Cohesive Swarming under Consensus

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***Abstract*— This paper studies the cohesion of multi-agent swarms moving under the control of a consensus filter. This paper's main result shows that swarming under consensus is cohesive. We establish specific bounds on the degree of cohesion and consensus as a function of the attraction/repulsion fields, swarm size, and connectivity in the communication network.**

I. Introduction

There has recently been considerable activity studying the swarming [1][2] of autonomous unmanned vehicles (AUV) or *mobile agents*. Most of this effort has used a Lagrangian framework [3] [4] which focuses on the relationship between individual agents. Nearly all of these papers assume the swarm consists of agents that have the same underlying vehicular dynamics. Vehicle movement is driven by a command that passes through either a single integrator [5][6][7] or double integrator dynamic [8]. The command input is usually the gradient of a potential field. This potential field can be automatically generated from proximity sensors detecting neighboring agents and obstacles. Potential fields associated with obstacles cause agents to move away from the obstacle. Potential fields generated by neighboring agents are based on long-range attraction and short distance repulsion between agents. This mechanism helps assure the cohesiveness of the swarm while minimizing the likelihood of agent collisions.

Potential fields, however, may also arise from *virtual objects* [9] that are not directly sensed by any individual agent. For example, a group of mobile robots attempting to find the source of a chemical plume, must use the aggregate of all local measurements of chemical concentration to determine the best direction for the swarm to move towards. The "source" of the chemical plume may be thought of as a *virtual* position that generates a potential field which draws all agents in the swarm toward that location. An individual agent's movements, therefore, are no longer determined locally by that agent's sensors. Those movements are guided by a vector that is the result of aggregating sensor information from agents throughout the entire swarm.

This sensor information is broadcast over the entire swarm using a communication network. In practice, these networks are multi-hop networks. It is already well known that ad hoc multi-hop networks have inherent capacity limitations [10]. So in recent years, there has been considerable interest in studying the impact that limited communication has on the performance of swarming in such systems. In much of this work [11][12] [13][14], the communication network is treated as a graph of limited connectivity and the question concerns the behavior of the swarm under limited network connectivity.

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This paper focuses on the use of consensus in swarm guidance and control. In particular, we study the interconnection of swarm dynamics with a consensus filter as shown in figure 1. The swarm dynamics used in this paper employ short range repulsion and long range attraction functions, similar to [4] and [6], to prevent agent collisions. The individual velocity is generated by integrating the mutual forces related to the sensed distance between neighboring agents. The consensus filter is based on the system introduced in [13]. In this study, the consensus filter estimates the swarm center and then computes the guidance direction from estimated center to a known target point.

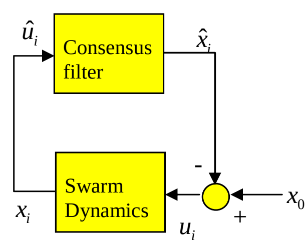

Fig. 1. Interconnection of Swarm and Consensus Filter

The paper derives uniform ultimate bounds on the swarm size and level of consensus through Lyapunov-based methodologies, similar to that done in [6] for the swarm and [13] for the consensus filter. The bounds are expressed as a function of the attraction/repulsion strength, number of network agents, and communication network connectivity. These results establish that the swarm is indeed cohesive under consensus filtering, though the level of consensus is a function of swarm size.

The remainder of the paper is organized as follows. The problem statement is stated in section II. The concepts of swarm error and consensus error are introduced in section III. The swarm and consensus filter stability analysis are presented in section IV and section V. We study the distribution of swarm agents in section VI. We then study the behavior of the consensus filter under integral action in section VII. Section VIII summarizes the paper.

II. Problem Statement

Consider a swarm of N dynamical agents that exchange information over a multi-hop communication network. Each agent is characterized by two types of states; its *physical state* representing the agent's position in the real world and its *consensus state* representing the agent's estimate of the swarm's geometric center. The physical state of the ith agent at time t is denoted as a vector $x_i(t)$ in Euclidean n-space, $\Re^n$. The trajectory of the ith agent's physical state is denoted by the function $x_i : \Re^+ \to \Re^n$ which satisfies the ordinary differential equation

$$\dot{x}_i(t) = u_i(t) + \sum_{j\sim i} g(\|x_i(t) - x_j(t)\|)(x_i(t) - x_j(t)) \quad (1)$$

for $i = 1, \ldots, N$. The vector $u_i(t) \in \Re^n$ is an external input and $g : \Re^+ \to \Re$ is a function from the positive real line, $\Re^+$, into the real line, $\Re$. We will use the notation g_{ij} to denote $g(\|x_i - x_j\|)$ and we let $\sum_{j\sim i} x_j$ to denote $\sum_{j=1,j\neq i}^{N} x_j$.

The summation in equation 1 represents long range physical interactions between agents. We assume that this interaction can be decomposed into a *repulsive* and *attractive* component. In particular, if we let $\rho : \Re^+ \to \Re^+$ and $\alpha : \Re^+ \to \Re^+$ denote the *repulsion* and *attraction* functions, then g may be written as

$$g(r) = \rho(r) - \alpha(r)$$

for any $r \in \Re^+$. This paper restricts its attention to attraction and repulsion functions of the form

$$\rho(r) = \frac{\rho_0}{r^2} \quad (2)$$

$$\alpha(r) = \frac{\alpha_0}{r} \quad (3)$$

for any $r \in \Re^+$ and where ρ_0 and α_0 are positive constants.

The consensus state of the ith agent at time t is denoted as a vector $\hat{x}_i(t) \in \Re^n$. The trajectory of the ith agent's consensus state is denoted by the function $\hat{x}_i : \Re^+ \to \Re^n$ which satisfies the *consensus filter* equation

$$\begin{aligned} \dot{\hat{x}}_i(t) &= (x_0(t) - \hat{x}_i(t)) + \sum_{j\sim i} A_{ij}(\hat{u}_j(t) - \hat{x}_i(t)) \\ &\quad + \sum_{j=1}^{N} A_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) \end{aligned} \quad (4)$$

for $i = 1, \ldots, N$. The vector $x_0(t) \in \Re^n$ is the state of the *target* at time t. The coefficient A_{ij} is the $ij-$th components of the matrix $\mathbf{I}_n + \mathbf{Adj}(G)$ where $\mathbf{I}_n$ is an $n \times n$ identity matrix and $\mathbf{Adj}(G)$ is the adjacency matrix of the graph G. The graph G models the communication connectivity within the swarm. Agent j is able to transmit its consensus state $\hat{x}_j$ and an input $\hat{u}_j$ to agent i if and only if $A_{ij} = 1$.

Figure 1 shows that the entire swarm may be viewed as an interconnection of the swarm dynamics (equation 1) and the consensus filter (equation 4). The swarm dynamic's input from the jth agent to the consensus filter's ith agent is the position of the jth agent, in other words $\hat{u}_j = x_j$. The consensus filter's input from the jth agent to the swarm dynamic's jth agent is the j'th agent's estimate of the swarm center (consensus state) relative to the target, in other words $u_j = x_0 - \hat{x}_j$. The consensus filter is trying to estimate the center of the swarm and the swarm is using those estimates to guide the swarm toward the target. The overall dynamics of this system may therefore be written as

$$\dot{x}_i = (x_0 - \hat{x}_i) + \sum_{j\sim i} g_{ij}(\|x_i - x_j\|)(x_i - x_j) \quad (5)$$

$$\begin{aligned} \dot{\hat{x}}_i &= (x_0 - \hat{x}_i) + \sum_{j\sim i} A_{ij}(\hat{x}_j - \hat{x}_i) \\ &\quad + \sum_{j=1}^{N} A_{ij}(x_j - \hat{x}_i) \end{aligned} \quad (6)$$

We're interested in establishing whether the swarm is **cohesive** and achieves **consensus**. Let $\overline{x}(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t)$ denote the **swarm center** at time t. Define the **swarm error** and **consensus error** of the ith agent at time t as

$$\begin{aligned} e_i(t) &= x_i(t) - \overline{x}(t) \in \Re^n, \\ \hat{e}_i(t) &= \hat{x}_i(t) - \overline{x}(t) \in \Re^n, \end{aligned}$$

respectively. Furthermore let $e(t) \in \Re^{Nn}$ and $\hat{e}(t) \in \Re^{Nn}$ denote the swarm and consensus error vectors,

$$\begin{aligned} e(t) &= \begin{bmatrix} e_1^T(t) & e_2^T(t) & \cdots & e_N^T(t) \end{bmatrix}^T \\ \hat{e}(t) &= \begin{bmatrix} \hat{e}_1^T(t) & \hat{e}_2^T(t) & \cdots & \hat{e}_N^T(t) \end{bmatrix}^T \end{aligned}$$

With regard to the previous notational conventions we say that the swarm defined by equations 5 and 6 is **cohesive** if and only if there exist positive real constants $\underline{R}$ and $\overline{R}$ such that $\limsup \|e(t)\| \leq \overline{R}$ and $\liminf \|e(t)\| \geq \underline{R}$. We say that the swarm achieves ϵ**-consensus** if there exists a positive real constant ϵ such that $\limsup \|\hat{e}(t)\| \leq \epsilon$. The objective of this paper is to establish whether the swarm defined in equations 5 and 6 is cohesive, achieves ϵ-consensus, and to provide bounds on the constants $\overline{R}$, $\underline{R}$, and ϵ as a function of the swarm parameters.

III. Error Equations

Since our analysis is concerned with the asymptotic behavior of the error vectors $e(t)$ and $\hat{e}(t)$, it will be convenient to transform the original system equations into a set of coupled error equations.

The derivative of the ith agent's swarm error is

$$\begin{aligned} \dot{e}_i &= \dot{x}_i - \frac{1}{N}\sum_{j=1}^{N} \dot{x}_j \\ &= (x_0 - \hat{x}_i) + \sum_{j\sim i} g_{ij}(\|x_i - x_j\|)(x_i - x_j) \\ &\quad - \frac{1}{N}\sum_{j=1}^{N}\left((x_0 - \hat{x}_j) + \sum_{k\sim j} g_{jk}(\|x_j - x_k\|)(x_j - x_k) \right) \\ &= -\hat{x}_i + \sum_{j\sim i} g_{ij}(\|x_i - x_j\|)(x_i - x_j) \end{aligned}$$

$$- \quad \frac{1}{N}\sum_{j=1}^{N}\left(-\hat{x}_j + \sum_{k\sim j} g_{jk}(\|x_j - x_k\|)(x_j - x_k)\right)$$

Note that $\sum_{j=1}^{N}\sum_{k\sim j} g_{jk}(\|x_j - x_k\|)(x_j - x_k) = 0$, $x_i - x_j = e_i - e_j$, and $\hat{x}_i - \hat{x}_j = \hat{e}_i - \hat{e}_j$ so the swarm error equation becomes

$$\dot{e}_i = \sum_{j\sim i} g_{ij}(\|e_i - e_j\|)(e_i - e_j) + \frac{1}{N}\sum_{j=1}^{N}(\hat{e}_j - \hat{e}_i) \quad (7)$$

for $i = 1, \dots, N$.

The derivative of the ith agent's consensus error is

$$\begin{aligned}
\dot{\hat{e}}_i &= \dot{\hat{x}}_i - \frac{1}{N}\sum_{j=1}^{N}\dot{\hat{x}}_j \\
&= (x_0 - \hat{x}_i) + \sum_{j\sim i} A_{ij}(\hat{x}_j - \hat{x}_i) \\
&\quad + \sum_{j=1}^{N} A_{ij}(x_j - \hat{x}_i) - \frac{1}{N}\sum_{j=1}^{N}(x_0 - \hat{x}_j) \\
&= \sum_{j=1}^{N} A_{ij}(\hat{x}_j - \hat{x}_i) + \sum_{j=1}^{N} A_{ij}(x_j - \hat{x}_i) \\
&\quad + \frac{1}{N}\sum_{j=1}^{N}(\hat{x}_j - \hat{x}_i) \\
&= \sum_{j=1}^{N} \overline{A}_{ij}(\hat{x}_j - \hat{x}_i) + \sum_{j=1}^{N} A_{ij}(x_j - \hat{x}_i)
\end{aligned}$$

where $\overline{A}_{ij} = A_{ij} + \frac{1}{N}$. Note that $\hat{x}_j - \hat{x}_i = \hat{e}_j - \hat{e}_i$ and $x_j - \hat{x}_i = e_j - \hat{e}_i$ so we can rewrite the consensus error state equation as

$$\dot{\hat{e}}_i = \sum_{j=1}^{N} \overline{A}_{ij}(\hat{e}_j - \hat{e}_i) + \sum_{j=1}^{N} A_{ij}(e_j - \hat{e}_i) \quad (8)$$

for $i = 1, \dots, N$.

It will be convenient to express equation (8) in matrix-vector form. In particular, let Δ_i denote the out-degree of the ith agent in the swarm's communication graph, G. Note that $\sum_{j=1}^{N} A_{ij} = 1 + \Delta_i$ and that $\sum_{j=1}^{N} \overline{A}_{ij} = \Delta_i + 2$. Further assume that there exist positive integers $\underline{\Delta}$ and $\overline{\Delta}$ such that $\underline{\Delta} \leq \Delta_i \leq \overline{\Delta}$ for $i = 1, \dots, N$. With these notational conventions we may rewrite equation 8 as

$$\begin{aligned}
\dot{\hat{e}}_i &= \sum_{j=1}^{N} \overline{A}_{ij}(\hat{e}_j - \hat{e}_i) + (e_i - \hat{e}_i) + \sum_{j\sim i} A_{ij}(e_j - \hat{e}_i) \\
&= -\hat{e}_i - \sum_{j\sim i} A_{ij}\hat{e}_i + \sum_{j=1}^{N} \overline{A}_{ij}(\hat{e}_j - \hat{e}_i) + e_i + \sum_{j\sim i} A_{ij} e_j
\end{aligned}$$

But note that

$$e_i = \frac{1}{N}\sum_{j=1}^{N}(x_i - x_j) = \frac{1}{N}\sum_{j=1}^{N}(e_i - e_j) = e_i - \frac{1}{N}\sum_{j=1}^{N} e_j$$

which implies that $\sum_{j=1}^{N} e_j = 0$. So we can rewrite our expression for $\dot{\hat{e}}_i$ as

$$\begin{aligned}
\dot{\hat{e}}_i &= -\hat{e}_i - \sum_{j\sim i} A_{ij}\hat{e}_i + \sum_{j\sim i} \overline{A}_{ij}(\hat{e}_j - \hat{e}_i) \\
&\quad - \sum_{j\sim i} e_j + \sum_{j\sim i} A_{ij} e_j \\
&= \left(-\frac{2N-1}{N} - 2\Delta_i\right)\hat{e}_i + \sum_{j\sim i} \overline{A}_{ij}\hat{e}_j + \sum_{j\sim i}(A_{ij} - 1)e_j
\end{aligned}$$

Then the vector form of consensus error equation is,

$$\dot{\hat{e}} = \mathbf{A}\hat{e} + \mathbf{B}e \quad (9)$$

where

$$\mathbf{A} = \begin{bmatrix} K_1\mathbf{I} & \overline{A}_{12}\mathbf{I} & \cdots & \overline{A}_{1N}\mathbf{I} \\ \overline{A}_{21}\mathbf{I} & K_2\mathbf{I} & \cdots & \overline{A}_{2N}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{A}_{N1}\mathbf{I} & \overline{A}_{N2}\mathbf{I} & \cdots & K_N\mathbf{I} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & (A_{12}-1)\mathbf{I} & \cdots & (A_{1N}-1)\mathbf{I} \\ (A_{21}-1)\mathbf{I} & \mathbf{0} & \cdots & (A_{2N}-1)\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ (A_{N1}-1)\mathbf{I} & (A_{N2}-1)\mathbf{I} & \cdots & \mathbf{0} \end{bmatrix}$$

$K_i = \left(-\frac{2N-1}{N} - 2\Delta_i\right)$, $\mathbf{0}$ is an $n \times n$ matrix of zeros and $\mathbf{I}$ is an $n \times n$ identity matrix.

IV. Uniform Ultimate Bound Analysis

The main result of this paper establishes bounds on the level of consensus (ϵ) and the swarm size ($\underline{R}$ and $\overline{R}$) as a function of the swarm parameters ρ_0, α_0, N, $\overline{\Delta}$ and $\underline{\Delta}$. This is accomplished by studying the uniform ultimate boundedness (UUB) of the swarm dynamics and consensus filters. This section presents 3 lemmas characterizing regions of the error space, $(e, \hat{e})$, in which suitably defined positive definite functions of the swarm error, e, or consensus error $\hat{e}$ have negative definite directional derivatives.

The following lemma studies the directional derivative of a positive definite function, $V(e)$ of the swarm error, e. The lemma provides sufficient conditions on the norm of $\|e\|$ for which we can show the directional derivative, $\dot{V}(e)$, of this function is negative. This lemma provides one part of the UUB analysis of the swarm under consensus

Lemma 4.1: Consider the system in equation 1 and let $V(e) = \frac{1}{2}e^T e$ for any $e \in \Re^{Nn}$. If there exists a positive real constant $\underline{\beta}$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \|x_i - x_j\| \geq \underline{\beta}\|e\| \quad (10)$$

and if

$$\|e\| \geq \frac{N(N-1)\rho_0}{\underline{\beta}\alpha_0} \quad (11)$$

then $\dot{V}(e) \leq 0$.

Proof: The directional derivative of $V(e)$ is

$$\begin{aligned}\dot{V}(e) &= \sum_{i=1}^{N} e_i^T \dot{e}_i \\ &= \sum_{i=1}^{N} \left(e_i^T \left(\sum_{j\sim i} g_{ij}(e_i - e_j) + \frac{1}{N}\sum_{j=1}^{N}(\hat{e}_j - \hat{e}_i) \right) \right) \\ &= \sum_{i=1}^{N} \left(\sum_{j\sim i} g_{ij}(\|e_i\|^2 - e_i^T e_j) + \frac{1}{N}\sum_{j=1}^{N} e_i^T(\hat{e}_j - \hat{e}_i) \right)\end{aligned}$$

The last term above is zero because $\sum_{i=1}^{N} e_i = 0$ so the above equation reduces to

$$\dot{V}(e) = \sum_{i=1}^{N}\sum_{j\sim i} g_{ij}(\|e_i\|^2 - e_i^T e_j)$$

Completing the square within the above summation yields

$$\dot{V}(e) = \sum_{i=1}^{N}\frac{1}{2}\left(\sum_{j\sim i} g_{ij}(\|e_i\|^2 - \|e_j\|^2 + \|e_i - e_j\|^2)\right)$$

Summing the first two termms over i equals zero and recall that $e_i - e_j = x_i - x_j$ so that the above equation reduces to

$$\dot{V}(e) = \sum_{i=1}^{N}\frac{1}{2}\sum_{j\sim i} g_{ij}\|x_i - x_j\|^2$$

Equations 2 and 3 allow us to reduce the above equation to

$$\dot{V}(e) = \frac{N(N-1)}{2}\rho_0 - \frac{\alpha_0}{2}\sum_{i=1}^{N}\sum_{j\sim i}\|x_i - x_j\| \tag{12}$$

By the assumption in equation 10 there exists $\underline{\beta}$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\|x_i - x_j\| \geq \underline{\beta}\|e\|,$$

so if

$$N(N-1)\rho_0 - \alpha_0\underline{\beta}\|e\| \leq 0. \tag{13}$$

then we can use equation 12 to show that $\dot{V}(e) \leq 0$. The inequality in equation 13 is simply a restatement of the lemma's second condtion (equation 11). $\diamondsuit$

Remark: Equation 10 of lemma 4.1 can be viewed as a lower bound on the average interagent distance.

The following lemma is an *instability* result that characterizes the set of $\|e\|$ for which $\dot{V}(e)$ is positive.

Lemma 4.2: Consider the system in equation 7 and let $V(e) = \frac{1}{2}e^T e$ where $e \in \Re^{Nn}$. If there exists $\overline{\beta} > 0$ such that

$$\overline{\beta}\|e\| \geq \sum_{i=1}^{N}\sum_{j=1}^{N}\|x_i - x_j\| \tag{14}$$

and if

$$\|e\| \leq \frac{N(N-1)\rho_0}{\overline{\beta}\alpha_0}$$

then $\dot{V}(e) \geq 0$.

Remark: Equation 14 of lemma 4.2 is an upper bound on the average interagent distance.

Proof: If there exists $\overline{\beta}$ satisfying inequality 14 and if we require

$$\frac{N(N-1)}{2}\rho_0 - \frac{\alpha_0\overline{\beta}}{2}\|e\| \geq 0 \tag{15}$$

then equation 12 in the proof of lemma 4.1 implies that $\dot{V} \geq 0$. $\diamondsuit$

The following lemma provides bounds on $\|e\|$ and $\|\hat{e}\|$ for which a positive definite function $V(\hat{e})$ of the consensus state has a negative definite directional derivative. Since the consensus error system is a linear system, this lemma is a straightforward UUB analysis.

Lemma 4.3: Consider the system defined in equation 9 and let $V : \Re^n \to \Re$ be the function $V(\hat{e}) = \frac{1}{2}\hat{e}^T\hat{e}$. Let $\underline{\Delta}$ denote the minimum out-degree of the swarm's communication graph. If

$$\|\hat{e}\| \geq \frac{N - 1 - \underline{\Delta}}{1 + \underline{\Delta}}\|e\| \tag{16}$$

then $\dot{V}(\hat{e}) \leq 0$.

Proof: Consider the consensus error dynamics in equation 9. Let $\lambda(\mathbf{A})$ and $\lambda(\mathbf{B})$ denote eigenvalues of system matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. The eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are real since these matrices are symmetric. An application of Gershgorin's theorem establishes that the eigenvalues of $\mathbf{A}$ lie in the union of the sets

$$\Omega_i(\mathbf{A}) = \left\{ z \in \mathcal{C} \;:\; \left| z + \left(\frac{2N-1}{N} + 2\Delta_i\right)\right| \leq \Delta_i + \frac{N-1}{N}\right\}$$

which means the eigenvalues of $\mathbf{A}$ are bounded as

$$-\frac{3N-2}{N} - 3\overline{\Delta} \leq \lambda(\mathbf{A}) \leq -1 - \underline{\Delta}.$$

where $\overline{\Delta}$ is the maximum out-degree of the swarm's communication graph, and $\underline{\Delta}$ is the minimum out-degree. A similar application of Gershgorin's theorem establishes that the eigenvalues of $\mathbf{B}$ lie in the union of sets

$$\Omega_i(\mathbf{B}) = \left\{ z \in \mathcal{C} \;:\; |z| \leq \sum_{j\sim i}|A_{ij} - 1| \right\}$$

which means the eigenvalues of $\mathbf{B}$ are bounded as

$$\underline{\Delta} + 1 - N \leq \lambda(\mathbf{B}) \leq N - 1 - \underline{\Delta}$$

Now consider the directional derivative

$$\dot{V}(\hat{e}) \;=\; \hat{e}^T\dot{\hat{e}} = \hat{e}^T\mathbf{A}\hat{e} + \hat{e}^T\mathbf{B}e.$$

We may use the aforementioned bounds on $\lambda(\mathbf{A})$ and $\lambda(\mathbf{B})$ to show that

$$\dot{V}(\hat{e}) \quad \leq \quad -(1+\underline{\Delta})\|\hat{e}\|^2 + (N-1-\underline{\Delta})\|e\|\|\hat{e}\| \quad (17)$$

If the righthand side of equation 17 is negative definite then $\dot{V}(\hat{e}) \leq 0$. Inequality 17 can be rearranged to yield equation 16. $\diamondsuit$

V. Cohesion Analysis

Establishing the cohesion of the swarm under consensus is accomplished by examining the regions identified in lemmas 4.1 to 4.3. This examination allows us to identify a compact region which is an attracting invariant set of the system.

Proposition 5.1: Consider the interconnected system given by equation 7 and 8. Assume there exist constants $\underline{\beta}$ and $\overline{\beta}$ such that

$$\underline{\beta}\|e\| \leq \sum_{i=1}^{N}\sum_{j=1}^{N}\|x_i - x_j\| \leq \overline{\beta}\|e\| \quad (18)$$

Let

$$\Omega_s^- = \left\{(e,\hat{e}) \in \Re^{2Nn} \ : \ \|e\| \geq \frac{N(N-1)\rho_0}{\underline{\beta}\alpha_0}\right\}$$
$$\Omega_s^+ = \left\{(e,\hat{e}) \in \Re^{2Nn} \ : \ \|e\| \leq \frac{N(N-1)\rho_0}{\overline{\beta}\alpha_0}\right\}$$
$$\Omega_c^- = \left\{(e,\hat{e}) \in \Re^{2Nn} \ : \ \|\hat{e}\| \geq \frac{N-1-\underline{\Delta}}{1+\underline{\Delta}}\|e\|\right\}$$

For any initial state $(e(0),\hat{e}(0)) \in \Re^{2Rn}$, the set

$$\Omega = (\Omega_s^+)^c \cap (\Omega_s^-)^c \cap (\Omega_c^-)^c$$

is an attracting invariant set.

Proof: The region identified in lemmas 4.1, 4.2, and 4.3 are precisely the sets Ω_s^-, Ω_s^+ and Ω_c^-, respectively. The set Ω is the intersection of the complements of these sets. From lemmas 4.1 and 4.2, we know the region $(\Omega_s^+)^c \cap (\Omega_s^-)^c$ must be an attracting invariant set. From lemma 4.3 we know that the region $(\Omega_c^-)^c$ is an attracting invariant set. Therefore the intersection of these two sets (the set Ω) is also an attracting invariant set and the proposition's proof is complete. $\diamondsuit$

Remark: Figure 2 provides a graphic illustration of proposition 5.1's proof. The boundary of sets Ω_s^-, Ω_s^+ and Ω_c^- are shown in figure 2 for a system in which $N = 20$, $\underline{\Delta} = 10$, $\rho_0 = 1$ and $\alpha_0 = 2$. The downward arrow shows the direction in which $V(\hat{e})$ is a monotone decreasing function of time.

The right to left (left to right) arrow shows the direction in which $V(e)$ is a monotone decreasing (increasing) function of time. The arrows point to the boundary that $\hat{e}$ or e is converging. The attracting invariant set Ω is the shaded region in the figure.

The following corollary of proposition 5.1 simply states that the swarm is cohesive under consensus.

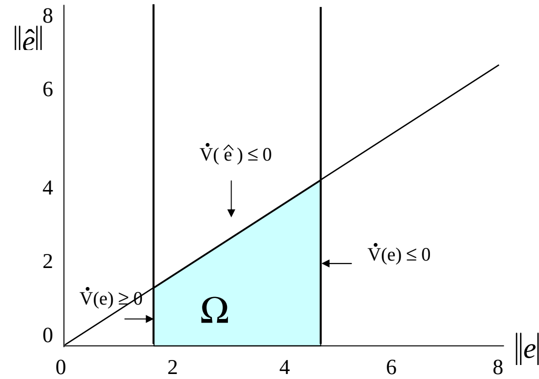

Fig. 2. Geometric Analysis of Interconnected System Cohesiveness ($\rho_0 = 1,\ \alpha_0 = 2,\ N = 20,\ \underline{\Delta} = 10$)

Corollary 5.2: Consider the interconnected system given by equation 7 and 8. Assume there exist constants $\underline{\beta}$ and $\overline{\beta}$ such that

$$\underline{\beta}\|e\| \leq \sum_{i=1}^{N}\sum_{j=1}^{N}\|x_i - x_j\| \leq \overline{\beta}\|e\|$$

Then the swarm is cohesive and achieves ϵ-consensus where

$$\overline{R} = \frac{N(N-1)\rho_0}{\underline{\beta}\alpha_0}, \qquad \underline{R} = \frac{N(N-1)\rho_0}{\overline{\beta}\alpha_0}$$
$$\epsilon = \frac{N-1-\underline{\Delta}}{1+\underline{\Delta}} \cdot \frac{N(N-1)\rho_0}{\underline{\beta}\alpha_0}$$

Proof: The variables $\overline{R}$ and $\underline{R}$ are the bounds on $\|e\|$ obtained in lemmas 4.1 and 4.2, respectively. The variable ϵ is obtained by inserting $\overline{R}$ into the upper bound for $\|\hat{e}\|$ in lemma 4.3. This corresponds to the upper righthand corner of the set Ω in figure 2. $\diamondsuit$

VI. Interagent Distance Analysis

As mentioned earlier, equation 18 in proposition 5.1 is an assumed upper and lower bound on the average inter-agent distance. This bound is expressed as a linear function of the vector 2-norm of the swarm error vector, which we can consider as a reasonable measure of the swarm's size. This section justifies the bound in equation 18 and shows how we can go about computing the constants $\overline{\beta}$ and $\underline{\beta}$.

We first claim that we can always bound the interagent distance as shown in equation 18. Let $\|x\|_1 = \sum_{i=1}^n |x_i|$ denote the 1-norm of vector $x \in \Re^n$. Let $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ denote the 2-norm of vector $x \in \Re^n$. It is already known from standard mathemtical analysis that we can always find constants c and C such that

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2$$

So now consider the swarm error vector $e \in \Re^{Nn}$ and note that

$$\begin{aligned} \|e\|_1 &= \sum_{i=1}^{N} \|e_i\|_1 && (19) \\ &\leq \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \|x_i - x_j\|_1 \\ &\leq \sum_{i=1}^{N} \frac{C}{N} \sum_{j=1}^{N} \|x_i - x_j\|_2 && (20) \end{aligned}$$

which implies that there exists a constant $\underline{\beta}$ such that

$$\underline{\beta}\|e\|_2 \leq \sum_i \sum_j \|x_i - x_j\|_2.$$

Also note that

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N} \|x_i - x_j\|_2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} \|e_i - e_j\|_2 \\ \leq & \sum_{i=1}^{N}\sum_{j=1}^{N} (\|e_i\|_2 + \|e_j\|_2) \\ \leq & \frac{N}{c}\left(\sum_{i=1}^{N} \|e_i\|_1 + \sum_{j=1}^{N} \|e_j\|_1\right) \\ = & \frac{2N}{c}\|e\|_1 && (21) \end{aligned}$$

which implies there exists a constant $\overline{\beta}$ such that

$$\overline{\beta}\|e\|_2 \geq \sum_{i=1}^{N}\sum_{j=1}^{N} \|x_i - x_j\|_2.$$

So we can conclude that we can always find constants $\underline{\beta}$ and $\overline{\beta}$ such that inequality 18 is true.

The determination of constants $\overline{\beta}$ and $\underline{\beta}$ may be accomplished by solving an associated optimization problem. In particular, consider the optimization problem

$$\begin{array}{ll} \text{minimize:} & J = \sum_{i=1}^{N}\sum_{j=1}^{N} \|e_i - e_j\|_2 \\ \text{with respect to:} & e_i \qquad (i = 1, \ldots, N) \\ \text{subject to:} & \sum_{i=1}^{N} \|e_i\|_2^2 = E^2 \\ & \sum_{i=1}^{N} e_i = 0 \end{array}$$

where E is a parameter to be chosen. This parameter represents the total squared distance between swarm agents. Essentially this problem is finding the smallest average interagent distance $\sum_{i=1}^{N}\sum_{j=1}^{N} \|x_i - x_j\|_2$ such that the total squared distance, $\|e\|_2$, is equal to E. Recall that we showed earlier $\sum_{i=1}^{N} e_i$ must always equal zero, so the final constraint in the optimization problem ensures that this condition is satisfied.

The solution to the previous optimization problem may be denoted as $\underline{J}(E)$ where E is the supplied parameter. Since E equals the swarm size $\|e\|_2$, the solutions to this optimization problem is generating the curve $\underline{J}(\|e\|_2)$ which we can then easily fit with a linear function of $\|e\|_2$, thereby identifying the constant $\underline{\beta}$ which enforces the lefthand side of inequality 18.

A similar approach may be used to determine $\overline{\beta}$. In this case, however, we seek to maximize J subject to the same constraints. The solutions to this set of maximization problems will generate solutions $\overline{J}(\|e\|_2)$ which we can again over bound with a linear function of $\|e\|_2$ to determine $\overline{\beta}$.

This optimization problem was solved for a specific swarm of size $N = 20$ using Matlab's `fmincon` function. The asterisks in figure 3 plot $\overline{J}(\|e\|_2)$ and $\underline{J}(\|e\|_2)$ versus $\|e\|_2$. The dashed lines represent the best fit linear functions of $\|e\|_2$ for the data. For this particular swarm we determined that $\overline{\beta} = 114$ and $\underline{\beta} = 40$.

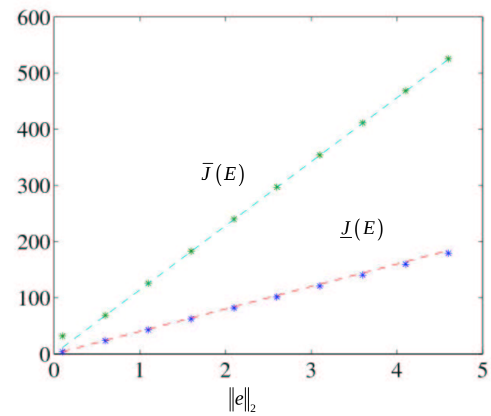

Fig. 3. $\overline{J}$ and $\underline{J}$ versus $\|e\|_2$

The distribution of swarm error vectors e computed by solving this optimization problem are shown in figure 4 for $\|e\|_2 = 5$. The left-hand figure corresponds to the low-energy configuration and the right-hand figure corresponds to the high-energy configuration. The high-energy configuration shows a configuration in which the agents have all grouped together into two distinct clusters. The low-energy configuration shows a set of agents that are uniformly spaced.

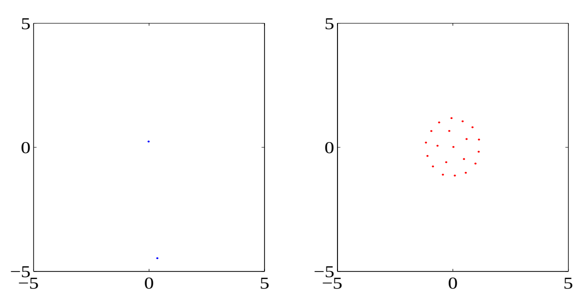

Fig. 4. Agent configurations associated with $\overline{J}^*$ (right) and $\underline{J}^*$ (left)

With the preceding bounds for $\overline{\beta}$ and $\underline{\beta}$ we can now verify the analysis results through simulation. A Matlab script was written to simulate the system equations in equations 5 and 6. This simulation was performed with 20 agents in which the repulsion coefficient ρ_0 and the attraction coefficient, α_0 were both equal to one. The communication graph was specified at time 0 and that graph was kept static over the length of the run. This simulation's communication graph had a maximum connectivity of about $\underline{\Delta} = 10$. The swarm was attempting to intercept a target that started at $(0, 150)$ and moved with a constant velocity of $(-10, -10)$. The swarm was initialized to be uniformly distributed over a rectangular region with side length 30 centered at the $(15, 15)$. The simulation was run for 100 time steps with a step size of $T = 0.02$.

Figure 5 shows the swarm and consensus errors ($x - \overline{x}$ and $\hat{x} - \overline{x}$) at the final simulation time. The righthand plot shows the final swarm error vectors and the lefthand plot shows the final consensus error vectors. In this case the final swarm size $\|e\| = 3.31$ and the final consensus error $\|\hat{e}\| = 0.48$. Note that the final swarm configuration is similar to the low-energy configuration shown in figure 4. This suggests that the associated swarm size should be closer to the lower bound in equation 18 than the upper bound.

Fig. 5. Final Swarm/Consensus Error Vectors ($\rho_0 = 1,\ \alpha_0 = 1,\ N = 20,\ \underline{\Delta} = 10$)

Figures 6 plot the the swarm and consensus errors convergence processes. In this particular simulation, the swarm size $N = 20, \rho_0 = 1, \alpha_0 = 1$ and $\underline{\Delta} = 10$. The figures show that the swarm error e and consensus error $\hat{e}$ converge to the invariant set Ω exponentially.

Fig. 6. Comparison with analytical bounds $\alpha_0 = 2.0$

Figure 7 is similar to the plot shown in figure 2. In this figure, however, not only do we plot Ω, but we show the final swarm and consensus errors achieved by the simulation. This simulation was run for 5000 steps with a step size $T = 0.02$. This final error vector is shown by the blue circle. The region Ω is marked by the dark dotted shaded region. The four plots in figure 7 show these regions and simulation data assuming $\alpha_0 = 2$ and with ρ_0 ranging from 0.5 to 2.0. In viewing the plots, we want to see if the experimental prediction lies within the set Ω. This happens in all cases with the simulation result usually resting at the far lefthand side of the set Ω.

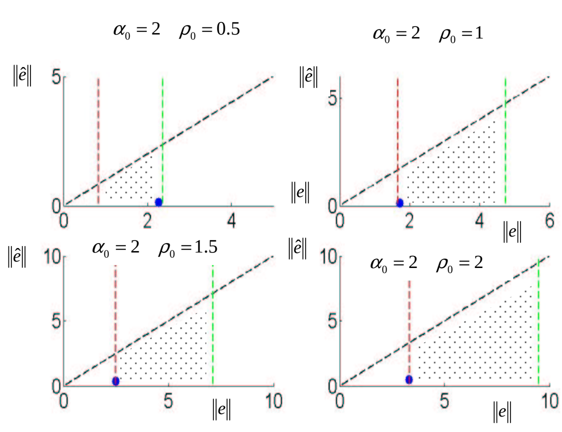

Fig. 7. Comparison with analytical bounds $\alpha_0 = 2.0$

VII. Consensus under Integral Action

As stated in the introduction, the consensus filter generates estimates of the swarm center which are then used by agents to guide the swarm to the target. The analytical bounds and simulation results presented above indicate that using the consensus filter in equation 4, the best we can hope for is ϵ-consensus where the size of ϵ is given in corollary 5.2. Obviously what we'd like to do is identify conditions under which we might drive ϵ to zero.

In general, we've found it is impossible to drive the consensus error $\hat{e}$ to zero for all agents. The best we can show is that under *integral action* we can force all agents to reach a consensus error that is identical for all agents. That error, however, will not be zero. This is done through the introduction of *integral action* in the consensus filter equation. The state equations for the consensus filter with integral action are shown below,

$$\begin{aligned} \dot{\hat{e}} &= \mathbf{A}\hat{e} + \mathbf{B}e + K\mathbf{I}z \\ \dot{z} &= -\mathbf{L}\hat{e} \end{aligned} \qquad (22)$$

where $z \in \Re^{Nn}$ is the integrated error, $K \in \Re$ is the integrator gain, $\mathbf{I}$ is an $Nn \times Nn$ identity matrix, and $\mathbf{L}$ is the Laplacian for the communication graph.

To see how integral action achieves perfect consensus, let's first consider vectors $\hat{e}_{\rm ss} \in \Re^{Nn}$ and $z_{ss} \in \Re^{Nn}$ such that

$$\begin{aligned} 0 &= \mathbf{A}\hat{e}_{ss} + \mathbf{B}e + K\mathbf{I}z_{ss} \\ 0 &= -\mathbf{L}\hat{e}_{ss} \end{aligned}$$

where e is the steady state swarm error vector. The augmented system equations may now be rewritten in matrix vector form as

$$\begin{bmatrix} \dot{\hat{e}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{e} \\ z \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} e$$

In the preceding equations, it should be apparent that $\hat{e}_{ss}$ and z_{ss} are equilibrium points of the unforced system (i.e. $e = 0$).

From the second equilibrium equation we see that $\hat{e}_{ss}$ must lie in the null space of the Laplacian, $\mathbf{L}$. This means that $\hat{e}_{ss} = \alpha\mathbf{1}$ where $\mathbf{1}$ is a vector of ones and α is any real constant. Inserting this into the above equation we see that $z_{ss} = -\frac{\alpha}{K}\mathbf{A1} - \frac{1}{K}\mathbf{B}e$. Since $\hat{e}_{ss} = \alpha\mathbf{1}$, we know that all agents converge to the same consensus error $\hat{e}_i$ under integral action. The magnitude of that error (e.g. α) will depend on the initial conditions of the system. In some sense we can say that the filter achieves perfect consensus because all agents agree upon the same error vector.

A Matlab script was written to simulate swarming under consensus with integral action. In this particular simulation, we set $K = 20$ with $\alpha_0 = 1$, $\rho_0 = 2$, $N = 20$, and $\underline{\Delta} = 10$. Figure 8 plots the swarm position error, the consensus error, and the integrator vector z as a function of time. The plots show that the consensus error clearly converges to a very small constant vector for all agents.

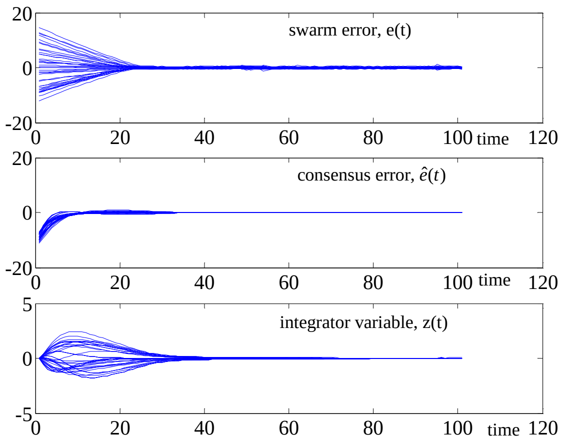

Fig. 8. Swarm/Consensus Time History (integral action)

VIII. Summary

This paper studied cohesive swarming under consensus filtering. Specific bounds were determined for average inter-agent distance, swarm size, and the level of consensus as a function of repulsion strength, attraction strength, number of agents, and communication network connectivity. The theoretical predictions of the analysis were corroborated with computer simulations of the system. This paper also studied the impact of integral action on the consensus filter.

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