound.

Recall that the dynamic quantizer constructs a sequence of intervals, $\{I_n\}$, such that $x_n \in I_n$. We then take the center of this interval as the reconstructed measurement, $\hat{x}_n$. This means that we can take the quantization error as half the interval's length

$$L_n = \frac{x_n^u - x_n^\ell}{2}$$

In particular, we're interested in L_n as $n \to \infty$.

L_n evolves according to the dynamic equation

$$L_{n+1} = \Theta_n L_n + M \quad (3)$$

where $\Theta_n = a$ if there is a dropout ($d_n = 1$) and $\Theta_n = a/Q$ if there is no droput ($d_n = 0$). So let's define the l-step transition function,

$$\Phi(n, n-l) = \begin{cases} \Theta_{n-1}\Theta_{n-2}\cdots\Theta_{n-l} & l \geq 1 \\ 1 & l < 1 \end{cases} \quad (4)$$

Using this definition for Φ in equation 3 allows us to conclude that

$$L_n = \sum_{i=0}^{n-1} \Phi(n, n-i)M + \Phi(n, 0)L_0 \quad (5)$$

We now determine an upper bound on $\Phi(n, n-i)$. Clearly if $i = k$, then the fact that $\{d_n\}$ satisfies the (m, k)-firm guarantee rule means that Θ_i should be a/Q for at least m times. Since $a/Q < a$, we can therefore see

$$\Phi(n, n-k) \leq \left(\frac{a}{Q}\right)^m a^{k-m}$$

Note that this bound is tight in the sense that there is a sequence of dropouts for which equality holds. Now consider $\Phi(n, n-i)$ where $i < k$. By the (m, k)-firm guarantee rule we can have at most $k - m$ dropouts in this situation. We may therefore bound $\Phi(n, n-i)$ as

$$\Phi(n, n-i) \leq \begin{cases} a^{k-m}(\frac{a}{Q})^{i-(k-m)} & \text{if } k-m < i < k \\ a^i & \text{if } i < k-m \end{cases}$$

As before this bound is tight in the sense that there is a sequence of dropouts for which equality holds. Combining the cases for $i = k$ and $i < k$, after some algebra we can infer that

$$\Phi(n, n-i) \leq \left(\frac{a^k}{Q^m}\right)^{i_k} a^{i_{\min}} \left(\frac{a}{Q}\right)^{i_{\max}} = \overline{\phi}_n \quad (6)$$

where

$$\begin{aligned} i_{\min} &= \min(i - i_k k, k - m), \\ i_{\max} &= \max(0, i - i_k k - (k-m)), \end{aligned}$$

and $i_k = \lfloor \frac{i}{k} \rfloor$, the largest integer that is less than or equal to i/k.

Applying equation 2 to the above bound (Eq. 6) implies that

$$\lim_{i\to\infty} \Phi(n+i, n) = 0$$

So we can see that

$$\begin{aligned} \lim_{n\to\infty} \sup L_n &= \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} \overline{\phi}_n M + \overline{\phi}_0 L_0 \right) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \overline{\phi}_n M \end{aligned}$$

where $\overline{\phi}_n$ is the upper bound on $\Phi(n, n-i)$ defined in equation 6. We now use the bound in equation 6 to simplify the preceding geometric series to the form

$$\begin{aligned} L &= \lim_{n\to\infty} \sup L_n \\ &= \frac{\left(\frac{a^{k-m+1}-1}{a-1} + \sum_{j=1}^{m-1} \frac{a^{j+k-m}}{Q^j}\right)}{\left(1 - \frac{a^k}{Q^m}\right)} M \quad (7) \\ &= M \frac{a^{k\epsilon}}{1-\chi^k} \left(\frac{a - a^{-k\epsilon}}{a-1} + \frac{\eta - \eta^{k(1-\epsilon)}}{1-\eta} \right) \quad (8) \end{aligned}$$

where $\eta = a/Q$ and $\chi = a/Q^{1-\epsilon}$. The expression in equation 8 represents a tight upper bound on the quantization error achieved by this dynamic quantizer under a dropout sequence satisfying an (m, k)-firm guarantee rule.

IV. Quantization Bound

This section establishes a lower bound required on the quantization level to assure a specified level of performance under an (m, k)-firm guarantee rule. We begin by rewriting our bound as

$$L = C(B_1 + B_2)$$

where C is the leading coefficient in equation 8 and B_1 and B_2 are the two terms in the parantheses. For stability we require $\chi < 1$, which implies that η is always less than one.

We now consider the limit as $Q \to \infty$. This corresponds to the perfect case in which there is no quantization. This limit is

$$\lim_{Q\to\infty} L = L_Q = a^{k\epsilon} M \left(\frac{a - a^{-k\epsilon}}{a-1} \right)$$

L_Q represents the best achievable performance under an (m, k) rule with no quantization.

If we begin quantizing data then we expect L to get larger than L_Q. In reviewing equation 8, we see that the term B_1 is independent of Q and the second term, B_2, depends heavily on Q. Furthermore, we see that B_2 goes to zero as $Q \to \infty$. Quantizing the data may therefore be seen as as introducing additional "noise" whose size is determined by the B_2 term. In particular, let's define γ as the **signal-to-quantization ratio** (SQR) defined as

$$\gamma = \frac{B_1}{B_2} = \frac{a - a^{-k\epsilon}}{a-1} \frac{1-\eta}{\eta - \eta^{k(1-\epsilon)}}$$

If we choose Q so that γ is greater than a specified γ_0, then

$$L \leq \frac{1}{1-\chi^k} L_Q (1 + \gamma_0)$$

Let's determine the range of quantization levels ensuring that the above inequality holds.

In particular, we can see that $\gamma > \gamma_0$ if

$$\frac{a - a^{-k\epsilon}}{a-1} > \gamma_0 \frac{\eta}{1-\eta}$$

Since $\eta = a/Q$, we can readily solve for Q to obtain

$$Q > a\left(1 + \gamma_0 \frac{a-1}{a - a^{-k\epsilon}}\right) = Q_{\min}$$

The righthand term in this inequality represents the minimum quantization level required to achieve the specified performance level in which $\gamma > \gamma_0$. We'll denote this quantization level as $Q_{\min}$.

This bound implies that if we have a specified performance level parameterized by γ_0 (signal-to-quantization noise level), then there is no real reason to quantize above $Q_{\min}$. Figure 2 plots $Q_{\min}$ as a function of the unstable pole location a. This plot assumes $\epsilon = 0$ (the worst case bound) and varies γ_0, the specified signal-to-quantization level. Note that for γ_0 on the order of 1 (which implies quantization noise is on the same level as the actual ideal performance, we see that relatively few bits are needed. As we increase, γ_0, the number of bits also increases in a manner that is proportional to γ_0.

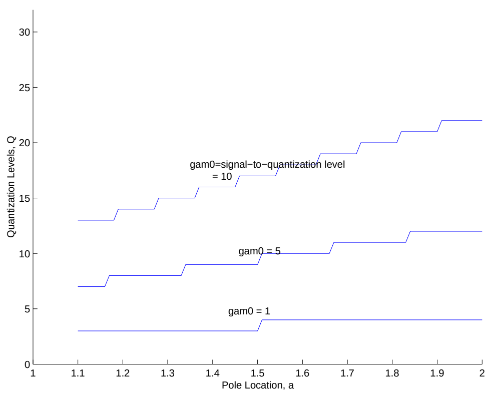

Fig. 2. $Q_{\min}$ versus pole location

V. Quantized Performance under (m, k) Firm Guarantees

Let's now consider how our performance varies with k (window size) and $\epsilon = 1 - m/k$ (dropout rate) associated with a particular (m, k)-firm guarantee rule. We know that if $Q > Q_{min}$, then the performance satisfies the inequality

$$L \leq \frac{L_Q}{1 - \chi^k}(1 + \gamma_0)$$

So to study the dependence on k and ϵ, we really only need to study how the term $\frac{L_Q}{1-\chi^k}$ varies with k and ϵ. This particular term may be rewritten as

$$\frac{L_Q}{1 - \chi^k} = \frac{a^{k\epsilon+1} - 1}{a - 1} \frac{1}{1 - \chi^k}$$

(assuming $M = 1$). The second term $1/(1-\chi^k)$ has a weak dependence on k since $\chi < 1$. The major sensitivity of this term is given by the numerator term $a^{k\epsilon+1}$. For unstable a ($|a| > 1$), we see that this grows exponentially with the size of $k\epsilon$. In other words, the primary variable of interest in the (m, k)-rule that effects performance is the product of k (window size) and ϵ dropout rate. This term is equal to $k - m$ which represents the largest number of possible consecutive dropouts that might occur in a window of size k. This particular consecutive sequence represents the worst possible growth on the uncertainty that can occur, so it is not surprising that our performance bound is strongly effected by this.

The fact that performance degrades with the exponent $k\epsilon$ suggests that we should keep k small and ϵ small. This has important consequence for real-time control. The m/k ratio can be construed as a utilization rate. Figure 3 plots the achievable performance of a real-time control system that quantizes at the $Q_{\min}$ level associated with $\gamma_0 = 10$ for various k window sizes between 2 and 5. We plot this performance for various utilization levels between 50 and 90 percent. Note that the performance of a system with $k = 5$ and 60 percent utilization is less than a similar system that assume $k = 2$ and has a 50 percent utilization, thereby reinforcing the intuition that smaller k windows should enable better performance at lower utilization levels.

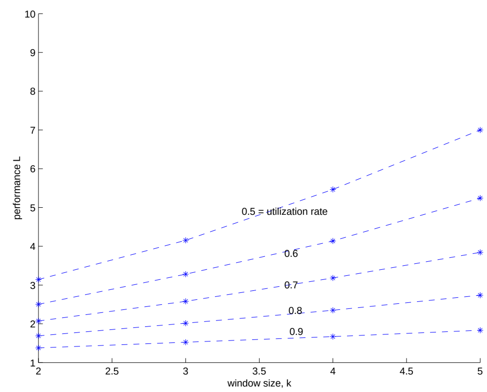

Fig. 3. Performance versus window size for various utilization rates

VI. Conclusions

This paper has studied the performance of a scalar quantized control system under an (m, k)-firm guarantee rule. The principal finding of this paper is a minimum quantization level, $Q_{\min}$ required to assure a specified level of performance. The second finding is that the primary variable degrading system performance under the (m, k)-rule is $k\epsilon$ where k is the window size and ϵ is the maximum dropout rate. This last observation suggests that real-time system engineers should design their systems to minimize k as much as possible.

Our findings are clearly limited by focusing on scalar feedback systems. We are currently studying methods for extending this paper's findings to multivariable quantized feedback systems using the dynamic bit assignment algorithm in [11].

References

[1] S. Tatikonda, *Control under Communication Constraints*, Ph.D. thesis, M.I.T., 2000.

[2] P. Ramanathan, "Overload management in real-time control applications using (m, k)-firm guarantee," *IEEE Transactions on Parallel and Distributed Systems*, vol. 10, pp. 549–559, 1999.

[3] W.S. Wong and R.W. Brockett, "Systems with finite communication bandwidth constraints- part i: state estimation problem," *IEEE Transactions on Automatic Control*, vol. 42(9), pp. 1294–1299, 1997.

[4] W.S. Wong and R.W. Brockett, "Systems with finite communication bandwidth constraints- part ii: stabilization with limited information feedback," *IEEE Transactions on Automatic Control*, vol. 44(5), pp. 1049–1053, 1999.

[5] N. Elia and S.K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46(9), pp. 1384 –1400, 2001.

[6] R.W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45(7), pp. 1279 –1289, 2000.

[7] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 48(2), pp. 304–307, 2003.

[8] J. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *Proc. of the Int. Symp. on the Mathematical Theory of Networks and Systems*, 2002.

[9] S. Tatikonda and S.K. Mitter, "Control under communication constraints," in *the 38th Annual Allerton Conference on Communications and Control Computation*, 2000, pp. 182–190.

[10] G.N. Nair and R.J. Evans, "Exponential stabilisability of finite dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[11] Q. Ling and M.D. Lemmon, "Stability of quantized control systems under dynamic bit assignment," in *American Control Conference*, Boston, MA, 2004.

[12] E. Verriest and M. Egerstedt, "Control with delayed and limited information: A first look," in *IEEE Conference on Decision and Control*, 2002.