Event-triggered Network Utility Maximization through Consensus Filtering

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***Abstract*— Network utility maximization (NUM) problems seek to maximize the aggregate utility network users receive for transmitting at a given data rate subject to limits on link throughput. Distributed solutions to the NUM problem assume one can directly measure link utilization; something that may not always be possible in practice. This paper examines the use of consensus filtering methods for the distributed estimation of link utilization in a distributed NUM algorithm. In particular, we establish sufficient conditions under which distributed network utility maximization using distributed consensus filtering converges to the problem's optimal solution.**

I. INTRODUCTION

A networked system is a collection of subsystems where individual subsystems exchange information over some communication channel. There are many large-scale networked system in the real world, such as the electrical power grid, wireless sensor networks, and the Internet. An interesting research problem seeks to optimize the overall system behavior subject to constraints generated by limited resources in the network. Network utility Maximization (NUM) problems maximize the aggregate utility by transmitting at a specified data rate subject to linear inequality constraints on link throughput. Many problems can be formulated as NUM problems, such as resource allocation, data gathering and power dispatch [1], [2]. Due to the complexity of these large-scale networked system, centralized optimization techniques may not be preferable since they require an unacceptably large amount of coordination and signaling. We are interested in distributed optimization algorithms, where subsystems solve the optimization problem collaboratively through communication between subsystems.

A variety of distributed algorithms have been proposed to solve the NUM problem. Early distributed algorithms [3] [2] suggest that the network's state will asymptotically converge to its optimal point if the communication between subsystems is frequent enough. The dual decomposition approach proposed by Low [4] is the most widely used among the existing algorithms. This approach shows that the message passing complexity might become unacceptable when the network size gets large. Recently, several other distributed optimization algorithms have been proposed. In [5], a subgradient based method is used to generate an approximate optimal solution for the unconstrained problem. Each agent in the network updates the decision vector containing all the decision variables. In [6], a randomized incremental subgradient method was proposed. In this distributed algorithm, communication is governed by a Markov chain. This algorithm assumes that local constraints can be implemented as a projection on the feasible set. Such a projection may be difficult to implement in practice. In both [5] and [6], information exchange happens each time the gradient or subgradient following update is applied and this may result in very expensive communication cost.

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Our recent work focuses on event-triggered distributed algorithms [7]. Under event-triggered communication schemes, links and users in a subsystem transmit information to each other when a local "error" signal exceeds a state dependent threshold. In this framework, agents transmit information sporadically rather than in a periodic way and the message passing complexity is greatly reduced. In order to solve the constrained optimization problem, we use the augmented Lagrangian method [8]. This approach transforms the constrained NUM problem into an unconstrained one by introducing penalty for infeasible data rates. One disadvantage of this method is that computational agents are needed to monitor the overall data flow going through each link in the network. Plugging such devices into the network may be expensive which makes this approach impractical for large networks.

The motivation for this work is to use a consensus filter estimating link utilization in the network. Consensus problems have a long history in distributed computing and management science [9]. In networked systems, "consensus" means all agents agree on a specified state according to some interaction rule. This interaction rule determines how an agent exchanges information with its neighbors in the network in order to reach an agreement. Consensus algorithms have broad application in networked systems, such as flocking, swarming, and formation control, see [10], [11], [12] and also in sensor networks [13], [14], [15]. This paper uses an event-triggered consensus filter to estimate link utilization. These estimates of link utilization are used by the distributed algorithm for solving the NUM problem. Our main result is that this closed loop system can generate an approximate optimal solution to the NUM problem.

The rest of this paper is organized as follows. Section II introduces the distributed NUM problem without direct measurement of link utilization. Section III formulates the system model for solving the NUM problem. Section IV studies the error for data rates and link states. Section V presents the main result of this paper. Section VI gives an example to verify our theoretical result. Section VII summarizes this paper.

II. Problem Statement

Consider a network with a set $\mathcal{S} = \{1, 2, \ldots, N\}$ users and $\mathcal{L} = \{1, 2, \ldots, M\}$ links. Let $A \in \{0, 1\}^{M \times N}$ denote the incidence matrix mapping the users in $\mathcal{S}$ onto the links in $\mathcal{L}$. Let a_{ji} denote the element in the jth row and ith column of A, so that a_{ji} is one if *link j is used by user i* and is zero otherwise. Let $\bar{a}_j^T$ denote the jth row of the incidence matrix, then $\bar{a}_j^T x$ denote the total data flow going through link j. Given a link $j \in \mathcal{L}$, let $\mathcal{S}_j \subset \mathcal{S}$ denote the set of all users using link j. In a similar way, given a user $i \in \mathcal{S}$, let $\mathcal{L}_i \subset \mathcal{L}$ denote the set of all links used by user i. If $j \in \mathcal{L}_i$, let $\mathcal{S}_j^{-i} = \mathcal{S}_j - \{i\}$, and $\mathcal{S}^{-i} = \bigcup_{j \in \mathcal{L}_i} \mathcal{S}_j - \{i\}$.

Let $x \in \mathbb{R}^N$ be the *data rate vector* whose ith component, $x_i \in \mathbb{R}$, is the data rate for the ith user. Let $U_i(\cdot) : \mathbb{R} \to \mathbb{R}^+$ be a continuous function where $i \in \mathcal{S}$ and $U_i(x_i)$ represents the *utility* user i receives for transmitting at data rate x_i. Let $c \in \mathbb{R}^M$ be the *link limit vector* whose jth component, $c_j \in \mathbb{R}$ is the largest total rate that can be carried by link j. The network utility maximization (NUM) problem seeks a data vector, $x \in \mathbb{R}^N$ that maximizes the summed utility of all network users subject to the total data rate in each link being less than or equal to the capacity limit c. Formally, this problem may be stated as

$$\begin{array}{rl} \text{Maximize:} & U(x) = \sum_{i=1}^N U_i(x_i) \\ \text{w.r.t:} & x_i \geq 0, \quad i = 1, 2, \ldots, N \\ \text{subject to:} & Ax \leq c. \end{array}$$

The function $U(\cdot) : \mathbb{R}^N \to \mathbb{R}$ represents the total network utility, constraint $Ax \leq c$ constrains the total flow through each link to be less than link limit vector, c, and the decision variable is the network rate vector, x.

An approximate solution to the NUM problem may be computed using the *augmented Lagrangian method* [8]. This approach converts the contrained NUM problem into a sequence of unconstrained optimization problems by augmenting the utility function, $U(x)$, with a penalty term that prescribes a high cost to infeasible data rates. The *augmented Lagrangian function* $L_p(\cdot;\cdot) : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is a function that takes values

$$L_p(x; w) = -\sum_{i \in \mathcal{S}} U_i(x_i) + \sum_{j \in \mathcal{L}} \psi_j(x; w),$$

for $x \in \mathbb{R}^N$, $w \in \mathbb{R}^M$ and where the real valued function $\psi_j(\cdot;\cdot) : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ takes values

$$\psi_j(x; w) = \begin{cases} 0 & \text{if } c_j - \bar{a}_j^T x \geq 0 \\ \frac{1}{2w_j}(\bar{a}_j^T x - c_j)^2 & \text{otherwise} \end{cases} . \quad (1)$$

for all $j \in \mathcal{L}$.

A primal algorithm based on the augmented Lagrangian method converges to an arbitrarily small neighborhood of the NUM problem's minimizer by approximately minimizing $L_p(x; w)$ for a sufficiently small weighting vector $w \in \mathbb{R}^M$ [8]. One may therefore use a standard gradient descent recursion

$$x[k+1] = \max\{0, x[k] - \gamma \nabla_x L_p(x[k], w)\}, \quad (2)$$

for $k = 0, 1, \ldots, \infty$, where γ is a step size. This recursive equation generates a sequence $\{x[k]\}_{k=0}^\infty$ that asymptotically approach a neighborhood of the solution to the NUM problem. This neighborhood may be made arbitrarily small by selecting the weighting vector w to be sufficiently small.

The update algorithm in equation (2) can be implemented in a distributed manner. In particular, the ith user's update equation takes the form

$$x_i[k+1] = \max\left\{0, x_i[k] - \gamma \frac{\partial L_p(x[k], w)}{\partial x_i}\right\}, \quad (3)$$

where

$$\frac{\partial L_p}{\partial x_i} = -\frac{\partial U_i(x_i[k])}{\partial x_i} + \sum_{j \in \mathcal{L}_i} \max\left\{0, \frac{\phi_j[k]}{w_j}\right\}, \quad (4)$$

$$\phi_j[k] = \bar{a}_j^T x[k] - c_j. \quad (5)$$

The variable, $\phi_j[k]$, is called the jth link's *state*. It represents the jth component of the vector $Ax[k] - c$. The information that user i needs in this recursion is its past data rate, $x_i[k]$, the sensitivity of its own utility function, $\partial U_i / \partial x_i$, and the weighted link states, ϕ_j, for all those links used by user i (i.e. $j \in \mathcal{L}_i$). The local data rate and utility sensitivity are clearly available to user i. The link states, however, must be forwarded to the user from the links $j \in \mathcal{L}_i$. This recursion is *distributed* in the sense that it only requires information from the user and those links that are being used by that user.

The distributed update shown in equation (3) assumes that some computational agent directly measures the link states, $\phi = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_M \end{pmatrix}^T \in \mathbb{R}^M$. In many applications, direct measurement of link utilization may not be possible. In particular, link utilization may need to be inferred from measured user rates. In a wireless communication channel, for instance, it may be difficult to measure the actual capacity of the channel since this is often a function of the number of users using that channel. In general, it may be easier to directly measure what the users generate, rather than how fully the link resources are utilized. Even in wired networks, direct measurement of link utility is based on indirect measurement of time delays, which may vary from user to user.

If direct measurement of link utilization is not possible, then individual agents must estimate link utilization based on the information received from other users using the same link. These link estimates, however, will vary from user to user, particularly if the time between received data varies over time. In this context, one can propose using a distributed consensus filter to estimate the link states, which are then used by the gradient update in equation (3) to update actual user rates. The issue addressed in this paper concerns stability of systems in which distributed consensus is used in a closed loop manner to update user data rates.

III. System Model

Figure 1 illustrates the system consisting of N users that are connected to a communication network through

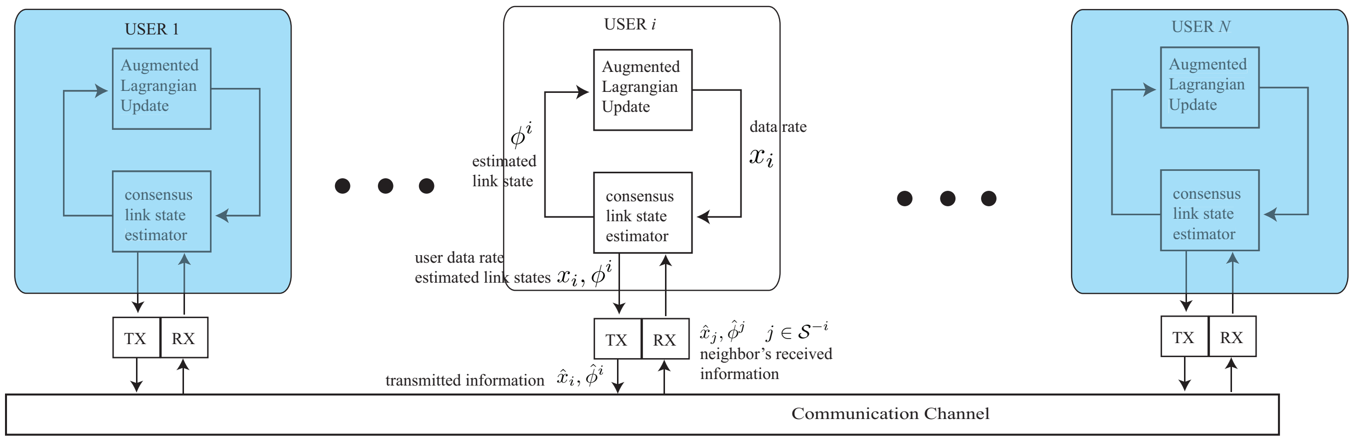

Fig. 1. System Model

transmitter (TX) and receiver (RX) subsystems. All systems are assumed to be synchronized to the same clock tick. The ith TX subsystem has a sequence $\mathbb{T}^i = \{\tau_k^i\}_{k=0}^{\infty}$ of increasing integers where τ_k^i denotes the kth consecutive time when the TX component transmits a message to the communication channel. We assume that all messages are transmitted with at most one clock-tick of delay and no data dropouts.

The information transmitted by the ith TX subsystem are user i's data rate x_i and his estimate for link utilization ϕ^i, where the element in ϕ^i is ϕ_j^i, for all $j \in \mathcal{L}_i$. The update rule is

$$x_i[k+1] = \left[x_i[k] + \gamma \frac{\partial U_i(x_i[k])}{\partial x_i} - \frac{\gamma}{\omega} \sum_{j \in \mathcal{L}_i} \left(\phi_j^i[k] \right)^+ \right]^+, \tag{6}$$

$$\begin{aligned} \phi_j^i[k+1] = \phi_j^i[k] + \gamma \left(\sum_{\ell \in \mathcal{S}_j^{-i}} \frac{1}{|\mathcal{S}_j|} \left(\hat{\phi}_j^\ell[k] - \phi_j^i[k] \right) \right) \\ - \gamma\alpha \left[\phi_j^i[k] - \left(x_i[k] + \sum_{\ell \in \mathcal{S}_j^{-i}} \hat{x}_\ell[k] - c_j \right) \right], \end{aligned} \tag{7}$$

where $\hat{\phi}_j^\ell$ is the latest link state received from user ℓ, $\hat{x}_\ell$ is the latest data rate received from user ℓ. $\mathcal{S}_j^{-i}$ denotes the other users who transmit information through each link $j \in \mathcal{L}_i$ and $|\mathcal{S}_j|$ denotes the number of users on each link $j \in \mathcal{L}_i$. $U_i(x_i)$ denotes the utility function for user i. The utility functions satisfy the following assumption [7].

Assumption 1: $U_i(x) : \Omega \to \mathbb{R}^+$ is continuous and
(a). $\partial U_i / \partial x > 0$;
(b). $\zeta < \partial^2 U_i / \partial^2 x < 0$.
where ζ is a lower bound for $\partial^2 U_i / \partial^2 x$, and $\Omega \subset R^+$ denotes a feasible set for the NUM problem.

In the update rules (6)-(7), we let $\gamma > 0$ denote a stepsize for the system, and we assume a common penalty parameter $\omega > 0$ for all link estimates. In order to ensure convergence of the method, γ and ω should be sufficiently small. The feedback gain $\alpha > 0$ in the consensus filter drives the link estimate, ϕ_j^i, to the optimal value. The notation $[f(x[k])]^+$ defines a positive projection ensuring data rates $x_i[k] \geqslant 0$ for all $k = 0, 1, \ldots, \infty$. The consensus filter used in (7) is based on the "equal neighbor model" proposed in [12]. This agreement algorithm using the equal neighbor model achieves asymptotic consensus when there is no difference between sampled data and actual data. Here, $\hat{\phi}_j^l$ and $\hat{x}_l$ denote the sampled data, and ϕ_j^i and x_i denote the actual data. In the following sections, we will show that when the difference is not exactly zero, but bounded above by some threshold, then asymptotic consensus can still be preserved.

The transmission (TX) subsystem of the ith user decides when to transmit the user data rate x_i and the estimated link state vector ϕ^i. Let $\mathbb{T}^i = \{\tau_k^i\}_{k=0}^{\infty}$ denote a sequence of integers where $\tau_k^i < \tau_{k+1}^i$ for $i \in \mathcal{S}$. The integer τ_k^i is the kth consecutive time instant when user i transmits the data state and link state of user i. This means we can define another signal, $\hat{x}^i[\cdot] : \mathbb{Z}^+ \to \mathbb{R}$ and $\hat{\phi}_j^i[\cdot] : \mathbb{Z}^+ \to \mathbb{R}$ that takes the values

$$\hat{x}_i[k] = x_i[\tau_k^i], \tag{8}$$
$$\hat{\phi}_j^i[k] = \phi_j^i[\tau_k^i], \tag{9}$$

for $k \in [\tau_k^i, \tau_{k+1}^i - 1]$ and all $k = 0, 1, \ldots, \infty$.

IV. Error Inequalities

Showing stability of the interconnected system is equivalent to showing that the error generated at each iteration step converges to zero asymptotically. Since the following analysis is concerned with the asymptotic behavior of the errors, it is convenient to transform the original system dynamics (6)-(7) into a set of coupled error inequalities.

Let x_i^* denote the data rate for user i that maximizes the utility of the network, and $\phi_j{}^*$ denote the link utilization when data rates for all users $i \in \mathcal{S}_j$ are optimal, where $j \in \mathcal{L}_i$. Obviously, we want to guarantee that all users $l \in \mathcal{S}_j$ agree on the same link state $\phi_j{}^*$ by use of the consensus filter. In other words, we should have $\phi_j^l[k] \to \phi_j{}^*$, as $k \to$

$+\infty$, for all $l \in \mathcal{S}_j$. The optimal value of x_i^* and ${\phi_j}^*$ should satisfy the following relationship:

$$\frac{1}{\omega}\sum_{j\in\mathcal{L}_i}\left(\phi_j^*\right)^+ = \frac{\partial U_i(x_i^*)}{\partial x_i}, \quad (10)$$

$${\phi_j}^* = \sum_{l\in\mathcal{S}_j} x_l^* - c_j. \quad (11)$$

Define the data rate error $\tilde{x}_i$ and link state error $\tilde{\phi}_j^i$ as

$$\begin{aligned}\tilde{x}_i &= x_i - x_i^*,\\ \tilde{\phi}_j^i &= \phi_j^i - \phi_j^*.\end{aligned}$$

From the update rule for data rate, (6), we obtain

$$\begin{aligned}\tilde{x}_i[k+1] &= \Bigg[x_i[k] + \gamma\frac{\partial U_i(x_i[k])}{\partial x_i}\\ &\quad -\frac{\gamma}{\omega}\left(\sum_{j\in\mathcal{L}_i}\phi_j^i[k]\right)^+\Bigg]^+ - x_i^*.\end{aligned}$$

With the non-expansive property of the Euclidean projection, we have

$$\begin{aligned}|\tilde{x}_i[k+1]| &\leqslant \Bigg|x_i[k] + \gamma\frac{\partial U_i(x_i[k])}{\partial x_i}\\ &\quad -\frac{\gamma}{\omega}\left(\sum_{j\in\mathcal{L}_i}\phi_j^i[k]\right)^+ - x_i^*\Bigg|.\end{aligned}$$

Substituting (10), and using the mean value theorem, we obtain

$$|\tilde{x}_i[k+1]| \leqslant \left(1+\gamma\frac{\partial U_i^2(\xi_i[k])}{\partial x_i^2}\right)|\tilde{x}_i[k]| + \frac{\gamma}{\omega}\sum_{j\in\mathcal{L}_i}\left|\tilde{\phi}_j^i[k]\right|,$$

with a stepsize $\gamma < -2/\zeta$ chosen to satisfy Assumption 1. For notational convenience, we let

$$\beta_i = 1+\gamma\frac{\partial U_i^2(\xi_i[k])}{\partial x_i^2}$$

where $\beta_i \in (0,1)$. Therefore, the data rate error satisfies the following inequality

$$|\tilde{x}_i[k+1]| \leqslant \beta_i|\tilde{x}_i[k]| + \frac{\gamma}{\omega}\sum_{j\in\mathcal{L}_i}|\tilde{\phi}_j^i[k]|. \quad (12)$$

Next we analyze the error of the link states, $\tilde{\phi}_j^i$, for $j \in \mathcal{L}_i$. According to the consensus algorithm (7), substituting (11), we obtain

$$\begin{aligned}\tilde{\phi}_j^i[k+1] &= \left(1-\gamma\alpha-\gamma\frac{|S_j|-1}{|S_j|}\right)\tilde{\phi}_j^i[k]\\ &\quad +\gamma\frac{1}{|S_j|}\sum_{\ell\in\mathcal{S}_j^{-i}}\tilde{\phi}_j^l[k] + \gamma\alpha\sum_{h\in\mathcal{S}_j}\tilde{x}_h[k]\\ &\quad +\gamma\alpha\sum_{\ell\in\mathcal{S}_j^{-i}}(\hat{x}_l[k]-x_l[k])\\ &\quad +\gamma\frac{1}{|S_j|}\sum_{\ell\in\mathcal{S}_j^{-i}}\left(\hat{\phi}_j^l[k]-\phi_j^l[k]\right).\end{aligned}$$

Then we have

$$\begin{aligned}|\tilde{\phi}_j^i[k+1]| &\leqslant \left(1-\gamma\alpha-\gamma\frac{|S_j|-1}{|S_j|}\right)|\tilde{\phi}_j^i[k]|\\ &\quad +\gamma\frac{1}{|S_j|}\sum_{\ell\in\mathcal{S}_j^{-i}}|\tilde{\phi}_j^l[k]| + \gamma\alpha\sum_{h\in S_j}|\tilde{x}_h[k]|\\ &\quad +\gamma\frac{1}{|S_j|}\sum_{\ell\in\mathcal{S}_j^{-i}}|\hat{\phi}_j^l[k]-\phi_j^l[k]|\\ &\quad +\gamma\alpha\sum_{\ell\in\mathcal{S}_j^{-i}}|\hat{x}_l[k]-x_l[k]|. \quad (13)\end{aligned}$$

Here, we characterize the relationship between the data rate error and link state error, (12)-(13). In order to ensure stability, we have to derive threshold conditions for $|\hat{\phi}_j^l[k]-\phi_j^l[k]|$ and $|\hat{x}_l[k]-x_l[k]|$, based on the link state error, $\tilde{\phi}_j^l[k]$. These thresholds are derived in the next section.

V. Main Result

This section shows (6)-(7) will converge to the optimal solution of the NUM problem. The main result establishes a threshold condition for message passing ensuring stability for the closed-loop system. In the following analysis, we will choose $\gamma > 0$ and $\omega > 0$ to be constants, which are sufficiently small.

The following lemma studies the data error vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)^T \in \mathbb{R}^N$. Consider the following system,

$$\begin{aligned}\tilde{x}[k+1] &= B\tilde{x}[k] + u[k]\\ y[k] &= D\tilde{x}[k]\end{aligned} \quad (14)$$

where $B = \text{diag}(\beta_1\cdots\beta_N) \in \mathbb{R}^{N\times N}$, $D \in \mathbb{R}^{\sum_{j=1}^M|\mathcal{S}_j|\times N}$ containing M blocks, and each block $D_j \in \mathbb{R}^{|\mathcal{S}_j|\times N}$ has same rows. For each block D_j, let d_{qh} denote the element in the qth row and hth column of D_j, then for $q = 1, \ldots, |\mathcal{S}_j|$ and $h = 1, \ldots, N$, we have

$$d_{qh} = \begin{cases}\gamma\alpha & \text{if } h \in \mathcal{S}_j\\ 0 & \text{otherwise}\end{cases}$$

Lemma 1: The system (14) is l_2 stable.

Proof: The transfer function for the discrete-time system (14) is denoted by $G(z)$, where $G(z) = D(zI-B)^{-1}$. The H_∞ norm for the system,

$$\begin{aligned}\|G(z)\|_{\mathcal{H}_\infty} &\leqslant \|D\|_\infty\|(zI-B)^{-1}\|_{\mathcal{H}_\infty}\\ &\leqslant N\gamma\alpha\|(zI-B)^{-1}\|_{\mathcal{H}_\infty}\\ &= N\gamma\alpha\left\|\text{diag}\left(\frac{1}{z-\beta_1},\ldots,\frac{1}{z-\beta_N}\right)\right\|_{\mathcal{H}_\infty}\\ &\leqslant \max_i\frac{N\gamma\alpha}{1-\beta_i}\\ &\leqslant \frac{N\gamma\alpha}{1-\beta}\\ &< \infty,\end{aligned}$$

where $\beta = \max_i\beta_i$, for $i = 1, \ldots, N$ and the second inequality holds since each link $j \in \mathcal{L}$ has at most N users. Therefore, the system is l_2 stable. ∎

Next, we derive threshold conditions for sampling. Let ρ and η be two constants such that $0<\rho<1$ and $0<\eta<1$. If

$$|\hat{\phi}_j^i[k]-\phi_j^i[k]| \leqslant \rho\left(\phi_j^i[k]\right)^+, \quad (15)$$

and

$$|\hat{x}_i[k]-x_i[k]| \leqslant \eta\left(\phi_j^i[k]\right)^+ \quad (16)$$

hold for all $j\in\mathcal{L}$ and $i\in\mathcal{S}$ and for $k=0,\ldots,\infty$, we can guarantee that

$$\begin{aligned}|\hat{\phi}_j^i[k]-\phi_j^i[k]| &\leqslant \rho|\tilde{\phi}_j^i[k]|,\\ |\hat{x}_i[k]-x_i[k]| &\leqslant \eta|\tilde{\phi}_j^i[k]|,\end{aligned}$$

since the optimal link state $\phi_j^*\leqslant 0$, as defined in (11). From (13), we know that for all $i\in\mathcal{S}$ and $j\in\mathcal{L}$,

$$\begin{aligned}|\tilde{\phi}_j^i[k+1]| \leqslant& \left(1-\gamma\alpha-\gamma\frac{|S_j|-1}{|S_j|}\right)|\tilde{\phi}_j^i[k]|\\ &+\gamma\left(\frac{1}{|S_j|}(\rho+1)+\alpha\eta\right)\sum_{\ell\in\mathcal{S}_j^{-i}}|\tilde{\phi}_j^l[k]|\\ &+\gamma\alpha\sum_{h\in S_j}|\tilde{x}_i[k]|. \quad (17)\end{aligned}$$

Consider the following system

$$\begin{aligned}\tilde{\phi}[k+1] &= E\tilde{\phi}[k]+v[k]\\ z[k] &= F\tilde{\phi}[k] \quad (18)\end{aligned}$$

where $\tilde{\phi}=(\tilde{\phi}_1,\ldots,\tilde{\phi}_M)^T\in\mathbb{R}^{\sum_{j=1}^M|\mathcal{S}_j|}$ with each $\tilde{\phi}_j\in\mathbb{R}^{|\mathcal{S}_j|}$. Notice that by ordering $\tilde{\phi}$ in this way, we actually put together the error for link j's state from all users $h\in\mathcal{S}_j$, for $j=1,2,\ldots,M$. In (18), $E=\mathrm{diag}(E_1,\ldots,E_M)\in\mathbb{R}^{\sum_{j=1}^M|\mathcal{S}_j|\times\sum_{j=1}^M|\mathcal{S}_j|}$, where each $E_j\in\mathbb{R}^{|\mathcal{S}_j|\times|\mathcal{S}_j|}$ is symmetric. Let e_{qh} denote the element in the qth row and hth column of E_j so that for $j=1,2,\ldots,M$,

$$e_{qh}=\begin{cases}1-\gamma\alpha-\gamma\frac{|S_j|-1}{|S_j|} & \text{if } q=h,\\ \gamma\left(\frac{1}{|S_j|}(\rho+1)+\alpha\eta\right) & \text{otherwise.}\end{cases}$$

$F\in\mathbb{R}^{N\times\sum_{j=1}^M|\mathcal{S}_j|}$ contains M blocks, where each block $F_j\in\mathbb{R}^{N\times|\mathcal{S}_j|}$. For each F_j, let f_{hq} denote the hth row and qth column of F_j, such that for $h=1,\ldots,N$ and for a $q\in\{1,\ldots,|\mathcal{S}_j|\}$, we have

$$f_{hq}=\begin{cases}\frac{\gamma}{\omega} & \text{if } j\in\mathcal{L}_h,\\ 0 & \text{otherwise.}\end{cases}$$

and q corresponds to that $\tilde{\phi}_j^h$ is the qth element of $\tilde{\phi}_j$.

The following lemma studies the link state error vector $\tilde{\phi}[k]\in\mathbb{R}^{\sum_{j=1}^M|\mathcal{S}_j|}$.

Lemma 2: Let ρ and η be two constants such that $0<\rho<1$ and $0<\eta<1$. If ρ and η satisfy

$$\begin{aligned}[1-(N-1)\eta]\alpha &> \rho\left(1-\frac{1}{N}\right)\\ \frac{1}{\eta} &> N-1\end{aligned}$$

then system (18) is l_2 stable.

Proof: The transfer function for the discrete-time system (18) is denoted by $H(z)$, where $H(z)=F(zI-E)^{-1}$. Since E is symmetric and nonsingular, we can find a nonsingular matrix P, such that $E=P\Lambda P^{-1}$, where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_{\sum_{j=1}^M|\mathcal{S}_j|})$. Then we have

$$\begin{aligned}&\|(zI-E)^{-1}\|_{\mathcal{H}_\infty}\\ =&\|(zI-P\Lambda P^{-1})^{-1}\|_{\mathcal{H}_\infty}\\ =&\|P^{-1}(zI-\Lambda)^{-1}P\|_{\mathcal{H}_\infty}\\ \leqslant&\|(zI-\Lambda)^{-1}\|_{\mathcal{H}_\infty}\\ =&\left\|\mathrm{diag}\left(\frac{1}{z-\lambda_1},\ldots,\frac{1}{z-\lambda_{\sum_{j=1}^M|\mathcal{S}_j|}}\right)\right\|_{\mathcal{H}_\infty}\\ \leqslant&\max_j\left|\frac{1}{1-\lambda_j}\right|,\end{aligned}$$

where $j\in\{1,2\ldots,\sum_{j=1}^M|\mathcal{S}_j|\}$. Since the following inequality holds,

$$\begin{aligned}\|E\|_\infty\leqslant&\max_j\left(1-\gamma\alpha+\gamma\rho(1-\frac{1}{|\mathcal{S}_j|})+(|\mathcal{S}_j|-1)\gamma\alpha\eta\right)\\ \leqslant&1-\gamma\alpha+\gamma\rho(1-\frac{1}{N})+(N-1)\gamma\alpha\eta,\end{aligned}$$

and from the condition that

$$[1-(N-1)\eta]\alpha>\rho\left(1-\frac{1}{N}\right)>0,$$

we obtain $\|E\|_\infty<1$. Because $\max_j|\lambda_j|\leqslant\|E\|_\infty$, we have $\max_j|\lambda_j|<1$, therefore

$$\|(zI-E)^{-1}\|_{\mathcal{H}_\infty} \leqslant \max_j\frac{1}{1-|\lambda_j|}.$$

Then we have

$$\begin{aligned}\|H(z)\|_{\mathcal{H}_\infty} &= \|F(zI-E)^{-1}\|_{\mathcal{H}_\infty}\\ &\leqslant \|F\|_\infty\max_j\frac{1}{1-|\lambda_j|}\\ &\leqslant \frac{M\gamma}{\omega}\max_j\frac{1}{1-|\lambda_j|}\\ &\leqslant \frac{M}{\omega}\frac{1}{\alpha-\rho(1-\frac{1}{N})-(N-1)\alpha\eta}\\ &< \infty,\end{aligned}$$

where the second inequality holds since there are most M links used by user $i\in\mathcal{S}$. Therefore, the system is l_2 stable. ■

The following lemma is a direct result of small gain theory [16]. It shows that when the systems (14) and (18) are connected in a feedback loop (one system's output is another's input, i.e., $u[k]=z[k]$ and $v[k]=y[k]$), then the closed loop system is l_2 stable.

Lemma 3: If conditions in Lemma 2 are satisfied, and the following inequality holds,

$$\left[1-(N-1)\eta-\frac{MN\gamma}{(1-\beta)\omega}\right]\alpha>\rho\left(1-\frac{1}{N}\right),$$

then system (19) is asymptotically stable, where

$$r[k+1] = \begin{bmatrix} B & F \\ D & E \end{bmatrix} r[k], \tag{19}$$

and $\beta = \max_i \beta_i$, for $i = 1, \ldots, N$.

Proof: By small gain theorem, we know that the induced gain for system (19) is less than 1 and hence the absolute value of all eigenvalues of T is less than 1. This implies all the eigenvalues stay inside the unit circle of z-plane. Therefore, system (19) is asymptotically stable. ■

Remark 1: In order to ensure stability, we have to choose γ and ω to satisfy

$$\frac{\gamma}{\omega} < \frac{1-\beta}{MN} < \frac{1}{MN}.$$

Next we will state the main theorem of this paper.

Theorem 1: Consider a network with fixed topology where the routing matrix A is of full rank. Assume the utility function U_i are twice differentiable, strictly increasing, and strictly concave. Assume a fixed setpsize $\gamma > 0$ and penalty parameter $\omega > 0$. Consider $\{\tau_k^i\}_{k=0}^{\infty}$ for each user $i \in \mathcal{S}$.

For each user $i \in \mathcal{S}$, let its data rate, $x_i[k]$, satisfy equation (6). For each link $j \in \mathcal{L}_i$, let the link state, $\phi_j^i[k]$, satisfy equation (7) with sampled link state and data rate from $l \in \mathcal{S}_j^{-i}$ given by (8)-(9). If conditions in Lemma 1-3 are satisfied, then the data rates will asymptotically converge to the optimal solution of the NUM problem.

Proof: We assume that the data rate errors, $\|\tilde{x}[0]\| = r_1[0]$, and the link state errors, $\|\tilde{\phi}[0]\| = r_2[0]$, where $r[k] = (r_1[k], r_2[k])^T$, with $r_1[k] \in \mathbb{R}^N$ and $r_2[k] \in \mathbb{R}^{\sum_{j=1}^{M} |S_j|}$. By comparison principle, we obtain

$$\begin{aligned} \|\tilde{x}[k]\| &\leqslant r_1[k], \\ \|\tilde{\phi}[k]\| &\leqslant r_2[k], \end{aligned}$$

for all $k = 0, 1 \ldots, \infty$. Hence, $\|\tilde{x}[k]\|$ and $\|\tilde{\phi}[k]\|$ are bounded below by zero and bounded above by $r[k]$ which is converging to zero according to Lemma 3. As a result of Pinching Theorem, we know that all the errors go to zero as time goes to infinity. Therefore, the date rates and link utilization computed by (6)-(7) converge to the optimal data rates, x_i^*, and the corresponding link utilization, ϕ_j^*, as defined in (10)-(11). ■

In order to solve the NUM problem, each user $i \in \mathcal{S}$ executes the following algorithm.

Algorithm 1: (1). Parameter Initialization: Let $k = 0$, $T = 0$, and choose suitable parameters γ, ω, α, η, ρ such that they satisfy the conditions in the theorem.

(2). State Initialization: Set the initial user rate x_i^0 so that x_i^0 lies in the feasible set. Set $\hat{x}_i(T) = x_i(T)$ and send $\hat{x}_i(T)$ to users $l \in \mathcal{S}_j^{-i}$, for all $j \in \mathcal{L}_i$. Upon receiving user state from $l \in \mathcal{S}_j^{-i}$, initialize link state

$$\phi_j^i(T) = x_i^0 + \hat{x}_l[T] - c_j$$

Set $\hat{\phi}_j^i(T) = \phi_j^i(T)$ and transmit $\hat{\phi}_j^i(T)$ to users $l \in \mathcal{S}_j^{-i}$, for all $j \in \mathcal{L}_i$.

(3). Update link state:

$$x_i[k+1] = \left[x_i[k] + \gamma \frac{\partial U_i(x_i[k])}{\partial x_i} - \frac{\gamma}{\omega} \left(\sum_{j \in \mathcal{L}_i} \phi_j^i[k] \right)^+ \right]^+$$

$$\begin{aligned} \phi_j^i[k+1] = \phi_j^i[k] &+ \gamma \left(\sum_{l \in \mathcal{S}_j^{-i}} \frac{1}{|S_j|} \left(\hat{\phi}_j^\ell[T] - \phi_j^i[k] \right) \right) \\ &- \gamma\alpha \left[\phi_j^i[k] - \left(x_i[k] + \sum_{l \in \mathcal{S}_j^{-i}} \hat{x}_\ell[T] - c_j \right) \right] \end{aligned}$$

where $\tau_k^i \in [T, T^+)$ and T^+ is the time instant when the following conditions is true:

(a). If $\phi_j^i(k) > 0$ and $\|\hat{\phi}_j^i[k] - \phi_j^i[k]\| > \rho\phi_j^i[k]$, broadcast $\phi_j^i[k]$ to all users $l \in \mathcal{S}_j^{-i}$ and set $\hat{\phi}_j^i[T^+] = \phi_j^i[k]$.

(b). If $\phi_j^i(k) > 0$ and $\|\hat{x}_i[k] - x_i[k]\| > \eta\phi_j^i[k]$, broadcast $x^i[k]$ to all users $l \in \mathcal{S}_j^{-i}$ and set $\hat{x}_i[T^+] = x_i[k]$.

(4). Increment Time: Set $T = T^+$ and $k = k + 1$ and then go to step (3).

Remark 2: Here, we find a nonlinear *event-triggered* condition (15)-(16) that determines when to transmit information. The main idea is that when link state $\phi_j^i < 0$, it says all data rates are in the feasible set, and we can then increase data rates in order to increase network utility. Otherwise, we have to inform other users that the link limit has been reached and require them to adjust their data rates.

VI. Simulation Results

To verify our results, let's consider the following example. This example is illustrated in Figure 2, where there are three users and two links in the network, i.e., $N = 3$, $M = 2$. Link j_1 is used by user i_1 and user i_2 with link capacity 1. Link j_2 is used by user i_1 and user i_3 with link capacity 2. The NUM problem is stated as follows:

$$\begin{aligned} \text{Maximize:} \quad & U(x) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} \\ \text{w.r.t:} \quad & x_i \geq 0, \quad i = 1, 2, 3 \\ \text{subject to:} \quad & x_1 + x_2 \leqslant 1 \\ & x_1 + x_3 \leqslant 2 \end{aligned}$$

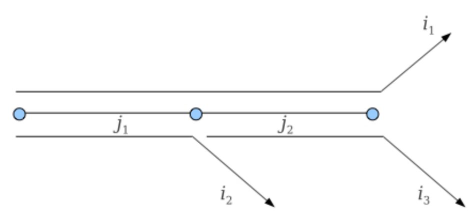

Fig. 2. Network topology

We choose the penalty parameter $\omega = 10^{-3}$ and the feedback gain $\alpha = 1/2$. According to Lemma 2, one threshold condition should satisfy $\eta < 1/2$, so we choose $\eta = 1/6$. Also, another threshold condition should satisfy $\rho < 1/2$, and we choose $\rho = 1/6$. We set the initial data rates for user i_1, i_2, i_3 are 0.4, 0.4 and 1.3, respectively. According to the theorem, we pick a constant stepsize $\gamma = 5 \times 10^{-5}$.

Figure 3 illustrates that user i_1 and i_2 will finally agree on link j_1's utilization and user i_1 and i_3 will finally agree on link j_2's utilization. In the first 3×10^5 iterations, the users have not agreed on the link states. Each user has a local estimate for the link states, based on information available to him. When this local estimate is negative, users' data rates is increasing in order to increase network utility. Once the local estimate is positive, user's data rate will be decreased, as we can see from Figure 3. Consensus on link utilization is achieved after 3×10^5 iterations. When the link states are zero, then the two links are fully utilized.

Figure 4 illustrates that data rates for all three users will converge to a small neighborhood of the optimal (analytical) solution: 0.26865, 0.73135 and 1.73135. The optimal data rate is denoted by the green straight line in each plot. The data rate for each user increases when the user thinks the link is not fully utilized. In other words, when the user's local estimate for the link state is negative, the data rate will increase, as we can see from Figure 3. In the first 3×10^5 iterations, data rates deviate from the optimal value since the three users have not agreed on the link states. Once the users reach an agreement on the link states, the date rates converge to the optimal solution very fast. Figure 5 shows the trajectory for the *augmented Lagrangian function* for the NUM problem. After 3×10^5 iterations, it converges to the minimum value. This implies the aggregate network utility is maximized with feasible data rates.

Figure 6 shows the channel utilization for transmitting data rates (the top plot) and link states (the middle and the bottom plots) is almost 0 after a number of iterations. We can see from the plots that the channel utilization is much higher in the first 3×10^5 iterations, which means users have to communicate more frequently in order to achieve consensus on the link states. Once the users reach an agreement on the link states (after 3×10^5 iterations in Figure 6), they could communicate less frequently just to ensure they stay in the small neighborhood of the optimal solution.

This example shows that *augmented Lagrangian method* under consensus filtering can generate an approximate solution for the NUM problem. By using "event-triggered" communication schemes, channel utilization is greatly reduced. One thing we should mention is that parameter choices here are conservative. For instance, the algorithm still converges with stepsize $\gamma = 5 \times 10^{-4}$.

VII. Final Remarks

Network utility maximization (NUM) problems seek to maximize the aggregate utility that network users receive for transmitting at a given data rate subject to limits on link throughput. Distributed solutions to the NUM problem require direct measurement of link utilization. This however may not be possible in practice. This paper examines the use of consensus filtering for the distributed estimation of link utilization in a distributed NUM algorithm. In particular, we find a nonlinear *event-triggered* condition such that distributed network utility maximization using distributed consensus filter converges to the problem's optimal solution. By using this *event-triggered* idea, message exchange between users in the network can be greatly reduced.

In the future work, we may consider transmission delays and dropouts in the network. We may also consider changing network topology and nondifferentiable utility functions in the NUM problem.

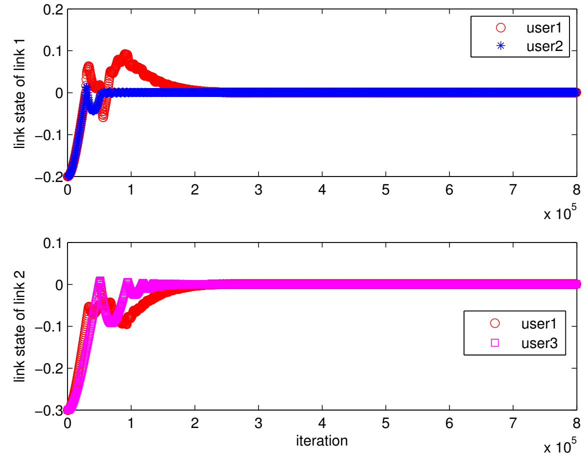

Fig. 3. Trajectories for estimation of link utilization

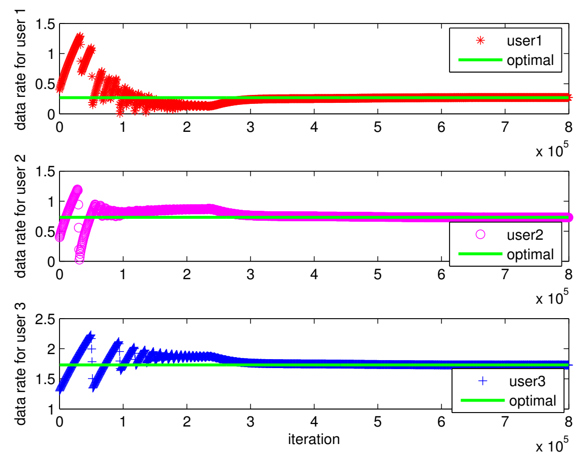

Fig. 4. Trajectories for three users' data rate

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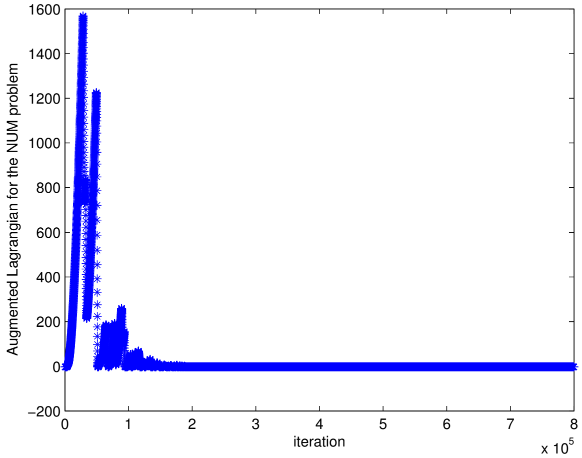

Fig. 5. Augmented Lagrangian function for the NUM problem

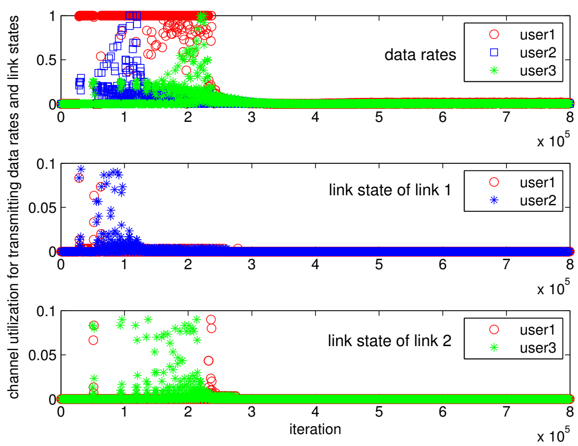

Fig. 6. Channel utilization for transmitting information

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