Weakly Coupled Event Triggered Output Feedback Control in Wireless Networked Control Systems

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***Abstract*—This paper examines output feedback control of wireless networked control systems where there are separate links between the sensor-to-controller and controller-to-actuator. The proposed triggering events only rely on local information so that the transmissions from the sensor and controller subsystems are not necessarily synchronized. This represents an advance over recent work in event-triggered output feedback control where transmission from the controller subsystem was tightly coupled to the receipt of event-triggered sensor data. The paper presents an upper bound on the optimal cost attained by the closed-loop system. Simulation results demonstrate that transmissions between sensors and controller subsystems are not tightly synchronized. These results are also consistent with derived upper bounds on overall system cost.**

I. Introduction

Large-scale wireless networked control systems (WNCS) are invaluable in many civil and military applications for monitoring and controlling in complex environment. An important issue for large-scale WNCS concerns energy efficiency. Sensor nodes need to operate on an extremely frugal energy budget, since they are battery driven and since battery replacement is not an option for large-scale WNCS with thousands of physically embedded nodes. To conserve power, it is important to manage wireless communication as such communication is a major source of power consumption [1]. There has been a great deal of prior work seeking to conserve power [2], [3] through energy efficient networking protocols. Another way of conserving power, however, is to make the application power aware, and attempt to minimize the application's use of the communication network, while still maintaining a desired level of control system performance. One recent method for realizing this goal is known as *event-triggered* sampling.

Event triggering can be seen as a communication protocol where information is transmitted only if some event occurs. In particular, information is transmitted when a measure of data 'novelty' exceeds a specified threshold. In contrast to more commonly used periodic transmission schemes, event-triggering tends to generate traffic patterns that are *sporadic* in nature. Prior experimental results has demonstrated that event-triggering can use fewer communication resources than periodic transmission schemes while maintaining the comparable performance levels [4]–[8]. The reason for this more efficient use of communication resources is that event-triggering makes use of on-line information to make transmission decisions. This method, therefore, can adapt its usage of the communication in channel to the importance of the data that it must transmit.

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The existing work in the event triggering literature, however, can not be easily applied to the large-scale WNCS with simple extension, not only because most prior work considered single sensor case, but also because the structure in the prior work was not suitable for the large-scale WNCS. In the large-scale WNCS, the controller can be far away from both the sensors and the actuators, which requires that the whole control loop has to be closed over network. However, most prior work assumed that only part of the control loop was closed over wireless network [9]–[11], which can not be used in the case when the controller is far away from both sensors and actuators. Another work in [12] did consider the case when the whole control loop is closed over the wireless network, but the transmissions from sensor-to-controller and controller-to-actuator were strongly synchronized (the transmission in one link triggered the transmission in the other link). The strong synchronization prohibits the extension from single sensor case to the multiple sensor case, since a large number of sensors would trigger very frequent transmissions from controller to actuator, which is neither desired nor necessary. Therefore, as a pre-step of applying event triggering to large-scale WNCS, we first present weakly coupled event triggered output feedback control system (the transmission in one link doesn't necessarily trigger the transmission in the other link) in this paper. Since the transmissions from sensor-to-controller and controller-to-actuator are only weakly coupled, the growing number of sensors will not necessarily cause the frequent transmissions from controller to actuator. The weakly coupled transmission structure, therefore, provides a path for applying the event triggering technique to the large-scale WNCS.

This paper designs local event-triggers for a wireless networked control system in which there are separate communication links from sensor-to-controller and controller-to-actuator. The event-triggers attempt to be optimal in the sense that the triggering sets of the sensor-to-controller link minimize the mean square estimation error discounted by communication cost and the triggering sets of the controller-to-actuator link are designed to minimize the mean square cost of the estimated system state discounted by the communication cost in that link. The sum of these two separate optimization problems provide an upper bound on the optimal mean square performance of the entire system. The events generated by this approach are only weakly coupled, rather than strongly coupled as seen in [12]. The results in this paper serve as a foundation for large-scale event-triggered output feedback control that will be studied in future papers.

II. Problem Statement

Consider a dynamic system whose control loop is closed over the network. A block diagram of the closed loop system is shown in Figure 1. This closed loop system consists of four components: a *plant subsystem*, a *sensor subsystem*, a *controller subsystem* and an *actuator subsystem*.

The plant subsystem consists of two parts: a plant and a sensor. The plant is a controllable and observable linear discrete time process whose state $x : \mathbb{Z}^+ \to \mathbb{R}$ ($\mathbb{Z}^+$ indicates nonnegative integer set) satisfies the difference equation

$$x(k) = Ax(k-1) + Bu_a(k-1) + w(k-1),$$

for $k = 1, 2, \cdots$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u_a \in \mathbb{R}^m$ is the actual control input applied into the plant which will be further explained when the actuator subsystem is introduced, and $w \in \mathbb{R}^n$ is a zero mean white Gaussian noise process with variance W. The initial state $x(0)$ is assumed to be a Gaussian random variable with mean μ_0 and variance Π_0. The sensor generates a measurement $y : \mathbb{Z}^+ \to \mathbb{R}^p$ which is an output with noise. The sensor measurement at time k is

$$y(k) = Cx(k) + v(k),$$

for $k \in \mathbb{Z}^+$, where $C \in \mathbb{R}^{p\times n}$, and $v : \mathbb{Z}^+ \to \mathbb{R}^p$ is another zero mean white Gaussian noise process with variance V. Notice that w, v and $x(0)$ are independent from each other. The corrupted measurement is fed into the sensor subsystem that decides when to transmit information to the controller subsystem.

The sensor subsystem shown in the right upper corner of Figure 1 consists of a *Kalman filter*, a *local observer* and an *event detector in sensor subsystem*. Let $\mathcal{Y}(k) = \{y(0), y(1), \cdots, y(k)\}$ denote the measurement information available at step k. The *Kalman filter* generates a state estimate $\overline{x}_{KF} : \mathbb{Z}^+ \to \mathbb{R}^n$ that minimizes the weighted mean square estimation error (MSEE) $E\left[\|x(k) - \overline{x}_{KF}(k)\|_Z^2 \,|\, \mathcal{Y}(k)\right]$ at each step conditioned on all of the sensor information received up to and including step k, where $Z \geq 0$ is a symmetric weighting matrix and $\|\theta\|_Z^2 = \theta^T Z\theta$. Let P_Z be the square root of Z (i.e. $Z = P_Z^T P_Z$). For the process under study the filter equation is

$$\begin{aligned}\overline{x}_{KF}(k) =& A\overline{x}_{KF}(k-1) + Bu_a(k-1) + L\,(y(k)\\ &-CA\overline{x}_{KF}(k-1) - CBu_a(k-1))\,,\end{aligned}$$

where $L = AXC^T(CXC^T + V)^{-1}$, and X satisfies the discrete linear Riccati equation

$$AXA^T - X - AXC^T(CXC^T + V)^{-1}CXA^T + W = 0.$$

The steady state estimation error $\overline{e}_{KF}(k) = x(k) - \overline{x}_{KF}(k)$ is a Gaussian random variable with zero mean and weighted variance $E(\overline{e}_{KF} Z \overline{e}_{KF}^T) = Q = (I - LC)X$.

Let $\{\tau_s^l\}_{l=1}^{\infty}$ denote a sequence of increasing times ($\tau_s^l \in [0, +\infty]$) when information is transmitted from the sensor to the controller subsystem. We require that τ_s^l is forward progressing, i.e. for any $k \geq 0$, there always exists a l such that $\tau_s^l \geq k$. Let $\overline{\mathcal{X}}(k) = \left\{\overline{x}_{KF}(\tau_s^1), \overline{x}_{KF}(\tau_s^2), \cdots, \overline{x}_{KF}(\tau_s^{l(k)})\right\}$ denote the filter estimates that are transmitted to the controller subsystem by step k where $l(k) = \max\left\{l : \tau_s^l \leq k\right\}$. We can think of this as the 'information set' available to the controller subsystem at time k. The local observer generates an a posteriori estimate $\overline{x}_{RO} : \mathbb{Z}^+ \to \mathbb{R}^n$ of the process state that minimizes the weighted MSEE, $E\left[\|x(k) - \overline{x}_{RO}(k)\|_Z^2 \,|\, \overline{\mathcal{X}}(k)\right]$, at time k conditioned on the information received up to and including time k. The a priori estimate of the local observer, $\overline{x}_{RO}^- : \mathbb{Z}^+ \to \mathbb{R}^n$, minimizes $E\left[\|x(k) - \overline{x}_{RO}^-(k)\|_Z^2 \,|\, \overline{\mathcal{X}}(k-1)\right]$, the weighted MSEE at time k conditioned on the information received up to and including step $k-1$. These estimates take the form

$$\overline{x}_{RO}^-(k) = A\overline{x}_{RO}(k-1) + Bu_a(k-1) \quad (1)$$

$$\overline{x}_{RO}(k) = \begin{cases} \overline{x}_{RO}^-(k), & \text{if } e_{KF,RO}^-(k) \in S_s; \\ \overline{x}_{KF}(k), & \text{otherwise}\,, \end{cases} \quad (2)$$

where $e_{KF,RO}^-(k) = \overline{x}_{KF}(k) - \overline{x}_{RO}(k)$, $S_s \subseteq \mathbb{R}^n$ is the triggering set in sensor subsystem, and $\overline{x}_{RO}^-(0) = \mu_0$.

The *event detector in sensor subsystem* detects the a priori gap $e_{KF,RO}^-(k)$ and compares the gap with the triggering set S_s. If the gap is inside the triggering set S_s, then no data is transmitted. Otherwise, the state estimate of Kalman filter $\overline{x}_{KF}(k)$ is sent to the controller subsystem.

The controller subsystem which is in the lower part of Figure 1 has three components: a *remote observer*, a *controller* and an *event detector in controller subsystem*. The remote observer has the same behavior as the local observer, and produces an a posteriori state estimate $\overline{x}_{RO}(k)$ which is fed into the controller. The controller generates a control input $u_c(k) = K\overline{x}_{RO}(k)$, where K is the controller gain. Notice that this control input is not the actual control input fed into the plant.

Let's define an increasing and forward progressing time sequence $\{\tau_c^j\}_{j=1}^{\infty}$, where τ_c^j is the jth time when the control input is sent to the actuator subsystem from the controller subsystem. The *event detector in the controller subsystem* transmits the current control input $u_c(k)$ to the actuator subsystem when $[\overline{x}_{RO}(k)\ u_a(k)]^T$ lies outside of a triggering set S_c. Once the current control input is sent to the actuator, an acknowledgement is transmitted to the sensor subsystem to let it know that the control input has been updated. When the sensor subsystem receives the acknowledgement, it uses the $\overline{x}_{RO}(k)$ generated by the local observer to obtain the new control input applying to the actuator subsystem.

The actuator subsystem has two parts: a zero order hold and an actuator. Let $u_a(k)$ denote the actual control input applied to the plant. When $u_c(\tau_c^j)$ is transmitted, the actuator subsystem updates $u_a(k)$ to be $u_c(\tau_c^j)$, and holds this value until the next transmission occurs. $u_a(k)$, therefore, takes the form

$$u_a(k) = u_c(\tau_c^j), \forall k \in [\tau_c^j, \tau_c^{j+1}).$$

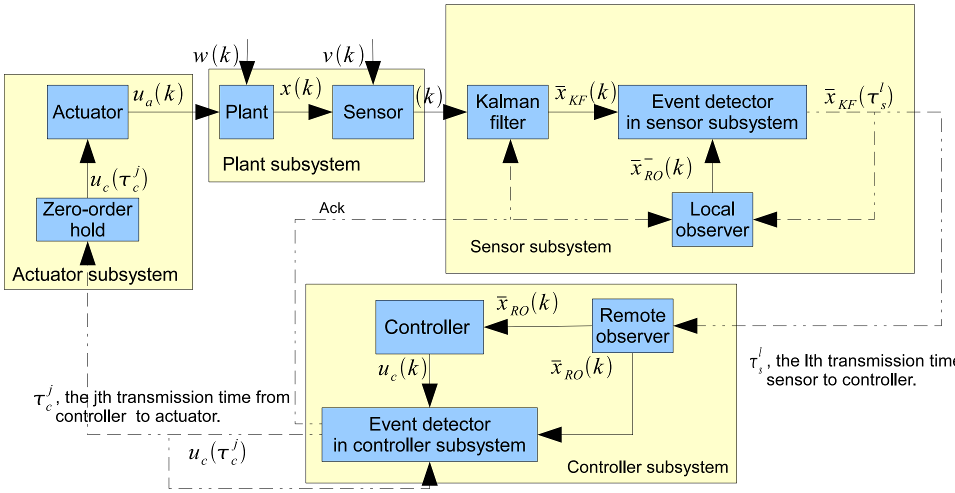

Fig. 1. Structure of the event triggered output feedback control systems

The average cost of the closed loop system is defined as

$$J(S_s, S_c) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\left(c(x(k), S_s, S_c)\right),$$

where the cost function c is

$$\begin{aligned} c(x(k), S_s, S_c) =& \|x(k)\|_Z^2 + \lambda_s 1(e_{KF,RO}(k) \notin S_s) \\ &+ \lambda_c 1\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k) \end{bmatrix} \notin S_c\right), \end{aligned}$$

where λ_s and λ_c are the communication prices for transmissions over the sensor-to-controller link and controller-to-actuator link, respectively. $1(\cdot)$ is the characteristic function.

Our objective is to design the triggering sets S_s and S_c to minimize the average cost $J(S_s, S_c)$, i.e.

$$J^* = \min_{S_s, S_c} J(S_s, S_c).$$

III. Main Results

The main result in this section derives event-triggers for the sensor and controller subsystem. The novel feature of these event triggers is that they are *weakly coupled* in the sense that transmissions over the sensor-to-controller link do not necessarily trigger transmissions over the controller-to-actuator link. By breaking the strong coupling between the two channels, we provide a path for extending event-triggered control to large-scale wireless networked systems.

The derivation of these weakly coupled triggering events is done by decomposing the average cost $J(S_s, S_c)$ into two parts. The first part is only a function of the sensor subsystem and represents the cost introduced by remote state estimation. The second part relies on information from the controller subsystem and the statistics of the state estimates generated by the sensor subsystem. This second part, therefore, represents the controller cost conditioned on the event-triggers of the sensor-subsystem. It is the conditioning on the sensor-subsystem in this second term which weakly couples the events generated by the sensor and controller subsystems.

The following lemma formally states the decomposition of the total system cost $J(S_s, S_c)$ that will be used later to derive the triggering events.

Lemma 3.1: Let $\overline{e}_{RO}(k) = x(k) - \overline{x}_{RO}(k)$. The average cost $J(S_s, S_c)$ may be written as

$$J(S_s, S_c) = J_s(S_s) + J_c(S_c, S_s),$$

where

$$\begin{aligned} J_s(S_s) =& \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left[\overline{e}_{RO}(k)\|_Z^2\right. \\ &\left.+\lambda_s 1(e_{KF,RO}(k) \notin S_s)\right] \\ J_c(S_c, S_s) =& \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left[\|\overline{x}_{RO}(k)\|_Z^2\right. \\ &\left.+\lambda_c 1\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k) \end{bmatrix} \notin S_c\right) \middle| S_s\right] \end{aligned}$$

Remark 3.2: The second cost J_c is conditioned on the sensor's triggering set, S_s, because the expectation taken in J_c must be computed with respect to the probability distribution of the state estimate, $\overline{x}_{RO}(k)$. As this random variable's distribution is a function of the sensor subsystem's triggering set, we see that the expectation in J_c must also be conditioned on S_s, thereby weakly coupling the cost of the controller subsystem to the cost of the sensor subsystem.

Proof: The key step in separating J into the two costs J_s and J_c relies on the fact that $\overline{x}_{RO}(k)$ and $\overline{e}_{RO}(k)$ are uncorrelated. This is shown in Lemma A.1. Realizing $x(k) = \overline{x}_{RO}(k) + \overline{e}_{RO}(k)$ together with the fact that $\overline{x}_{RO}(k)$ and

$\overline{e}_{RO}(k)$ are uncorrelated, $J(S_s, S_c)$ can be rewritten as

$$J(S_s,S_c) = \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left(\|\overline{e}_{RO}(k)\|_Z^2 + \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_s 1(e_{KF,RO}(k)) + \lambda_c 1\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k) \end{bmatrix} \notin S_c\right)\right) = J(S_s) + J(S_c, S_s),$$

■

Since both J_s and J_c rely on S_s, the S_s which minimizes J_s doesn't necessarily minimizes J. We can see, however that the minimum cost J^* is bounded above by

$$J^* \le J(S_s^\dagger, S_c^\dagger) = J_s^\dagger + J_c^\dagger(S_s^\dagger) \tag{3}$$

where $S_s^\dagger$ is the optimal sensor triggering set that minimizes $J_s(S_s)$. The sensor cost achieved by $S_s^\dagger$ is $J_s^\dagger = \min_{S_s} J_s(S_s)$. In a similar way we can see $S_c^\dagger$ as the controller's event-triggering strategy that minimizes the controller cost $J_c(S_c, S_s^\dagger)$ assuming the sensor is the optimal event-trigger $S_s^\dagger$. In this case the controller's cost becomes $J_c^\dagger = \min_{S_c} J_c(S_c, S_s^\dagger)$. The problem of search for the optimal triggering sets can now be obtained by solving two coupled optimization problems. The first optimization problem seeks the sensor triggering set, $S_s^\dagger$, that minimizes $J_s(S_s)$. The second optimization problem seeks an optimal triggering set, S_c that minimizes the cost $J_c(S_c, S_s^\dagger)$. The next two subsections present methods for solving these two optimization problems.

A. The optimal and suboptimal triggering sets in sensor subsystem

This subsection first characterizes the optimal triggering set minimizing the estimation cost J_s. Determining the optimal triggering set from this characterization has high computational complexity both in terms of terms of computation time and space (memory). We therefore present a suboptimal triggering set whose computational complexity is more tractable and bounds the cost achieved by this suboptimal trigger from above. The results presented in this subsection were originally described in [13], so we only review the main results below.

Lemma 3.3: If there exists a bounded function $h_s : \mathbb{R}^n \to \mathbb{R}$ and a finite number J_s' such that

$$J_s' + h_s(e_{KF,RO}^-(k)) = G_s\left(h_s(e_{KF,RO}^-(k))\right) \tag{4}$$

where

$$G_s\left(h_s(\theta)\right) = \min_{S_s} \left\{ E(h_s(e_{KF,RO}^-(k+1)) | e_{KF,RO}(k) = \theta) + c_s(\theta, S_s) \right\},$$

then the optimal average cost of remote state estimation is

$$J_s^\dagger = tr(QZ) + J_s', \tag{5}$$

and the optimal triggering set in sensor subsystem

$$S_s^\dagger = \left\{ \theta : E(h_s(e_{KF,RO}^-(k+1)) | e_{KF,RO}(k) = \theta) + \|\theta\|_Z^2 \le \lambda_s + E(h_s(e_{KF,RO}^-(k+1)) | e_{KF,RO}(k) = 0) \right\}. \tag{6}$$

To compute $S_s^\dagger$, we need to compute the function h_s which satisfies Equation (4). This equation can only be solved numerically, which means that the function h_s can only be expressed in a numerical way, and we need a large table to store the function h_s. Computation and storage of such a large table will be impractical for many applications. We therefore introduce a suboptimal triggering set for sensor subsystem whose computational complexity is tractable and the cost can be bounded from above.

Lemma 3.4: Given a quadratic triggering set

$$S_s = \{e_{KF,RO}^-(k) : \|e_{KF,RO}^-(k)\|_{H_s}^2 \le \lambda_s - \zeta_s\}, \tag{7}$$

where the $n \times n$ matrix $H_s \ge 0$ satisfies the Lyapunov inequality

$$\frac{A^T H_s A}{1+\delta_s^2} - H_s + \frac{Z}{1+\delta_s^2} \le 0,$$

for some $\delta_s^2 \ge 0$, and

$$\zeta_s = \frac{\delta_s^2 \lambda_s + tr(H_s R)}{1+\delta_s^2},$$

where $R = P_Z^{-1} L (CAQA^T C^T + CWC^T + V) L^T (P_Z^{-1})^T$, then

$$J_s(S_s) \le \overline{J}_s(S_s) = tr(QZ) + \min\{tr(H_s R) + \zeta_s, \lambda_s\} \tag{8}$$

Remark 3.5: For any A and $Z \ge 0$, there always exists an $H_s > 0$ and a $\delta_s^2 \ge 0$ such that the Lyapunov inequality (3.4) holds. We should notice that the greatest singular value of A, $\overline{\sigma}(A)$, is always greater than or equal to the absolute value of any eigenvalue of A. So if we set δ_s^2 to be the value such that $\overline{\sigma}(A) \le \sqrt{1+\delta_s^2}$, then $A/\sqrt{1+\delta_s^2}$ is always stable, and there always exists $H_s \ge 0$ such that (3.4) holds for any semi-positive definite matrix Z.

B. Optimal and suboptimal triggering sets in controller subsystem

This subsection first studies the optimal triggering set for the controller subsystem and the corresponding minimum average cost of control. As was found in the preceding subsection, direct computation of the optimal triggering set is complex. We therefore introduce a suboptimal event-trigger and bound the performance obtained by this trigger.

Since minimizing $J_c(S_c, S_s)$ with respect to S_c is a discrete time average optimal control problem, the method in [14] can be applied to our problem. So we have the lemma below to state the optimal triggering set and cost in controller subsystem.

Lemma 3.6: Given $S_s^\dagger$, if there exists a bounded function $h_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and a bounded function $J_c' : \mathbb{S}^n \to \mathbb{R}$ ($\mathbb{S}^n$ indicates the collection of all subsets of $\mathbb{R}^n$) such that

$$J_c'(S_s^\dagger) + h_c\left(\overline{x}_{RO}(k), u_a(k-1)\right) = \min_{S_c} \left\{ E\left[h_c\left(\overline{x}_{RO}(k+1), u_a(k)\right) | \overline{x}_{RO}(k), u_a(k-1), S_c\right] + C_c\left(\overline{x}_{RO}(k), u_a(k-1), S_c\right) \right\}, \tag{9}$$

then

$$J_c^\dagger(S_s^\dagger) = J_c'(S_s^\dagger), \tag{10}$$

and the optimal triggering set in controller subsystem is

$$S_c^\dagger = \left\{ \begin{bmatrix} \theta \\ \eta \end{bmatrix} : E\left[h_c\left(\begin{bmatrix} \overline{x}_{RO}(k+1) \\ u_a(k) \end{bmatrix} \right) \middle| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right. \right.$$
$$\left. = \begin{bmatrix} \theta \\ \eta \end{bmatrix} \right] \le E\left[h_c\left(\begin{bmatrix} \overline{x}_{RO}(k+1) \\ u_a(k) \end{bmatrix} \right) \right|$$
$$\left. \left. \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} = \begin{bmatrix} \theta \\ K\theta \end{bmatrix} \right] + \lambda_c \right\}$$

To determine the optimal triggering set in controller subsystem $S_c^\dagger$, one must find the function, h_c, which satisfies Equation (9). This equation would be numerically solved to obtain a concrete representation for the function of h_c and hence the controller's event-trigger, $S_c^\dagger$. Due to its concrete representation, the event-trigger would require a great deal of memory to store. We therefore introduce a suboptimal triggering set for the controller subsystem which is easy to computer and store. The next lemma describes this suboptimal triggering set and provides an upper bound on its cost.

Lemma 3.7: Let S_s in Equation (7) be the triggering set in sensor subsystem, $A_u = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$, $A_c = \begin{bmatrix} A+BK & 0 \\ K & 0 \end{bmatrix}$, $Z_a = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$, and $H_s = P_{Hs}^T P_{Hs}$. Given a quadratic triggering set of controller subsystem

$$S_c = \left\{ \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} : \left\| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right\|_{H_c}^2 + \zeta_c \le \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c \right\}, \tag{11}$$

where $H_c \ge Z_a$ and controller gain K satisfy

$$A_u^T H_c A_u + (1+\delta_c^2)(Z_a - H_c) \le 0, \tag{12}$$
$$A_c^T H_c A_c + (1-\rho_c^2)(Z_a - H_c) \le 0, \tag{13}$$

for some constant $\delta_c^2 \ge 0$ and $0 \le \rho_c^2 \le 1$, and

$$\zeta_c = \frac{\delta_c^2 + \rho_c^2 - 1}{\delta_c^2 + \rho_c^2}\lambda_c, \tag{14}$$

the optimal controller cost is bounded from above by

$$J_c(S_s, S_c) \le \overline{J}_c(S_c, S_s)$$
$$= \frac{\delta_c^2}{\delta_c^2 + \rho_c^2}\lambda_c + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s), \tag{15}$$

where $\overline{\sigma}(\cdot)$ indicates the greatest singular value, and $H_{c,lu}$ is the left upper $n \times n$ sub-matrix of H_c.

The upper bound on $J_c(S_c, S_s)$, $\overline{J}_c(S_c, S_s)$, is greater when the uncontrolled system is more unstable. It is easy to see that $\overline{J}_c(S_c, S_s)$ is monotonically increasing with respect to δ_c^2 which can be seen as a measure of how unstable the matrix A_u is. The $(n+m) \times (n+m)$ matrix A_u indicates the dynamic behavior of $[\overline{x}_{RO} \ u_a]^T$ during the interval between two consecutive transmissions from controller subsystem to actuator subsystem.

$\overline{J}_c(S_c, S_s)$ is less if the controller is more aggressive. $\overline{J}_c(S_c, S_s)$ is monotonically decreasing with respect to ρ_c^2 which indicates how aggressive the controller is. The greater the ρ_c^2 is, the more aggressive the controller is. So a more aggressive controller will provides a greater ρ_c^2, and hence a less $\overline{J}_c(S_c, S_s)$.

The more precious the communication resource between controller and actuator subsystem is, the greater upper bound we will have. This is easy to see from Equation (15), since higher communication price λ_c implies more precious communication resource.

A larger triggering set in sensor subsystem S_s results in a larger upper bound on the average cost of control. In Section III, we have said that $J_c(S_c, S_s)$ is influenced by the triggering set in sensor subsystem S_s. The second term in $\overline{J}_c(S_c, S_s)$ involves all the parameters in S_s, so it is the impact of S_s on the upper bound on the average cost of controller. This term $\overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s)$ reflects how big S_s (centered about the origin) is. If S_s is big, the value of the second term of $\overline{J}_c(S_c, S_s)$ is great, which results in a great upper bound on the average cost of controller.

From the results in Equation (3), Lemma 3.3, and 3.6, we can give the optimal weakly coupled triggering sets in sensor and controller subsystems, and an upper bound on the optimal cost.

Theorem 3.8: The optimal triggering set in sensor subsystem $S_s^\dagger$ defined in Lemma 3.3 minimizes $J_s(S_s)$, and the optimal triggering set in controller subsystem $S_c^\dagger$ defined in Lemma 3.6 minimizes $J_c(S_c, S_s^\dagger)$. The optimal cost of the closed loop system J^* is bounded from above by $J^\dagger = J_s^\dagger + J_c^\dagger(S_s^\dagger)$, where $J_s^\dagger$ and $J_c^\dagger(S_s^\dagger)$ are described in Equation (5) and (10), respectively.

From the analysis following Lemma 3.3 and 3.6, we know that the optimal triggering set $S_s^\dagger$ and $S_c^\dagger$ are hard to compute and store. So suboptimal triggering sets and an upper bound on the cost of closed loop system triggered by these triggering sets are derived, which are computationally effective and easy to store. From the results in Lemma 3.1, 3.4 and 3.7, we can have the theorem below.

Theorem 3.9: Given the triggering set in sensor subsystem S_s defined in Equation (7) and the triggering set in controller subsystem S_c defined in Equation (11), the average cost $J(S_s, S_c)$ given by the two weakly coupled triggering sets is bounded from above by $\overline{J}(S_s, S_c) = \overline{J}_s(S_s) + \overline{J}_c(S_c, S_s)$, where $J_s(S_s)$ and $J_c(S_c, S_s)$ are defined in Equation (8) and (15), respectively.

IV. Simulation Results

In this section, an example is used to demonstrate Theorem 3.9. We first calculate the triggering sets S_s and S_c according to Equation (7) and (11), and search for the controller gain K such that Inequality (13) is satisfied. The system, then, is run with the calculated controller gain K, and the transmission is triggered with the computed triggering sets. Next, the average cost given by simulation is compared with the upper bound given in Theorem 3.9 to demonstrate Theorem 3.9. Finally, we show the number of transmission times in sensor subsystem, the number of transmission times in controller subsystem, and the number of times when both sensor and controller transmit (concurrent transmission times) to illustrate that the transmission in sensor subsystem doesn't necessarily trigger the transmission in controller subsystem, or vice versa.

Let's consider the system with A to be $\begin{bmatrix} 0.4 & 0 \\ 0 & 1.01 \end{bmatrix}$, B to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and C to be $\begin{bmatrix} 0.1 & 1 \end{bmatrix}$. The variances of the system noises are $W = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$, and $V = 0.3$. The weight matrix Z is chosen to be an identity matrix.

Given $\delta_s^2 = 1.5$, $\lambda_s = 3$, $\delta_c^2 = 1.02$ and $\rho_c = 0.3$, we can obtain the triggering set in sensor subsystem S_s as below

$$\left\{ e_{KF,RO}^- : e_{KF,RO}^{-T} \begin{bmatrix} 2.5641 & 0 \\ 0 & 4.0543 \end{bmatrix} e_{KF,RO}^- \leq 0.8414 \right\},$$

the triggering set in controller subsystem S_c as below

$$\left\{ \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} : \left\| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right\|_{M_c}^2 \leq 0.9008\lambda_c \right\},$$

where $M_c = \begin{bmatrix} 1.3315 & -0.2836 & -0.3512 \\ -0.2836 & 3.6377 & 2.6808 \\ -0.3512 & 2.6808 & 13.7606 \end{bmatrix}$, and the controller gain $K = [-0.1967 \ \ -0.3133]$. The closed loop system is run for 3000 steps with different λ_c. We first compare the average cost given by our simulation (J) with the upper bound given by Theorem 3.9 (J_{up}), and then have a look at the transmission times in sensor subsystem, the transmission times in controller subsystems and the concurrent transmission times.

The left hand side plot of Figure 2 shows that the average cost given by simulation (J) is always bounded from above by the upper bound given by Theorem 3.9 (J_{up}). The x-axis of this plot indicates the communication price in controller subsystem λ_c, and the y-axis is the average cost. we can see that for any λ_c, the average cost J (blue star) is always bounded from above by the upper bound given by Theorem 3.9 (black cross), which demonstrates Theorem 3.9.

The right hand side plot of Figure 2 shows that the transmission in sensor subsystem doesn't always trigger the transmission in controller subsystem, or vice versa. The x-axis of this plot is the communication price in controller λ_c, and the y-axis indicates the transmission times. We can see that the number of concurrent transmission times (pink circle) is always less or equal to both the numbers of transmission times in sensor and controller subsystems, which indicates that the transmission in sensor subsystem doesn't always trigger the transmission in controller subsystem, or vice versa.

V. Conclusion and Future work

This paper presents weakly coupled triggering events in event triggered output feedback system with the whole control loop closed over wireless network. By 'weakly coupled', we mean that the triggering events in both sensor and controller only use local information to decide when to transmit data, and the transmission in one link doesn't necessarily trigger the transmission in other link. We also show that with the triggering events and controller we designed, the cost of the closed loop system is bounded from above, and an explicit upper bound on the cost is obtained. Our simulation results demonstrate the proposed weakly coupled triggering events and the upper bound on the cost of the closed loop system. This paper serves as a foundation for our future work which will study the multi sensor systems. We are interested in how the cost increases with respect to the number of sensors, and the methods to bound the increasing rate.

Appendix

Lemma A.1: $\overline{x}_{RO}(k)$ and $\overline{e}_{RO}(k)$ are uncorrelated with each other.

Proof: From the dynamics of the closed system, we can derive that

$$\overline{e}_{RO}^-(k) = A\overline{e}_{RO} + w(k-1)$$
$$\overline{e}_{RO}(k) = \begin{cases} \overline{e}_{RO}^-(k), & e_{KF,RO}^- \in S_s; \\ \overline{e}_{KF}(k), & \text{otherwise.} \end{cases}$$

From the equations above, we can see that $\overline{e}_{RO}(k)$ is a linear combination of $\overline{e}_{KF}(\tau_s^{l(k)})$, $w(\tau_s^{l(k)})$, $w(\tau_s^{l(k)}+1), \cdots, w(k)$.

From Equation (1) and (2), we can see that $\overline{x}_{RO}(k)$ is a linear combination of $\overline{x}_{KF}(\tau_s^{l(k)})$, $\overline{x}_{KF}(\tau_s^{l(k)-1}), \cdots, \overline{x}_{KF}(\tau_s^1)$.

Since $\overline{e}_{KF}(\tau_s^{l(k)})$, $w(\tau_s^{l(k)})$, $w(\tau_s^{l(k)}+1)$, $\cdots$, $w(k)$ is uncorrelated with $\overline{x}_{KF}(\tau_s^{l'})$ for any $l' \leq l(k)$, we can conclude that $\overline{x}_{RO}(k)$ and $\overline{e}_{RO}(k)$ are uncorrelated with each other. ■

A. Proof of Lemma 3.7

Before the proof of Lemma 3.7, we would like to state a lemma which will be used in the proof of Lemma 3.7.

Lemma A.2: Given any S_c. If there exists a function f : $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ bounded from below and a finite constant $\overline{J}_s'$ such that

$$\overline{J}_c' + f\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right) \geq C_c\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}, S_c\right) + E\left[f\left(\begin{bmatrix} \overline{x}_{RO}(k+1) \\ u_a(k) \end{bmatrix}\right) \middle| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}, S_c\right], \quad (16)$$

then $J_c(S_c) \leq \overline{J}_c'$.

Proof: See [15]. ■

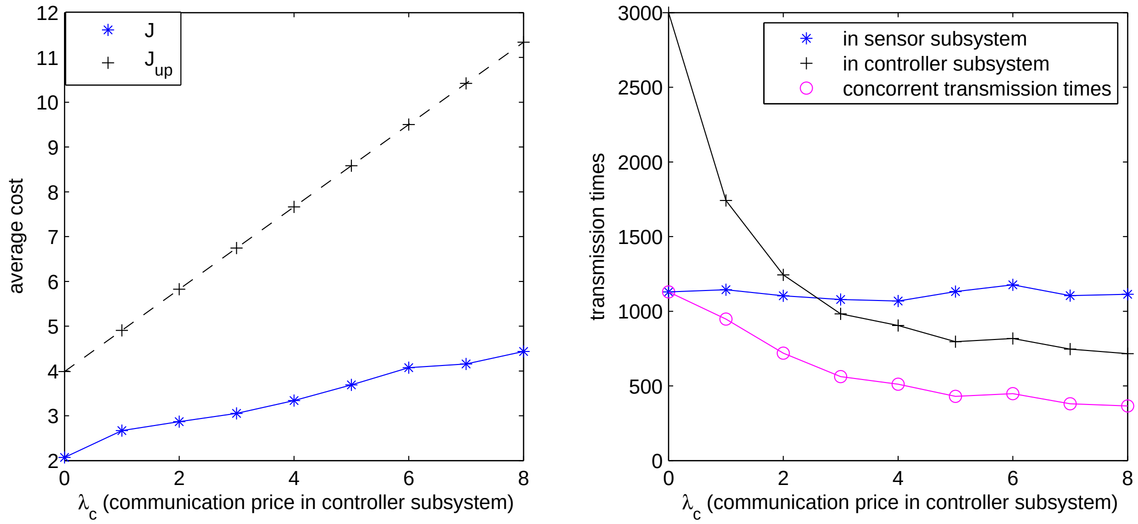

Fig. 2. Simulation results about average cost and transmission times

1) Proof of Lemma 3.7:

Proof: According to Lemma A.2, as long as we can find a function f bounded from below such that the Inequality (16) is satisfied with $\overline{J}_c' = \overline{J}_c$, Lemma 3.7 is true.

Let $f\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right) = \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{H_c}^2 + \zeta_c$. We will consider two cases, and check whether the Inequality (16) holds for both cases. If yes, then Lemma 3.7 is proven.

The first case is when

$$\left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{H_c}^2 + \zeta_c \leq \overline{x}_{RO}(k)^T Z \overline{x}_{RO}(k) + \lambda_c. \quad (17)$$

In this case, the controller subsystem doesn't transmit at step k. The right hand side of Inequality (16) can be rewritten as

$$\begin{aligned} &= \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{A_u^T H_c A_u}^2 + E(\|e_{KF,RO}^-(k+1)\|_{H_{c,lu}}^2) \\ &\quad \cdot 1(e_{KF,RO}^-(k+1) \notin S_s) + \zeta_c + \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{Z_a}^2 \\ &\leq \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{A_u^T H_c A_u}^2 + \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{Z_a}^2 \\ &\quad + \zeta_c + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s) \\ &\leq \overline{J}_c + f\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right) \end{aligned}$$

The first inequality is from Equation (7), and the second inequality is from Equation (12), (17) and (14).

The second case is when $\left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{H_c}^2 + \zeta_c > \overline{x}_{RO}(k)^T Z \overline{x}_{RO}(k) + \lambda_c$. In this case, the controller subsystem transmit information. So the right hand side of Inequality (16) can be rewritten as

$$\begin{aligned} &= \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{A_c^T H_c A_c}^2 + E(\|e_{KF,RO}^-(k+1)\|_{H_{c,lu}}) \\ &\quad \cdot 1(e_{KF,RO}^-(k+1) \notin S_s) + \zeta_c + \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{Z_a}^2 + \lambda_c \\ &\leq \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{A_c^T H_c A_c}^2 + \left\|\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right\|_{Z_a}^2 \\ &\quad + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s) + \zeta_c + \lambda_c \\ &\leq \overline{J}_c + f\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}\right). \end{aligned}$$

The first inequality is from Equation (7), and the second inequality is from Equation (13) and (14).

■

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