Periodic Communication Logics for the Decentralized Control of Multi-Agent Systems

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Abstract— **This paper studies a system consisting of several dynamical subsystems (agents) that coordinate their actions by broadcasting their local state information over a communication network. It is assumed each agent can only observe its local state and that the medium-access control (MAC) protocol guarantees collision free broadcasts. Finally we assume that all agents have an internal dynamical model of their neighbors, so it is possible to estimate the neighboring agent's state between consecutive broadcasts from that agent. This paper examines open-loop communication logics that seek to minimize the entire group's aggregate state estimation error while minimizing the average broadcast rate. This paper's main result shows that open-loop communication logics requiring periodic broadcasts are optimal with respect to the aforementioned performance measure.**

I. Introduction

This paper studies a system consisting of multiple discrete-time dynamical subsystems (also called agents) that must coordinate their local behaviors in pursuit of a global objective. Each agent measures its local state and broadcasts this state to all members of the group with a specified cost of λ. It is assumed that each agent has a dynamical model of its neighbors, so it can estimate a neighbor's local state in between consecutive broadcasts from that neighbor. A *communication logic* is a protocol that each agent uses to decide when it should broadcast its state information to the group. We say the communication logic is *open-loop* if the broadcast decision is not related to the current state of the system. We say the communication logic is *closed-loop* if the broadcast decision is conditioned on the current state of the system. This paper examines open-loop communication logics that are "optimal" in the sense that they minimize the average error in an agent's estimate of its neighbor's state discounted by a communication cost. The paper's main result proves that optimal open-loop logics require agents to periodically broadcast their state across the group. We then experimentally compare the performance of this optimal open-loop logic against a recently proposed optimal closed-loop logic [1].

In our framework, every agent uses an estimator to predict its neighbor's state in between consecutive broadcasts from that neighbor. Our problem, therefore is similar to that considered in [2]. As to communication logics examined in [2], an individual agent decides to broadcast when the local estimation error exceeds a given threshold. This "threshold-based" communication logic is *closed-loop* because the broadcast decisions are made on the basis of the estimator's performance. Yook [2] investigated the system performance achievable under this threshold-based logic. The stochastic threshold-based communication logic is exploited in [3]. In [3], the broadcast decision is a Poisson process whose rate depends on the estimation error. Both of performance measures [2], [3], however, were not discounted by the communication cost. An optimized communication logic problem is presented in [1]. In [1] broadcast decision are made in a way that optimizes the mean square estimation error discounted by the communication cost. The optimal closed-loop decision executes under a deterministic threshold-based manner, in which the agent broadcasts when the measured estimation error exceeds a specified level.

The closed-loop logic studied in [1] requires that each agent be able to measure its local estimation error in real time. There are, however, many applications where this may not be possible. One obvious situation occurs in multi-robotic formation control. In this application, an individual robot only has local measurements of its position and velocity relative to a local coordinate frame. The robot's knowledge of its error relative to a global coordinate frame must be obtained from remote sensors observing the robot's movements relative to its neighbors. In this situation, it may be impossible for the individual agents to make broadcast decisions on the basis of their current estimation error, since they can't observe that error locally and immediately. In these applications, it may make more sense to use an open-loop communication logic.

This paper, therefore, studies "optimal" open-loop communication logics that minimizes the weighted sum of the estimation error discounted by the broadcast cost. In particular we find that the optimal open-loop communication logic requires periodic transmission of an agent's state. Unlike, the logic considered in [1], our communication logic does not broadcast on the basis of the current state estimation error. Broadcast decisions are solely based on the time since the last broadcast. A simulation comparison shows that the difference of the periodic-based logic performance and the threshold-based logic of [1] can be relatively small.

The remainder of this paper is organized as follows. Section II formally states the problem. Section III states and proves the paper's main result. Section IV presents simulation results comparing the performance of the proposed periodic communication logic with a threshold-based communication logic.

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II. Problem Statement

Consider a set of N interconnected discrete-time feedback control systems in which the ith subsystem's state, $\mathbf{x}_i$, satisfies the following difference equation,

$$\mathbf{x}_i[k+1] = \mathbf{A}\mathbf{x}_i[k] + \mathbf{B}\sum_{j=0, j\neq i}^{N-1} \hat{\mathbf{x}}_j^{(i)}[k] + \mathbf{w}_i[k] \tag{1}$$

where $\mathbf{x}_i[k]$ is in $\Re^n$ and $\mathbf{x}_i[0] = 0$. $\mathbf{A}$ and $\mathbf{B}$ are matrices of appropriate dimension. $\mathbf{w}_i[k]$ is a zero-mean white noise process with variance σ^2 and $\mathbf{E}\left[\mathbf{w}_i[k]\mathbf{w}_i^T[k]\right] = \sigma_a^2\mathbf{I}$ ($\sigma_a = \frac{\sigma}{n}$). The ith agent's estimate of the jth agent's state is denoted as $\hat{\mathbf{x}}_j^{(i)}$.

Let $u_j[k] \in \{0,1\}$ denote agent j's decision at time k to broadcast its state $\mathbf{x}_j[k]$ to all other agents in the system. In particular, we let $u_j[k] = 1$ if agent j broadcasts its state and let it be zero otherwise. Assuming that $\hat{\mathbf{x}}_j^{(i)}[0] = \mathbf{x}_j[0]$, then the ith agent's state estimate for neighbor j satisfies the following difference equation

$$\hat{\mathbf{x}}_j^{(i)}[k+1] = \mathbf{A}\hat{\mathbf{x}}_j^{(i)}[k] + \mathbf{B}\sum_{\ell=0, \ell\neq j}^{N-1} \hat{\mathbf{x}}_\ell^{(i)}[k] \tag{2}$$

if the jth agent's control decision is to stay quiet ($u_j[k] = 0$). If the jth agent broadcasts its state ($u_j[k] = 1$) then the ith agent's estimate of agent j's state satisfies the difference equation

$$\hat{\mathbf{x}}_j^{(i)}[k+1] = \mathbf{A}\mathbf{x}_j^{(i)}[k] + \mathbf{B}\sum_{\ell=0, \ell\neq j}^{N-1} \hat{\mathbf{x}}_\ell^{(i)}[k] \tag{3}$$

The ith agent's error in estimating the jth agent's state is denoted as

$$\tilde{\mathbf{x}}_j^{(i)}[k] = \mathbf{x}_j[k] - \hat{\mathbf{x}}_j^{(i)}[k]$$

Since the ith agent only has ability to observe its own state $\mathbf{x}_i$, the estimation error $\tilde{\mathbf{x}}_j^{(i)}$ is not available at the ith agent. In order to predict the estimation error at the ith agent, the jth agent imitates the estimation processes in equation (2) and (3) like,

$$\hat{\mathbf{x}}_j^{(j)}[k+1] = \mathbf{A}\hat{\mathbf{x}}_j^{(j)}[k] + \mathbf{B}\sum_{\ell=0, \ell\neq j}^{N-1} \hat{\mathbf{x}}_\ell^{(j)}[k] \tag{4}$$

and,

$$\hat{\mathbf{x}}_j^{(j)}[k+1] = \mathbf{A}\mathbf{x}_j[k] + \mathbf{B}\sum_{\ell=0, \ell\neq j}^{N-1} \hat{\mathbf{x}}_\ell^{(j)}[k] \tag{5}$$

The jth agent uses the knowledge of the estimation error $\tilde{\mathbf{x}}_j^{(j)}[k] = \mathbf{x}_j[k] - \hat{\mathbf{x}}_j^{(j)}[k]$ to make the broadcast decision $u_j[k]$.

Subtracting the estimator equation (eqn's 4 and 5) from the true state equation (1) yields the following equation for the state estimation error, $\tilde{\mathbf{x}}_j^{(j)}$,

$$\tilde{\mathbf{x}}_j^{(j)}[k+1] = \begin{cases} \mathbf{A}\tilde{\mathbf{x}}_j^{(j)}[k] + \mathbf{w}_j[k] & \text{if } u_j[k] = 0 \\ \mathbf{w}_j[k] & \text{if } u_j[k] = 1 \end{cases} \tag{6}$$

It is assumed that the jth agent imitates the estimator at the ith agent well without any error. So it is said that $\tilde{\mathbf{x}}_j^{(j)} = \tilde{\mathbf{x}}_j^{(i)}$.

Let $\{u_j[k]\}$ denote the sequence of broadcast decisions made by the jth agent and consider the finite-horizon cost functional,

$$J[\overline{u}_j \,|\, T] = \mathbf{E}\left[\sum_{k=0}^{T-1}\left(\tilde{\mathbf{x}}_j^{(i)}[k]^T\tilde{\mathbf{x}}_j^{(i)}[k](1-u_j[k]) + u_j[k]\lambda\right)\right]$$

where λ is the stage cost for broadcasting across the network, T is the horizon's length, $\overline{u}_j = \{u_j[k]\}_{k=0}^{T-1}$. Our problem is to find the communication decisions $\overline{u}_j$ that minimize the cost functional $J[\overline{u}_j \,|\, T]$ for a given T

III. Main Result

In the open loop communication logics, $u_j[k]$ is independent of the current value of estimation error $\tilde{\mathbf{x}}_j^{(i)}[k]$. Therefore, we consider the finite-horizon cost functional,

$$J[\overline{u}_j \,|\, T] = \sum_{k=0}^{T}\Big((1-u_j[k])\tilde{P}_j[k] + \lambda u_j[k]\Big) \tag{7}$$

where,

$$\tilde{P}_j[k] = \mathbf{E}\left[(\tilde{\mathbf{x}}_j[k])^{T-1}(\tilde{\mathbf{x}}_j[k])\right] \tag{8}$$

is the variance of the estimation error at time k. We drop the $^{(i)}$ superscript on the variance of estimation error $\tilde{\mathbf{x}}_j^{(i)}[k]$ because each agent has the identical estimator for the agent j.

Let the sequence $\{k_i\}_{i=1}^{M}$ denote the time instants when agent j transmits its state, where $0 \leq k_i \leq T-1$ for $i = 1, \ldots, M$. Denote the interval between k_i and k_{i+1} as the estimator's ith *stage* and let the *stage cost* be defined as

$$C_i(m_i) = \lambda + \sum_{d=0}^{m_i-2} \tilde{P}_j[k_i + d] \tag{9}$$

where $m_i = k_{i+1} - k_i$ is the interval between consecutive transmissions. The total cost over the horizon $[0, T-1]$ may therefore be written as

$$J[\overline{u}_j \,|\, T] = \sum_{i=1}^{M} C_i(m_i) \tag{10}$$

Our sequence of control decisions, $\{u_j[k]\}_{k=0}^{T-1}$ may therefore be characterized by the sequence $\{m_i\}_{i=1}^{M}$. The following lemma provides a useful expression for the stage cost.

Lemma 3.1: The stage cost $C_i(m_i)$ in equation (9) is

$$C_i(m_i) = \lambda + \sigma_a^2 \sum_{r=0}^{m_i-2} (m_i - 1 - r) Q_r \tag{11}$$

where $Q_r = \text{trace}\left[(\mathbf{A}^r)^T(\mathbf{A}^r)\right]$.

Proof: The following proof drops the superscript, (i), on the estimation error for notational convenience. For $0 \leq$

$d \leq m_i - 2$, the estimation error variance may be rewritten as

$$\begin{aligned}
&\tilde{P}_j[k_i+d] \\
&= \mathbf{E}\left[(\tilde{\mathbf{x}}_j[k_i+d])^T(\tilde{\mathbf{x}}_j[k_i+d])\right] \\
&= \mathbf{E}\left[\|\mathbf{A}\tilde{\mathbf{x}}_j[k_i+d-1]+\mathbf{w}_j[k_i+d-1]]\|^2\right] \\
&= \sum_{r=0}^{d}\mathbf{E}\left[\|\mathbf{A}^r\mathbf{w}_j[k_i+d-r]\|^2\right] \\
&= \sum_{r=0}^{d}\mathbf{E}\left[\text{trace}\left((\mathbf{A}^r)^T(\mathbf{A}^r)\mathbf{w}_j\mathbf{w}_j^T\right)\right] \\
&= \sum_{r=0}^{d}\text{trace}\left((\mathbf{A}^r)^T(\mathbf{A}^r)\mathbf{E}\left[\mathbf{w}_j\mathbf{w}_j^T\right]\right) \\
&= \sigma_a^2\sum_{r=0}^{d}Q_r
\end{aligned}$$

Substituting the above expression for $\tilde{P}_j[k_i+d]$ into equation (9) yields,

$$\begin{aligned}
C_i(m_i) &= \lambda+\sum_{d=0}^{m_i-2}\tilde{P}_j[k_i+d] \\
&= \lambda+\sigma_a^2\sum_{d=0}^{m_i-2}\sum_{r=0}^{d}Q_r \\
&= \lambda+\sigma_a^2\sum_{r=0}^{m_i-2}(m_i-1-r)Q_r
\end{aligned}$$

which completes the proof. ■

The following theorem shows there exists an optimal m^* that minimizes the average stage cost $C_i(m_i)/m_i$.

Theorem 3.2: If we let $\frac{C_i(m_i)}{m_i}$ denote the average stage cost associated with a given interval m_i, then there exists a unique interval m^* such that,

$$\frac{C_i(m^*)}{m^*} \leq \frac{C_i(m_i)}{m_i}$$

for all $m_i \neq m^*$.

Proof: From lemma 3.1, we know that $\frac{C_i(m_i)}{m_i} \leq \frac{C_i(m_i+1)}{m_i+1}$ if and only if

$$\begin{aligned}
&(m_i+1)\left(\lambda+\sigma_a^2\sum_{r=0}^{m_i-2}(m_i-1-r)Q_r\right) \\
&\quad\leq m_i\left(\lambda+\sigma_a^2\sum_{r=0}^{m_i-1}(m_i-r)Q_r\right)
\end{aligned}$$

which can be rewritten as

$$\lambda \leq \sigma_a^2\sum_{r=0}^{m_i-1}(r+1)Q_r \tag{12}$$

So $\frac{C_i(m_i)}{m_i} \leq \frac{C_i(m_i+1)}{m_i+1}$ if and only m_i satisfies inequality (12).

In a similar way, lemma 3.1 shows that $\frac{C_i(m_i)}{m_i} \leq \frac{C_i(m_i-1)}{m_i-1}$ if and only if

$$\begin{aligned}
&(m_i-1)\left(\lambda+\sigma_a^2\sum_{r=0}^{m_i-2}(m_i-1-r)Q_r\right) \\
&\quad\leq (m_i)\left(\lambda+\sigma_a^2\sum_{r=0}^{m_i-3}(m_i-2-r)Q_r\right)
\end{aligned}$$

which can be rewritten as

$$\lambda \geq \sigma_a^2\sum_{r=0}^{m_i-2}(r+1)Q_r \tag{13}$$

So $\frac{C_i(m_i)}{m_i} \leq \frac{C_i(m_i-1)}{m_i-1}$ if and only if m_i satisfies inequality (13).

Let m^* denote any integer that satisfies both inequality (12) and (13). Does such an integer exist and if so, is it unique? To answer this question let $\underline{M}_\lambda$ and $\overline{M}_\lambda$ denote the set of all m that satisfy equations (12) and (13), respectively. In other words,

$$\begin{aligned}
\underline{M}_\lambda &= \left\{m \mid \lambda \leq \sigma_a^2\sum_{r=0}^{m-1}(r+1)Q_r\right\} \\
\overline{M}_\lambda &= \left\{m \mid \lambda \geq \sigma_a^2\sum_{r=0}^{m-2}(r+1)Q_r\right\}
\end{aligned}$$

We let $\overline{m} = \max\overline{M}_\lambda$ and $\underline{m} = \min\underline{M}_\lambda$. We shall prove that $m^* = \underline{m} = \overline{m}$.

We can easily show that $\underline{m} \geq \overline{m}$. Let's suppose $\overline{m} \neq \underline{m}$, so there exists $c > 0$ such that $\underline{m} - \overline{m} = c$. There are then three possibilities for the average cost at these two values of m. We have that $\frac{C_i(\overline{m})}{\overline{m}} = \frac{C_i(\underline{m})}{\underline{m}}$ or $\frac{C_i(\overline{m})}{\overline{m}} > \frac{C_i(\underline{m})}{\underline{m}}$ or $\frac{C_i(\overline{m})}{\overline{m}} < \frac{C_i(\underline{m})}{\underline{m}}$. The last two cases cannot occur. The inequality in the third case, for example implies that inequality (13) is satisfied which means that $\underline{m} \in \overline{M}_\lambda$. But $\overline{m} = \max\overline{M}_\lambda$ and $\underline{m} > \overline{m}$, which means there is an element of $\overline{M}_\lambda$ which is greater than $\overline{m}$. This contradicts the maximal nature of $\overline{m}$ and so the third case can't occur. A similar argument can be used to dispose of the second case.

If the first case is true then we know that $\frac{C_i(m)}{m} = \frac{C_i(\overline{m})}{\overline{m}} = \frac{C_i(\underline{m})}{\underline{m}}$ for any $\overline{m} \leq m \leq \underline{m}$. In particular, let's consider $m = \underline{m} - 1$ In this case we see that

$$\begin{aligned}
&(\underline{m}-1)\left(\lambda+\sigma_a^2\sum_{r=0}^{\underline{m}-2}(\underline{m}-1-r)Q_r\right) \\
&\quad= \underline{m}\left(\lambda+\sigma_a^2\sum_{r=0}^{\underline{m}-3}(\underline{m}-2-r)Q_r\right)
\end{aligned}$$

This equation can be rewritten as

$$\begin{aligned}
\lambda = \sigma_a^2\Bigg(&\sum_{r=0}^{\underline{m}-3}\underline{m}Q_r+(\underline{m}-1)Q_{\underline{m}-2} \\
&-\sum_{r=0}^{\underline{m}-3}(\underline{m}-1-r)Q_r\Bigg)
\end{aligned}$$

which can, in turn, be simplified to

$$\begin{aligned}\lambda &= \sigma_a^2 \left(\sum_{r=0}^{\underline{m}-3} (1+r)Q_r + (\underline{m}-1)Q_{\underline{m}-2} \right) \\ &= \sigma_a^2 \sum_{r=0}^{\underline{m}-2} (1+r)Q_r\end{aligned}$$

Note that this equality implies that $\underline{m}-1$ is an element of the set $\underline{M}_\lambda$. Since $\underline{m}-1 < \underline{m}$, then clearly $\underline{m}$ cannot be the minimal element of $\underline{M}_\lambda$. So we have a contradiction and we know that $\underline{m} = \overline{m} = m^*$. ∎

The following theorem states that a periodic communication logic minimizes the finite-horizon cost. In the following statement, we drop the $_j$ subscript on the communication logic $\overline{u}_j$ for notational simplicity.

Theorem 3.3: The problem's cost functional, $J[\overline{u}_j \,|\, T]$ is minimized by a communication logic that periodically transmits the agent state information. The optimal period, m^*, satisfies inequalities (12) and (13).

Proof: Let $\overline{u}$ denote a sequence of transmission decisions consisting of M transmissions. Let m_i denote the ith transmission interval for $\overline{u}$. Let u^* denote a sequence of transmission decisions consisting of M' transmissions. Let m_i^* denote the ith transmission interval for u^*. Further assume that $m_i^* = m^*$ for all i where m^* satisfies inequalities (12) and (13).

The cost achieved under $\overline{u}$ may be written as

$$J[\overline{u} \,|\, T] = \sum_{i=1}^{M} \frac{C_i(m_i)}{m_i} m_i$$

From theorem 3.2 we know that $\frac{C_i(m_i)}{m_i} \geq \frac{C_i(m^*)}{m^*}$ so our preceding expression for the cost may be written as

$$\begin{aligned}J[\overline{u} \,|\, T] &\geq \sum_{i=1}^{M} \frac{C_i(m^*)}{m^*} m_i \\ &= \frac{C_i(m^*)}{m^*} \sum_{i=1}^{M} m_i = T \frac{C_i(m^*)}{m^*}\end{aligned}$$

Note that the last expression is the cost achieved under u^*. So we can conclude that $J[\overline{u} \,|\, T] \geq J[u^* \,|\, T]$, which implies that the periodic logic is optimal. ∎

IV. Simulation Results

This section presents simulation results comparing the performance of a simple system under three different communication logics. The first logic is the *periodic* protocol studied above. The second logic is a *random* logic in which the agent transmits with a probability p at each time instant. The third logic is the optimal *threshold* logic proposed in [1].

The results shown in figure 1 are for a scalar system in which $\mathbf{A} = 0.95$, with a process noise variance $\sigma^2 = 6$, and a communication cost $\lambda = 100$. Theorem 3.2 was used to compute the period of our periodic communication logic. This period was found to be $m^* = 7$. Figure 1 plots the average cost $\frac{J}{T}$ versus the observed probability that an agent transmits. This probability was obtained by simply dividing the number of transmissions over the total time. For our periodic logic using $m^* = 7$, this means the minimum cost should occur at a transmission probability of 0.142. Figure 1 indeed shows that the minimum cost is achieved by our proposed periodic logic and that this occurs at the predicted probability level. The proposed periodic logic performed slightly better than the random logic and performed slightly worse than the threshold-based logic.

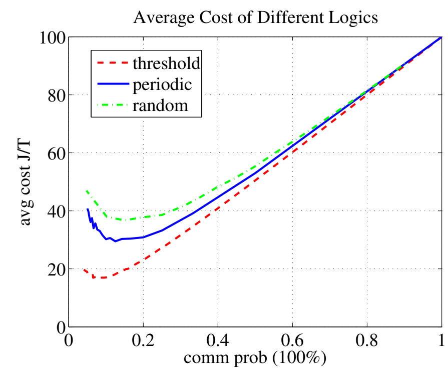

Fig. 1. Average cost in different communication policies

It is not unexpected that the optimal open-loop communication logic has worse performance than the optimal closed-loop logic. There are many applications, however, where a closed-loop logic cannot be used because the individual agent cannot directly measure its state estimation error. In these situations the open-loop periodic logic is a practical alternative whose performance levels (in this example at least) are only slightly worse than the closed-loop logics achieved performance.

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