Stability of Quantized Control Systems under Dynamic Bit Assignment

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Abstract

In recent years there have been several papers characterizing the minimum number of quantization levels required to assure closed loop stability. This minimum bit rate is usually achieved through time-varying quantization policies. Many networks, however, prefer a constant bit rate configuration [1], so it is useful to characterize the stability of quantized feedback systems under constant bit rate quantization. This paper first derives a lower bound on the number of quantization levels required for closed loop stability under constant bit rates. We then introduce a novel dynamic bit allocation policy that achieves this bound.

I. Introduction

In recent years there has been a considerable amount of work studying the stability of quantized feedback control systems [2] [3] [4] [5] [6] [7] [8] [9] [10] [11]. These papers may be classified into two groups; static and dynamic quantization policies. Static policies [2] [3] [4] [5] presume that data quantization at time k is only dependent on the data at time k. Such policies are sometimes said to be memoryless. In dynamic policies [6] [7] [8] [9] [10] [11] data quantization at time k depends on data at time instants less than or equal to k. The major advantage of static quantization policies is the simplicity of their coding/decoding schemes. In [2], however, it was proven that static policies with a finite number of quantization levels cannot achieve asymptotic stability. A finite number of quantization levels can only achieve practical stability

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(i.e. states converge into a bounded set)[4] [6]. When an infinite number of quantization levels are available, sufficient bounds for asymptotic stability under static quantization were derived in [5] using robust stability methods. It was shown in [3] that the least dense static quantizer with an infinite number of quantization levels is the logarithmic quantizer.

Dynamic policies have been shown to achieve asymptotic stability with a finite number of quantization levels. These policies presume that the state, $x[k] \in R^N$, at time instant k lies inside a set $P[k]$ called the *uncertainty set*. If $P[k]$ converges to 0, i.e. every point in $P[k]$ converges to 0, then the system is asymptotically stable. The basic approach was introduced in [7] [8]. In these papers the uncertainty set, $P[k]$, is partitioned into M^N small rectangles. Denote the small rectangles as $P_i[k]$ ($i = 0, 1, \cdots, N^M - 1$). If $x[k] \in P_j[k]$ then the index j is transmitted. This uncertainty set is then propagated to set $P[k+1]$ using what we know of the plant's dynamics. These papers provided sufficent conditions for the convergence of the sequence, $\{P[k]\}$, to zero.

A generalization of the approach in [7] was presented in [9] [10]. Suppose the eigenvalues of the quantized system are denoted as λ_i for $i = 1, \ldots, N$ and assume $P[k]$ is shaped like a rectangle. Let the i^{th} side of $P[k]$ be equally partitioned into $2^{R_i[k]}$ parts, i.e. $R_i[k]$ bits are assigned to the i^{th} dimension ($R_i[k]$ must be an integer). The total number of bits is $R[k] = \sum_{i=1}^{N} R_i[k]$, i.e. there are $Q[k] = 2^{R[k]}$ quantization levels at time k. The approach discussed in [9] [10] assumes a time-varying bit rate policy in which $R[k]$ varies over time, but has an average value $\overline{R} = \lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} R[k]$. In [9] [10] it is shown that the quantized system is asymptotically stabilizable if and only if

$$\overline{R} > \sum_{i=1}^{N} \max(0, \log_2 |\lambda_i|) \tag{1}$$

Constant bit rate policies require that $R_i[k] = R_i$ for $i = 1, \ldots, N$ and where R_i is independent of k. In [9] a sufficient condition for asymptotic stability under constant bit rates was given as

$$R = \sum_{i=1}^{N} R_i > \sum_{i=1}^{N} \max\left(0, \lceil \log_2 |\lambda_i| \rceil\right). \tag{2}$$

where $\lceil\cdot\rceil$ means $\lceil x \rceil = \min\{n | n > x, n \in \mathcal{N}\}$. There can be a significant gap between the bounds in equations 1 and 2, so it is natural to ask whether there exists a tighter bound than the one in equation 2 for the constant bit rate case. That is precisely the question addressed in this note.

This paper shows that a lower bound on the number of quantization levels required to stabilize the system is given by the equation

$$Q = 2^R \geq \left\lceil \prod_{i=1}^{N} \max(1, |\lambda_i|) \right\rceil . \tag{3}$$

We then introduce a **dynamic bit assignment policy** that actually achieves this bound. This bit assignment is done as follows. Suppose $P[k]$ is a parallelogram, there are $Q = 2^R$ quantization levels and Q is an integer. At every step, only the "longest" side of $P[k]$ (in the sense of a weighted length) is equally partitioned into Q parts; the other sides aren't partitioned. Because no side is always the longest, the bit assignments are dynamic rather than static. The paper's main contribution proves that the lower bound in equation 3 is realized by this policy.

II. Quantized Feedback Control System

This paper studies a quantized feedback control system with dropouts, which is shown in figure 1. The plant is a discrete-time linear system whose state equations are

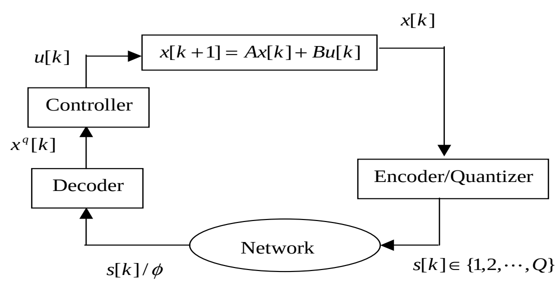

Fig. 1. Quantized feedback control system

$$\begin{cases} x[k+1] &= Ax[k] + Bu[k] \\ u[k] &= Kx^q[k] \end{cases} \tag{4}$$

The state $x[k] \in R^N$ is quantized and encoded into a symbol $s[k]$ from a discrete set $\{1, 2, \cdots, Q\}$. Throughout this paper, the terms "quantizer" and "encoder" are used interchangeably. $s[k]$ is transmitted to the decoder over a communication network. Because the network is non-deterministic, a portion of the transmitted symbols may be dropped. A dropped symbol is denoted by receiving ϕ at the decoder. The decoder uses the received symbols to compute an estimate,

$x^q[k]$, of the plant's true state, $x[k]$. The controller uses this estimate, $x^q[k]$ to compute the control signal $u[k]$.

We are interested in the following notion of deterministic stability,

$$\lim_{k\to\infty} \|x[k]\|_2 = 0, \tag{5}$$

for all $x[0] \in R^N$ where $\|\cdot\|_2$ denotes the Euclidean 2-norm. We study stability under the following assumptions

1) (A, B) is controllable. $A = diag(J_1, J_2, \cdots, J_p)$ where J_i is an $n_i \times n_i$ real matrix with a single real eigenvalue λ_i or a pair of conjugate eigenvalues λ_i and $\overline{\lambda}_i$. All eigenvalues λ_i are assumed to be unstable, i.e. $|\lambda_i| > 1$.
2) The initial condition $x[0]$ lies in a parallelogram $P[0]$.
3) Transmitted symbols, $s[k]$, are dropped at the rate of ε symbols per transmission. The precise definition of ε will be found in equation 7. We assume that the encoder and decoder both know whether a dropout has occurred.
4) Both the encoder and the decoder know the system matrices (A and B), the coding-decoding policy and the control law. They also agree upon the initial uncertainty set, i.e. the parallelogram which $x[0]$ lies in.

We take the matrix, A, (assumption 1) to be in its real Jordan canonical form. Since any system may be reduced to this form through a similarity transformation, we may therefore assume $A = diag(J_1, J_2, \cdots, J_p)$. When $|\lambda_i| < 1$, the subsystem corresponding to J_i is stable. We can exclude the stable subsystem and consider only the unstable lower dimensional subsystem. This paper therefore assumes that $|\lambda_i| > 1$ ($i = 1, \cdots, p$).

Assumption 2 requires that the initial state is known to lie within a specified parallelogram $P[0]$. This set may be written as

$$P[0] = x^q[0] + U[0]$$

where $x^q[0]$ is the center of $P[0]$ and $U[0]$ is a parallelogram centered at the origin and defined in equations 10-11.

Assumption 3 comes from the non-determinism of the network. We introduce a dropout indicator $d[k]$,

$$d[k] = \begin{cases} 1, & \text{the symbol at time } k \text{ is dropped} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

We assume that the dropout model satisfies

$$\varepsilon = \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} d[i+k_0], \tag{7}$$

for all $k_0 \geq 0$ where ε is the "average" dropout rate and the convergence in equation 7 is uniform with respect to k_0.

Assumption 4 requires that the coder and the decoder deal with the same initial uncertainty, and share the same coding-decoding policy and control law so that the symbol produced by the encoder can be correctly interpreted by the decoder. This is a strong assumption for it requires that the encoder and decoder are "synchronized". Maintaining such synchronization in a fault-tolerant manner requires further study, but that study is not done in this paper.

III. Preliminary Results

This section introduces notational conventions and outlines a proof for the bound in equation 3. For the matrix A in assumption 1, let

$$\gamma(A) = \prod_{i=1}^{p} \left(\max(1, |\lambda_i|)\right)^{n_i} \tag{8}$$

We assume all eigenvalues of A are unstable. So $\gamma(A) = |\det(A)|$, where $\det(\cdot)$ is the determinant of a matrix.

The state $x[k]$ at time k is quantized with respect to a parallelogram representing the quantization "uncertainty". These uncertainty sets are represented as

$$P[k] = x^q[k] + U[k] \tag{9}$$

where $x^q[k] \in R^N$ is the center of $P[k]$ and $U[k]$ is a parallelogram with its center at the origin. The parallelogram $U[k]$ is formally represented by a set of vectors $\{v_{i,j}[k] \in R^{n_i}\}$ where $i = 1, \ldots, p$ and $j = 1, \ldots, n_i$. The "side" of the parallelogram associated with the ith Jordan block in A is denoted as the convex hull

$$S_i[k] = \mathrm{Co}\left\{ v \;:\; v = \sum_{j=1}^{n_i} (\pm\frac{1}{2}) v_{i,j}[k] \right\} \tag{10}$$

The entire parallelogram, $U[k]$, may therefore be expressed as the Cartesian product of the sides, $S_i[k]$. In other words

$$U[k] = \prod_{i=1}^{p} S_i[k] \tag{11}$$

The volume of U is defined as $\mathrm{vol}(U) = \int_{x\in U} 1 \cdot dx$ The "size" of $U[k]$ is measured by its diameter $d_{\max}(U[k])$. The diameter of U is defined as

$$d_{\max}(U) = \sup_{x,y\in U} \|x - y\|_2 \tag{12}$$

where $\|\cdot\|_2$ denotes Euclidean 2-norm of a vector. The quantization error is defined as $e[k] = x[k] - x^q[k]$. By equation 9, we know $e[k] \in U[k]$. When a quantization policy is used, we will generate a sequence of uncertainty sets, $\{U[k]\}$. The following lemma asserts that the convergence of the diameter of $U[k]$ is equivalent to the asymptotic stability of the system.

Lemma 3.1: The system in equation 4 is asymptotically stable if and only if the sequence of uncertainty sets, $\{U[k]\}$, satisfies

$$\lim_{k\to\infty} d_{\max}(U[k]) = 0. \tag{13}$$

Lemma 3.1 can be proven in a manner analogous to that found in Lemma 3.5.1 of [10].

A lower bound on the number of quantization levels required to stabilize the feedback control system is stated below in theorem 3.2. We only sketch the proof of this theorem as the proof's method directly follows that used in [11].

Theorem 3.2: Under assumptions 1 - 4, if the quantized feedback system in equation 4 can be asymptotically stabilized, then the number of quantization levels, Q, satisfies

$$Q \geq \left\lceil \gamma(A)^{\frac{1}{1-\varepsilon}} \right\rceil \tag{14}$$

Sketch of Proof: The volume of $U[k]$ (in the worst case) is updated by

$$\mathrm{vol}(U[k+1]) \begin{cases} \geq \frac{|\det(A)|}{Q}\mathrm{vol}(U[k]), & d[k] = 0 \\ = |\det(A)|\mathrm{vol}(U[k]), & d[k] = 1 \end{cases}$$

Because of asymptotic stability, lemma 3.1 implies $vol(U[k]) \to 0$ as $k \to \infty$. This volume limit, together with the dropout rate of ε, yields

$$\frac{|\det(A)|}{Q^{1-\varepsilon}} < 1 \tag{15}$$

Because $\gamma(A) = |\det(A)|$ and Q is an integer, we obtain the lower bound in equation 14. $\diamondsuit$

IV. Main results

This section presents the *dynamic bit assignment policy* (algorithm 4.1) and states a theorem (theorem 4.1) asserting that the lower bound is achieved by this bit assignment policy.

The following algorithm dynamically quantizes the state $x[k]$ for the feedback system in equation 4 under assumptions 1- 4. The algorithm updates a parallelogram, $P[k]$ containing the state at time k. This parallelogram, $P[k]$, is characterized by, $x^q[k]$, the center of the parallelogram, and $U[k]$, the uncertainy set. The uncertainty set $U[k]$ is formed from a set of vectors $\{v_{i,j}[k] \in R^{n_i}\}$ ($i = 1, \ldots, p$ and $j = 1, \ldots, n_i$) according to equations 10-11. The uncertainty set $U^{(I,J)}[k]$ is a modification of $U[k]$ that is formed from the vectors $\left\{v'_{i,j}[k]\right\}$ where $v'_{i,j} = v_{i,j}$ if $(i,j) \neq (I,J)$ and $v'_{i,j} = v_{i,j}/Q$ if $(i,j) = (I,J)$. The basic variables updated by this algorithm are therefore the collection of vectors $\{v_{i,j}[k]\}$ and $x^q[k]$. The quantized signal that is sent between the encoder and decoder at time k is denoted at $s[k]$. This quantized signal is equal to one of Q discrete symbols. The following algorithm consists of two tasks that are executed concurrently, the *encoder* and *decoder* tasks. Each task's first step starts its execution at the same time instant.

Algorithm 4.1: **Dynamic Bit Assignment:**

Encoder/Decoder initialization:

Initialize $x^q[0]$ and $\{v_{i,j}[0]\}$ so that $x[0] \in x^q[0] + U[0]$ and set $k = 0$.

Encoder Task:

1) **Select** the indices (I,J) by

$$(I,J) = \arg\max_{i,j} \|J_i v_{i,j}[k]\|_2 \,.$$

2) **Quantize** the state $x[k]$ by setting $s[k] = s$ if and only if

$$x[k] \in x^q[k] + x_s^{(I,J)} + U^{(I,J)}[k]$$

where

$$x_s^{(I,J)} = \left[\begin{array}{ccccccc} 0 & \cdots & 0 & v^T & 0 & \cdots & 0 \end{array}\right]^T \tag{16}$$

and $v = \frac{-Q+(2s-1)}{2Q} v_{I,J}[k]$ for $s = 1, \ldots, Q$.

3) **Transmit** the quantized symbol $s[k]$ and wait for acknowledgement

4) **Update** the variables

$$\begin{aligned} v_{i,j}[k+1] &= J_i v_{i,j}[k] \\ x^q[k+1] &= (A+BK)x^q[k] \end{aligned}$$

5) **If decoder ack received:**

$$v_{I,J}[k+1] := \frac{1}{Q} v_{I,J}[k+1]$$
$$x^q[k+1] := x^q[k+1] + A x^{(I,J)}_{s[k]}$$

where $x^{(I,J)}_{s[k]}$ is defined in equation 16.

6) Update time, $k := k+1$ and return to step 1.

Decoder Task:

1) **Update** the variables

$$v_{i,j}[k+1] = J_i v_{i,j}[k]$$
$$x^q[k+1] = (A+BK)x^q[k]$$

2) **Wait** for quantized data, $s[k]$, from encoder.
3) **If data received:**

$$v_{I,J}[k+1] := \frac{1}{Q} v_{I,J}[k+1]$$
$$x^q[k+1] := x^q[k+1] + A x^{(I,J)}_{s[k]}$$

where $x^{(I,J)}_{s[k]}$ is defined in equation 16. Then send *ack* back to the encoder.

4) Update time index, $k := k+1$, and return to step 1.

Remark: This algorithm assumes the variables $\{v_{i,j}[k]\}$ and $x^q[k]$ are "synchronized" at the beginning of the kth time interval. Furthermore, we assume the "ack" from decoder to the encoder is reliably transmitted.

Remark: The decision in step 1 of the encoder algorithm is made on the uncertainty set at time $k+1$, rather than k. This was motivated by preliminary studies which showed that using the kth uncertainty set may perform poorly when some of the λ_i are large.

Theorem 4.1: Let $Q = \left\lceil \gamma(A)^{\frac{1}{1-\varepsilon}} \right\rceil$. The feedback system in equation 4 is asymptotically stable under the quantizer in algorithm 4.1. Furthermore for any $\Delta\eta > 0$, there exists a finite $\lambda_{\Delta\eta} > 0$ such that

$$d_{\max}(U[k]) \leq \lambda_{\Delta\eta} \left(\eta + \Delta\eta\right)^{k/N} \tag{17}$$

where $\eta = \frac{\gamma(A)}{Q^{1-\varepsilon}}$

In order to improve readability, we move the proof of theorem 4.1 to the appendix, section VI.

Remark: We now compare the two sufficient stability conditions in equations 2 and 3. For convenience, we rewrite these two conditions as

$$Q = 2^R \geq \prod_{i=1}^{N} 2^{\lceil \log_2(|\lambda_i|) \rceil} \tag{18}$$

$$Q \geq \lceil \prod_{i=1}^{N} |\lambda_i| \rceil \tag{19}$$

Considering the ceiling operations above, we know that the bound in equation 19 is equal to or smaller than that in equation 18. We use the following example to illustrate this difference more clearly. Let $A = \begin{bmatrix} 1.8 & 0 \\ 0 & 1.1 \end{bmatrix}$. The bound in equation 2 is $Q \geq 4$. The bound in equation 14 is $Q \geq 2$. So the latter bound is better.

We offer an intuitive explanation for this difference. The quantization policy in [10] deals seperately with the two subsystems

$$x_1[k+1] = 1.8x_1[k] + b_1 u[k] \tag{20}$$

$$x_2[k+1] = 1.1x_2[k] + b_2 u[k] \tag{21}$$

Every subsystem is unstable and therefore needs at least 2 quantization levels. So by equation 2, we need at least $2 \times 2 = 4$ quantization levels. Although the two subsystems are unstable, however, it can be seen that they are not *too unstable*. If we assign 2 quantization levels to every subsystem, there exists excess stability margin because $\frac{1.8}{2} < 1$ and $\frac{1.1}{2} < 1$. This paper's dynamic bit assignment policy considers the two subsystems as a whole. It should be possible to combine the two stability margins together so that fewer quantization levels are required. This is precisely the case in this example. Figure 2 shows the reponse of the quantized system under our dynamic bit assignment method in which *only 1 bit* is used to quantize the feedback. The plot clearly shows that this system converges to zero. The "chatter" in this plot arises from the fact that the algorithm quantizes only one component of the state at a time.

V. Conclusions

This paper derived a lower bound on the minimum number of quantization levels required to stabilize a closed loop system. This bound is a minor extension on a previous bound in [10]. We also proposed a dynamic bit assignment policy that achieves this lower bound. The major contribution of this paper lies with the proposed quantization policy.

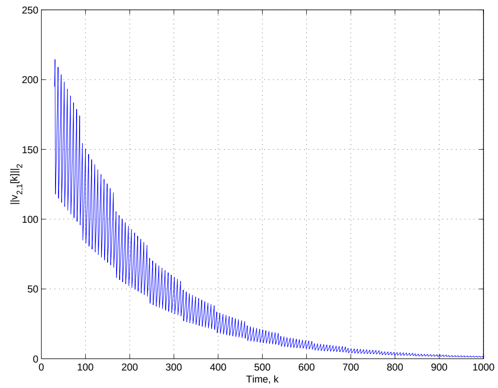

Fig. 2. Response of Quantized System

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VI. Appendix

The following lemma follows from basic algebra, so its proof is omitted.

Lemma 6.1: Let J_i be as defined in assumption 1. For any non-zero $v_i \in R^{n_i}$,

$$\lim_{k\to\infty} \frac{\|J_i^{k+1} v_i\|_2}{\|J_i^k v_i\|_2} \quad = \quad |\lambda_i|$$

By algorithm 4.1, we know $v_{i,j}[k]$ is a scaled version of $J_i^k v_{i,j}[0]$. Therefore lemma 6.1 guarantees that for any $\varepsilon_0 > 0$, there exists K_1 such that

$$(1-\varepsilon_0)|\lambda_i| \le \frac{\|J_i v_{i,j}[k]\|_2}{\|v_{i,j}[k]\|_2} \le (1+\varepsilon_0)|\lambda_i|, \tag{22}$$

for $k \ge K_1$ and any i and j.

Define the average dropout rate as

$$\overline{\varepsilon}_{l,k} = \frac{1}{l}\sum_{i=0}^{l-1} d[k+i] \tag{23}$$

Since $\overline{\varepsilon}_{l,k} \to \varepsilon$ as $l \to \infty$, we know that for any $\delta_0 > 0$, there exists $M > 0$ such that

$$\varepsilon - \delta_0 \le \overline{\varepsilon}_{l,k} \le \varepsilon + \delta_0, \tag{24}$$

for all $l \ge M$ and all k.

We prove that the uncertainty set $U[k]$ converges to zero by first showing that the "volume" of this set (as defined by the product of side lengths $p[k] = \prod_{i=1}^{p}\prod_{j=1}^{n_i} \|v_{i,j}[k]\|_2$) converges exponentially to zero.

Lemma 6.2: Assume $Q \ge \lceil \gamma(A)^{\frac{1}{1-\varepsilon}} \rceil$ and let $\eta = \frac{\gamma(A)}{Q^{1-\varepsilon}}$. For any $\Delta\eta > 0$, there exist constants $p_{\Delta\eta}$ and K_3 such that for all $k \ge K_3$

$$p[k] \le p_{\Delta\eta}(\eta + \Delta\eta)^k \tag{25}$$

Proof: For any small numbers ε_0, δ_0, there exists K_3 and M such that equation 22 and 24 hold. So we limit our attention to $k \ge K_3$ and $l \ge M$. From time k to $k+l-1$, there are $(1-\overline{\varepsilon}_{l,k})l$ sucessfully quantized measurements. So

$$p[k+l] = \frac{1}{Q^{(1-\overline{\varepsilon}_{l,k})l}} \prod_{i=1}^{p}\prod_{j=1}^{n_i} \|J_i^l v_{i,j}[k]\|_2.$$

Equations 22 and 24 let us bound $p[k+l]$ as

$$p[k+l] \le \left(\frac{\gamma(A)}{Q^{1-\varepsilon-\delta_0}}(1+\varepsilon_0)^N\right)^l p[k]$$

Note that if $Q \ge \lceil \gamma(A)^{1/(1-\varepsilon)} \rceil$, then $\eta < 1$. Choose K_3 and M large enough to make ε_0 and δ_0 arbitrarily small. We can couple this choice with the fact that $\eta < 1$ to infer that $\frac{\gamma(A)}{Q^{1-\varepsilon-\delta_0}}(1+\varepsilon_0)^N < \min(1, \eta + \Delta\eta)$. If we let

$$p_{\Delta\eta} = (\max(p[K_3], \cdots, p[K_3+M-1])) \left(\tfrac{\gamma(A)}{Q^{1-\varepsilon-\delta_0}}(1+\varepsilon_0)^N\right)^{-M-K_3},$$

then $p[k] \leq p_{\Delta\eta}\left(\eta + \Delta\eta\right)^k$ for $k \geq K_3$. $\diamondsuit$

For the preceding lemma to imply that $U[k]$ goes to zero, we must establish that each side of the parallelgram gets quantized an infinite number of times. In particular let $T_{i,j}$ denote the time instants when side $v_{i,j}$ was successfully quantized. In other words,

$$T_{i,j} = \{k \,:\, I_k = i, J_k = j, d[k] = 0\}$$

Define $\mathcal{T}_\infty = \{(i,j) \,:\, \mathrm{card}(T_{i,j}) = \infty\}$ where $\mathrm{card}(I)$ is the cardinality of set I. The following lemma shows that $\mathrm{card}(T_{i,j}) = \infty$,

Lemma 6.3: If $v_{i,j}[0] \neq 0$, then $\mathrm{card}(T_{i,j}) = \infty$

Proof: This lemma is proven by contradiction. Suppose $v_{I,J}[0] \neq 0$ but $\mathrm{card}(T_{I,J}) < \infty$, then there exists a large number K_u such that $K_u \geq K_1$ and such that the side $v_{I,J}$ is never quanitzed after time K_u. The update rule for $v_{I,J}$ in our algorithm requires $v_{I,J}[k+1] = J_I v_{I,J}[k]$ for all $k \geq K_u + 1$. Applying lemma 6.1 to this equation yields $\|J_I v_{I,J}[k]\|_2 \geq c_0((1-\epsilon_0)|\lambda_I|)^{k-K_u}$ for all $k \geq K_u$ where $c_0 = \|J_I v_{I,J}[K_u]\|_2$. By choosing ϵ_0 small enough, we can guarantee $(1-\epsilon_0)|\lambda_i| > 1$ for all $i = 1, \ldots, p$, which implies that $\|J_I v_{I,J}[k]\|$ is bounded below by a monotone increasing function of k.

Now consider any other side $v_{i,j}$ where $(i,j) \neq (I,J)$ and $\mathrm{card}(T_{i,j}) = \infty$. Define $K_{i,j} = \min\{k \,|\, k \in T_{i,j}, k \geq K_u\}$. In other words, $K_{i,j}$ is the first time instant after K_u when side $v_{i,j}$ is quantized again. From our algorithm, we know that

$$\|v_{i,j}[K_{i,j}+1]\|_2 \;=\; \frac{1}{Q}\|J_i v_{i,j}[K_{i,j}]\|_2 \geq \frac{1}{Q}\|J_I v_{I,J}[K_u]\|_2 = c_Q \tag{26}$$

where $c_Q = c_0/Q \neq 0$. For $k \geq K_{i,j}+1$, if $v_{i,j}[k]$ is not successfully quantized, then

$$\|v_{i,j}[k+1]\|_2 = \|J_i v_{i,j}[k]\|_2 \geq (1-\epsilon_0)|\lambda_i|\|v_{i,j}[k]\|_2 \tag{27}$$

If $v_{i,j}[k]$ is successfully quantized then

$$\|v_{i,j}[k+1]\|_2 = \frac{1}{Q}\|J_i v_{i,j}[k]\|_2 \geq \frac{1}{Q}\|J_I v_{I,J}[K_u]\|_2 = c_Q \tag{28}$$

Combining equations 26, 27, and 28, in addition to $(1-\epsilon_0)|\lambda_i| > 1$, guarantees $\|v_{i,j}[k]\|_2 \geq c_Q$ for all $k \geq K_{i,j}+1$. Now define the product of part of the side lengths as $p'[k] = \prod_{(i,j)\in\mathcal{T}_\infty} \|v_{i,j}[k]\|_2$ and let $\overline{K} = \max_{(i,j)\in\mathcal{T}_\infty} K_{i,j} + 1$ By equation 28 we know that for $k \geq \overline{K}$

$$p'[k] \geq c_Q^{N'} \tag{29}$$

where $N' = \mathrm{card}(\mathcal{T}_\infty)$. Equation 29 is an eventual lower bound on $p'[k]$.

We may repeat the procedure used in lemma 6.2 to obtain an upper bound on $p'[k]$ of the form

$$p'[k] \leq p'_{\delta\eta'}(\eta' + \Delta\eta')^k \tag{30}$$

where $\Delta\eta' > 0$ is any chosen tolerance, $p'_{\Delta\eta'}$ is a constant, and $\eta' = \frac{1}{Q}\prod_{(i,j)\in\mathcal{T}_\infty}|\lambda_i| < \frac{\gamma(A)}{Q^{1-\epsilon}} < 1$. We choose $\Delta\eta'$ small enough so that $\eta' + \Delta\eta' < 1$. Thus $\lim_{k\to\infty} p'[k] = 0$, which contradicts the eventual lower bound in equation 29. $\diamondsuit$

This note assumes that $v_{i,j}[0] \neq 0$ for all i, j. So lemma 6.3 guarantees $\mathrm{card}(T_{i,j}) = \infty$ for all i, j. Thus there must exist $K_2 > K_1$ such that

$$\mathrm{card}\left(T_{i,j} \cap [K_1, K_2]\right) \geq M \tag{31}$$

for all i, j, where $[K_1, K_2]$ is the set of integers from K_1 to K_2. In the following discussion, we assume $k \geq K_2$ and we let $(i_0, j_0) = \arg\min_{i,j} \|J_i v_{i,j}[k]\|_2$. We define

$$\bar{l}(i_0, j_0, M, k) = \min\left\{m \,:\, \mathrm{card}\left([k-m, k-1] \cap T_{i_0,j_0}\right) = M\right\} \tag{32}$$

where $\bar{l}(i_0, j_0, M, k)$ is the shortest length of time prior to time instant k in which the side v_{i_0,j_0} was quantized exactly M times.

The following lemma establishes the "fairness" of the algorithm by showing that $\bar{l}(i_0, j_0, M, k)$ is uniformly bounded with respect to i_0, j_0, and k.

Lemma 6.4: There exists a constant l_M such that for $k \geq K_2$

$$\bar{l}(i_0, j_0, M, k) \leq l_M. \tag{33}$$

Proof: Throughout this proof, we denote $\bar{l}(i_0, j_0, M, k)$ as $\bar{l}$. Let's first consider $(i, j) \neq (i_0, j_0)$. Let $l_{i,j}$ denote the number of times side $v_{i,j}$ was successfully quantized in the interval $[k-\bar{l}, k-1]$. Then the update equations in our algorithm imply that

$$J_i v_{i,j}[k] = \frac{1}{Q^{l_{i,j}}} J_i^{\bar{l}}(J_i v_{i,j}[k-\bar{l}]) \tag{34}$$

By inequality 22, we obtain

$$\|J_i v_{i,j}[k]\|_2 \leq \frac{((1+\varepsilon_0)|\lambda_i|)^{\bar{l}}}{Q^{l_{i,j}}} \left\|J_i v_{i,j}[k-\bar{l}]\right\|_2 \tag{35}$$

When $(i,j) = (i_0, j_0)$, we know that side v_{i_0,j_0} was updated exactly M times during $[k - \bar{l}, k-1]$. So the algorithm's update equations imply that

$$J_{i_0} v_{i_0,j_0}[k] = \frac{1}{Q^M} J_{i_0}^{\bar{l}} \left(J_{i_0} v_{i_0,j_0}[k-\bar{l}] \right) \tag{36}$$

Using inequality 22 in equation 36 yields

$$\|J_{i_0} v_{i_0,j_0}[k]\|_2 \quad \geq \quad \frac{((1-\varepsilon_0)|\lambda_{i_0}|)^{\bar{l}}}{Q^M} \|J_{i_0} v_{i_0,j_0}[k-\bar{l}]\|_2 \tag{37}$$

From the definitions of (i_0, j_0) and $\bar{l}$, we also know that

$$\left\|J_{i_0} v_{i_0,j_0}[k-\bar{l}]\right\|_2 \quad \geq \quad \left\|J_i v_{i,j}[k-\bar{l}]\right\|_2 \tag{38}$$

$$\|J_{i_0} v_{i_0,j_0}[k]\|_2 \quad \leq \quad \|J_i v_{i,j}[k]\|_2 \tag{39}$$

Inserting equations 35, 37 and 38 into equation 39, yields,

$$((1-\varepsilon_0)|\lambda_{i_0}|)^{\bar{l}} \frac{1}{Q^M} \leq ((1+\varepsilon_0)|\lambda_i|)^{\bar{l}} \frac{1}{Q^{l_{i,j}}} \tag{40}$$

There are at most $\bar{l}(\varepsilon + \delta_0)$ dropouts during $[k-\bar{l}, k-1]$. So $l_{i,j}$ satisfies the inequality,

$$\sum_{(i,j)\neq(i_0,j_0)} l_{i,j} \quad \geq \quad \bar{l} - \bar{l}(\varepsilon+\delta_0) - M = (1-\varepsilon-\delta_0)\bar{l} - M \tag{41}$$

Multiply inequality 40 over all (i,j) not equal to (i_0, j_0) and use equation 41 to obtain

$$((1-\varepsilon_0)|\lambda_{i_0}|)^{\bar{l}(N-1)} \frac{1}{Q^{(N-1)M}} \leq \left((1+\varepsilon_0)^{N-1} \prod_{(i,j)\neq(i_0,j_0)} |\lambda_i| \right)^{\bar{l}} \frac{1}{Q^{\bar{l}(1-\varepsilon-\delta_0)-M}}$$

The above inequality may be solved with respect to $\bar{l}$ to show that $\bar{l} \leq l_{i_0}$ where

$$l_{i_0} = \frac{MN\ln(Q)}{(N-1)\ln\left(\frac{1-\varepsilon_0}{1+\varepsilon_0}\right) + N\ln(|\lambda_{i_0}|) + \ln\left(\frac{Q^{1-\varepsilon}}{\gamma(A)}\right) - \delta_0 \ln(Q)}$$

Letting $l_M = \max_{i_0} l_{i_0}$ gives the desired bound. $\diamondsuit$

The following lemma establishes that the sides are *balanced* in the sense that the ratio $\|v_{i_1,j_1}[k]\|_2 / \|v_{i_2,j_2}[k]\|_2$ is uniformly bounded for all i_1, j_1, i_2, j_2, and $k \geq K_2$.

Lemma 6.5: For $k \geq K_2$ and all i_1, j_1, i_2, and j_2, there exists a finite constant r such that

$$\frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r. \tag{42}$$

Proof: For any i_1, i_2, j_1, and j_2, equation 22 implies that

$$\frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq \alpha \frac{\|J_{i_1} v_{i_1,j_1}[k]\|_2}{\|J_{i_2} v_{i_2,j_2}[k]\|_2}, \tag{43}$$

where $\alpha = \frac{1+\varepsilon_0}{1-\varepsilon_0} \max_{i_1,i_2} \frac{|\lambda_{i_1}|}{|\lambda_{i_2}|}$.

Following the arguments used in the preceding lemma, we know that

$$\begin{aligned}
\frac{\|J_i v_{i,j}[k]\|_2}{\|J_{i_0} v_{i_0,j_0}[k]\|_2} &\leq \frac{\frac{|\lambda_i|^{\bar{l}}}{Q^{l_{i,j}}}(1+\varepsilon_0)^{\bar{l}}\|J_i v_{i,j}[k-\bar{l}]\|_2}{\frac{|\lambda_{i_0}|^{\bar{l}}}{Q^M}(1-\varepsilon_0)^{\bar{l}}\|J_{i_0} v_{i_0,j_0}[k-\bar{l}]\|_2} \\
&\leq Q^M \left(\frac{|\lambda_i|}{|\lambda_{i_0}|}\right)^{\bar{l}} \left(\frac{1+\varepsilon_0}{1-\varepsilon_0}\right)^{\bar{l}} \leq r_0
\end{aligned}$$

where

$$r_0 = Q^M \left(\max_{i_1,i_2} \frac{|\lambda_{i_1}|}{|\lambda_{i_2}|}\right)^{l_M} \left(\frac{1+\varepsilon_0}{1-\varepsilon_0}\right)^{l_M},$$

and l_M is the bound in lemma 6.4.

At time k we know $\|J_{i_0} v_{i_0,j_0}[k]\|_2$ is the smallest among $\|J_i v_{i,j}[k]\|_2$, so

$$\frac{\|J_{i_1} v_{i_1,j_1}[k]\|_2}{\|J_{i_2} v_{i_2,j_2}[k]\|_2} \leq r_0, \tag{44}$$

for all i_1, i_2, j_1, and j_2. Let $r = r_0 \alpha$ to obtain the desired bound. $\diamondsuit$

Proof of theorem 4.1: This theorem follows from the direct application of lemmas 6.5 and 6.2. Let $K_0 = \max(K_2, K_3)$. At the beginning, we will limit our attention to $k \geq K_0$ so that lemmas 6.5 and 6.2 are true. Lemma 6.5 shows that $\frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r$, for all i_1, j_1, i_2, and j_2. Choose v_{i_1,j_1} to be the longest side, to obtain $\frac{\max_{m,n}\|v_{m,n}[k]\|_2}{\|v_{i,j}[k]\|_2} \leq r$ which we may rewrite as

$$\|v_{i,j}[k]\|_2 \geq \frac{1}{r} \max_{m,n} \|v_{m,n}[k]\|_2 \tag{45}$$

The above relationship, the definition of $p[k]$, and lemma 6.2 yield

$$\max_{m,n} \|v_{m,n}[k]\|_2 \leq r \sqrt[N]{p_{\Delta\eta}} \eta^{\frac{k}{N}} \tag{46}$$

$U[k]$ is a parallelogram with sides $v_{i,j}[k]$. The triangle inequality implies

$$d_{\max}(U[k]) \leq \sum_{i=1}^{P} \sum_{j=1}^{n_i} \|v_{i,j}[k]\|_2 \leq N \max_{m,n} \|v_{m,n}[k]\|_2$$

Substituting equation 46 into the above bound on $d_{\max}(U[k])$ yields $d_{\max}(U[k]) \leq \lambda_0(\eta + \Delta\eta)^{\frac{k}{N}}$ where $\lambda_0 = Nr \sqrt[N]{p_{\Delta\eta}}$. By choosing

$$\lambda_{\Delta\eta} = \max\left(\max_{m\in[1,K_0-1]} \left(d_{max}\left(U[m]\right)(\eta+\Delta\eta)^{-\frac{m}{N}}\right), \lambda_0\right)$$

we can guarantee that equation 17 holds for all k. $\diamondsuit$