Swarming under Perfect Consensus using Integral Action

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***Abstract*— Prior work [1] studied the cohesiveness of swarm dynamics when a consensus £lter [2] is guiding swarm movement. In that earlier work it was shown that the degree of consensus achieved was dependent on the swarm's size. This paper proves that if the swarm's communication graph is regular, then the introduction of integral action into the consensus £lter achieves perfect consensus regardless of swarm size.**

I. INTRODUCTION

Swarming is a collective behavior in which a group of distinct dynamical agents begin to move as a single entity. Swarming behavior has fascinated those interested in how collective actions can *emerge* from the sel£sh behavior of individuals in the group. Early examples may be found in physics and biology [3], [4] [5] [6] [7]. Control scientists have recently begun studying multi-agent systems for applications involving cooperative groups of unmanned autonomous vehicles (UAV's). The cooperative behaviors includes moving in formation [8] [9] [10] [11], aggregating in swarms [12] [13], and exploring hazardous environments [14] [15].

A popular approach to investigate agent interactions uses *Lagrangian* models. Broadly speaking, Lagrangian models can be divided further into two types; swarming and ¤ocking. The term "swarming" is often reserved for kinematic models in which swarm members are treated as point masses. The standing assumption in this case is that viscous forces are large enough so that an agent's acceleration is only signi£cant over a short period of time. The ith agent's state, $x : \Re \rightarrow \Re^n$, therefore, satis£es a £rst order differential equation $\dot{x}_i(t) = F_i(t)$. The function F_i is the control input signal. On the other hand, the term "¤ocking" pertains to a group of agents whose states satisfy a second order differential equation, $\ddot{x}_i(t) = F_i(t)$, in which individuals react to external forces by accelerating. This is clearly distinct from the "swarming" case in which inertial forces are neglected. In both cases, the control input can be written as

$$F_i(t) = \sum_{j \in N_i} f(x_i, x_j) + u_i$$

where N_i is the set of the ith agent's neighbors, $f_i : \Re^n \times \Re^n \rightarrow \Re^n$ models the inter-agent forces and u_i is an exogenous input. Early work in swarming [12] and ¤ocking [13] assumed that N_i was the entire group. Other groups [8] [16] considered ¤ocking in groups in which N_i only captured nearest neighbor interactions. In nearly all of these works, inter-agent forces are modeled as a mixture of short-range repulsion and long-range attraction forces of the form

$$f(x_i, x_j) = \rho(\|x_i - x_j\|)(x_i - x_j) - \alpha(\|x_i - x_j\|)(x_i - x_j) \quad (1)$$

where $\rho : \Re^+ \rightarrow \Re^+$ and $\alpha : \Re^+ \rightarrow \Re^+$ represent repulsive and attractive forces between agents, respectively.

This paper considers *swarm dynamics* in which N_i represents the entire group and inter-agent forces are governed by the repulsive-attractive forces shown in equation 1. The novelty in this work is its focus on external inputs u_i that are generated by a *consensus £lter*. In other words we study the interconnection of a swarm with a consensus £lter as shown in £gure 1. The consensus £lter was originally introduced by Olfati-Saber and Shamma [2] as an extension of consensus protocols [17] used in distributed estimation. In this paper, the consensus £lter generates a collective estimate of the swarm's center and agents use that estimate to guide their movements. The primary question addressed in this paper concerns the cohesiveness of the swarm under consensus and the level of consensus achieved.

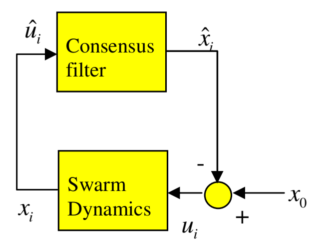

Fig. 1. Interconnection of Swarm and Consensus Filter

Olfati-Saber et al. [18] brie¤y discussed using consensus protocols to guide cooperative multi-agent systems with regard to multi-vehicle formation control [11]. Recently, a detailed stability analysis of vehicle swarms under consensus was done by Lemmon and Sun [1] in which it was shown that the level of consensus was dependent on the swarm's size. In that earlier paper it was conjectured that the introduction of integral action into the consensus £lter could achieve near perfect consensus regardless of swarm size. This paper follows up on that conjecture to prove that integral action indeed achieves perfect consensus provided the group's communication graph is regular.

The remainder of the paper is organized as follows. Section II reviews the main results from [1] that are relevant

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to this paper and introduces the consensus £lter with integral action. The paper's main results will be found in section III with simulation results presented in section IV. Section V summarizes the paper.

II. Prior Results

Our earlier paper [1] studied a swarm of N dynamical agents, whose system equations are,

$$\begin{aligned} \dot{x}_i &= (x_0 - \hat{x}_i) + \sum_{j\sim i} g(\|x_i - x_j\|)(x_i - x_j) \qquad (2) \\ \dot{\hat{x}}_i &= (x_0 - \hat{x}_i) + \sum_{j\sim i} A_{ij}(\hat{x}_j - \hat{x}_i) + \sum_{j=1}^{N} A_{ij}(x_j - \hat{x}_i) \end{aligned}$$

where x_i the *physical* state of the ith agent and $\hat{x}_i$ is the *consensus* state of the ith agent. The target state is denoted as x_0. The coef£cient A_{ij} is the ij-th component of the matrix $\mathbf{I}_N + \mathbf{Adj}(G)$ where $\mathbf{I}_N$ is an $N \times N$ identity matrix and $\mathbf{Adj}(G)$ is the adjacency matrix of the undirected graph, G. The graph G models the communication connectivity within the swarm. Agent j is able to transmit its consensus state $\hat{x}_j$ and its current state x_j to agent i if and only if $A_{ij} = 1$. We let $\sum_{j\sim i} x_j$ denote $\sum_{j=1,j\neq i}^{N} x_j$. The function $g : \Re^+ \rightarrow \Re$ can be written as

$$g(r) = \rho(r) - \alpha(r)$$

for any $r \in \Re^+$ where $\rho : \Re^+ \rightarrow \Re^+$ and $\alpha : \Re^+ \rightarrow \Re^+$ denote the *repulsion* and *attraction* function, respectively. In particular, the original paper [1] and this paper assume that

$$\rho(r) = \frac{\rho_0}{r^2}, \quad \alpha(r) = \frac{\alpha_0}{r}$$

for any $r \in \Re^+$ where ρ_0 and α_0 are positive real constants.

We de£ne the *swarm error* and *consensus error* of the ith agent at time t as

$$\begin{aligned} e_i(t) &= x_i(t) - \overline{x}(t) \\ \hat{e}_i(t) &= \hat{x}_i(t) - \overline{x}(t) \end{aligned}$$

where

$$\overline{x}(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t)$$

denotes the *swarm center*. We let $\mathbf{e}(t)$ and $\hat{\mathbf{e}}(t)$ denote vectors in $\Re^{Nn}$ whose $(in+j)^{\text{th}}$ element is the jth component of the ith agent's swarm and consensus error, respectively.

In the paper [1] it was shown that the norm of the swarm error may be bounded as,

$$\underline{R} = \frac{N(N-1)\rho_0}{\overline{\beta}\alpha_0} \leq \|\mathbf{e}\| \leq \frac{N(N-1)\rho_0}{\underline{\beta}\alpha_0} = \overline{R} \qquad (3)$$

where $\overline{\beta}$ and $\underline{\beta}$ are positive real constants. These constants are associated with the swarm's internal energy [1]. It was also shown that the norm on the consensus error is bounded as

$$\|\hat{\mathbf{e}}\| \geq \frac{N-1-\underline{\Delta}}{1+\underline{\Delta}}\|\mathbf{e}\| \qquad (4)$$

where $\underline{\Delta}$ denotes the minimum out-degree of the communication graph, G. The results in equation 4 indicate that the level of consensus will be bounded below by the swarm size which will always be nonzero. As a result the swarm in [1] will not achieve perfect consensus.

In our earlier paper [1], simulation results were presented suggesting that the introduction of integral action into the consensus £lter might achieve near perfect consensus in which $\|\hat{\mathbf{e}}\|$ is nearly zero. This is similar to the low pass consensus £lter introduced in [2]. The consensus £lter with integral action satis£es the following equations,

$$\begin{aligned} \dot{\hat{x}}(t) &= (x_0(t) - \hat{x}_i(t)) + \sum_{j\sim i} A_{ij}(x_j(t) - \hat{x}_i(t)) \\ &\quad + \sum_{j\sim i} A_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) + K z_i \\ \dot{z}_i &= \sum_{j=1}^{N} A_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) \qquad (5) \end{aligned}$$

for $i = 1, \ldots, N$. The entire system is formed by combining the swarm dynamics in equation 2 with the modi£ed consensus £lter above in equation 5.

Let $\mathbf{z} \in \Re^{Nn}$ be the vector of integrated errors. We can use equations 2 and 5 to obtain the following state equation for the consensus error $\hat{\mathbf{e}}$,

$$\begin{aligned} \dot{\hat{\mathbf{e}}} &= (\mathbf{A} \otimes \mathbf{I}_n)\,\hat{\mathbf{e}} + (\mathbf{B} \otimes \mathbf{I}_n)\,\mathbf{e} + K\mathbf{I}_{Nn}\mathbf{z} \qquad (6) \\ \dot{\mathbf{z}} &= -\mathbf{L} \otimes \mathbf{I}_n \hat{\mathbf{e}} \qquad (7) \end{aligned}$$

where $\mathbf{A} \otimes \mathbf{B}$ is the Kronecker product of matrix $\mathbf{A}$ and $\mathbf{B}$. We can rewrite these equations in matrix vector form as

$$\begin{bmatrix} \dot{\hat{\mathbf{e}}} \\ \dot{\mathbf{z}} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \otimes \mathbf{I}_n \right) \begin{bmatrix} \hat{\mathbf{e}} \\ \mathbf{z} \end{bmatrix} + \left(\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \otimes \mathbf{I}_n \right) \mathbf{e} \qquad (8)$$

In the above equation

$$\begin{aligned} \mathbf{A} &= \mathbf{X} + \mathbf{L}(G) \\ \mathbf{B} &= \mathbf{Adj}(G) + \mathbf{I}_N - \mathbf{1}\mathbf{1}^T \\ \mathbf{L} &= \mathbf{Deg}(G) - \mathbf{Adj}(G) \\ \mathbf{X} &= \frac{1}{N}\mathbf{1}\mathbf{1}^T - 2\mathbf{I}_N - 3\,(\mathbf{Adj}(G)) \end{aligned}$$

where $\mathbf{Adj}(G)$ and $\mathbf{Deg}(G)$ are the adjacency and degree matrix of graph G, respectively. The matrix $\mathbf{L}$ is the Laplacian matrix of G.

III. Consensus Error Analysis

This section contains the paper's main result, which is a theorem establishing conditions under which the consensus £lter achieves perfect consensus. The proof of this result requires the following two technical lemmas. The £rst lemma characterizes the eigenvalues of the system matrix

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix}.$$

The second lemma characterizes the similarity transformation taking $\mathbf{\Phi}$ to its diagonal canonical form. The proofs for both lemmas will be found in the appendix (section VI).

Lemma 3.1: Assume the communication graph, G, is connected then $\mathbf{\Phi}$ has exactly one zero eigenvalue and all other eigenvalues have real parts strictly less than zero.

Lemma 3.2: Let $\mathbf{\Lambda}$ be a diagonal complex-valued matrix whose diagonal elements are the eigenvalues of $\mathbf{\Phi} \otimes \mathbf{I}$. Let $\mathbf{U}$ denote the similarity transformation such that $\mathbf{\Phi} \otimes \mathbf{I} = \mathbf{U\Lambda U}^{-1}$.

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{2N} \\ \underline{\mathbf{u}}_1 & \underline{\mathbf{u}}_2 & \cdots & \underline{\mathbf{u}}_{2N} \end{bmatrix} \otimes \mathbf{I}_n,$$

$$\text{and} \qquad \mathbf{U}^{-1} = \begin{bmatrix} \mathbf{v}_1^T & \underline{\mathbf{v}}_1^T \\ \mathbf{v}_2^T & \underline{\mathbf{v}}_2^T \\ \vdots & \vdots \\ \mathbf{v}_{2N}^T & \underline{\mathbf{v}}_{2N}^T \end{bmatrix} \otimes \mathbf{I}_n$$

where $\mathbf{u}_i$, $\underline{\mathbf{u}}_i$, $\mathbf{v}_i$ and $\underline{\mathbf{v}}_i \in \Re^N$ for $i = 1, \ldots, 2N$. The matrices $\mathbf{U}$ and $\mathbf{U}^{-1}$ have following properties,

1) $\mathbf{u}_{2N} = u \cdot \mathbf{1}^T \in \Re^N$, where u is a constant.
2) $\mathbf{v}_{2N} = 0 \cdot \mathbf{1}^T \in \Re^N$, and
   $\underline{\mathbf{v}}_{2N} = v \cdot \mathbf{1}^T \in \Re^N$, where v is a constant.
3) $K\mathbf{v}_i^T = \lambda_i \underline{\mathbf{v}}_i^T \qquad i = 1, \cdots, 2N-1$
4) $\sum_{i=1}^{2N-1} \mathbf{u}_i \underline{\mathbf{v}}_i^T = -\mathbf{u}_{2N}\underline{\mathbf{v}}_{2N}^T$
5) $\underline{\mathbf{v}}_{2N}^T \underline{\mathbf{u}}_{2N} = 1$
6) $\mathbf{A}\mathbf{u}_{2N} + K\underline{\mathbf{u}}_{2N} = 0$

From equation 3, we know eventually $\|\mathbf{e}(t)\| \leq \overline{R}$. Assume that $\mathbf{e}(t)$ satisfies this inequality at time 0. Assuming that initial state satisfies $\hat{\mathbf{e}}(0) = \mathbf{z}(0) = 0$, we can use the consensus error equations 6-7 to see

$$\begin{bmatrix} \hat{\mathbf{e}}(t) \\ \mathbf{z}(t) \end{bmatrix} = \int_0^t \mathbf{U} e^{\mathbf{\Lambda}(t-\tau)} \mathbf{U}^{-1} \begin{bmatrix} \mathbf{B} \otimes \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{e}(\tau) d\tau$$

We define the vector $\|\hat{\mathbf{e}}_{\text{ss}}\|$

$$\|\hat{\mathbf{e}}_{\text{ss}}\| = \lim_{t\to\infty} \sup \{\|\hat{\mathbf{e}}(\tau)\| \ : \ t \geq \tau\} \tag{9}$$

The following theorem provides an upper bound on $\|\hat{\mathbf{e}}_{\text{ss}}\|$.

Theorem 3.3:

$$\|\hat{\mathbf{e}}_{\text{ss}}\| \leq \left\| \left(\frac{1}{c} \mathbf{1}\mathbf{1}^T \right) \mathbf{B} \right\| \overline{R} \tag{10}$$

where the scalar c is

$$c = -\mathbf{1}^T \mathbf{A} \mathbf{1}$$

Proof: For notational convenience we let $\overline{\mathbf{A}} = \mathbf{A} \otimes \mathbf{I}$ and $\overline{\mathbf{B}} = \mathbf{B} \otimes \mathbf{I}$. Then

$$\begin{aligned} \|\hat{\mathbf{e}}(t)\| &= \left\| \int_0^t [\mathbf{I}_{Nn} \ \mathbf{0}_{Nn}] \mathbf{U} e^{\mathbf{\Lambda}(t-\tau)} \mathbf{U}^{-1} \begin{bmatrix} \overline{\mathbf{B}} \\ \mathbf{0} \end{bmatrix} \mathbf{e}(\tau) d\tau \right\| \\ &\leq \left\| \int_0^t \mathbf{u} e^{\Lambda(t-\tau)} \mathbf{v}^T \overline{\mathbf{B}} d\tau \right\| \overline{R} \end{aligned}$$

where

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{2N} \end{bmatrix} \otimes \mathbf{I}_n \\ \mathbf{v} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{2N} \end{bmatrix} \otimes \mathbf{I}_n \end{aligned}$$

Since $\mathbf{v}_{2N} = \mathbf{0}$, the lowest $n \times n$ sub-matrix in the matrix $\mathbf{v}^T \overline{\mathbf{B}}$ is zero. So we can conclude

$$\begin{aligned} \|\hat{\mathbf{e}}_{ss}\| &\leq \left\| \mathbf{u} \begin{bmatrix} -\frac{1}{\lambda_1} & & & \\ & \ddots & & \\ & & -\frac{1}{\lambda_{2N-1}} & \\ & & & r \end{bmatrix} \otimes \mathbf{I}_n \mathbf{v}^T \overline{\mathbf{B}} \right\| \overline{R} \\ &= \left\| \left(\sum_{i=1}^{2N-1} -\frac{\mathbf{u}_i \mathbf{v}_i^T}{\lambda_i} + r\mathbf{u}_{2N}\mathbf{v}_{2N}^T \right) \mathbf{B} \right\| \overline{R} \\ &= \left\| \sum_{i=1}^{2N-1} -\frac{\mathbf{u}_i \mathbf{v}_i^T}{\lambda_i} \mathbf{B} \right\| \overline{R} \\ &= \left\| \left(\frac{1}{K} \mathbf{u}_{2N} \underline{\mathbf{v}}_{2N}^T \right) \mathbf{B} \right\| \overline{R} \\ &= \left\| \left(\frac{1}{K} uv \mathbf{1}\mathbf{1}^T \right) \mathbf{B} \right\| \overline{R} \end{aligned}$$

where the last couple equalities follow from lemma 3.2.

Multiplying $\underline{\mathbf{v}}_{2N}^T$ on the left-hand side and using the sixth item in lemma 3.2 yields,

$$\underline{\mathbf{v}}_{2N}^T \mathbf{A} \mathbf{u}_{2N} + K = 0$$

The above equation is equivalent to,

$$uv \cdot \mathbf{1}^T \mathbf{A} \mathbf{1} + K = 0$$

Therefore,

$$\|\hat{\mathbf{e}}_{\text{ss}}\| \leq \left\| \left(\frac{1}{-\mathbf{1}^T \mathbf{A} \mathbf{1}} \mathbf{1}\mathbf{1}^T \right) \mathbf{B} \right\| \overline{R}$$

and the proof is complete. ■

The following theorem represents the main result of this paper. It states and proves a bound on $\|\hat{\mathbf{e}}_{ss}\|$ for swarming under consensus with integral action.

Theorem 3.4: Let $\overline{\Delta}$ and $\underline{\Delta}$ denote the maximum and minimum out-degree of the communication graph, respectively. Then

$$\|\hat{\mathbf{e}}_{ss}\| \leq \frac{\overline{\Delta} - \underline{\Delta}}{N(1 + \underline{\Delta})} \overline{R} \tag{11}$$

Remark: A regular graph is one in which $\underline{\Delta} = \overline{\Delta}$, so that all nodes have the same out degree. Theorem 3.4 means that if the graph, G, is regular then the consensus error will be zero in swarms under consensus with integral action regardless of the swarm's size.

Proof: Let

$$\mathbf{M} = \left(\frac{1}{c} \mathbf{1}\mathbf{1}^T \right) \mathbf{B},$$

where the scalar c is $-\mathbf{1}^T \mathbf{A} \mathbf{1}$. The norm of $\mathbf{M}$ can be bounded as

$$\begin{aligned} \|\mathbf{M}\|^2 &\leq \frac{1}{c^2} \mathbf{x}^T \mathbf{B}^T \mathbf{1}\mathbf{1}^T \cdot \mathbf{1}\mathbf{1}^T \mathbf{B} \mathbf{x} = \frac{N}{c^2} \mathbf{x}^T \mathbf{B}^T \mathbf{1}\mathbf{1}^T \mathbf{B} \mathbf{x} \\ &= \frac{N}{c^2} \left(\mathbf{1}^T \mathbf{B} \mathbf{x} \right)^2 \end{aligned}$$

where $\mathbf{x} \in \Re^N$ is a nonzero vector such that $\|\mathbf{x}\| = 1$ and $\sum_{i=1}^N x_i = 0$.

From the de£nition of matrix $\mathbf{B}$ we can show that

$$\mathbf{1}^T\mathbf{B}\mathbf{x} = -\sum_{i=1}^{N}(N-1-\Delta_i)x_i$$

where x_i is the i^{th} element of vector $\mathbf{x}$.

By construction $\sum_{i=1}^{N} x_i = 0$ so we can partition $\mathbf{x}$ so that $x_i \leq 0$ $(i = 1, \cdots, \ell)$ and $x_i \geq 0$ $(i = \ell+1, \cdots, N)$. Then,

$$\left|\mathbf{1}^T\mathbf{B}\mathbf{x}\right| \leq \overline{\Delta}\sum_{i=1}^{\ell} x_i - \underline{\Delta}\sum_{i=\ell+1}^{N} x_i = (\overline{\Delta}-\underline{\Delta})\sum_{i=1}^{\ell} x_i$$

Application of Cauchy's Inequality $N\left(\sum_{i=1}^{\ell} x_i\right)^2 \leq \sum_{i=1}^{\ell} x_i^2$ yields,

$$\begin{aligned} \|\mathbf{M}\|^2 &\leq \frac{N}{c^2}\left((\overline{\Delta}-\underline{\Delta})\sum_{i=1}^{\ell} x_i\right)^2 \leq \frac{1}{c^2}(\overline{\Delta}-\underline{\Delta})\sum_{i=1}^{\ell} x_i^2 \\ &\leq \frac{1}{c^2}(\overline{\Delta}-\underline{\Delta})\sum_{i=1}^{N} x_i^2 = \frac{1}{c^2}(\overline{\Delta}-\underline{\Delta})\,\|\mathbf{x}\| \\ &= \frac{1}{c^2}(\overline{\Delta}-\underline{\Delta}) \end{aligned}$$

From the de£nition of $\mathbf{A}$ we obtain

$$c = N + \sum_{i=1}^{N}\Delta_i$$

which we can combine in the above inequality to obtain the theorem's result. ■

The following corollary characterizes the degree of consensus achieved with and without integral action when swarming under consensus.

Corollary 3.5: The ratio of the minimum consensus errors $\hat{\mathbf{e}}_{\text{int}}$ and $\hat{\mathbf{e}}_{\text{no}-\text{int}}$ achieved with and without integral action, respectively is

$$\frac{\|\hat{\mathbf{e}}\|_{\text{int}}}{\|\hat{\mathbf{e}}\|_{\text{no}-\text{int}}} \leq \frac{\overline{\Delta}-\underline{\Delta}}{N(N-1-\underline{\Delta})} \qquad (12)$$

Remark: This corollary bounds the decrease in the consensus error when we add integral action. The result shows that consensus error can be small in poorly connected graphs ($\overline{\Delta}$ is small) provided the swarm is large enough.

Proof: This follows directly from equation 4 and equation 11 in theorem 3.4. ■

IV. Simulation

A matlab script was written to simulate swarming under consensus with integral action. In the following simulations, the integrator gain is $K = 20$ and the swarm size is $N = 20$. The repulsion/attraction strengths are $\rho_0 = 1$ and $\alpha_0 = 2$, respectively. Every simulation ran for 6000 time steps with a step size of $T = 0.02$.

We £rst simulated swarming under consensus with integral action on the two communication graphs shown in £gure 2. The left-hand £gure corresponds to a regular graph with degree 8. The right-hand £gure corresponds to a connected

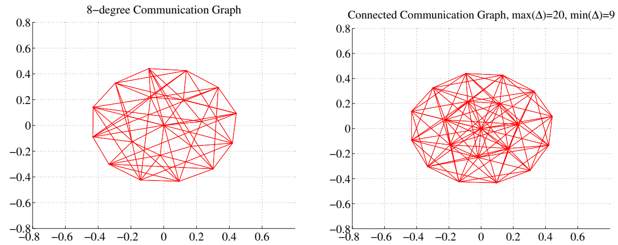

Fig. 2. Communication Graph, N= 20 (left) 8-degree (right) connected graph

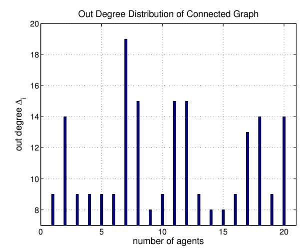

Fig. 3. Out Degree Distribution of Connected Graph

graph with $\overline{\Delta} = 19$ and $\underline{\Delta} = 8$. The degree distribution for this graph is shown in £gure 3.

Even though the connected graph has an agent that is connected to all other agents, the entire swarm is unable to achieve perfect consensus. The regular graph, on the other hand, achieved perfect consensus as is shown in £gure 4. This £gure plots the log of the norm squared consensus error, $\|\hat{\mathbf{e}}\|^2$ as a function of time. In this particular simulation the swarm size, $\|\mathbf{e}\|$ was bounded above by 1.6662. The solid line in £gure 4 is the consensus error for the regular graph and the dashed line is the consensus error for the other graph. In the regular graph, the consensus error reached a minimum level of $\|\hat{\mathbf{e}}_{ss}\| = 5.9174e-014$, which is essentially zero. The minimum consensus error achieved over the other graph was several orders of magnitude larger.

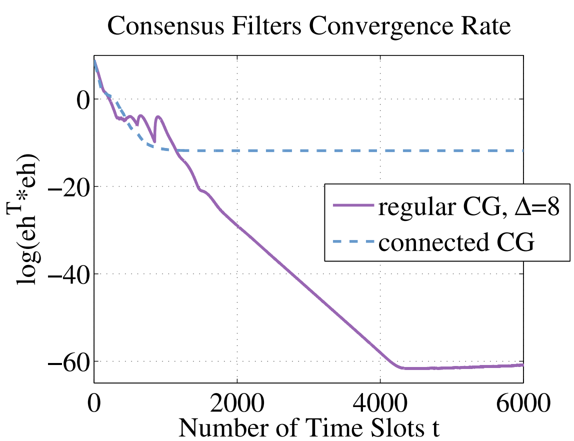

Fig. 4. Consensus Error Bound with Integral Action, N=20

As noted above, even if the graph is not regular, integral action can dramatically improve the level of consensus. Figure 5 shows the comparison of minimum consensus error with /without integral action on the same communication graph. In this paritcular graph the node out-degrees were bounded between $\underline{\Delta} = 3$ and $\overline{\Delta} = 7$. Without integral action the minimum consensus error was about 0.9851 (dashed line). With an integral gain of $K = 20$, the same system achieved a minimum consensus level of 0.0102. The £gure veri£es that integral action can decrease the consensus error signi£cantly.

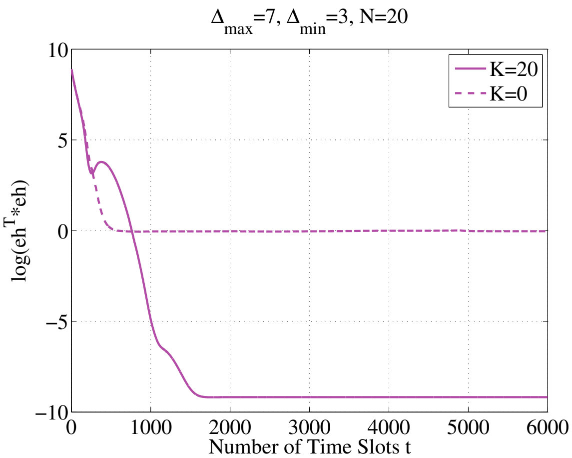

Fig. 5. Consensus Error Equilibrium with /without Integral Action

The following simulation results experimentally evaluate the tightness of the bounds presented in theorem 3.4 and corollary 3.5. Theorem 3.4's proof used the following bound

$$\left|\mathbf{1}^T\mathbf{B}\mathbf{x}\right| \leq (\overline{\Delta} - \underline{\Delta}) \sum_{i=1}^{\ell} x_i \quad \text{and,}$$

$$(\overline{\Delta} - \underline{\Delta}) \sum_{i=1}^{\ell} x_i^2 \leq (\overline{\Delta} - \underline{\Delta}) \sum_{i=1}^{N} x_i^2$$

in which a unit vector $\mathbf{x}$ satisfying $\sum_{i=1}^{N} x_i = 0$ was partitioned into its positive and negative components ($x_i \leq 0$ $(i = 1, \cdots \ell)$ and $x_i \geq 0$ $(i = \ell + 1, \cdots, N)$). The bound clearly gets tight when $\underline{\Delta}$ is close to $\overline{\Delta}$.

Figure 6 illustrates the relationship between $(\overline{\Delta} - \underline{\Delta})$ and the bound on $\|\hat{\mathbf{e}}_{ss}\|$. This £gure plots $\|\hat{\mathbf{e}}_{ss}\|$ for two different graphs. In the £rst graph (solid line) there is a large spread in the node out-degrees ($\overline{\Delta} = 13$ and $\underline{\Delta} = 5$). In this case the consensus error is predicted to be less than 0.3176 by theorem 3.4. The actual minimum consensus error, however, was only 0.0154. In the second graph (dashed line), there is a small spread in the node out-degrees ($\underline{\Delta} = 10$ and $\overline{\Delta} = 11$). For this case, theorem 3.4 predicts a consensus error that is less than 0.0216 with the actual norm being 0.0134. These results show close agreement between the predictions made in theorem 3.4 and actual simulated results.

V. Summary

This paper studied the effect of integral action on the consensus £lter. When compared to consensus errors in swarms without integral action, we found that adding integral action

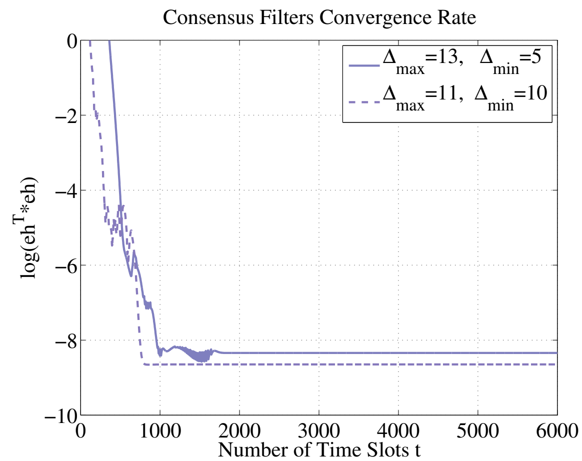

Fig. 6. Consensus Error Bound with different $\overline{\Delta}$ and $\underline{\Delta}$

dramatically improved the level of consensus. In particular, we found that if the communication graph is regular, then the swarm could achieve perfect consensus under integral action.

VI. Proofs

Proof: **Lemma 3.1:** Any eigenvalue λ of $\mathbf{\Phi}$ must satisfy the characteristic equation $\chi(\mathbf{\Phi}) = 0$ so that

$$\begin{aligned} 0 &= \chi(\mathbf{\Phi}) = \det \begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & -K\mathbf{I} \\ \mathbf{L} & \lambda\mathbf{I} \end{bmatrix} \\ &= \det(\lambda\mathbf{I} - \mathbf{A}) \det\left(\lambda\mathbf{I} + K\mathbf{L}(\lambda\mathbf{I} - \mathbf{A})^{-1}\right) \\ &= \det(\lambda(\lambda\mathbf{I} - \mathbf{A}) + K\mathbf{L}) \end{aligned} \quad (13)$$

The rank of Laplacian matrix $\mathbf{L}(G)$ is $N - 1$ when the graph G is connected, so that $\det(K\mathbf{L}) = 0$. This implies that $\mathbf{\Phi}$ has at least one zero eigenvalue.

We now show that the eigenvalues of $\mathbf{\Phi}$ cannot have positive real parts. Assume that λ is an eigenvalue of $\mathbf{\Phi}$ such that $\text{Re}(\lambda) > 0$. If λ is complex, its complex conjugate, $\overline{\lambda}$ must also be an eigenvalue of $\mathbf{\Phi}$. So consider any vector $\mathbf{x} \neq 0$ in $\Re^N$ and let $\mathbf{x}^+$ denote its conjugate transpose, then

$$\begin{aligned} & \mathbf{x}^+(\lambda(\lambda\mathbf{I} - \mathbf{A}) + K\mathbf{L})\mathbf{x} \\ + \; & \mathbf{x}^+(\overline{\lambda}(\overline{\lambda}\mathbf{I} - \mathbf{A}) + K\mathbf{L})\mathbf{x} \\ = \; & \left(\lambda^2 + \overline{\lambda}^2\right) \mathbf{x}^+\mathbf{x} - \left(\lambda + \overline{\lambda}\right) \mathbf{x}^+\mathbf{A}\mathbf{x} + K\mathbf{x}^+\mathbf{L}\mathbf{x} \\ = \; & 2\left(\text{Re}(\lambda)\right)^2 \mathbf{x}^+\mathbf{x} - 2\text{Re}(\lambda)\mathbf{x}^+\mathbf{A}\mathbf{x} + K\mathbf{x}^+\mathbf{L}\mathbf{x} \end{aligned} \quad (14)$$

In [1], Gershgorin's theorem was used to show that the eigenvalues of $\mathbf{A}$ were real and negative. Therefore the facts that $\mathbf{A}$ is negative de£nite, $\mathbf{L}$ is positive semide£nite and $\text{Re}(\lambda) > 0$ can be used to deduce that the right-hand side of equation 14 is greater than zero which means that λ cannot satisfy the characteristic equation and so if λ is complex, it cannot have positive real part. A similar argument can be used to show that λ cannot be positive if it is real.

Finally, we show that $\mathbf{\Phi}$ has at most one zero eigenvalue. We consider matrix $\mathbf{\Phi}^T$,

$$\mathbf{\Phi}^T = \begin{bmatrix} \mathbf{A} & -\mathbf{L} \\ K\mathbf{I} & \mathbf{0} \end{bmatrix}$$

and let λ_j be the zero eigenvalue of $\mathbf{\Phi}^T$ with associated eigenvector $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T \end{bmatrix}^T \in \Re^{2N}$ in which $\mathbf{u}_1, \mathbf{u}_2 \in \Re^N$. then

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{u}_1 - \mathbf{L}\mathbf{u}_2 \quad (15) \\ \mathbf{0} &= -K\mathbf{u}_1 \quad (16) \end{aligned}$$

Equation 16 implies that $\mathbf{u}_1 = 0$ and $\mathbf{u}_2$ is the eigenvector resulting in $\mathbf{L}\mathbf{u}_2 = \mathbf{0}$. $\mathbf{u}_2$ will be any vector belonging to the null-space of matrix $\mathbf{L}$. Because $\mathbf{L}$ is the Laplacian of graph G, the dimension of $\mathbf{L}$'s null space is exactly one, thereby completing the proof. ■

Proof: **Lemma 3.2:** It is straightforward that

$$\mathbf{U}^{-1} \begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \otimes \mathbf{I}_n = \mathbf{\Lambda}\mathbf{U}^{-1} = \begin{bmatrix} \mathbf{x} \\ \vdots \\ \mathbf{x} \\ \mathbf{0} \end{bmatrix} \in \Re^{2Nn \times 2Nn}$$

and,

$$\begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \otimes \mathbf{I}_n \mathbf{U} = \mathbf{U}\mathbf{\Lambda} = \begin{bmatrix} \mathbf{x} & \cdots & \mathbf{x} & \mathbf{0} \end{bmatrix} \in \Re^{2Nn \times 2Nn}$$

where $\mathbf{x}$ is any complex vector satisfying dimension requirement.

1) For the eigenvalue of $\lambda_{2N} = 0$, we have,

$$\begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2N} \\ \underline{\mathbf{u}}_{2N} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^{2N}$$

Hence, $\mathbf{u}_{2N}$ is the null space vector of Laplacian matrix $\mathbf{L}$, which completes the proof of the £rst item.

2) The second item's proof is similar to the £rst one. For the eigenvalue of $\lambda_{2N} = 0$, we have,

$$\begin{bmatrix} \mathbf{v}_{2N}^T & \underline{\mathbf{v}}_{2N}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \in \Re^{1 \times 2N}$$

So that, $\mathbf{v}_{2N} = \mathbf{0}$ and $\underline{\mathbf{v}}_{2N} = v \cdot \mathbf{1}$.

3) For the eigenvalue of $\lambda_i \neq 0$, we have,

$$\begin{bmatrix} \mathbf{v}_i^T & \underline{\mathbf{v}}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} = \lambda_i \begin{bmatrix} \mathbf{v}_i^T & \underline{\mathbf{v}}_i^T \end{bmatrix}$$

It is easy to show,

$$K\mathbf{v}_i^T = \lambda_i \underline{\mathbf{v}}_i^T \qquad i = 1, \cdots, 2N-1$$

4) Since $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$, it means the matrix block

$$\begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_{2N} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{v}}_1^T \\ \vdots \\ \underline{\mathbf{v}}_{2N}^T \end{bmatrix} = \mathbf{0} \in \Re^{N \times N}$$

It turns out,

$$\sum_{i=1}^{2N-1} \mathbf{u}_i \underline{\mathbf{v}}_i^T = -\mathbf{u}_{2N} \underline{\mathbf{v}}_{2N}^T$$

5) Because of $\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$, we have the last element

$$\mathbf{v}_{2N}^T \mathbf{u}_{2N} + \underline{\mathbf{v}}_{2N}^T \underline{\mathbf{u}}_{2N} = 1$$

in terms of the property of $\mathbf{v}_{2N} = \mathbf{0}^T$, the above equation is equivalent to, $\underline{\mathbf{v}}_{2N}^T \underline{\mathbf{u}}_{2N} = 1$.

6) It is easy to shown based on the equation,

$$\begin{bmatrix} \mathbf{A} & K\mathbf{I} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2N} \\ \underline{\mathbf{u}}_{2N} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^{2N \times 1}$$

■

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