Convergence of Consensus Filtering under Network Throughput Limitations

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***Abstract*— Consensus filters [1] provide a distributed way of computing data aggregates in embedded sensor networks. Prior work has suggested that the rate at which such filters achieve consensus is proportional to the number of neighbors. This conclusions, however, is simplistic because it ignores the intrinsic throughput limitation of multi-hop networks. This paper examines the convergence behavior of consensus filters under such throughput limitations. We consider a time-slotted frequency division multiple access (FDMA) network assuming a regular network. Under these assumptions we show that throughput limits can be modeled as delays. We study the impact these delays have on the time and energy that consensus filters require to achieve ϵ-consensus.**

I. Introduction

In many sensor network applications, it is important for agents to have a global aggregate of the network's sensor measurements. Consensus filtering [1] provides one way of computing such aggregates in a distributed manner. These filters achieve consensus when all agents within the network agree upon the same value for the aggregated variable. The rate at which such filters achieve consensus is proportional to the number of neighbors each agent can communicate with. This suggests that as we increase the connectivity within the network's communication graph, we increase the rate at which the algorithm achieves consensus. This conclusion, however, is simplistic because it ignores the intrinsic throughput limitation of multi-hop communication networks.

Network throughput limitations [2] have a major impact on the consensus filter's convergence rate. A direct consequence of limited throughput capacity is longer communication delay or latency. Due to message collisions, it is impossible for a receiver to collect all its neighbors' information instantaneously. There is always a finite probability that some of the neighboring data will be corrupted and will have to be resent. Resending data will delay message delivery in a way that can adversely effect the relative stability of the consensus filter, and in a way that decrease the filter's convergence rate.

It is well known [3] that if there are no delays then the consensus filter's convergence rate increases with network connectivity. But as discussed above, increased network connectivity will also increase the average delay in message delivery. There is, therefore, a fundamental trade off between network connectivity and message delay that leads to an "optimal" level of network connectivity. The purpose of this paper is to examine that trade off in greater detail.

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Considerable research, e.g., [3], [4], [5], [6], [7], [8], [9], [10] and [11] has been done on consensus protocols for the purpose of either aggregation of statistics or averaging of computation load. In most of the prior work the consensus communication model is described as a disc model. Each member is allowed to communicate with neighbors within a specified communication radius. Hence, the behavior of the consensus filter is studied under the assumption of limited network connectivity. Sufficient conditions for asymptotic consensus convergence have been derived using different mathematical tools, e.g., Lyapunov methodology [3], convex optimization [8] and matrix analysis [9]. The convergence rates of these protocols are characterized by the second-smallest eigenvalue of the Laplacian matrix associated with the communication graph. In [3], it is shown that the convergence rate is proportional to the connectivity of the communication network. This rate can also be changed by manipulating weights on the edges of the communication graph [7] [11].

Real-life communication networks have limited resources such as the number of channels and channel bandwidth. These resource constraints can delay message delivery in ways that can adversely effect the consensus filter stability. Consensus with time-delay was first explored in [3]. This work, however, only established an upper bound on the delay time for the system's asymptotic stability. This paper does much more by studying the relation between network topology, time delay, and convergence rate. Probably the work most related to our approach will be found in the communications literature. Results similar to those derived in this paper will be found in [12] where the throughput is maximized subject to the packet collision in wireless networks. More recently a series of papers [13], [14], [15] and [16] have studied connectivity in packet radio networks for different optimization objectives. The results obtained in these papers are similar to our results in that they also try to identify network topologies that optimize message throughput.

In this paper we still use the consensus filter model proposed in [1]. The paper answers questions on the performance of consensus filters in ad-hoc wireless network scenarios. The main contribution of this paper is the analysis of the convergence rate of consensus under throughput limits. We show that there is an optimal level of communication connectivity which maximizes the filter's convergence rate and energy efficiency. Section II discusses the delay with which messages are delivered under a communication model commonly found in multi-hop wireless sensor networks. Section III characterizes the convergence rate of the consensus filter under such message delays. Section IV characterizes the minimum energy required to achieve ϵ-consensus. Simulation results in support of the paper's analysis will be found in section V.

II. Problem Statement

The consensus problem studied in this paper has it origins in the distributed filter framework introduced by [1]. Consider a sensor network of size N. The *consensus* state of this network is a function $x_i : \Re \to \Re^n$ $(i = 1, \ldots, N)$ that satisfies the consensus filter equations [1]

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i) + \sum_{j \in (\mathcal{N}_i \cup \{i\})} a_{ij} (u_j - x_i)$$

where $u_i : \Re \to \Re^n$ is the filter's ith input. This equation can be rewritten in matrix-vector form as

$$\dot{\mathbf{x}} = -(\mathbf{I}_N + \mathbf{\Delta} + \mathbf{L})\,\mathbf{x} + (\mathbf{I}_N + \mathbf{Adj}(G))\,\mathbf{u} \tag{1}$$

where $\mathbf{x}$ is the vector of consensus states, $\mathbf{u}$ is the vector of filter inputs, $\mathbf{I}_N$ is an $N \times N$ identity matrix, $\mathbf{Adj}(G)$ is the adjacency matrix of the undirected graph, G. $\mathbf{L}$ is the graph G's Laplacian matrix and $\mathbf{\Delta}$ is a diagonal matrix whose diagonal elements are the outdegrees of the graph's nodes. The graph G models the communication connectivity within the nodes. Node j is able to transmit its consensus state x_j to node i if and only if the ijth component of $\mathbf{Adj}(G)$ is one (i.e.,$a_{ij} = 1$).

We are concerned with the impact that limits on network throughput have on the consensus filter's performance. This paper therefore confines its attention to the consensus dynamics given by,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t - \overline{\tau}) + \mathbf{r} \tag{2}$$

where $\mathbf{A} = -(\mathbf{I}_N + \mathbf{\Delta} + \mathbf{L})$, $\mathbf{r}$ is a constant input vector, and $\overline{\tau}$ is the average delay with which messages are transmitted throughout the network.

We consider the delay τ arising in a time-slotted wireless communication network in frequency division multiple access (FDMA). We consider a wireless communication network in which nodes transmit in time slots. When a node wants to transmit its local state, it select with equal probability one frequency in a fixed set of Q frequencies. The network, therefore, is a time-slotted network using frequency division multiple access (FDMA) to the wireless medium. Suppose node x_i transmits over the m^{th} subchannel to a node x_j. the communication delay $\tau_{ij} > 0$ measures the average time slots taken for accomplishing a successful transmission. Regular communication graphs have been shown to be an efficient network topology for distributed consensus [1]. They have also been shown to arise naturally in the swarming under consensus framework [17]. We therefore assume that each node has the same number, Δ, of neighbors. Based on this assumption we can then derive a delay, $\overline{\tau}$, for equation 2 which is the statistical average over all information flows.

Because we're considering wireless radio networks, each node's broadcast through a selected subchannel is simultaneously transmitted to all of the node's Δ neighbors at the same time. The wireless medium, however, is shared with other nodes in the network so that two messages may collide with each other at the designated receiver. When this occurs, the message from both transmitters is destroyed. In addition to this we assume that nodes cannot receive and transmit at the same time, so that transmitted messages will not be successfully received if the destination is also transmitting. To decrease the likelihood of these message collisions, each node broadcasts with a probability p in every time slot. On average, the time it takes for a node to successfully gather all of its Δ neighbors' messages is denoted as τ. After receiving the Δ messages, the given node updates its consensus state.

The average communication delay $\overline{\tau}$ can be written as

$$\overline{\tau} = \gamma \cdot \frac{1}{Rp}$$

where R is the probability of a successful receiving all of the neighbors' messages, and p is the broadcast probability. For a regular network with Δ-connectivity, the receiving probability R is,

$$R = \left(1 - \frac{p}{Q}\right)^{\Delta} \tag{3}$$

The average time delay, therefore, is a function of the graph's connectivity, Δ. Obviously, if a node knows how many neighbors it has, then it will select p to minimize the delay $\overline{\tau}$. It can be easily shown that the broadcast probability that minimizes the average delay will be $p^* = \frac{Q}{1+\Delta}$. This leads to the following average delay

$$\overline{\tau}^* = \frac{\gamma}{Q} \frac{(1+\Delta)^{1+\Delta}}{\Delta^{\Delta}} \tag{4}$$

The minimum communication delay $\overline{\tau}^*$ in equation 4 is proportional to the communication degree. This implies that as the network connectivity increases, there will be a subsequent increase in message latency. The following sections explore how this increase in message delay may effect the time and energy required to achieve ϵ-consensus.

III. Convergence Rate

This section considers the convergence rate of the consensus filter under message delay. Section II modeled throughput limitations in a time-slotted FDMA wireless network as an average message delay. Olfati-saber et al. [3] derived a upper bound on the maximum delay for which the consensus filter is asymptotically stable. This leads to the following interval of stable delays for the consensus filter,

$$\overline{\tau} \in \left(0,\ -\frac{\pi}{2\lambda_{max}(\mathbf{A})}\right) \tag{5}$$

For a regular network, we can use Gershgorin's theorem to show that the eigenvalues of $\mathbf{A} = -(\mathbf{I}_N + \mathbf{\Delta} + \mathbf{L})$ lie in the interval,

$$-3\Delta - 1 \le \lambda(\mathbf{A}) \le -\Delta - 1 \tag{6}$$

Inserting equation 6 into equation 5 yields the following lemma which is stated without formal proof.

Lemma 3.1: If $\overline{\tau} \in \left(0, \frac{\pi}{2(1+\Delta)}\right)$, then consensus filter in equation 2 is asymptotically stable.

Using the upper end of the interval in lemma 3.1 in our expression for the optimal delay, $\overline{\tau}^*$, in equation 4 yields the following bounds on the network connectivity, Δ, required for the consensus filter's stability.

Lemma 3.2: If the communication degree Δ satisfies

$$Q - 1 \leq \Delta \leq \sqrt{\frac{\pi Q}{2e\gamma}} - 1 \quad (7)$$

then the consensus filter in equation 2 is asymptotically stable.

The convergence rate of the consensus filter can be analyzed through a direct application of Laplace transforms. Taking the Laplace transform on the consensus filter dynamics (eq:2) yields,

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}e^{-\gamma\overline{\tau}s}\right)^{-1}\left(\mathbf{x}(0) + \frac{\mathbf{r}}{s}\right) \quad (8)$$

where $\mathbf{X}(s)$ denotes the Laplace transform of $\mathbf{x}(t)$, and $\mathbf{x}(0)$ is the initial state. The eigen decomposition of the symmetric matrix $\mathbf{A}$ yields $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where $\mathbf{\Lambda}$ is the diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal, and $\mathbf{U}$ is an orthogonal matrix. $\mathbf{X}(s)$ may therefore be written as

$$\mathbf{X}(s) = \mathbf{U}\,\mathrm{diag}\left(\frac{1}{s - \lambda_i e^{-\overline{\tau}s}}\right)\mathbf{U}^T\left(\mathbf{x}(0) + \frac{\mathbf{r}}{s}\right) \quad (9)$$

We use the $P_{2,1}$ *Pade* approximation to approximate the time delay term $e^{-\overline{\tau}s}$ to get the locations of the system poles. This allows us to approximate the diagonal terms in equation 9 as

$$\begin{aligned} \frac{1}{s - \lambda_i e^{-\overline{\tau}s}} &= \frac{1}{s - \lambda_i \frac{6 - 4\overline{\tau}s + (\overline{\tau}s)^2}{6 + 2\overline{\tau}s}} \\ &= \frac{6 + 2\overline{\tau}s}{s^2 + \frac{6 + 4\lambda_i\overline{\tau}}{2\overline{\tau} - \overline{\tau}^2\lambda_i}s - \frac{6\lambda_i}{2\overline{\tau} - \overline{\tau}^2\lambda_i}} \end{aligned} \quad (10)$$

Let $L_i(s)$ denote the characteristic polynomial for the i^{th} subsystem. For our system, $L_i(s)$ is

$$L_i(s) = s^2 + \frac{6 + 4\lambda_i\overline{\tau}}{2\overline{\tau} - \overline{\tau}^2\lambda_i}s - \frac{6\lambda_i}{2\overline{\tau} - \overline{\tau}^2\lambda_i} \quad (11)$$

The roots of $L_i(s)$ describe the convergence rate of the consensus filter in equation 2. The properties of the characteristic polynomial's, $L_i(s)$, roots are studied in the following lemma.

Lemma 3.3: Suppose s_1 and s_2 are the two roots of the characteristic polynomial, $L_i(s)$, in equation 11 and let $Re(s_1) \leq Re(s_2)$. For any fixed λ_i the following statements are true.

- If s_1 and s_2 are real roots, then s_2 is a monotonically decreasing function of the communication degree Δ.
- If s_1, s_2 are a pair of conjugate complex roots, then $Re(s_2)$ is a monotonically increasing function of the communication degree Δ.
- The root $Re(s_2)$ achieve its minimum value for that value of $\Delta^*(\lambda_i)$ that renders the discriminant of the quadratic function, $L_i(s)$, equal to zero.

Proof: See the appendix ■

The lemma tells us that for a given eigenvalue λ_i, there is a corresponding communication degree, $\Delta^*(\lambda_i)$, which minimizes $Re(s_2)$. Different eigenvalues have different optimal degrees. The system's overall performance, however, is a result of all of the eigenvalues of $\mathbf{A}$. The following lemma examines what happens when we consider all of the eigenvalues of $\mathbf{A}$.

Lemma 3.4: Assume that the eigenvalues of matrix $\mathbf{A}$ are sorted in non-decreasing order so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N < 0$.

- Let $s_\ell(\lambda_i) = Re(s_2|\lambda_i, \Delta_{min})$ for eigenvalue λ_i. When $\Delta_{\min} = \min\{Q-1, 2\}$, then $s_\ell(\lambda_N) \leq s_\ell(\lambda_{N-1}) \leq \cdots s_\ell(\lambda_1) < 0$.
- Let $s_r(\lambda_i) = Re(s_2|\lambda_i, \Delta_{max})$ for eigenvalue λ_i. When $\Delta_{max} = N-1$, then $s_r(\lambda_N) \geq s_r(\lambda_{N-1}) \geq \cdots s_r(\lambda_1)$
- $\Delta^*(\lambda_N) \leq \Delta^*(\lambda_{N-1}) \leq \cdots \leq \Delta^*(\lambda_1)$.

Figure 1 illustrates the conclusions of lemma 3.3 for a particular system in which we let $Q = 3$ and $N = 20$. In this figure, the x-axis is the communication degree, Δ, and the y-axis is the real part of the root s_2. The red dashed line plots $Re(s_2)$ as a function of the system eigenvalue λ_N and the green dotted line plots $Re(s_2)$ as a function of λ_1. The solid blue line shows the maximum of $Re(s_2)$ over all system eigenvalues, λ_i for $i = 1, \ldots, N$. Lemma 3.3 asserts that if the discriminant of the quadratic function $L_i(s)$ greater than zero, then the maximum $Re(s_2)$ is a decreasing function of Δ. If this discriminant is negative, then the maximum $Re(s_2)$ is an increasing function of Δ. The point where the discriminant vanishes is precisely that point where the maximum of $Re(s_2)$ is minimized.

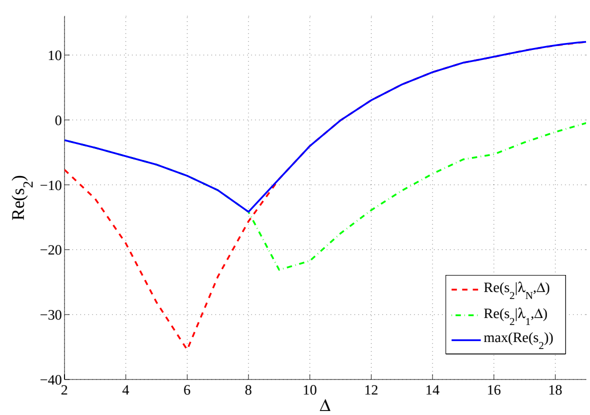

Fig. 1. Property of the eigenvalues of $\mathbf{A}$

Figure 1 can also be used to visualize the conclusions of lemma 3.4. Let $Re(s_2|\lambda_i, \Delta)$ denote the largest real part of the characteristic equation's (eq:11) root when the system eigenvalue is λ_i. The figure shows that when $\Delta = Q-1$, the dashed line representing $Re(s_2|\lambda_N, Q-1)$ is smaller than

$Re(s_2|\lambda_1, Q-1)$. On the other hand when $\Delta = N = 20$, then the roles of these two quantities is reversed, just as asserted in the first two statements in lemma 3.4. The solid blue line in figure 1 draws the largest eigenvalue $\max Re(s_2)$ as a function of the node degree Δ. smallest $Re(s_2)$ is at the intersection of the two curves for $Re(s_2|\lambda_1)$ and $Re(s_2|\lambda_N)$. At this point consensus filter should exhibit the fastest convergence rate. For the system in figure 1, this point occurs when the blue line achieves its minimum value. This occurs when the node outdegree is 8. A precise characterization of this "optimal" degree, Δ^*, is given in the following theorem.

Theorem 3.5: Consider the consensus filter given by equation 2 whose delay $\overline{\tau}$ as a function of network connectivity Δ satisfies equation 4. Let $\mathrm{Re}(s_2|\lambda_i, \Delta)$ denote the maximum real part of the roots of the system's characteristic equation (eq:11) where λ_i is the ith eigenvalue of $\mathbf{A}$ and Δ is the network's connectivity. The optimal network connectivity is approximated by,

$$\begin{aligned} \Delta^* &= \min \max_{(\lambda_i, \Delta)} \{\mathrm{Re}(s_2|\lambda_i, \Delta)\} \\ &= \left\lfloor \sqrt{\frac{0.3Q}{\gamma e}} - 1 \right\rfloor . \end{aligned}$$

Proof: see the appendix ■

Remark 3.6: Theorem 3.5 shows that the optimal degree is inversely proportional to the square root of the time slot's duration, γ. A ten fold increase in γ decreases the optimal degree Δ^* by a factor of three. This means that if we increase the network's throughput by using transmitters/receivers that transmit at a higher rate (smaller γ) then we can increase the optimal Δ^* and thereby improve the filter's convergence rate.

The consensus filter's convergence rate is directly dependent to the location of the characteristic polynomial's roots. Since these roots are a function of the network connectivity Δ, we can see that the consensus filter's convergence rate is also a function of Δ. The following theorem provides a more precise characterization of this relationship.

Theorem 3.7: Consider the consensus dynamics (eq:2)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t-\tau) + \mathbf{r}$$

where the eigenvalues of $\mathbf{A}$ satisfy the inequality (6). Assume the initial condition is $\mathbf{x}(0)$, and the steady state is $\mathbf{x}(\infty)$, then

$$\|\mathbf{x}(t) - \mathbf{x}(\infty)\| \leq C(\Delta) e^{-J(\Delta)t} \left(\|\mathbf{x}(0)\| + 2k\mathbf{r} \right) \quad (12)$$

where $k = \frac{\gamma e}{Q}$ and

$$C = \frac{(2k(1+\Delta) + k^2(1+\Delta)^3)\sqrt{3 - k(1+\Delta)J}}{3 - 4k(1+\Delta)^2 - 3k^2(1+\Delta)^4}$$

If $\Delta \leq \sqrt{\frac{0.3}{k}} - 1$

$$J = \frac{3k(1+\Delta)^3 + 2(1+\Delta)}{2 + k(1+\Delta)^2} \quad (13)$$

$$(14)$$

otherwise,

$$J = \frac{3 - 2k(1+\Delta)(3\Delta+1)}{2k(1+\Delta) + k^2(1+\Delta)^2(3\Delta+1)} \quad (15)$$

Proof: See the appendix ■

When $\Delta > \sqrt{\frac{0.3}{k}} - 1$, the system has complex conjugate roots so that the consensus state trajectory is oscillatory. This oscillatory behavior is undesirable, so we confine our attention to cases where the system only has two real root (i.e.,$\Delta \leq \sqrt{\frac{0.3}{k}} - 1$). Therefore in the remainder of this paper we confine our attention to the case where the exponent J satisfies equation 13. Note that the exponent J in equation 13 is a monotone increasing function of the node degree Δ. Since this equation is only valid for a finite interval of Δ, then the optimal degree occurs at the upper edge of this interval. In other words the *optimal* degree that maximizes the consensus filter's convergence rate is $\Delta^* = \sqrt{\frac{0.3Q}{\gamma e}} - 1$ for a given transmission rate (as fixed by the time slot length γ and the number Q of subchannels).

IV. Energy Efficiency

The preceding section identified network connectivities that minimize the "time" to ϵ-consensus. This may be the preferred problem setting in situations such as the swarming under consensus model [17] where the consensus filter is used to guide the movement of a dynamical swarm. In other sensor network applications, however, we may be more interested in minimizing the "energy" required to achieve ϵ-consensus. This section uses the results of the preceding analysis to study energy-efficient consensus filtering.

We restrict our attention to the disk model for wireless radio network in which each disk has an equal number of regularly spaced neighbors. This assumption is overly simplistic for randomly deployed sensor networks. However for the swarming under consensus model [17] [18] it was shown that agent's usually converge to swarms in which agents are regularly spaced with nearly constant distances between neighbors.

We therefore assume that the N agents are uniformly distributed with a density of ρ. If we let r denote the radio's transmission radius, then the average number of agents in a given disk will be $1 + \Delta = \rho\pi r^2$. The average power at the receivers (located at the edge of the disk) will be $P_R = P_T r^{-\alpha}$ where α is the path loss exponent and P_T is the transmitter power. If we let $\overline{P}_R$ denote the minimum received power required to assure successful reception of the transmitted messsage, then the transmitter will need to have a transmission power of

$$P_T = (\rho\pi)^{-\alpha} \overline{P}_R (1+\Delta)^{\frac{\alpha}{2}} \quad (16)$$

to assure that all neighbors in the disk successfully receive the message.

Let T_c denote the time it takes for the consensus filter to achieve ϵ-consensus from an initial state $\mathbf{x}(0)$ with constant input $\mathbf{r}$. From equation 12, we know that

$$\|\mathbf{x}(T_c) - \mathbf{x}(\infty)\| \leq \epsilon \cdot \left(\|\mathbf{x}(0)\| + 2\frac{\gamma e}{Q}\mathbf{r} \right) .$$

We may therefore approximate the time to ϵ-consensus as

$$T_c \geq -\frac{1}{J(\Delta)} \ln \frac{\epsilon}{C(\Delta)} \tag{17}$$

If $\epsilon \ll 1$, then we can treat the $\ln(\epsilon / C(\Delta))$ term as a constant that is independent of Δ. The parameter dominating the convergence time is therefore $T_c \propto 1/J(\Delta)$.

Determining which Δ minimizes T_c is what we considered in the preceding section. If, however, we're more interested in energy-efficient communication, then we should think of minimizing the total energy

$$\begin{aligned} \text{Total Energy} &= P_T \cdot T_c \\ &\propto \frac{(1+\Delta)^{\alpha/2}}{J(\Delta)} \equiv E(\Delta) \end{aligned} \tag{18}$$

required to achieve ϵ-consensus. The following theorem provides a characterization of the energy efficient Δ^*.

Theorem 4.1: Consider the optimization problem,

$$\begin{array}{ll} \text{minimize}: & E(\Delta) \\ \text{with respect to:} & \Delta \\ \text{subject to:} & Q-1 \leq \Delta \leq \sqrt{\frac{\pi}{2k}} - 1 \end{array}$$

where $k = \frac{\gamma e}{Q}$ and $E(\Delta)$ is defined in equation 18.

The optimal degree Δ^* solving the above problem is

$$\Delta^* = \begin{cases} \max\left\{ \left\lfloor \sqrt{\frac{0.3Q}{\gamma e}} - 1 \right\rfloor, \ Q-1 \right\} ; & \alpha \leq \alpha_0 \\ \max\{2, Q-1\} ; & \alpha > \alpha_0 \end{cases} \tag{19}$$

where

$$\alpha_0 = \begin{cases} 4\left(\frac{\ln 1.26 \frac{2+9k}{2+27k}}{\ln \frac{0.1}{3k}} + 0.5 \right); & Q \leq 3 \\ 4\left(\frac{\ln 1.26 \frac{2+kQ^2}{2+3kQ^2}}{\ln \frac{0.3}{kQ^2}} + 0.5 \right); & Q > 3 \end{cases} \tag{20}$$

Proof: See the appendix. ∎

Note that the energy-efficient Δ^* has an interesting threshold behavior in that it is either $\left\lfloor \sqrt{\frac{0.3Q}{\gamma e}} - 1 \right\rfloor$ or $Q-1$. In relatively lossy environments ($\alpha > \alpha_0$), we optimize energy efficiency by adopting a sparsely connected network. In a relatively lossless environment ($\alpha \leq \alpha_0$), we achieve better energy-efficient by increasing network connectivity to that level that minimizes the time, T_c, to ϵ-consensus.

V. SIMULATION

A Matlab script was written to simulate the behavior of consensus filters on regular graphs with various node out-degrees, Δ. The following simulations assumed $N = 40$ nodes with $Q = 2$ sub-channels and a nominal time-slot length of $\gamma = 0.004$ seconds.

Figure 2 plots $-J(\Delta)$ for different network topologies. The system diverges when $-J(\Delta) > 0$, which occurs when $\Delta > 9$. Lemmas 3.3 and 3.4 assert that $-J(\Delta)$ is a decreasing function for $\Delta < 6$ and an increasing function for $\Delta > 6$. Hence the "optimal" out-degree that minimizes the time, T_c, to ϵ-convergence is 6, which is clearly seen in figure 2.

Simulated responses of the consensus filter are shown in figure 3. In this figure, the x-axis is the consensus time index and the y-axis is the norm of the total consensus error. The different curves correspond to different node degrees. The figure shows that the fastest convergence is achieved with $\Delta = 6$. As Δ increases beyond, the trajectory becomes highly oscillatory eventually becoming unstable when $\Delta > 9$. These results closely match the analytical predictions that were displayed in figure 2.

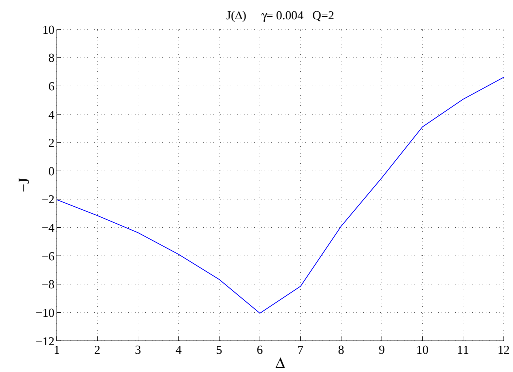

Fig. 2. The convergence exponent of consensus system

Note: It can be difficult to construct regular graphs for any given N. In the simulations shown in figure 3, we chose graphs whose degree sequence showed that 90% of the nodes had the same out degree.

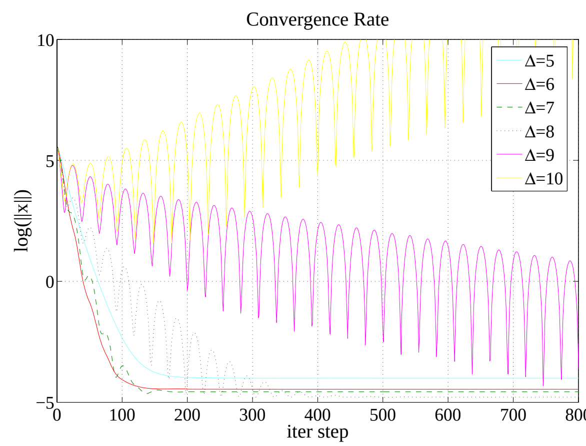

Fig. 3. The optimal Δ

Figure 4 summarizes results from several simulations that were used to study energy-efficient consensus for networks with $N = 40$ nodes and $Q = 2$ sub-channels. Again we choose the time slot duration $\gamma = .004$ seconds and measured the total energy required to achieve ϵ-consensus where $\epsilon = 10^{-6}$. Figure 4 plots the total energy cost versus network degree, Δ for path loss exponents of α of 2, 3, and 4. These plots show that the energy cost is a concave function over the interval $\Delta \in [2, \Delta^*]$. This energy is monotonically increasing for $\Delta > \Delta^*$. As a result, the optimal communication degree, Δ^* will lie at one of two boundary points of the concave region, either when $\Delta = Q - 1$ (2 in our example) or Δ^*. From theorem 4.1, we expect the smaller Δ to be chosen when $\alpha > \alpha_0$. For this case, α_0 will be about 2.8, thereby suggesting that the optimal degree, Δ^* should be equal to $Q - 1 = 2$ when $\alpha = 3$. This is indeed what we see in figure 4. Moreover, for $\alpha < \alpha_0$ we would expect to choose Δ^* to be 6 (that level that maximizes the consensus filter's convergence rate. Figure 4 shows that this $\Delta^* = 6$ for $\alpha = 2$ and 3, which is again consistent with our analytical predictions.

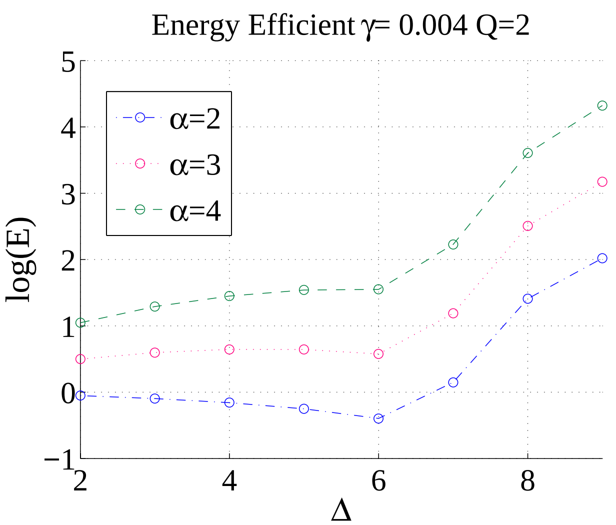

Fig. 4. Energy Efficient Convergence

VI. Conclusion

In this paper, we studied the relations among communication topology, time delay and convergence rate. We have designed techniques to calculate the "good" communication connectivity for maximizing the consensus convergence rate, subject to communication throughput limitation. Moreover, we proposed an energy efficient consensus optimization problem and explored the variation of consensus performance under different the wireless communication conditions.

VII. Appendix

Mathematical Preliminaries: The discriminant of $L_i(s)$ in equation (11) is

$$R_i = -2(\overline{\tau}\lambda_i)^2 + 24\lambda_i\overline{\tau} + 9 \qquad (21)$$

and the roots s_1, s_2 of $L_i(s)$ are,

$$s_1 = -\frac{3 + 2\lambda_i\overline{\tau} + \sqrt{R}}{2\overline{\tau} - \overline{\tau}^2\lambda_i}, \quad s_2 = -\frac{3 + 2\lambda_i\overline{\tau} - \sqrt{R}}{2\overline{\tau} - \overline{\tau}^2\lambda_i}$$

When $R_i \geq 0$, s_1 and s_2 are both real and $s_1 \leq s_2$. When $R_i < 0$, then s_1, s_2 are a pair of conjugate complex roots in which $Re(s_2) = -\frac{3+2\lambda_i\overline{\tau}}{2\tau - \overline{\tau}^2\lambda_i}$.

proof of Lemma 3.3

Proof:

- if $R_i \geq 0$, we want to show s_2 monotonically decreases with Δ increasing.

First, we simplify $\sqrt{R_i}$ as,

$$\begin{aligned} \sqrt{R_i} &= \sqrt{-2(\overline{\tau}\lambda_i)^2 + 24\overline{\tau}\lambda_i + 9} \\ &\approx 3 + 4\overline{\tau}\lambda_i - 3(\overline{\tau}\lambda_i)^2 \end{aligned}$$

then we have,

$$\begin{aligned} y &= -Re(s_2) \approx \frac{3(\overline{\tau}\lambda_i)^2 - 2\overline{\tau}\lambda_i}{2\overline{\tau} - \overline{\tau}^2\lambda_i} \\ &= \frac{3\overline{\tau}\lambda_i^2 - 2\lambda_i}{2 - \overline{\tau}\lambda_i} = \frac{8/\overline{\tau}}{2 - \overline{\tau}\lambda_i} + \frac{4}{\overline{\tau}} - 3\lambda_i \end{aligned}$$

which is monotonically increasing with λ_i. Hence, we consider $\lambda_i = -(1 + \Delta)$, and obtain,

$$\begin{aligned} y &= \frac{3\overline{\tau}(1+\Delta)^2 + 2(1+\Delta)}{2 + \overline{\tau}(1+\Delta)} \\ &= \frac{3\frac{\gamma}{Q}\frac{(1+\Delta)^{3+\Delta}}{\Delta^\Delta} + 2(1+\Delta)}{2 + \frac{\gamma}{Q}\frac{(1+\Delta)^{2+\Delta}}{\Delta^\Delta}} \\ &= \frac{3x\frac{(1+\Delta)^{3+\Delta}}{\Delta^\Delta} + 2(1+\Delta)}{2 + x\frac{(1+\Delta)^{2+\Delta}}{\Delta^\Delta}} = \frac{u(\Delta)}{v(\Delta)} \end{aligned}$$

where $x = \frac{\gamma}{Q}$. To show $-Re(s_2)$ is monotonically increasing is equivalent to showing $u'v - v'u > 0$. We first get

$$\begin{aligned} u' &= 3x\frac{(1+\Delta)^{3+\Delta}}{\Delta^\Delta}(\ln\frac{1+\Delta}{\Delta} + \frac{2}{1+\Delta}) + 2 \\ &= 3xA((1+\Delta)^2B + (1+\Delta)) + 2 \\ v' &= x\frac{(1+\Delta)^{2+\Delta}}{\Delta^\Delta}(\ln\frac{1+\Delta}{\Delta} + \frac{1}{1+\Delta}) \\ &= xA(B(1+\Delta) + 1) \end{aligned}$$

and then

$$\begin{aligned} u'v - v'u &= 2A(1+\Delta)(2(1+\Delta)B + 3)x + 4 \\ &> 0 \end{aligned}$$

where we let $A = \frac{(1+\Delta)^{(1+\Delta)}}{\Delta^\Delta} > 0$ and $B = \ln\frac{1+\Delta}{\Delta} > 0$, which completes the proof.

- For $R_i < 0$, the real part of the conjugate roots is

$$Re(s_2) = -\frac{3 + 2\lambda_i\overline{\tau}}{2\overline{\tau} - \overline{\tau}^2\lambda_i} = -\frac{7}{2\overline{\tau} - \overline{\tau}^2\lambda_i} + \frac{2}{\overline{\tau}}$$

which decreases with λ_i increasing for a given $\overline{\tau}$. Hence, we consider the bound by letting $\lambda_i = -1 - 3\Delta$. Therefore, let

$$\begin{aligned} y &= -Re(s_2) = \frac{u(\Delta)}{v(\Delta)} \\ &= \frac{3 - 2\frac{\gamma}{Q}\frac{(1+\Delta)^{(1+\Delta)}(3\Delta+1)}{\Delta^\Delta}}{2\frac{\gamma}{Q}\frac{(1+\Delta)^{(1+\Delta)}}{\Delta^\Delta} + \left(\frac{\gamma}{Q}\right)^2\frac{(1+\Delta)^{(2+2\Delta)}}{\Delta^{2\Delta}}(3\Delta+1)} \\ &= \frac{3 - 2x\frac{(1+\Delta)^{(1+\Delta)}}{\Delta^\Delta}(3\Delta+1)}{2x\frac{(1+\Delta)^{(1+\Delta)}}{\Delta^\Delta} + x^2\frac{(1+\Delta)^{(2+2\Delta)}}{\Delta^{2\Delta}}(3\Delta+1)} \end{aligned}$$

where $x = \frac{\gamma}{Q}$ is not dependent on Δ. Let $A = \frac{(1+\Delta)^{(1+\Delta)}}{\Delta^{\Delta}}$ and $B = \ln\frac{1+\Delta}{\Delta}$, then

$$\begin{aligned} u(\Delta) &= 3 - 2xA(1+3\Delta) \\ v(\Delta) &= 2xA + x^2A^2(1+3\Delta) \end{aligned}$$

Taking derivative of y related to Δ gives $\frac{dy}{d\Delta} = \frac{u'v - v'u}{v^2}$ where

$$\begin{aligned} u' &= -2xA(B(1+3\Delta)+3) \\ v' &= 2xAB + x^2A^2(B(1+3\Delta)+3) \end{aligned}$$

Therefore, we obtain,

$$u'v - v'u = 3A^2(3 - B(1+3\Delta))x^2 - 6ABx$$

For $\Delta > 1$, $3 - B(1+3\Delta) < 0$. Thus $\frac{dy}{d\Delta} < 0$, since $x > 0$, or y decreases with Δ increasing. Therefore, the real part of the conjugate roots is monotonically increasing.

■

proof of Lemma 3.4

Proof:

- Similar to the proof in the first part of lemma 3.3, we can show s_2 increases with increasing λ_i. Hence, it is straightforward to prove $s_\ell(\lambda_N) \le \cdots \le s_\ell(\lambda_1)$. Moreover, the numerator of the $Re(s_2)$ is

$$\begin{aligned} &-3 - 2\lambda_i\overline{\tau} + \sqrt{-2(\overline{\tau}\lambda_i)^2 + 24\lambda_i\overline{\tau} + 9} \\ &= -3 - 2\lambda_i\overline{\tau} + \sqrt{(3+2\lambda_i\overline{\tau})^2 + 6\overline{\tau}\lambda_i(2-\overline{\tau}\lambda_i)} < 0 \end{aligned}$$

  since $\lambda_i < 0$ and $0 < \overline{\tau} \le \frac{\pi}{2\lambda_N}$.
- The proof is similar to the first item's proof.
- In terms of the lemma 3.3, the optimal degree Δ^* associated with a given λ_i is obtained by solving that the discriminant of $L_i(s)$ equals to zero, such that $R_i = 0$. This leads to

$$\lambda_i\overline{\tau}(\Delta) = -0.364 \tag{22}$$

  for all λ_i associated with different communication degrees. $\overline{\tau}$ is determined by the degree Δ, which increases with increasing Δ. Therefore, in equation (22), an increase in Δ leads to an increase in $\overline{\tau}$ and a decrease in λ_i.

■

proof of theorem 3.5

Proof: As illustrated in figure 1, the optimal degree Δ^* for the fastest convergence is located at the intersection of the two curves for λ_1 and λ_N. In the system's stable region, we solve the equation

$$\frac{3 + 2\overline{\tau} - \lambda_1\sqrt{R_1}}{2\overline{\tau} - \overline{\tau}^2\lambda_1} = \frac{3 + 2\overline{\tau}\lambda_N}{2\overline{\tau} - \overline{\tau}^2\lambda_N}$$

to obtain $\Delta^* = \sqrt{\frac{0.3Q}{\gamma e}} - 1$. ■

proof of theorem 3.7

Proof: Taking the inverse Laplace transform on equation 9 yields,

$$\begin{aligned} \mathbf{x}(t) = &\ \mathbf{U}\mathcal{L}^{-1}\left\{diag\left(\frac{1}{s - \lambda_i e^{-\overline{\tau}s}}\right)_{N\times N}\right\}\mathbf{U}^T \cdot \\ &\left(\mathbf{x}(0) + diag\left(\frac{2\overline{\tau} - \overline{\tau}^2\lambda_i}{\lambda_i}\right)_{N\times N}\mathbf{r}\right) \\ &+\mathbf{U}\mathcal{L}^{-1}\left\{diag\left(-\frac{2\overline{\tau} - \overline{\tau}^2\lambda_i}{s\lambda_i}\right)\right\}\mathbf{U}^T\mathbf{r} \end{aligned}$$

The second term in the above equation represents the system steady state $\mathbf{x}(\infty)$, so the consensus error can be written as,

$$\begin{aligned} &\|\mathbf{x}(t) - \mathbf{x}(\infty)\| \\ \le &\left\|\mathcal{L}^{-1}\left\{diag\left(\frac{1}{s - \lambda_i e^{-\overline{\tau}s}}\right)\right\}\right\| \cdot \\ &\left\|\mathbf{x}(0) + diag\left(\frac{2\overline{\tau} - \overline{\tau}^2\lambda_i}{\lambda_i}\right)\mathbf{r}\right\| \\ \le &\ Ce^{\max\{Re(s)\}t}\left\|\mathbf{x}(0) + 2\frac{\gamma e}{Q}\mathbf{r}\right\| \end{aligned}$$

where $\max\{Re(s)\}$ is the maximal value of the roots of all subsystem $L_i(s) = s - \lambda_i e^{-\overline{\tau}s} = 0$ (eq:11), and C is a function of the communication degree. The expression of $\max\{Re(s)\}$ and C is obtained easily from lemma 3.3 and theorem 3.5. ■

proof of theorem 4.1

Proof: Denote $x = 1 + \Delta$. According to the second item of lemma 3.3, we know the cost function $E(\Delta)$ is monotonically increasing with Δ when $x > \sqrt{\frac{0.3Q}{\gamma e}}$. Therefore, we only consider optimization problem in the region that $Q \le x \le \sqrt{\frac{0.3Q}{\gamma e}}$. We use the approximation of $\sqrt{R_i}$ in the first item of lemma 3.3 again to have,

$$\begin{aligned} J(x) &= \frac{3 - 2kx^2 - \sqrt{9 - 24kx^2 - 2k^2x^4}}{2kx + k^2x^3} \\ &\approx \frac{3 - 2kx^2 - (3 - 4kx^2 - 3k^2x^4)}{2kx + k^2x^3} \\ &= 3x - \frac{4x}{2 + kx^2} \end{aligned}$$

and hence the energy cost function can be written as

$$E(\Delta) = \frac{x^{\frac{\alpha}{2}}(2 + kx^2)}{2x + 3kx^3} = \left(\frac{t}{k}\right)^{\beta}\frac{2+t}{2+3t}$$

where $t = kx^2$, and $\beta = \frac{\alpha}{4} - \frac{1}{2}$. Since $2 \le \alpha \le 4$, then $0 \le \beta \le \frac{1}{2}$. Taking the second-order derivative of E related to t gives

$$\begin{aligned} \frac{d^2E}{dt^2} = &\ \frac{1}{k^\beta(2+3t)^3} \cdot \left[4(\beta^2 - \beta)t^{\beta-2}\right. \\ &+(22\beta^2 - 26\beta)t^{\beta-1} + (6\beta^2 - 18\beta + 12)t^{\beta} \\ &\left.+9(\beta^2 - \beta)t^{\beta+1}\right] < 0 \end{aligned}$$

for $0 \le t \le 0.3$. Hence, E is a concave function of t, which implies that $\min_x E = \min\{E(\max\{Q, 3\}), E(\sqrt{\frac{0.3}{k}})\}$.

Let $E(\max\{Q,3\}) = E(\sqrt{\frac{0.3}{k}})$, we have

$$\alpha_0 = 4\left(\frac{\ln 1.26\frac{2+9k}{2+27k}}{\ln\frac{0.1}{3k}} + 0.5\right) \qquad (Q-1 \leq 2);$$

$$\alpha_0 = 4\left(\frac{\ln 1.26\frac{2+kQ^2}{2+3kQ^2}}{\ln\frac{0.3}{kQ^2}} + 0.5\right) \qquad (Q-1 > 2)$$

To summarize, the optimal degree $\Delta^* = x^* - 1$ is

$$\Delta^* = \begin{cases} \max\left\{\left\lfloor\sqrt{\frac{0.3}{k}} - 1\right\rfloor,\ Q-1\right\} & \alpha \leq \alpha_0 \\ \max\{2, Q-1\} & \alpha > \alpha_0 \end{cases}$$

■

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