Input-to-state stabilizability of quantized linear control systems under feedback dropouts

Qiang Ling and Michael D. Lemmon

***Abstract*— This paper studies the input-to-state stabilizability of quantized linear control systems with external noise under feedback dropouts. A vector of feedback measurements is quantized prior to being transmitted over a communication channel. The transmitted data may be dropped by the channel. The channel dropouts are governed by a stationary model, which is quite general to include many realistic dropout models. This paper derives a lower bound on the constant bit rates which can stabilize the system under the given dropout condition. A dynamic quantization policy is shown which can stablize the system at that lower rate bound. So the minimum constant stabilizing bit rate has be obtained. The achieved theoretical results are also verified through an example.**

I. INTRODUCTION

In recent years there has been increasing interest in implementing the feedback loop of a control system over a non-deterministic digital communication network [1]. This may have many benefits, such as lower cost, higher reliability, and easier maintenance. These advantages are, however, achieved at the cost of loss of perfect feedback information.

- Due to the network non-determinism, the feedback information may be dropped or erased sometimes.
- Due to the digital nature of the network, all data must be quantized before transmission, which will incur error of feedback information, i.e., quantization error.

Then the results built upon the perfect feedback assumption have to be re-evaluated. As the most important property of control systems, stability is the first to check. A major concern about such systems is stabilizability, i.e., *whether the originally stabilizable system can still be stabilized under the given network dropout and quantization conditions*. Here stability is measured by input-to-state stability (ISS) in the almost sure sense, which quantitatively characterizes the system's robustness against the input noise and the initial condition[2] [1]. In order to stabilize a linear system, not only the controller but also quantization and dropout compensation policies will be designed.

Quantization requires the transmitted real-valued signal to be represented with a finite number of bits, and incurs quantization error, which can significantly affect stability and performance of control systems. The most important parameter of quantization is the number of quantization levels Q (per sample or packet), or the number of quantization bits R (which is related to Q by $R = log_2(Q)$). The number of quantization bits R is proportional to the occupied network bandwidth (under the constant sampling periods). So R are often abused as *"bit rate"*. In order to save network bandwidth, it is perferred to use as low as possible bit rate to satisfy control requirements like stabilizability of control systems. Sometimes the number of quantization bits per packet is time-varying [2] and the number of quantization bits R is understood in the average sense. Under a given dropout condition, *what is the minimum R to stabilize a control system* is the major question to be answered in the present paper.

Much research on quantized control systems has been done in the last two decades [4]. Many results on quantized control systems assume that the quantization bits (or symbols) are **errorlessly** (dropout-freely) transmitted, which may be violated in the situation of sharing network among many control and non-control systems. The quantization policies can be categorized into two groups, static one and dynamic one. *Static quantization policies* take a constant quantization range, map each bit to a specific subset of that range in a fixed(static) way. The attraction of static policies is the simplicity of their coding/decoding schemes. Their main drawback is that an infinite number of quantization bits are required to ensure asymptotic stability of noise-free control systems [5] [6]. When only a finite number of quantization bits are availabe, the best to expect is the ultimate boundedness of the state, instead of asymptotic stability [7] [8]. Under the condition that an infinite number of quantization bits are allowed, the lowest quantization density of memoryless policies is given in [9] [10].

Compared with static policies, *dynamic quantization policies* may choose a time-varying quantization range and their mapping between the quantization bits and the subsets of the quantization range can also be time-varying. Although more complicated, the dynamic policies can asymptotically stabilize noise-free linear systems with a finite number of quantization bits [11] [12] . The minimum number of quantization bits to maintain asymptotic stability is given in [13] [14], where variable length coding strategies are chosen and the number of quantization bits is understood in the average sense. Under the fixed length coding constraint, a similar result is obtained in [15]. For quantized systems with

Qiang Ling is with Department of Automation, University of Science and Technology of China, Hefei, Anhui 230027, China; *Email:* qling@ustc.edu.cn; *Phone:* (86)551-360-0504.

Michael D. Lemmon is with Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556; *Email:* lemmon@nd.edu; *Phone:* (1)574-631-8309.

[1]Among many types of stability [3], this paper investigates almost sure stability because sample path properties are more important than the average or moment behaviour in real applications.

[2]It is helpful for improving communication efficiency to transmit a constant number of bits in all packets, i.e., implement the fixed length coding.

bounded exogenous noise, bounded-input-bounded-output (BIBO) stability, instead of asymptotic stability, is pursued and the minimum bit rate to achieve such stability is derived [13] [16][17] while the input-to-state stability is investigated in [18]. Due to their efficiency, dynamic quantization polies are chosen in the present paper to stabilize quantized systems with bounded noise.

Feedback dropout is less considered in the previous quantization literature. It seems intuitively pleasing that when the dropout rate is low, the stabilizability of the quantized systems would be preserved under feedback dropouts. In [19], the feedback dropout is modeled as an i.i.d. process and it was asserted that the *almost sure stabilizability* of quantized linear systems can be guaranteed if the average bit rate, $\overline{R}$ satisfies

$$\overline{R} > \sum_{i=1}^{n} max(0, |log_2 \lambda_i|) \tag{1}$$

where λ_i ($i = 1, \cdots, n$) are the eigenvalues of the open-loop system matrix. The above statement is, however, proven to be incorrect in [20]. Furthermore, it is shown that the system state almost surely diverges for any $\overline{R}$ [20]. In order to reslove this diverging issue, one may

- **Choose a weaker notion of stability**, such as mean square stability [21] [17] [22], under the given i.i.d. dropout condition.
- **Put constraints on the dropout sequences**. In [23], the BIBO stabilizability of quantized systems is preserved under some dropout conditions different from the above i.i.d. process.

Because mean square stable systems may still generate sample paths with arbitrarily large state magnitude, the first approach listed above may not be appropriate for real applications. This paper, therefore, mainly focuses on the second approach.

Because the dropout sequence in [19] is i.i.d., the probability of any number of consecutive dropouts is non-zero, i.e., it is almost sure that dropout patterns with any number of consecutive dropouts will occur. The consecutive dropouts are the main reason to drive the state to diverge from its equilibrium point (the origin)[20]. One may ask a question "*whether do the real networks allow arbitrarily long consecutive dropouts?*" Fortunately the answer is **NO**. Real-time system engineers always work hard to avoid consecutive dropouts. They proposed different constraints on dropouts and made their best to guarantee these constaints. One important constraint is *skip-over* policy [24], which requires that there are at least s successes between 2 failures (dropouts). Another one is *(m,k)*-firm guarantee rule [25] [26], which requires that at least m out of k consecutive attempts succeed, i.e. at most $k - m$ dropouts are allowed for k consecutive steps. A new constraint proposed in [27] considers the effect of dropouts on system performance and searches the "*optimal*" stochastic policy with the minimum performance degradation under the given average dropout rate. The present paper will give a dropout model (or a constraint on the dropout sequences) more general than the aforementioned ones later, and proves the quantized system can be almost surely stabilized under that dropout condition at the minimum constant rate, which is an extension of the results in [18] by explicitly taking the feedback dropouts into account.

The rest of this paper is organized as follows. Section II presents the mathematical model of the quantized linear system, and more importantly, the stabilizing dropout model. Under the given dropout condition, we derive a lower bound R_{min} on the constant bit rates to stabilize the quantized system in Section II. That lower bound R_{min} is shown to be achievable, i.e., it is the minimum stabilizing constant bit rate. The achievability of R_{min} is shown by constructing a quantizer in Section III. Simulation results of an example system are also included to demonstrate the correctness of the theoretical results in Section III. Some final remarks are included in Section IV. To improve readability, we move all technical proofs to the appendix, Section V.

II. Mathematical Models

A. Model of the Quantized Linear System

This paper focuses on the system in Fig. 1. In Fig. 1, $x[k] \in R^n$ is the state at time instant k ($\in \overline{Z^-}$). For simplicity, we assume the state $x[k]$ is directly accessible by the quantizer (encoder). $x[k]$ is quantized into one of Q ($= 2^R$) symbols, $s[k]$ and sent over the digital communication network. Note that the fixed length coding is employed here due to its bandwidth efficiency. The transmitted symbol $s[k]$ is either received by the decoder with 1 step delay or dropped (with a dropout represented by receiving the empty signal ϕ). It is assumed that there exists reliable ACK to notify the transmitter (encoder/quantizer) regarding dropouts. Define a dropout indicator

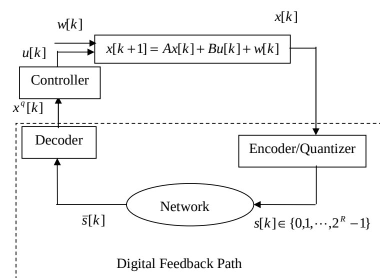

Fig. 1. A quantized linear system

$$d[k] = \begin{cases} 1, & \text{Dropout at time } k \\ 0, & \text{Success at time } k \end{cases} \tag{2}$$

$\{d[k]\}$ is referred to as "*dropout sequence*". The input of the decoder in Fig. 1 is

$$\overline{s}[k] = \begin{cases} s[k-1], & d[k] = 0 \\ \phi, & d[k] = 1 \end{cases}$$

The decoder uses all received symbols $\{\overline{s}[k], \overline{s}[k-1], \cdots, \overline{s}[0]\}$ to estimate the state $x[k]$. The state estimate is denoted as $x^q[k]$, which can also be viewed as a quantized version of $x[k]$. The control input $u[k] \in R^m$ is then constructed from $x^q[k]$. In Fig. 1, the input signal, $w[k] \in R^n$, represents an exogenous bounded noise signal satisfying

$$\sup_{k\geq 0} \|w[k]\| \leq 0.5W \tag{3}$$

where $\|\cdot\|$ denotes the infinity norm of a vector.

For mathematical convenience, we write down the difference equation of the linear system in Fig. 1 as

$$\begin{cases} x[k+1] &= Ax[k] + Bu[k] + w[k] \\ u[k] &= Gx^q[k] \end{cases} \tag{4}$$

The system is assumed to be stabilizable (under the perfect feedback). So there must exist a stabilizing gain G. The matrices A, B and G are of appropriate dimensions.

The system in eq. 4 has bounded noise input $\{\omega[k]\}$. We are interested in the input-to-state stability (ISS) of the system [2]

$$\|x[k]\| \leq \beta'(\|x[0]\|, k) + \gamma'(\sup_{j\geq 0}\|w[j]\|), \forall k \geq 0 \tag{5}$$

where $\gamma'(\cdot)$ is a $\mathcal{K}$ function which is continous, strictly increasing and $\gamma'(0) = 0$, $\beta'(\cdot,\cdot)$ is a $\mathcal{KL}$ function which is a $\mathcal{K}$ function by fixing its second arguement k and is a decreasing function to converge to 0 as $k \to \infty$ after fixing the first argument $x[0]$.

The control input $u[k]$ in eq. 4 is computed from the quantized state $x^q[k]$. The quantization error is defined as $e[k] = x[k] - x^q[k]$. $e[k]$ surely affects stabilizability of the quantized system 4. It can be shown that the input-to-state stability in eq. 5 is equivalent to the following equation [28]

$$\|e[k]\| \leq \beta(\|e[0]\|, k) + \gamma(\sup_{j\geq 0}\|w[j]\|), \forall k \geq 0 \tag{6}$$

where $\beta(\cdot,\cdot)$ is a $\mathcal{KL}$ function and $\gamma(\cdot)$ is a $\mathcal{K}$ function. Therefore this paper establishes the input-to-state stabilizability of the system in eq. 4 through proving eq. 6.

Assumption 1: The system matrix in eq. 4, A, takes a real Jordan canonical form, i.e.,

$$A = diag(J_1, J_2, \cdots, J_P) \tag{7}$$

where J_i is an $n_i \times n_i$ real matrix with a single real eigenvalue λ_i (of the multiplicity of n_i) or a pair of conjugate eigenvalues λ_i and λ_i^* (of the multiplicity of $n_i/2$). $|\lambda_i| \geq 1, \forall i$.

For notational convenience, we define

$$\alpha(A) = \prod_{i=1}^{P} |\lambda_i|^{n_i} \tag{8}$$

B. Dropout model

Based on the dropout indicator $d[k]$ in eq. 2, we define the local dropout rate as

$$\varepsilon_l[k] = \frac{1}{l}\sum_{i=0}^{l-1} d[k+i] \tag{9}$$

It is obvious that $0 \leq \varepsilon_l[k] \leq 1$. For any $l \in \mathcal{N}$, $\sup_{k\geq k_0} \varepsilon_l[k]$ exists, is bounded between 0 and 1, and is non-increasing w.r.t. k_0. So the limit $\overline{\varepsilon}_l = \lim_{k_0\to\infty}\sup_{k\geq k_0}\varepsilon_l[k]$ must exist. Again, $0 \leq \overline{\varepsilon}_l \leq 1$. Similarly we can show another limit must exist

$$\varepsilon' = \lim_{l_0\to\infty}\sup_{l\geq l_0}\overline{\varepsilon}_l \tag{10}$$

We call ε' in eq. 10 the average dropout rate, which may be different from the ordinary definition of the average dropout rate $\overline{\varepsilon} = \lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l-1} d[k]$ For example, $\{d[k]\} = \{101100111000\cdots\}$ gives $\varepsilon' = 1$ v.s. $\overline{\varepsilon} = 0.5$.

Assumption 2: There exists $0 \leq \hat{\varepsilon} < 1$ such that

$$\lim_{l_0\to\infty}\sup_{l\geq l_0}\left(\lim_{k_0\to\infty}\sup_{k\geq k_0}\varepsilon_l[k]\right) \leq \hat{\varepsilon}, \text{ almost surely.} \tag{11}$$

It can be verified that many real-time constraints, such as the *skip-over* policy [24], the *(m,k)*-firm guarantee rule [25], satisfy eq. 11. Under the dropout condition in eq. 11, we can place the following upper bound on local dropout rates. Its proof is straightforward and omitted here.

Corollary 2.1: Assume the dropout condition in eq. 11. For any small number $\delta > 0$, we can find large enough M_δ and k_δ such that it is almost sure that

$$\varepsilon_{M_\delta}[k] \leq (\hat{\varepsilon} + \delta), \forall k \geq k_\delta \tag{12}$$

Under the dropout condition in eq. 11, *what is the smallest R to stabilize the system?* The following Lemma presents a lower bound on all constant bit rates to stabilize the system in eq. 4. Its proof closely follows that of Proposition 3.2 in [19] and is omitted here.

Lemma 2.2: For dropout sequences satisfying eq. 11, if the quantized system in eq. 4 can be almost surely stabilized under a constant bit rate of R, then

$$R \geq R_{min} = \left\lfloor \frac{1}{1-\hat{\varepsilon}} log_2\left(\alpha(A)\right)\right\rfloor + 1 \tag{13}$$

where $\alpha(A)$ is defined in eq. 8, and $\lfloor\cdot\rfloor$ stands for the flooring operation over a real number.

The lower bound R_{min} on stabilizing bit rates in Lemma 2.2 can be achieved by the quantizer in Section III. So R_{min} in Lemma 2.2 is the minimum stabilizing bit rate.

III. Main results

A. Mathematical preliminaries of quantization policies

In order to construct the desirable quantizer, we need the preliminaries in the following subsection.

1) Coordinate transformation: When the quantized system 4 have complex eigenvalues, the coordinate transformation in [13] is needed.

$$z[k] = H^k x[k] \tag{14}$$

where the transformation matrix H is defined as $H = diag(H_1, H_2, \cdots, H_P)$. Each H_i is associated with one of the Jordan blocks J_i in eq. 7. Specifcally, $H_i = I_{n_i}$ if λ_i (the eigenvalue of J_i) is real and $H_i = diag\left(r(\theta_i)^{-1}, \cdots, r(\theta_i)^{-1}\right)$ with $r(\theta_i) = \begin{bmatrix} cos(\theta_i) & sin(\theta_i) \\ -sin(\theta_i) & cos(\theta_i) \end{bmatrix}$ if λ_i is complex and $\lambda_i = |\lambda_i|e^{j\theta_i}$. By [13], eq. 14 transforms eq. 4 into

$$z[k+1] = HAz[k] + H^{k+1}Bu[k] + \overline{w}[k] \tag{15}$$

where $\overline{w}[k] = H^{k+1}w[k]$. By the boundedness of $w[k]$ and the structure of H, we know $\overline{w}[k]$ is still bounded,

$$\|\overline{w}[k]\| \leq 0.5\overline{W} = 0.5 \times (2W). \quad (16)$$

Considering the structure of H, we infer from eq. 14 that $0.5\|x[k]\| \leq \|z[k]\| \leq 2\|x[k]\|$ for any $k \geq 0$. So the input-to-state stability of eq. 4 (with the noise input of $\{w[k]\}$) is equivalent to that of eq. 15. The present paper, therefore, focuses on the boundedness of $z[k]$. We use $z^q[k]$ to denote the quantized version of $z[k]$, or the estimate of $z[k]$, at time k. The quantization error is represented as $e[k] = z^q[k] - z[k]$. As argued in Section II, $\{z[k]\}$ satisfy the ISS requirement in eq. 5 if and only if $\{e[k]\}$ can satisfy eq. 6.

2) Uncertainty set: Any closed set in R^N can be over-bounded by a rectangle P, which is characterized by its center z^P and its side length vector $L = [L_1, L_2, \cdots, L_N]^T$. A rectangle with the center of the origin and the side length vector L is denoted as $rect(L)$. So P can be expressed as $P = z^P + rect(L)$.

Corresponding to the block diagonal structure of A in eq. 7, we relabel L with a 2-dimensional index as $L = [L_{1,1}, \cdots, L_{1,n_1}, \cdots, L_{P,1}, \cdots, L_{P,n_P}]^T$, where $L_{i,j}$ corresponds to the $m-th$ entry of L with $m = \sum_{l=1}^{i-1} n_l + j$.

The system in eq. 15 is perturbed by the unknown bounded noise $\{\overline{w}[k]\}$. Although it is impossible to exactly know the state $z[k]$, we can know the set which $z[k]$ lies within. That set is referred to as the *"uncertainty set"*. The uncertainty set is usually a closed set. We can, therefore, over-bound it with a rectangle $P[k]$. Without confusion, $P[k]$ is also called the *"uncertainty set"* at time k. It is reasonable to estimate $z[k]$ by the center of $P[k]$, $z^q[k]$ (the quantized version of $z[k]$). The quantization (estimation) error is

$$e[k] = z^q[k] - z[k] \in rect(L[k])$$

where $L[k]$ is the side length vector of $P[k]$. By the above equation, we know $\{e[k]\}$ satisfies eq. 6 if and only if

$$\|L[k]\| \leq \beta_L(\|L[0]\|, k) + \gamma_L(\sup_{j\geq 0} \|w[j]\|), \forall k \geq 0 \quad (17)$$

where $\beta_L(\cdot,\cdot)$ is a $\mathcal{KL}$ function and $\gamma_L(\cdot)$ is a $\mathcal{K}$ function.

3) Evolution of uncertainty sets: As time moves forward, we need to update $P[k]$, more specifically $z^q[k]$ and $L[k]$. The general updating rule is given as followes. Suppose $z[k] \in P[k] = z^q[k] + rect(L[k])$ and the control at time k is $u[k]$. We want to determine a rectangle $P[k+1] = z^q[k+1] + rect(L[k+1])$ within which $z[k+1]$ lies.

By eq. 4 and the property of $\{w[k]\}$ in eq. 3, we can update $z^q[k]$ and $L[k]$ as

$$\begin{aligned} z^q[k+1] &= HAx^q[k] + H^{k+1}Bu[k] & (18) \\ L[k+1] &= KL[k] + \left[\overline{W}, \overline{W}, \cdots, \overline{W}\right]^T & (19) \end{aligned}$$

where H is defined in eq. 14, and $K = diag\,(K_1, K_2, \cdots, K_P)$ with

$$K_i = \begin{bmatrix} |\lambda_i| & 1 & 0 & \cdots & 0 \\ 0 & |\lambda_i| & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & |\lambda_i| \end{bmatrix}_{n_i\times n_i} \text{ when } \lambda_i \text{ is real,}$$

$$K_i = \begin{bmatrix} |\lambda_i|I & E & 0 & \cdots & 0 \\ 0 & |\lambda_i|I & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & |\lambda_i|I \end{bmatrix}_{n_i\times n_i} \text{ for complex}$$

λ_i and $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

B. A stabilizing quantizer at $R = R_{min}$

Lemma 2.2 places a lower bound R_{min} on all constant bit rates which may be able to stabilize the system in eq. 4. *Can we stabilize the system under that dropout condition at the minimum constant bit rate $R = R_{min}$?* This section gives an affirmative answer to the above question. A quantizer at $R = R_{min}$ is constructed. It is proven that under the dropout condition in eq. 11, the proposed quantizer can stabilize the system in eq. 4. So we know the lower bound R_{min} is **achievable**, and is, therefore, the minimum stabilizing constant bit rate.

Now we start to build the desired quantizer. Let $Q = 2^{R_{min}}$. Considering the definition of R_{min} in eq. 13, we can find a positive parameter ρ such that

$$Q^{1-\hat{\varepsilon}} > \alpha(A)\left(1 + Q\frac{3}{\rho}\right)^n \quad (20)$$

We first assume both the encoder and the decoder agree upon that

$$z[0] \in P[0] = z^q[0] + rect(L[0]) \quad (21)$$

The quantizer chooses the "longest" side at $k = 0$ by the following rule

$$(I_k, J_k) = arg\max_{i,j} \left(Q^2\rho\right)^j L_{i,j}[k] \quad (22)$$

Partitioning side (I_k, J_k) into Q equal parts, we get a modified side length vector $L^{I_k,J_k}[k]$

$$L_{i,j}^{I_k,J_k}[k] = \begin{cases} L_{i,j}[k], & (i,j) \neq (I_k, J_k) \\ L_{i,j}[k]/Q, & (i,j) = (I_k, J_k) \end{cases}$$

Now the original set $P[k] = z^q[k] + U[k]$ is partitioned into Q smaller sets $P_s[k]$ $(s = 0, \cdots, Q-1)$

$$P_s[k] = z_s^q[k] + rect(L^{(I_k,J_k)}[k])$$

where $z_s^q[k] = z^q[k] + z_s^{(I_k,J_k)}$ and $z_s^{(I_k,J_k)}$ is an n-dimensional vector with the (I_k, J_k)-th element equal to $\frac{-Q+(2s+1)}{2Q} L_{I_k,J_k}[k]$ and other elements of 0.

Because $P[k] = \cup_{s=0}^{Q-1} P_s[k]$ and $z[k] \in P[k]$, there must exist $s_0 \in \{0, \cdots, Q-1\}$ such that $z[k] \in P_{s_0}[k]$. Set $s[k] = s_0$, code $s[k]$ into R_{min} bits (or a symbol with Q levels) and send these bits to the decoder through the network. Upon receiving $s[k]$, decoder sends ACK back to the encoder to confirm the receipt of $s[k]$. Due to ACK, the encoder and the decoder always agree upon the information of $z[k]$: either $z[k] \in z^q[k] + rect(L[k])$ (when $s[k]$ is dropped, i.e., $d[k]=1$) or $z[k] \in z^q_{s[k]}[k] + rect(L^{(I_k,J_k)}[k])$ (when $s[k]$ is successfully transmitted, i.e., $d[k]=0$). Based on the system equation 18, the encoder and decoder update the state set, $P[k+1]$ $(= z^q[k+1] + rect(L[k+1]))$, as

$$\begin{cases} \text{When } d[k]=1: \\ \begin{cases} L[k+1] = KL[k] + [\overline{W}, \cdots, \overline{W}]^T \\ z^q[k+1] = HAz^q[k] + H^{k+1}Bu[k] \end{cases}, \\ \text{When } d[k]=0: \\ \begin{cases} L[k+1] = & KL^{I_k,J_k}[k] + [\overline{W}, \cdots, \overline{W}]^T \\ z^q[k+1] = & HAz^q[k] + H^{k+1}Bu[k] \\ & +HAz^{(I_k,J_k)}_{s[k]} \end{cases}. \end{cases} \quad (23)$$

where the control variable is computed as

$$u[k] = G\left(H^{-k}z^q[k]\right). \quad (24)$$

The quantization policy is summarized into the following algorithm.

Algorithm 1: **Quantization algorithm:**

Encoder/Decoder initialization:
Initialize $z^q[0]$ and $L[0]$ so that $z[0] \in z^q[0] + rect(L[0])$ and set $k=0$.

Encoder Algorithm:

1) **Select** the indices (I_k, J_k) by eq. 22.
2) **Quantize** the state $z[k]$ by setting $s[k]=s$ if $z[k] \in z^q[k] + z_s^{(I_k,J_k)} + rect(L^{(I_k,J_k)}[k])$.
3) **Transmit** the quantized symbol $s[k]$ and wait for ACK. If ACK is received before time $k+1$, $d[k]=0$; otherwise, $d[k]=1$.
4) **Update** $z^q[k+1]$ and $L[k+1]$ by eq. 23 immediately before time $k+1$. Update time index, $k=k+1$ and return to step 1.

Decoder Algorithm:

1) **Compute** control for time k by eq. 24.
2) **Wait** for the quantized data, $s[k]$, from the encoder. If $s[k]$ is received before time k, send ACK to decoder and set $d[k]=0$; otherwise, set $d[k]=1$.
3) **Update** $z^q[k+1]$ and $L[k+1]$ by eq. 23 immediately before time $k+1$. Update time index, $k=k+1$ and return to step 1.

Remark: Note that when a symbol $s[k]$ is dropped, it will not be re-transmitted. Instead, a new symbol at the next time $s[k+1]$ is generated from the new state $x[k+1]$ and transmitted.

Under the quantizer in Algorithm 1, the quantized system in eq. 4 is input-to-state stable in the almost sure sense. That result is formally presented by Theorem 3.1. Its proof is moved to Section V to improve readability.

Theorem 3.1: Let $R_{min} = \left\lfloor \frac{1}{1-\varepsilon} log_2\left(\alpha(A)\right) \right\rfloor + 1$ and $Q = 2^{R_{min}}$. The dropout model in eq. 11 is assumed. The quantized linear system in eq. 4 is almost surely input-to-state stable under the quantizer in Algorithm 1.

Remark: Algorithm 1 can guarantee the input-to-state stability in the almost sure sense at the *minimum* constant bit rate R_{min}. Compared with the ultimate state boundedness in the prior literature [23], input-to-state stability describes more precisely the dependence of the state on the bounded noise and the initial condition [18]. Moreover, the input-to-state stability in Theorem 3.1 unifies both the asymptotic stability of noise-free quantized systems [15] and the BIBO stability of quantized systems perturbed by bounded noise [23] at the minimum bit rate. Compared with [18], Theorem 3.1 explicitly takes the dropouts into account.

C. Simulation results

Here we verify the obtained theoretical results through an example system. Its parameters are $A = \begin{bmatrix} 1.1 & 1 & 0 \\ 0 & 1.1 & 1 \\ 0 & 0 & 1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $G = [-1.29, -3.56, -3.27]$. The dropout sequence is governed by a $(2,3)$-firm model, i.e., among any 3 consecutive packets, at least 2 ones are transmitted successfully. So $\hat{\varepsilon} = 1/3$, $R_{min} = 1$ and $Q=2$. According to eq. 20, we choose $\rho = 109.1$. Initial conditions are $L[0] = [1,1,1]^T$, $x[0] = [0,0,0]^T$, $x^q[0] = [0,0,0]^T$. The simulation results for $W=1$ and $W=0$ are shown in Fig. 2. Note that the zoom-in versions of the two figures are also shown. It can be seen that the quantization error $\|e[k]\|$ is bounded by $\|L[k]\|$ (confirming there is no overflowing in quantization), and $\|L[k]\|$ and $\|x[k]\|$ are bounded for $W=1$ and are asymptotically converging to 0 for $W=0$ (verifying Theorem 3.1).

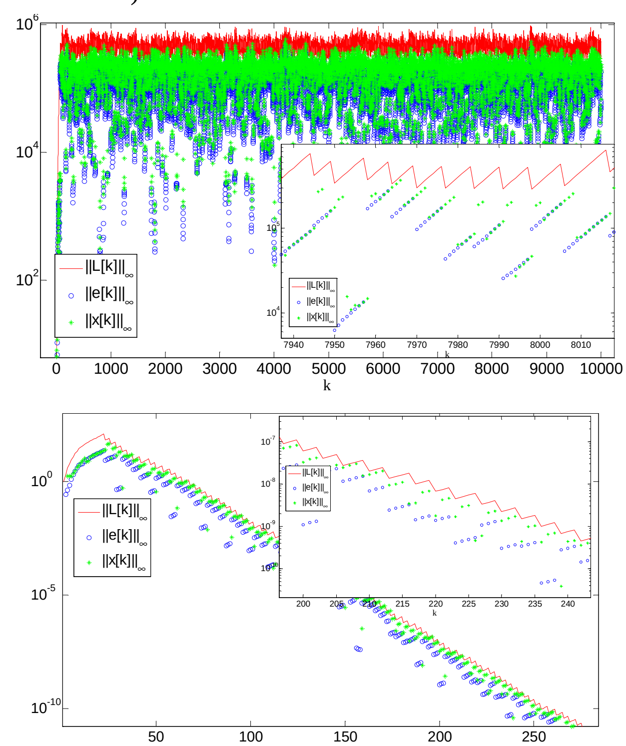

Fig. 2. $\|L[k]\|$, $\|e[k]\|$ and $\|x[k]\|$ with : (top) $W=1$; (bottom) $W=0$.

IV. Conclusions

This paper studies input-to-state stabilizability of quantized systems with feedback dropouts and bounded noise at constant bit rates. It derives a lower bound on the constant bit rates required to stabilize the system. That lower bound can be achieved by a dynamic quantization policy. Due to its achievability, that lower bound is actually the minimum constant bit rate to stabilize the quantized system.

In this paper, only boundedness of the state is of interest. The achieved bound on the quantization error may be used to measure the quantization policy's performance. The bound in this paper may, however, be too loose to adequately measure that performance. Future work will look for a better bound.

REFERENCES

[1] O. Beldiman, G. Walsh, and L. Bushnell, "Predictors for networked control systems," in *American Control Conference*, 2000, pp. 2347–2351.

[2] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41(9), pp. 1283–1294, 1996.

[3] F. Kozin, "A survey of stability of stochastic systems," *Automatica*, vol. 5, pp. 95–112, 1969.

[4] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview," *IEEE Proceedings*, vol. 95(1), pp. 108–137, 2007.

[5] D. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Transactions on Automatic Control*, vol. 35(8), pp. 916–924, 1990.

[6] H. Ishii and B. Francis, "Quadratic stabilization of sampled-data systems with quantization," *Automatica*, vol. 39(10), pp. 1793–1800, 2003.

[7] J. Baillieul, "Feedback designs in information-based control," 2002.

[8] F. Fagnani and S. Zampieri, "Stability analysis and synthesis for scalar linear systems with a quantized feedback," *IEEE Transactions on Automatic Control*, vol. 48(9), pp. 1569–1584, 2003.

[9] N. Elia and S. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46(9), pp. 1384 –1400, 2001.

[10] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50(11), pp. 1698–1711, 2005.

[11] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45(7), pp. 1279 –1289, 2000.

[12] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 48(2), pp. 304–307, 2003.

[13] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49(7), pp. 1056–1068, 2004.

[14] G. Nair and R. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[15] Q. Ling and M. Lemmon, "Stability of quantized control systems under dynamicc bit assignment," *IEEE Trans. on Automatic Control*, vol. 50(5), pp. 734–740, 2005.

[16] G. Nair and R. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal of Control and Optimization*, vol. 43(2), pp. 413–436, 2004.

[17] N. C. Martins, M. Dahleh, and N. Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Transactions on Automatic Control*, vol. 51(3), pp. 438–447, 2006.

[18] D. Liberzon and D. Nesic, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Transactions on Automatic Control*, vol. 52(5), pp. 767–781, 2007.

[19] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49(7), pp. 1196–1201, 2004.

[20] A. Matveev and A. Savkin, "Comments on "control over noisy channels" and relevant negative results," *IEEE Transactions on Automatic Control*, vol. 50(12), pp. 2105–2110, 2005.

[21] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication linkpart i: scalar systems," *IEEE Transactions on Information Theory*, vol. 52(8), pp. 3369–3395, 2006.

[22] S. Tatikonda and S. Mitter, "Authors' reply," *IEEE Transactions on Automatic Control*, vol. 50(12), p. 2110, 2005.

[23] Q. Ling and M. Lemmon, "Stability of quantized linear systems with bounded noise under dynamic bit assignment," in *IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 2004.

[24] G. Koren and D. Shasha, "Skip-over: algorithms and complexity for overloaded systems that allow skips," in *IEEE Real-Time Systems Symposium*, 1995, pp. 110–117.

[25] P. Ramanathan, "Overload management in real-time control applications using (m, k)-firm guarantee," *IEEE Transactions on Parallel and Distributed Systems*, vol. 10, pp. 549–559, 1999.

[26] G. Bernat and A. Burns, "Combining (n/m)-hard deadlines and dual priority scheduling," in *the 18th IEEE Real-Time Systems Symposium*, San Francisco, CA, USA, 1997, pp. 46–57.

[27] D. Liu, X. Hu, M. Lemmon, and Q. Ling, "Firm real-time system scheduling based on a novel qos constraint," *IEEE Transactions on Computers*, vol. 55(3), pp. 320–333, 2006.

[28] S. Tatikonda, "Control under communication constraints," Ph.D. dissertation, M.I.T., 2000.

V. APPENDIX:PROOF OF THEOREM 3.1

Theorem 3.1 is proven through constructing the following bound on $L[k]$

$$\|L[k]\| \leq c_1\|L[0]\|\eta^k + c_2 W, \forall k \tag{25}$$

where c_1 and c_2 are two constants to be determined, and η is a positive constant less than 1.

The dropout sequence satisfies the condition in eq. 11. By Corollary 2.1 and eq. 13, we can place the following upper bound on the local dropout rate $\varepsilon_l[k]$.

Lemma 5.1: There exists $\delta > 0$, $N \in \mathcal{N}$ and $k_1 \in \mathcal{N}$ to almost surely guarantee that, for $\forall l \geq N, \forall k \geq k_1$,

$$\begin{cases} \varepsilon_l[k] \leq \hat{\varepsilon} + \delta \\ Q^{1-\hat{\varepsilon}-\delta} \geq \alpha(A)\left(1 + \frac{3Q}{\rho}\right)^n \end{cases} \tag{26}$$

$$\eta = \sqrt[n]{\frac{\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n}{Q^{1-\hat{\varepsilon}-\delta}}} < 1. \tag{27}$$

By comparing $\overline{W}$ and $\|L[k_1]\|$, we see there are two cases: (i). $\overline{W} \geq \|L[k_1]\|$; (ii). $\overline{W} < \|L[k_1]\|$. We will find upper bounds on $\|L[k]\|$ $(k \geq k_1)$ for both cases, respectively. By combining these bounds, together with a bound on $\|L[k]\|$ for $k < k_1$, we will get eq. 25.

A. When $\overline{W} \geq \|L[k_1]\|$

Define

$$r_{i,j}[k] = \begin{cases} \max\left(L_{i,j}[k], \rho\overline{W}\right), & j = n_i \\ \max\left(L_{i,j}[k], \rho r_{i,j+1}[k]\right), & j < n_i \end{cases} \tag{28}$$

$$p[k] = \prod_{i=1}^{P}\prod_{j=1}^{n_i} r_{i,j}[k] \tag{29}$$

It is straigtforward to get

$$\begin{cases} r_{i,j}[k] \geq L_{i,j}[k] \\ r_{i,j}[k] \geq \rho^{n_i-j+1}\overline{W} \geq \rho\overline{W} \geq \overline{W} \end{cases}. \tag{30}$$

where the parameter ρ is defined in eq. 20. There are two bounds on the growth rate of $r_{i,j}[k]$.

Lemma 5.2: For $\forall k, \forall i = 1, \cdots, P; j = 1, \cdots, n_i$,

$$\frac{r_{i,j}[k+1]}{r_{i,j}[k]} \leq |\lambda_i|\left(1 + \frac{3Q}{\rho}\right). \tag{31}$$

Sketch of proof: We prove eq. 31 for a complex λ_i as an example. Mathematical induction method is applied. When $j = n_i$, Algorithm 1 tells us

$$\begin{aligned} L_{i,n_i}[k+1] &\leq |\lambda_i| L_{i,n_i}[k] + \overline{W} \\ &= |\lambda_i| L_{i,n_i}[k] + \frac{1}{\rho}\left(\rho\overline{W}\right) \\ &< |\lambda_i|\left(1+\frac{3Q}{\rho}\right) r_{i,n_i}[k] \end{aligned}$$

Similar procedure can be applied to $j = n_i - 1$. Now suppose eq. 31 holds for $j \leq j_0$ ($\leq n_i - 1$). We want to verify eq. 31 for $j = j_1 = j_0 - 1$. Assume j_1 is even. By Algorithm 1, we know

$$\begin{aligned} L_{i,j_1}[k+1] &\leq |\lambda_i| L_{i,j_1}[k] + L_{i,j_1+1}[k] + L_{i,j_1+2}[k] + \overline{W} \\ &\leq |\lambda_i|\left(1+\frac{3Q}{\rho}\right) r_{i,j_1}[k] \end{aligned} \quad (32)$$

If $r_{i,j_1}[k+1] = L_{i,j_1}[k+1]$, eq. 32 yields eq. 31. Otherwise $r_{i,j_1}[k+1] = \rho r_{i,j_1+1}[k+1]$, and we get the conclusion from $r_{i,j_1}[k] \geq \rho r_{i,j_1+1}[k]$ and the assumption that eq. 31 holds for $j \geq j_1 + 1 = j_0$. The other cases, such as real λ_i and complex λ_i with odd j_1, can be similarly proven. $\diamondsuit$

The upper bound in Lemma 5.2 is quite loose. The following Lemma presents a tighter one.

Lemma 5.3: Suppose side (I_k, J_k) is the longest at time k according to the criterion in Algorithm 1. When $d[k] = 0$ and $L_{I_k,J_k}[k] \geq Q^2 \rho^{n_{I_k} - J_k + 1}\overline{W}$,

$$\frac{r_{I_k,J_k}[k+1]}{r_{I_k,J_k}[k]} \leq \frac{|\lambda_i|}{Q}\left(1+\frac{3Q}{\rho}\right). \quad (33)$$

Sketch of proof: Under the condition of $L_{I_k,J_k}[k] \geq Q^2\rho^{n_{I_k}-J_k+1}\overline{W}$, we can show that $r_{I_k,J_k}[k] = L_{I_k,J_k}[k]$ based on the selection rule of (I_k, J_k) (i.e., $\left(Q^2\rho\right)^{J_k} L_{I_k,J_k}[k] \geq \left(Q^2\rho\right)^{j} L_{I_k,j}[k], j = J_k+1, \cdots, n_{I_k}$). By the updating rule of $L_{i,j}[k]$, we know $L_{I_k,J_k}[k+1] \geq \frac{|\lambda_i|}{Q} L_{I_k,J_k}[k]$, which, together with the previous lower bound on $L_{I_k,j}[k+1]$ ($j \geq J_k+1$), the selection rule of (I_k, J_k) and Lemma 5.2, produces

$$\begin{cases} L_{I_k,J_k}[k+1] = r_{I_k,J_k}[k+1] \\ L_{I_k,J_k}[k+1] \leq \frac{|\lambda_i|}{Q}\left(1+\frac{3Q}{\rho}\right) r_{I_k,J_k}[k] \end{cases}$$

So eq. 33 is reached. $\diamondsuit$

By eq. 29, $p[k]$ is just the product of all $r_{i,j}[k]$. Combining Lemmas 5.2 and 5.3, we get

Lemma 5.4:

$$p[k+1] \leq \alpha(A)\left(1+\frac{3Q}{\rho}\right)^n p[k] < Qp[k], \forall k. \quad (34)$$

When $d[k] = 0$ and $p[k] \geq \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right)$,

$$p[k+1] \leq \frac{1}{Q}\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n p[k] \quad (35)$$

Sketch of proof: Eq. 34 simply comes from Lemma 5.2. When $p[k] \geq \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right)$, we can show $L_{I_k,J_k}[k] \geq Q^2\rho^{n_{I_k}-J_k+1}\overline{W}$ by the contradition method. Combining eq. 31 and 33 gives eq. 35. $\diamondsuit$

Now we partition the time instants into windows with the duration of N (see Lemma 5.1 for the definition of N). We get an upper bound on $p[mN + k_1]$ ($m = 0, 1, \cdots$).

Lemma 5.5: It is almost sure that

$$p[mN+k_1] \leq Q^N \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right), \forall m \geq 0. \quad (36)$$

Proof: We prove it by mathematical induction. Eq. 36 is trivially true for $m = 0$ because $\|L[k_1]\| \leq \overline{W}$. Suppose eq. 36 holds for $m = m_1 - 1$. Now we try to prove it works for $m = m_1$. By Lemma 5.1, we know it is almost sure that

$$\varepsilon_N[k] \leq \hat{\varepsilon} + \delta, \forall k \geq k_1.$$

Let $T = N - \lfloor N(\hat{\varepsilon}+\delta)\rfloor$. There are at least T successfully transmitted packets from time $(m_1-1)N+1+k_1$ to m_1N+k_1. Denote the time instants of successful transmissions as $k_1, k_2, \cdots, k_T$. If at one of these instants, to say k_j,

$$p[k_j] < \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right) \quad (37)$$

By implementing eq. 34 from $k = k_j$ to $k = m_1N + k_1$, we get

$$\begin{aligned} p[m_1N+k_1] &\leq Q^{m_1N+k_1-k_j} p[k_j] \\ &\leq Q^N \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right) \end{aligned}$$

If for all $k \in \{k_1, k_2, \cdots, k_T\}$, eq. 37 is false, i.e.,

$$p[k_j] \geq \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right), \forall k_j \quad (38)$$

Implementing eq. 35 at $k = k_j$ ($j = 1, \cdots, T$) and eq. 34 at other time instants yields

$$\begin{aligned} p[m_1N+k_1] &\leq \frac{\left(\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n\right)^N}{Q^T} p[(m_1-1)N+k_1] \\ &\leq \left(\frac{\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n}{Q^{1-\hat{\varepsilon}-\delta}}\right)^N p[(m_1-1)N+k_1] \\ &\leq p[(m_1-1)N+k_1] \end{aligned}$$

By the assumption that eq. 36 holds at $m = m_1 - 1$, we know from the above inequality that eq. 36 is also valid for $m = m_1$. The proof has been completed. $\diamondsuit$.

For $mN + k_1 \leq k < (m+1)N + k_1$, we can implement eq. 34 from $mN + k_1$ to k, together with Lemma 5.5, to reach

Corollary 5.6: It is almost sure that

$$p[k] \leq Q^{2N} \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right), \forall k \geq k_1 \quad (39)$$

$p[k]$ is the product of n terms, $r_{i',j'}[k](i' = 1, \cdots, P; j' = 1, \cdots, n_{i'})$. Among these terms, we consider a particular one with $i' = i, j' = j$. With the lower bounds of $r_{i',j'}[k](i' \neq$

i or $j' \neq j$) in eq. 30 and the upper bound of $p[k]$ in Corollary 5.6, we obtain

Proposition 5.7: For $\forall k \geq k_1$,

$$L_{i,j}[k] \leq r_{i,j}[k] \leq Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1} \right) \right) \overline{W}. \quad (40)$$

B. When $\overline{W} < \|L[k_1]\|$

There exist k_2 ($k_2 > k_1$) such that $\|L[k_1]\|\eta^{k_2 - k_1} \geq \overline{W}$ and $\|L[k_1]\|\eta^{k2 - k_1 + 1} < \overline{W}$, where η is defined in eq. 27.

1) Under the condition $k \leq k_2$,: we redefine $r_{i,j}[k]$ and $p[k]$ into $r'_{i,j}[k]$ and $p'[k]$ as

$$\begin{cases} r'_{i,j}[k] = \begin{cases} \max(L_{i,n_i}[k], \rho\eta^{k-k_1}\|L[k_1]\|), & j = n_i \\ \max(L_{i,j}[k], \rho r_{i,j+1}[k]), & j < n_i \end{cases} \\ p'[k] = \prod_{i=1}^{P} \prod_{j=1}^{n_i} r'_{i,j}[k] \end{cases}$$

Similar to Lemmas 5.2 and 5.3, we can get

Lemma 5.8: For $\forall k \in \{k_1, k_1 + 1, \cdots, k_2\}$,

$$\begin{cases} \frac{r'_{i,j}[k+1]}{r'_{i,j}[k]} \leq |\lambda_i| \left(1 + \frac{3Q}{\rho}\right) \\ p'[k+1] \leq \alpha(A) \left(1 + \frac{3Q}{\rho}\right)^n p[k] < Q p'[k] \end{cases}. \quad (41)$$

When $L_{I_k, J_k}[k] \geq Q^2 \rho^{n_{I_k} - J_k + 1} \eta^{k - k_1} \|L[k_1]\|$ and $d[k] = 0$,

$$\frac{r'_{I_k, J_k}[k+1]}{r'_{I_k, J_k}[k]} \leq \frac{|\lambda_i|}{Q} \left(1 + \frac{3Q}{\rho}\right). \quad (42)$$

When $p'[k] \geq \prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1} \eta^{k - k_1} \|L[k_1]\|\right)$ and $d[k] = 0$,

$$p'[k+1] \leq \frac{1}{Q} \alpha(A) \left(1 + \frac{3Q}{\rho}\right)^n p'[k] \quad (43)$$

Under the condition in eq. 26,

$$p'[k] \leq Q^{2N} \prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1} \eta^{k - k_1} \|L[k_1]\|\right) \quad (44)$$

Similar to eq. 30, we get, for $\forall k \in \{k_1, k_1 + 1, \cdots, k_2\}$,

$$\begin{cases} r'_{i,j}[k] \geq L_{i,j}[k] \\ r'_{i,j}[k] \geq \rho^{n_i - j + 1} \eta^{k - k_1} \|L[k_1]\| > \eta^{k - k_1} \|L[k_1]\| \end{cases}. \quad (45)$$

Considering the definition of $p'[k]$ and applying eq. 45 to eq. 44, we get when $k_1 \leq k \leq k_2$,

$$L_{i,j}[k] \leq Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \eta^{k - k_1} \|L[k_1]\|. \quad (46)$$

2) Under the condition $k \geq k_2 + 1$: Starting from $k \geq k_1$, the time intants are grouped into epoches with the duration of N. Let $m_0 = \lfloor (k_2 - k_1)/N \rfloor$. Because $\|L[k_1]\|\eta^{k_2 + 1 - k_1} \leq \overline{W}$ and $k_2 + 1 - k_1 \leq (m_0 + 1)N$, we know

$$\eta^{m_0 N} \|L[k_1]\| \leq \frac{1}{\eta^N} \overline{W}$$

Because $m_0 N + k_1 \leq k_2$, eq. 46 is applicable and yields

$$\begin{aligned} & L_{i,j}[m_0 N + k_1] \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \eta^{m_0 N} \|L[k_1]\| \quad (47) \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \frac{\overline{W}}{\eta^N} \quad (48) \end{aligned}$$

Define $\overline{W}' = \frac{Q^{2N}}{\eta^N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \overline{W}$. Note that $\overline{W}' \geq \overline{W}$. Similar to $r_{i,j}[k]$ and $p[k]$, we define, for $k \geq m_0 N + k_1$,

$$r''_{i,j}[k] = \begin{cases} \max\left(L_{i,j}[k], \rho\overline{W}'\right), & j = n_i \\ \max\left(L_{i,j}[k], \rho r''_{i,j+1}[k]\right), & j < n_i \end{cases} \quad (49)$$

$$p''[k] = \prod_{i=1}^{P} \prod_{j=1}^{n_i} r''_{i,j}[k] \quad (50)$$

So we can repeat the previous procedure for the case of $\|L[k_1]\| \leq \overline{W}$ to get a result similar to eq. 40

$$\begin{aligned} & L_{i,j}[k] \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \overline{W}' \\ = \quad & \left(Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \right)^2 \frac{1}{\eta^N} \overline{W} \quad (51) \end{aligned}$$

for $\forall k \geq m_0 N + k_1$. Of course, the above inequality holds for $k \geq k_2$ due to $k_2 \geq m_0 N + k_1$.

C. Final proof to Theorem 3.1

From time 0 to k_1, we can easily deduce the following inequality on $\|L[k]\|$ by the updating rule of $L[k]$

$$\|L[k+1]\| \leq (\alpha(A) + 2) \|L[k]\| + \overline{W}$$

So it is straightforward to reach

$$\|L[k]\| \leq (\alpha(A) + 2)^{k_1} \|L[0]\| + (\alpha(A) + 2)^{k_1} \overline{W}, \quad (52)$$

for $\forall k \in \{0, 1, \cdots, k_1\}$. Eq. 40, 46, 51 and 52 provide 4 upper bounds on $L_{i,j}[k]$ under 4 different conditions. These 4 bounds can be bounded by $c_1 \|L[0]\| \eta^k + \frac{1}{2} c_2 \overline{W}$ from above for $\forall k \geq 0$ with

$$\begin{cases} c_1 = & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \eta^{-k_1} (\alpha(A) + 2)^{k_1} \\ c_2 = & 2 \left(Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j + 1}\right) \right) \right)^2 \\ & \times \eta^{-k_1 - N} (\alpha(A) + 2)^{k_1} \end{cases}$$

Then it is almost sure that

$$\begin{aligned} \|L[k]\| \quad & \leq \quad c_1 \|L[0]\| \eta^k + \frac{1}{2} c_2 \overline{W} \\ & \leq \quad c_1 \|L[0]\| \eta^k + c_2 W, \quad \forall k \geq 0 \end{aligned}$$

where the relationship $\overline{W} = 2W$ is utilized. The proof of Theorem 3.1 has been completed. $\diamondsuit$