# Stabilizing bit-rate of perturbed event triggered control systems ⋆

**Lichun Li** * **Xiaofeng Wang** ** **Michael Lemmon** ***

* *Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: lli3@nd.edu)*
** *Department of Mechanical Science Engineering, University of Illinois, Urbana, IL 61801, USA (e-mail: wangx@illinois.edu)*
*** *Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: lemmon@nd.edu)*

**Abstract:** Event triggering is a sampling method where sampling occurs only if data 'novelty' exceeds a threshold. Prior work has demonstrated that event triggered systems have longer average sampling periods than periodic sampled systems with comparable system performance. Based on this fact, it is claimed that event triggered systems make more efficient use of communication resources than periodic sampled systems. If, however, we account for the number of bits in each sample and the maximum acceptable delay of this sample, it is possible that the bit-rates generated by event triggered systems are greater than that of periodic sampled systems. Our prior work in Li et al. [2012] has established, in noise-free cases, the condition under which the stabilizing bit-rates for quantized event-triggered systems converge asymptotically to a finite rate as the system approaches its equilibrium point. In some cases, it was shown that this limiting bit rate may actually be zero. This paper extends that earlier work to quantized event-triggered systems with essentially bounded disturbances. Conditions on triggering event, quantization error and maximum delay are established to assure the input-to-state stability (ISS). The stabilizing bit-rate is, then, shown to be always bounded by a continuous, positive definite, increasing function with respect to the norm of the state. Since the system is ISS, the stabilizing bit-rate can be bounded from above by a function of time. This result provides a guide on how to assign communication resource to the control system. If we set external disturbance to be 0, the results in Li et al. [2012] are recovered.

## 1. INTRODUCTION

State-dependent event-triggered control systems are systems that transmit the system state over the feedback channel when the difference between the current state and last sampled-state exceeds a state-dependent threshold. These systems were originally viewed as embedded computational systems in Tabuada and Wang [2006]. In this case, one was interested in reducing how often the system state was sampled, as a means of reducing processor utilization. The concept of event-triggering can be easily extended to networked control systems and wireless sensor-actuator networks (as explained in Hespanha et al. [2007] and Akyildiz and Kasimoglu [2004], respectively), in which case the sampled state is *transmitted* over a communication channel.

Early interest in event-triggered control was driven by experimental results suggesting that these systems could have longer inter-sampling intervals than comparably performing periodic sampled-data systems (see Sandee et al. [2007], Tabuada [2007], Wang and Lemmon [2009a]). In extending this idea to networked control systems, one might suppose that event-triggering can also reduce the system's usage of the communication channel since it might reduce the frequency at which feedback states are transported across the channel. This extension, however, is complicated by the fact that the communication channel is discrete in nature. Sampled states must first be quantized into a finite number of bits before being transmitted across the channel. Moreover, the transmitted bits must be delivered with a delay that does not de-stabilize the system. So an accurate measure of channel usage is the bit rate as defined by the number of bits per sampled state divided by the acceptable delay in message delivery. It means that the system's *stabilizing bit rate* (i.e., the bit rate assuring closed-loop stability) rather than the inter-transmission interval (i.e. the time between consecutive transmissions of the sampled state) provides a more realistic measure of channel usage in event-triggered networked control systems.

Prior work in state-dependent event-triggered control has used two different techniques to bound the inter-transmission times and acceptable delays. The method used in Tabuada [2007] bounds the minimum inter-transmission as a function of the open-loop system's Lipschitz constant. This work goes on to show that system stability is preserved for sufficiently small delays. More accurate measures of inter-transmission intervals were obtained in Anta and Tabuada [2009] using scaling properties of homogeneous systems. Quantitative bounds on both the

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inter-transmission time and maximum acceptable delay were obtained for self-triggered $\mathcal{L}_2$ systems in Wang and Lemmon [2009b] and networked control systems (see Wang and Lemmon [2011b]). The results in Wang and Lemmon [2009b, 2011b] are significant because they show how the delay and inter-transmission time scale as a function of the last sampled state. These scaling properties led to the characterization in Wang and Lemmon [2011a] of event-triggered systems whose inter-transmission times exhibited *efficient attentiveness* (i.e. the inter-transmission intervals asymptotically approach infinity as the state approaches its equilibrium point). The approach used in this paper builds upon the techniques used in Wang and Lemmon [2011a]. This new paper characterizes how stabilizing bit rates scale as the system state approaches the equilibrium point when there are disturbances.

This paper's bounds on stabilizing bit rates is reminiscent of earlier work on dynamic quantization. Prior work showed that static quantization maps required an infinite number of bits to achieve asymptotic stability (see Delchamps [1990]). With a finite number of bits, the best one can achieve is ultimate boundedness when using static maps (see Wong and Brockett [1999]). This led to the development of dynamic quantization maps in which the quantization map is dynamically varied to track state uncertainty (see Brockett and Liberzon [2000]). For linear systems, one was able to obtain bounds on the bit rate that were necessary and sufficient for stability, assuming a single sample delay (see Tatikonda and Mitter [2004] and Hespanha et al. [2002]). In the case of nonlinear systems, lower bounds on the quantization rate were obtained (see Liberzon and Hespanha [2005]). The quantization maps developed in this paper are dynamic maps, similar to those used in Liberzon and Hespanha [2005]. The different thing is that our work is based on the event triggered sampling with state dependent delay while Liberzon and Hespanha [2005] considered periodic sampling with one period delay. This paper shows that the bit-rate sufficient to guarantee input-to-state stability for a nonlinear system with essentially bounded disturbance is always bounded from above by a continuous, positive definite, and increasing function with respect to the norm of state. It indicates that the farther the state is away from the origin, the higher the bit-rate may be used to stabilize the control system. If we set the disturbance to be 0, the results in Li et al. [2012] are recovered.

## 2. MATHEMATICAL PRELIMINARIES

Throughout this paper the linear space of real $n$-vectors will be denoted as $\mathbb{R}^n$ and the set of non-negative reals will be denoted as $\mathbb{R}^+$. The infinity norm of a vector $x \in \mathbb{R}^n$ will be denoted as $\|x\|$. Given the real-valued function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, we let $x(t)$ denote the value $x$ takes at time $t \in \mathbb{R}^+$. The $\mathcal{L}$ infinity norm of a function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as $\|x\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t\geq 0} \|x(t)\|$. This function is said to be essentially bounded if $\|x\|_{\mathcal{L}_\infty} = M < \infty$ and the linear space of all essentially bounded real-valued functions will be denoted as $\mathcal{L}_\infty$. A subset $\Omega \subset \mathbb{R}^n$ is said to be compact if it is closed and bounded.

We say function $g$ has *non-negative* order, if $\lim_{s\to 0} g(s) < \infty$. With this definition, we have the following two lemmas.

*Lemma 1.* Let $g : [0, \iota] \to \mathbb{R}^+$ be a continuous, positive definite function with non-negative order for some $\iota \geq 0$. There must exist continuous, positive definite, increasing functions $\underline{h}$ and $\overline{h}$ defined on $[0, \iota]$ such that

$$\underline{h}(s) \leq g(s) \leq \overline{h}(s), \forall s \in [0, \iota],$$
$$\lim_{s\to 0} g(s) = \lim_{s\to 0} \underline{h}(s) = \lim_{s\to 0} \overline{h}(s).$$

**Proof.** See Lemma 4.3 in Khalil and Grizzle [1992]. □

*Lemma 2.* Let $g_i : [0, \iota] \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, positive definite, and strictly increasing function with respect to both arguments with $g_i(0, 0) = 0$ for $i = 1, 2, 3$. Given $\bar{w} \in [0, \infty)$, if $g_1(s, \bar{w}) < s + g(\bar{w})$ for some positive definite function $g$, and $\frac{g_2(s,\bar{w})}{g_3(s,\bar{w})}$ has non-negative order, i.e. $\lim_{s\to 0} \frac{g_2(s,\bar{w})}{g_3(s,\bar{w})} \leq c_1 < \infty$, then

$$\lim_{s\to 0} \frac{g_2(s, \bar{w})}{g_3(|s - g_1(s, \bar{w})|, \bar{w})} \leq c_2 < \infty. \quad (1)$$

Moreover, in the case of $\bar{w} = 0$, if $c_1 = 0$, then $c_2 = 0$.

**Proof.** If $\bar{w} > 0$, we know that $g_2(0, \bar{w}) > 0$ and $g_3(|0 - g_1(0, \bar{w})|, \bar{w}) > 0$. So, (1) is true.

If $\bar{w} = 0$, since $g_1(s, 0) < s$, there exists a constant $\epsilon \in (0, 1)$ such that $g_1(s, 0) \leq \epsilon s$ for all $s \in [0, \iota]$.

$$\lim_{s\to 0} \frac{g_2(s, 0)}{g_3(s - g_1(s, 0), 0)} \leq \lim_{s\to 0} \frac{g_2(s, 0)/g_3(s, 0)}{g_3((1 - \epsilon)s, 0)/g_3(s, 0)} \leq c_2 < \infty.$$

If $c_1 = 0$, then $c_2 = 0$. □

A given real valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(x) > 0$ for all $x \neq 0$. The function $V$ is said to be radially unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$. A function $\alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is class $\mathcal{KL}$ if $\beta(\cdot, t)$ is class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Let $\Omega$ be a closed and bounded subset of $\mathbb{R}^n$. We say $f(\cdot) : \Omega \to \mathbb{R}^n$ is Lipschitz on $\Omega$ if for any $x, y \in \Omega$, we know there exists a constant $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Consider a system whose state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the initial value problem,

$$\dot{x}(t) = f(x(t), w(t)), \quad x(0) = x_0 \quad (2)$$

where $w(\cdot) : [0, \infty) \to \mathbb{R}^m$ is an essentially bounded real function.

Let $x = 0$ be an equilibrium point for (2) and $\Upsilon \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \Upsilon \to \mathbb{R}$ be a continuously differentiable function such that

$$\underline{\alpha}(\|x\|) \leq V \leq \overline{\alpha}(\|x\|), \quad (3)$$
$$\frac{\partial V}{\partial x} f(x, w) \leq -\alpha(\|x\|) + \gamma(\|w\|), \quad (4)$$

for all $(x, w) \in \Upsilon \times \mathbb{R}^m$, where $\underline{\alpha}$, $\overline{\alpha}$ are class $\mathcal{K}_\infty$ functions, and $\alpha$, $\gamma$ are class $\mathcal{K}$ functions, then the system (2) is input-to-state stable (ISS). The function $V$ is called *ISS-Lyapunov* function.

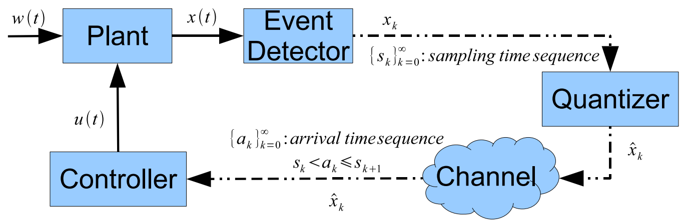


Fig. 1. Event-triggered control system with quantization

## 3. PROBLEM STATEMENT

The system under study is a networked event-triggered control system with quantization. Figure 1 is a block diagram showing the components of this system.

The *plant*'s state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is an absolutely continuous function satisfying the initial value problem,

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad x(0) = x_0 \tag{5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ is locally Lipschitz in $x$, $u$ and $w$ with $f(0,0,0) = 0$.The control signal $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^m$ is generated by the *controller* in figure 1. The disturbance $w(\cdot) : \mathbb{R}^+ \to \mathbb{R}^q$ is an $L_\infty$ disturbance with $\|w\|_{L_\infty} = \bar{w}$. The vector $x_0$ is the plant's initial condition.

The system state, $x(t)$, at time $t$ is measured by the *event detector*. The event detector decides when to hand over the system state to the *quantizer*. The sequence of *sampling times* is denoted as $\{s_k\}_{k=0}^{\infty}$. For notational convenience, the $k$th consecutively sampled state $x(s_k)$ will be denoted as $x_k$. The $k$th *inter-sampling* interval is defined as $T_k = s_{k+1} - s_k$.

Upon receiving the sampled state, $x_k$, the *quantizer* converts this real vector into a finite bit representation. This quantized state is denoted as $\hat{x}_k \in \mathbb{R}^n$. The finite nature of the representation is modeled as a *quantization error*

$$\bar{e}_q(\|x_k\|, \bar{w}) \geq \|x_k - \hat{x}_k\| \tag{6}$$

where $\bar{e}_q(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to both arguments and satisfying $\bar{e}_q(0,0) = 0$.

We define the *gap* between the current state and quantized state as $e_k(t) = x(t) - \hat{x}_k$. We assume that quantization is done instantaneously and that the quantizer transmits the quantized sampled state, $\hat{x}_k$, across the *channel*. The transmission times are therefore equivalent to the sampling times generated by the event trigger. The rest of the paper uses the terms transmission and sampling in an interchangeable way. The sampling times $\{s_k\}$ are generated by the event trigger so that the gap is always less than a state-dependent *threshold function*

$$\|e_k(t)\| \leq \theta(\|\hat{x}_k\|, \bar{w}) \tag{7}$$

for all $t \in (s_k, s_{k+1}]$ where $k = 0, 1, \ldots, \infty$. The function $\theta(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to both arguments and satisfying $\theta(0,0) = 0$.

We assume that the quantized state, $\hat{x}_k$, is always successfully delivered to the controller. The channel, however, is assumed to introduce a finite delay into the delivery time. In particular, the arrival time of the $k$th sampled state $\hat{x}_k$ at the controller is denoted as $a_k \in \mathbb{R}^+$. This time is strictly greater than $s_k$. The delay of the $k$th message is $D_k = a_k - s_k$. We need to assume some orderliness to the transmission and delivery of such messages. In particular, we require that the transmission times, $s_k$, and arrival times, $a_k$, satisfy the following order $s_k < a_k \leq s_{k+1}$ for $k = 0, 1, \ldots, \infty$. Such a sequence of transmissions and arrivals will be said to be *admissible*.

Upon the arrival of the $k$th quantized state, $\hat{x}_k$, at the controller, a control input is computed and then held until the next quantized state is received. In other words, the control signal takes the form

$$u(t) = u_k = K(\hat{x}_k) \tag{8}$$

for $t \in [a_k, a_{k+1})$. The function $K(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz, and satisfies $K(0) = 0$. As has been done in Tabuada [2007], this paper assumes that $K$ is chosen so the system

$$\dot{x}(t) = f(x(t), K(x(t) + e(t)), w(t)), \tag{9}$$

is input-to-state stable with respect to the signals $e, w \in \mathcal{L}_\infty$. This means, of course, that there exists a function $V(\cdot) : \Upsilon \to \mathbb{R}^+$ satisfying (3) and

$$\frac{\partial V}{\partial x} f(x, w) \leq -\alpha(\|x\|) + \gamma_1(\|e\|) + \gamma_2(\|w\|), \tag{10}$$

for all $x \in \Upsilon$ where $\Upsilon \subseteq \mathbb{R}^n$ is a domain containing the origin, $\alpha$, $\gamma_1$ and $\gamma_2$ are class $\mathcal{K}$ functions. Note that this can be a very restrictive assumption since such $K$ may not always exist (see Angeli et al. [2000]).

## 4. INPUT-TO-STATE STABILITY

This section characterizes a threshold function, a quantization error function and a maximum delay such that the event-triggered system described in section 3 is ISS.

In our system, the error signal $e$ in (10) is defined as

$$e(t) = e_k, \forall t \in [a_k, a_{k+1}), \forall k = 0, 1, \cdots, \infty,$$

and equation (10), then, takes the form of

$$\frac{\partial V}{\partial x} f(x, w) \leq -\alpha(\|x\|) + \gamma_1(\|e_k\|) + \gamma_2(\bar{w}), \tag{11}$$

for all $t \in [a_k, a_{k+1})$, all $k = 0, 1, \cdots, \infty$ and all $x \in \Upsilon$. In the case of no delay, if we set our triggering event (7) to be $\|e_k(t)\| \leq \theta(\|\hat{x}_k\|, \bar{w}) \leq \xi(\|x\|, \bar{w})$ where

$$\xi(\|x\|, \bar{w}) = \gamma_1^{-1}(\varsigma\alpha(\|x\|) + \gamma_3(\bar{w})), \tag{12}$$

for some $\varsigma \in (0,1)$, and some class $\mathcal{K}$ function $\gamma_3$, then equation (11) is changed to

$$\frac{\partial V}{\partial x} f(x, w) \leq -(1-\varsigma)\alpha(\|x\|) + \gamma_2(\bar{w}) + \gamma_3(\bar{w}), \tag{13}$$

for all $t \in [a_k, a_{k+1})$, all $k = 0, 1, \cdots, \infty$ and all $x \in \Upsilon$, and there must exist some class $\mathcal{KL}$ function $\beta$ and class $\mathcal{K}$ function $\gamma$ such that

$$\|x(t)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\bar{w})\}, \forall t \geq 0.$$

Extending the idea described above to the case when there is delay, we show that the system is ISS with our designed threshold function $\theta$, quantization error $\bar{e}_q$ and maximum delay in theorem 3. Before stating the theorem, we introduce some conventions that will be used in the rest of this paper.

With the event triggered control system described in section 3, the plant for this system is characterized by the function $f$ on the right hand side of equation (5). We assume that this function is Lipschitz on *compacts*. In particular, this means if we let $\Omega_k$ be a compact set containing $\hat{x}_k$ and all possible trajectories of $x(t)$ for any $t \in [a_k, a_{k+1})$, then

$$\|f(x, K(\hat{x}_k), w)\| \le \psi(\hat{x}_k, K(\hat{x}_k), \bar{w}) + L_{\Omega_k}\|e_k\| \quad (14)$$

where $\psi_k(\hat{x}_k, K(\hat{x}_k), \bar{w}) = \|f(\hat{x}_k, K(\hat{x}_k), 0)\| + \bar{L}_{\Omega_k}\bar{w}$,and $e_k(t) = x(t) - \hat{x}_k$ is the gap function defined earlier. We can think of $L_{\Omega_k}$ and $\bar{L}_{\Omega_k}$ as the Lipschitz constants with respect to $x$ and $w$ over compact set $\Omega_k$.

Define a ball set $B_1 = \{x : \|x\| \le \max\{\beta(\|x_0\|, 0), \gamma(\bar{w})\}\}$. Since $\alpha$ is a continuous class $\mathcal{K}$ function, it is easy to show that there exists a function $g(\bar{w})$ such that

$$\alpha(\|x\| + g(\bar{w})) \le \alpha(\|x\|) + \frac{\gamma_3(\bar{w})}{\varsigma}, \forall x \in B_1.$$

Let $\xi'(s) = \gamma_1^{-1}(\varsigma\alpha(s))$. It's easy to see that $\xi'(\|x\| + g(\bar{w})) \le \xi(\|x\|, \bar{w})$ for all $x \in B_1$. Define $\underline{\xi}(\|\hat{x}_k\|, \bar{w})$ as

$$\begin{aligned}\underline{\xi}(s, \bar{w}) = &\sup\{\epsilon : \epsilon \le \min\{\xi'(s + g(\bar{w}) - \epsilon), s + g(\bar{w})\},\\ &\forall s \in [0, \eta]\}, \qquad (15)\end{aligned}$$

where $\eta = 2\max\{\beta(\|x_0\|, 0), \gamma(\bar{w})\} + g(\bar{w})$. With $\eta$ as the radius, we define another ball set $B_2 = \{x : \|x\| \le \eta\}$.

With these preliminaries we now state the main theorem of this section.

*Theorem 3.* Consider the system described in section 3. Assume that the threshold function $\theta$ and quantization error $\bar{e}_q$ satisfy

$$\theta(\|\hat{x}_k\|, \bar{w}) < \underline{\xi}(\|\hat{x}_k\|, \bar{w}), \quad (16)$$

$$\bar{e}_q(\|x_k\|, \bar{w}) < \min\{\theta(\|\hat{x}_k\|, \bar{w}), \|x_k\| + g(\bar{w})\}, \quad (17)$$

for all $x_k \in B_1$ and all $\hat{x}_k \in B_2$. The closed-loop event triggered system is ISS, if the actual channel delay $D_k = a_k - s_k$ is always no greater than $\Delta_k = \min\{\underline{T}_k, \hat{D}_k\}$ for all $k = 0, 1, \ldots, \infty$, where

$$\begin{aligned}\underline{T}_k = &\frac{1}{L_{\Omega_k}}\left(\ln\left(1 + \frac{L_{\Omega_k}\theta(\|\hat{x}_k\|, \bar{w})}{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1}, \bar{w})}\right)\right.\\ &\left.- \ln\left(1 + \frac{L_{\Omega_k}\bar{e}_q(\|x_k\|, \bar{w})}{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1}, \bar{w})}\right)\right) \qquad (18)\end{aligned}$$

$$\begin{aligned}\hat{D}_k = &\frac{1}{L_{\Omega_{k-1}}}\left(\ln\left(1 + L_{\Omega_{k-1}}\frac{\underline{\xi}(\|\hat{x}_{k-1}\|, \bar{w})}{\psi(\hat{x}_{k-1}, K(\hat{x}_{k-1}), \bar{w})}\right)\right.\\ &\left.- \ln\left(1 + L_{\Omega_{k-1}}\frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\psi(\hat{x}_{k-1}, K(\hat{x}_{k-1}), \bar{w})}\right)\right), \qquad (19)\end{aligned}$$

$\Psi_{k,k-1}$ takes the form of

$$\begin{aligned}&\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1}, \bar{w})\\ =&|\psi(\hat{x}_k, K(\hat{x}_k), \bar{w}) - \psi(\hat{x}_k, K(\hat{x}_{k-1}), \bar{w})|\\ &+ \psi(\hat{x}_k, K(\hat{x}_{k-1}), \bar{w}),\end{aligned}$$

and $\Omega_k$ takes the form of

$$\Omega_k = \{x \in \mathbb{R}^n : \|x\| \le \|\hat{x}_k\| + \underline{\xi}(\|\hat{x}_k\|, \bar{w})\}.$$

**Proof.** We first show that $\hat{x}_k$ is always bounded. Once this is shown, it's easy to see that $\Omega_k$ is always bounded, and hence we have finite $L_{\Omega_k}$ for all $k = 0, 1, \cdots, \infty$. Based on the fact that $L_{\Omega_k}$ is always finite, we, then, prove that equation (13) holds for all $t \ge 0$, and the ISS can be shown.

Let's start from showing that $x_k$ is always bounded. If this is true, then $\hat{x}_k$ must be bounded, too. Now, let's assume that there exists an integer $k'$ such that

$$\|x_k\| \le \max\{\beta(\|x_0\|, s_k), \gamma(\bar{w})\}, \forall k = 0, 1, \cdots, k' \quad (20)$$

$$\|x_{k'+1}\| > \max\{\beta(\|x_0\|, s_{k'+1}), \gamma(\bar{w})\}. \quad (21)$$

In the following, we will show that equation (21) and (20) contradict with each other, and hence demonstrate that $x_k$ is always bounded.

The first step is to show that $\|e_k(t)\| < \underline{\xi}(\|\hat{x}_k\|, \bar{w})$ for all $t \in [s_k, s_{k+1}]$ and all $k = 0, 1, \cdots, k'$. We know that for all $t \in [s_k, s_{k+1}]$, $\|e_k(t)\| \le \theta(\|\hat{x}_k\|, \bar{w})$. Since $x_k \in B_1$, and (17) holds, $\|\hat{x}_k\| \le \|x_k\| + \bar{e}_q(\|x_k\|, \bar{w}) \le 2\|x_k\| + g(\bar{w})$. So, $\hat{x}_k \in B_2$. From (16), we know that $\|e_k(t)\| \le \theta(\|\hat{x}_k\|, \bar{w}) < \underline{\xi}(\|\hat{x}_k\|, \bar{w})$.

The second step is to show that $\|e_k(t)\| < \underline{\xi}(\|\hat{x}_k\|, \bar{w})$ for all $t \in [s_{k+1}, a_{k+1})$ and all $k = 0, 1, \cdots, k'$. To do so, we first need to show that during interval $[s_k, a_k)$ for $k = 0, 1, \cdots, k'$, no sampling occur. In other words, $D_k \le T_k$ for $k = 0, 1, \cdots, k'$. From (14) the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \le \|\dot{e}_k(t)\| \le \psi(\hat{x}_k, K(\hat{x}_{k-1}), \bar{w}) + L_{\Omega_k}\|e_k(t)\|,$$

for all $t \in [s_k, a_k)$. According to the comparison principle, we have

$$\begin{aligned}\|e_k(a_k)\| \le &\frac{\psi(\hat{x}_k, K(\hat{x}_{k-1}), \bar{w})}{L_{\Omega_k}}(e^{L_{\Omega_k}D_k} - 1)\\ &+ \bar{e}_q(\|x_k\|, \bar{w})e^{L_{\Omega_k}D_k}.\end{aligned}$$

For interval $[a_k, a_{k+1})$, the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \le \|\dot{e}_k(t)\| \le \psi(\hat{x}_k, K(\hat{x}_k), \bar{w}) + L_{\Omega_k}\|e_k(t)\|.$$

With $\|e_k(a_k)\|$ as the initial condition, we have

$$\begin{aligned}\|e_k(s_{k+1})\| \le &\frac{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1}, \bar{w})}{L_{\Omega_k}}(e^{L_{\Omega_k}T_k} - 1)\\ &+ \bar{e}_q(\|x_k\|, \bar{w})e^{L_{\Omega_k}T_k}. \qquad (22)\end{aligned}$$

From $\|e_k(s_{k+1})\| = \theta(\|\hat{x}_k\|, \bar{w})$, it can be derived that $T_k \ge \underline{T}_k$. Since $D_k \le \Delta_k \le \underline{T}_k$, we see that $D_k \le T_k$, i.e. the sequence of transmissions and arrivals are admissible.

To complete the second step, we assume that there exist $D' < D_{k+1} \le \Delta_{k+1}$, such that $\|e_k(s_{k+1} + D')\| > \underline{\xi}(\|\hat{x}_k\|, \bar{w})$. With the same technique as above, we show

$$\begin{aligned}\|e_k(s_{k+1} + D')\| \le &\frac{\psi(\hat{x}_k, K(\hat{x}_k), \bar{w})}{L_{\Omega_k}}(e^{L_{\Omega_k}D'} - 1)\\ &+ \theta(\|\hat{x}_k\|, \bar{w})e^{L_{\Omega_k}D'}.\end{aligned}$$

Since $\|e_k(s_{k+1} + D')\| > \underline{\xi}(\|\hat{x}_k\|, \bar{w})$, we can derive that $D' \ge \hat{D}_{k+1} > \Delta_{k+1}$, which contradicts the assertion that $D'_{k+1} < \Delta_{k+1}$. So, we conclude that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|, \bar{w})$ for all $t \in [s_{k+1}, a_{k+1})$ and all $k = 0, 1, \cdots, k'$.

By now, we have shown that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|, \bar{w})$ for all $t \in [s_k, a_{k+1})$, and all $k = 0, 1, \cdots, k'$. The next, we will show that $\underline{\xi}(\|\hat{x}_k\|, \bar{w}) < \xi(\|x(t)\|, \bar{w})$ for all $t \in [0, a_{k'+1})$. From (15), we can derive that $\underline{\xi}(\|\hat{x}_k\|, \bar{w}) \le \xi(\|x\|, \bar{w})$. for all $t \in [s_k, a_{k+1})$, and all $k = 0, 1, \cdots, k'$. Now, we know that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|, \bar{w}) \le \xi(\|\hat{x}_k\|, \bar{w})$ for all $t \in [s_k, a_{k+1})$ and all $k = 0, 1, \cdots, k'$. Together with (11) and (12), it is easy to show that (13) holds for all $t \in [s_k, a_{k+1})$ and all $k = 0, 1, \cdots, k'$, i.e. there exists an ISS-Lyapunov function $V$ for all $t \in [0, a_{k'+1})$.

Therefore, $\|x(t)\| \le \max\{\beta(\|x_0\|, t), \gamma(\bar{w})\}$ for all $t \in [0, a_{k'+1})$, and hence $\|x_{k'+1}\| \le \max\{\beta(\|x_0\|, s_{k'+1}), \gamma(\bar{w})\}$. It contradicts our assumption in (21), which demonstrates that $\|x_k\| \le \max\{\beta(\|x_0\|, s_k), \gamma(\bar{w})\}$, for all $k =$

$0, 1, \cdots, \infty$. If $\|x_k\|$ is bounded, $\hat{x}_k$ is also bounded, and hence so is $\Omega_k$. From the fact that the system is locally Lipschitz, we can conclude that $L_{\Omega_k}$ is always bounded.

Since we have proven $L_{\Omega_k}$ is always bounded, we can follow the same idea to show that (13) is true for all $t \geq 0$, and hence the ISS stability of the system is shown. □

*Remark 4.* Inequality (16) and (17) assure that $\underline{T}_k$ and $\hat{D}_k$ are positive, and hence guarantee that $T_k$ and $\Delta_k$ are always positive.

*Remark 5.* If we set the quantization error, network delay to be 0, then the minimum inter-sampling intervals $\underline{T}_k$ in our work and the work in Wang and Lemmon [2011a] are the same. If we set the quantization error to be 0, the minimum inter-sampling intervals and the maximum delays in our work and Wang's work in Wang and Lemmon [2009b] are in similar forms.

## 5. STABILIZING BIT-RATES

A stabilizing bit-rate is the bit-rate which is sufficient to guarantee the ISS stability of the system. This section shows that the stabilizing bit-rates are always bounded from above by a continuous increasing function with respect to the norm of the state. Since $\|x\|$ is always bounded from above by $\max\{\hat{\beta}(\|x_0\|, t), \hat{\alpha}(\bar{w})\}$ for some class $\mathcal{KL}$ function $\hat{\beta}$ and class $\mathcal{K}$ function $\hat{\alpha}$, there must exist a class $\mathcal{KL}$ function $\tilde{\beta}$ and a class $\mathcal{K}$ function $\tilde{\alpha}$ such that the stabilizing bit-rate is bounded from above by $\max\{\tilde{\beta}(\|x_0\|, t), \tilde{\alpha}(\bar{w})\}$. This gives us a guide on how to assign the communication bandwidth to the control system ahead of time.

Before talking about the stabilizing bit-rate, we first give a quantization map for the system given quantization error $\bar{e}_q(\|x_k\|, \bar{w})$. Since at sampling time $s_k$, both sensor and controller understand that $\|e_{k-1}(s_k)\| = \theta(\|\hat{x}_{k-1}\|, \bar{w})$, we only need to quantize the surface of the box $\{e_{k-1} : \|e_{k-1}(s_k)\| \leq \theta(\|\hat{x}_{k-1}\|, \bar{w})\}$. First, we use $\lceil \log_2 2n \rceil$ bits to identify which side $e_{k-1}$ lies on, and then we cut this side uniformly into $\left\lceil \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|x_k\|, \bar{w})} \right\rceil^{n-1}$ parts. If $e_{k-1}(s_k)$ lies on one of the small parts, then $e_{k-1}(s_k)$ will be quantized as the center of this part, and $\hat{x}_k$ can be calculated as the sum of $\hat{x}_{k-1}$ and the quantized $e_{k-1}(s_k)$. In all, the number of bits used at the $k$th sampling is

$$N_k = \lceil \log_2 2n \rceil + (n-1) \left\lceil \log_2 \left\lceil \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|x_k\|, \bar{w})} \right\rceil \right\rceil \quad (23)$$

We should notice that the number of bits transmitted at each time can be different, since we fix the quantization error instead of the number of bits.

From theorem 3, we know that as long as the network delay $D_k$ is no greater than the maximum delay $\Delta_k$, the closed loop system is ISS. If we define $\underline{r}_k$ as

$$\underline{r}_k = \frac{N_k}{\Delta_k}, \quad (24)$$

$\underline{r}_k$ is sufficient to stabilize the system, and we call it the stabilizing bit-rate of the $k$th transmission.

For convenience of the rest of this paper, we define $\phi_c(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$\psi(\hat{x}_k, K(\hat{x}_k), 0) \leq \phi_c(\|\hat{x}_k\|) \quad (25)$$

and $\phi_u(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$u_k = \|K(\hat{x}_k)\| \leq \phi_u(\|\hat{x}_k\|). \quad (26)$$

With these preliminaries, we show that the stabilizing bit-rate is bounded from above by an increasing function with respect to (w.r.t.) the norm of the state. This is done by showing that $N_k$ is bounded from above by an increasing function, and that $\Delta_k$ is bounded from below by decreasing functions.

### *5.1 Increasing upper bound on $N_k$ w.r.t. $\|x_{k-1}\|$*

*Theorem 6.* If all the conditions in theorem 3 hold, and

$$\lim_{s \to 0} \frac{\theta(s, \bar{w})}{\bar{e}_q(s, \bar{w})} < \infty, \quad (27)$$

then $N_k$ is bounded from above by an increasing function $\overline{N}_k(\|\hat{x}_{k-1}\|)$ with respect to $\|\hat{x}_{k-1}\|$, i.e. $N_k \leq \overline{N}_k(\|\hat{x}_{k-1}\|)$, where

$$\overline{N}_k(s) = \lceil \log_2 2n \rceil + (n-1) \lceil \log_2 \lceil h_1(s, \bar{w}) \rceil \rceil, \quad (28)$$

and $h_1(s, \bar{w})$ is a continuous increasing function satisfying

$$\frac{\theta(s, \bar{w})}{\bar{e}_q(|s - \theta(s, \bar{w})|, \bar{w})} \leq h_1(s, \bar{w}), \quad (29)$$

$$\lim_{s \to 0} \frac{\theta(s, \bar{w})}{\bar{e}_q(|s - \theta(s, \bar{w})|, \bar{w})} = \lim_{s \to 0} h_1(s, \bar{w}). \quad (30)$$

**Proof.** First, we show that there exists a continuous, positive definite, and increasing function satisfying (29) and (30). We notice that

$$\frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|x_k\|, \bar{w})} \leq \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(|\|\hat{x}_{k-1}\| - \theta(\|\hat{x}_{k-1}\|, \bar{w})|, \bar{w})}. \quad (31)$$

Since both $\theta$ and $\bar{e}_q$ are continuous, positive definite, and strictly increasing w.r.t both arguments, and $\theta(s, \bar{w}) < s + g(\bar{w})$ (from (16) and (15)), according to lemma 2, we have

$$\lim_{s \to 0} \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(|\|\hat{x}_{k-1}\| - \theta(\|\hat{x}_{k-1}\|, \bar{w})|, \bar{w})} < \infty. \quad (32)$$

Besides, we know that both $\theta$ and $\bar{e}_q$ are continuous and positive definite. According to lemma 1, there must exist a continuous, positive definite, and increasing function satisfying (29) and (30).

From (31) and (29), it is easy to see that $N_k \leq \overline{N}_k(\|\hat{x}_{k-1}\|)$. Since $h_1(\|\hat{x}_{k-1}\|, \bar{w})$ is an increasing function w.r.t $\|\hat{x}_{k-1}\|$, we can see that $\overline{N}_k(\|\hat{x}_{k-1}\|)$ is also increasing w.r.t. $\|\hat{x}_{k-1}\|$. □

Theorem 6 indicates that we may need more bits to quantize the state information if the last sampled state goes far away from the origin. Next, we will show that the maximum delay has a decreasing lower bound w.r.t. the norm of the state.

### *5.2 Decreasing lower bound on $\Delta_k$ w.r.t. $\|\hat{x}_{k-1}\|$*

We show that there exists a decreasing lower bound w.r.t. $\|\hat{x}_{k-1}\|$ on $\Delta_k$ by showing that both $\hat{D}_k$ and $\underline{T}_k$ have decreasing lower bounds w.r.t. $\|\hat{x}_{k-1}\|$.

*Lemma 7.* If all the conditions in theorem 3 hold and

$$\lim_{s \to 0} \frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}} \bar{w}}{\theta(s, \bar{w})} < \infty, \quad (33)$$

then $\hat{D}_k$ is bounded from below by a positive definite, decreasing function $\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|)$ w.t.t. $\|\hat{x}_{k-1}\|$, i.e. $\hat{D}_k \geq \underline{\hat{D}}_k$ where

$$\underline{\hat{D}}_k(s) = \frac{1}{L_{\Omega_{k-1}}} \ln\left(1 + L_{\Omega_{k-1}} \frac{1-c_1}{h_2(s,\bar{w}) + L_{\Omega_{k-1}} c_1}\right) \quad (34)$$

$c_1 \in (0,1)$ satisfies

$$\frac{\theta(s,\bar{w})}{\underline{\xi}(s,\bar{w})} \leq c_1, \forall s \in B_2, \quad (35)$$

and $h_2(s,\bar{w})$ is a continuous, positive definite, increasing function satisfying

$$\frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(s,\bar{w})} \leq h_2(s,\bar{w}), \quad (36)$$

$$\lim_{s\to 0} \frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(s,\bar{w})} = \lim_{s\to 0} h_2(s,\bar{w}). \quad (37)$$

**Proof.** We first show that there exist $c_1$ satisfying (35) and $h_2(s,\bar{w})$ satisfying (36) and (37). From (16), we have $0 \leq \frac{\theta(s,\bar{w})}{\underline{\xi}(s,\bar{w})} < 1$. So, we can always find a constant $c_1 \in (0,1)$ such that (35) holds. From (16) and (33), we know that

$$\frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(s,\bar{w})} \leq \frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\theta(s,\bar{w})} < \infty.$$

Since both $\phi_c$ and $\underline{\xi}$ are continuous and positive definite, according to lemma 1, there exists a continuous, positive definite and increasing function $h_2$ satisfying (36) and (37).

We, then, show that $\hat{D}_k \geq \underline{\hat{D}}_k$. From (25), we know that

$$\hat{D}_k \geq \frac{1}{L_{\Omega_{k-1}}} \ln \Bigg(1 + L_{\Omega_{k-1}} \frac{1 - \frac{\theta(\|\hat{x}_{k-1}\|,\bar{w})}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}}{\frac{\phi_c(\|\hat{x}_{k-1}\|) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})} + L_{\Omega_{k-1}} \frac{\theta(\|\hat{x}_{k-1}\|,\bar{w})}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}}\Bigg).$$

The inequality holds because of (25). Since (35) and (36) hold, we show that $\hat{D}_k \geq \underline{\hat{D}}_k$.

We notice that $\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|)$ is a decreasing function w.r.t. $L_{\Omega_{k-1}}$ which is a increasing function w.r.t. $\hat{x}_{k-1}$. Together with the fact that $h_2(s,\bar{w})$ is increasing w.r.t. $s$, we conclude that $\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|)$ is decreasing w.r.t. $\|\hat{x}_{k-1}\|$. □

The next lemma shows that there exists a decreasing lower bound on $\underline{T}_k$ w.r.t. $\|\hat{x}_{k-1}\|$.

*Lemma 8.* If all the conditions in theorem 3 and equation (33) are satisfied, and

$$\lim_{s\to 0} \frac{\phi_u(s)}{\theta(s,\bar{w})} < \infty, \quad (38)$$

then $\underline{T}_k$ is bounded from below by a continuous, positive definite, increasing function $\underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$, i.e. $\underline{T}_k \geq \underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$, where

$$\underline{\hat{T}}_k(s) = \frac{1}{L_{\Omega'_k}} \ln\left(1 + L_{\Omega'_k} \frac{1-c_2}{h(s,\bar{w}) + L_{\Omega'_k} c_2}\right) \quad (39)$$

with $h(s,\bar{w}) = 3h_3(s',\bar{w}) + 2L'_{\Omega'_k}(h_4(s',\bar{w}) + h_5(s,\bar{w}))$, $L'_{\Omega'_k}$ is the Lipschitz constant of $f$ with respect to $u$, and $s' = s + \theta(s,\bar{w}) + \bar{e}_q(s + \theta(s,\bar{w}),\bar{w})$ $c_2 \in (0,1)$ is a constant satisfying

$$\frac{\bar{e}_q(\|x_k\|,\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})} \leq c_2, \forall x_k \in B_1, \quad (40)$$

$h_3(s,\bar{w})$ is a continuous, positive definite, increasing function w.r.t. $s$ satisfying

$$\frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\theta(s,\bar{w})} \leq h_3(s,\bar{w}), \quad (41)$$

$$\lim_{s\to 0} \frac{\phi_c(s) + \overline{L}_{\Omega_{k-1}}\bar{w}}{\theta(s,\bar{w})} = \lim_{s\to 0} h_3(s,\bar{w}), \quad (42)$$

$h_4(s,\bar{w})$ is a continuous, positive definite, increasing function w.r.t. $s$ satisfying

$$\frac{\phi_u(s)}{\theta(s,\bar{w})} \leq h_4(s,\bar{w}), \quad (43)$$

$$\lim_{s\to 0} \frac{\phi_u(s)}{\theta(s,\bar{w})} = \lim_{s\to 0} h_4(s,\bar{w}), \quad (44)$$

$h_5(s,\bar{w})$ is a continuous, positive definite, increasing function w.r.t. $s$ satisfying

$$\frac{\phi_u(s)}{\theta(\underline{h}(|s-\theta(s,\bar{w})|,\bar{w}),\bar{w})} \leq h_5(s,\bar{w}), \quad (45)$$

$$\lim_{s\to 0} \frac{\phi_u(s)}{\theta(\underline{h}(|s-\theta(s,\bar{w})|,\bar{w}),\bar{w})} = \lim_{s\to 0} h_5(s,\bar{w}), \quad (46)$$

$\underline{h}(s,\bar{w})$ is a continuous, positive definite, increasing function w.r.t. $s$ satisfying

$$|s - \bar{e}_q(s,\bar{w})| \leq \underline{h}(s,\bar{w}), \quad (47)$$

$$\lim_{s\to 0} |s - \bar{e}_q(s,\bar{w})| = \lim_{s\to 0} \underline{h}(s,\bar{w}), \quad (48)$$

and

$$\Omega'_k = \{x : \|x\| \leq \|\hat{x}_{k-1}\| + \theta(\|\hat{x}_{k-1}\|,\bar{w}) + \bar{e}_q(\|\hat{x}_{k-1}\| + \theta(\|\hat{x}_{k-1}\|,\bar{w}))\}.$$

**Proof.** We first show that there exist $c_2$, $h_3$, $h_4$, $h_5$ and $\underline{h}$ such that (41)-(48) hold. With the same technique used to prove the existence of $c_1$, we show the existence of $c_2$. With the same technique used to prove the existence of $h_2$, we show the existence of $h_3$, $h_4$ and $\underline{h}$. With the same technique used to prove the existence of $h_1$, we show the existence of $h_5$.

We, then, show that $\underline{T}_k \geq \underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$. From (25) and (40), we know that

$$\underline{T}_k \geq \frac{1}{L_{\Omega_k}} \ln\left(1 + L_{\Omega_k} \frac{1-c_2}{h'(\|x_k\|,\|\hat{x}_k\|,\bar{w})}\right)$$

where

$$h'(\|x_k\|,\|\hat{x}_k\|,\bar{w}) = 3\frac{\phi_c(\|\hat{x}_k\|) + \bar{L}_{\Omega_k}\bar{w}}{\theta(\|\hat{x}_k\|,\bar{w})} + L_{\Omega_k}c_2 + 2L'_{\Omega_k}\frac{\phi_u(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|,\bar{w})} + 2L'_{\Omega_k}\frac{\phi_u(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}.$$

From (6), (7) and (47), it is easy to see that

$$\|\hat{x}_k\| \geq \big|\|x_k\| - \bar{e}_q(\|x_k\|,\bar{w})\big| \geq \underline{h}(|\|\hat{x}_{k-1}\| - \theta(\|\hat{x}_{k-1}\|,\bar{w})|,\bar{w}),$$
$$\|\hat{x}_k\| \leq \|\hat{x}_{k-1}\| + \theta(\|\hat{x}_{k-1}\|,\bar{w}) + \bar{e}_q(\|\hat{x}_{k-1}\| + \theta(\|\hat{x}_{k-1}\|,\bar{w}),\bar{w})$$

Together with (41), (43), (45), we show that $h'(\|x_k\|,\|\hat{x}_k\|,\bar{w}) \leq h(\|\hat{x}_{k-1}\|,\bar{w}) + L_{\Omega_k}c_2$, and hence $\underline{T}_k \geq \underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$.

$\underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$ is a decreasing function w.r.t. $\|\hat{x}_{k-1}\|$, since $h(s,\bar{w})$ and $L_{\Omega'_k}$ are increasing w.r.t. $\|\hat{x}_{k-1}\|$. □

With lemma 7 and 8, we now can say that the maximum delay is bounded from below by a continuous, positive definite, and decreasing function.

*Theorem 9.* If all the condition in theorem 3 hold, and equation (33) and (38) are satisfied, then there exists a continuous, positive definite, decreasing function $\underline{\Delta}_k(\|\hat{x}_{k-1}\|)$ which bounds $\Delta_k$ from below, i.e. $\Delta_k \geq \underline{\Delta}_k(\|\hat{x}_{k-1}\|)$, where

$$\underline{\Delta}_k(\|\hat{x}_{k-1}\|) = \min\{\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|), \underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)\}, \qquad (49)$$

and $\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|)$ and $\underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$ are defined in (34) and (39), respectively.

### *5.3 Increasing upper bound on $\underline{r}_k$ w.r.t. $\|\hat{x}_{k-1}\|$*

With theorem 6 and 9, it is easy to find an upper bound on $\underline{r}_k$ which is increasing w.r.t. $\|\hat{x}_{k-1}\|$.

*Theorem 10.* If all the condition in theorem 3 hold, and equation (27), (33) and (38) are satisfied, then there exists a continuous, positive definite, increasing function $\overline{r}_k(\|\hat{x}_{k-1}\|)$ which bounds $\underline{r}_k$ from above, i.e. $\underline{r}_k \leq \overline{r}_k(\|\hat{x}_{k-1}\|)$, where

$$\overline{r}_k(\|\hat{x}_{k-1}\|) = \frac{\overline{N}_k(\|\hat{x}_{k-1}\|)}{\underline{\Delta}_k(\|\hat{x}_{k-1}\|)},$$

where $\overline{N}_k(\|\hat{x}_{k-1}\|)$ and $\underline{\Delta}_k(\|\hat{x}_{k-1}\|)$ are defined in (28) and (49), respectively.

*Remark 11.* Theorem 10 indicates that the further the state is away from the origin, the higher the stabilizing bit-rate will be. Moreover, since there exist some class $\mathcal{KL}$ function $\hat{\beta}$ and class $\mathcal{K}$ function $\hat{\alpha}$ such that $\|\hat{x}_{k-1}\| \leq \max\{\hat{\beta}(\|x_0\|, t), \hat{\alpha}(\overline{w})\}$, we are also able to find how the upper bound on the stabilizing bit-rate, which measures the maximum of the minimum communication resource sufficient to stabilize the system, varies with respect to time. This piece of information gives us a guide on how to assign communication resource to the control system. We should notice that this assignment can be time varying.

*Remark 12.* If there is no disturbance, the results in Li et al. [2012] are recovered. Li et al. [2012] focused on the asymptotic behavior of the stabilizing bit-rate. It's easy to see that if all the conditions in theorem 10 hold, the upper bound on $\underline{r}_k$ is always finite. Moreover, if $\lim_{s\to 0} \frac{\phi_c(s)}{\theta(s)} = 0$, $\lim_{s\to 0} \frac{\phi_u(s)}{\theta(s)} = 0$ and $\lim_{x\to 0} L_{\Omega_k} = 0$, $\underline{r}_k$ goes to 0 when the state approaches the origin.

## 6. SIMULATION RESULTS

This section uses a nonlinear system to demonstrate theorem 3 and 10. We are interested in both cases when the system is disturbed or not.

Now, let's consider a nonlinear system

$$\dot{x}_1 = x_1^3 + x_2^3 + u_1 + w_1$$
$$\dot{x}_2 = -x_1^3 + x_2^3 + u_2 + w_2$$

with $u_1 = -3\hat{x}_1^3$, $u_2 = -3\hat{x}_2^3$, $\overline{w} = 1$ and $x_0 = [1; 1]$. It's easy to see that $\phi_c(s) = 2s^3$, $\phi_u(s) = 2s^3$ and $L = 14\|x\|$. We give the ISS-Lyapunov function as $V = x_1^4 + x_2^4$. It can be shown that (13) is satisfied with $\Upsilon = \{x : \|x\| \leq 1\}$, if we set $\xi(s, \overline{w}) = \left(\frac{0.4s^6 + 20\overline{w}^2}{1216}\right)^{0.25}$. From (15), we have $\underline{\xi}(s, \overline{w}) = 0.2s^{1.5} + 0.1\overline{w}$. From (16) and (17), we choose three pairs of threshold functions and quantization errors. The first pair is $\theta_1(s, \overline{w}) = 0.075s^{1.5} + 0.05\overline{w}$ and $\bar{e}_{q1}(s, \overline{w}) = 0.025s^{1.5} + 0.017\overline{w}$, $\theta_2(s, \overline{w}) = 0.15s^3 + 0.05\overline{w}$ and $\bar{e}_{q2}(s, \overline{w}) = 0.05s^3 + 0.017\overline{w}$.

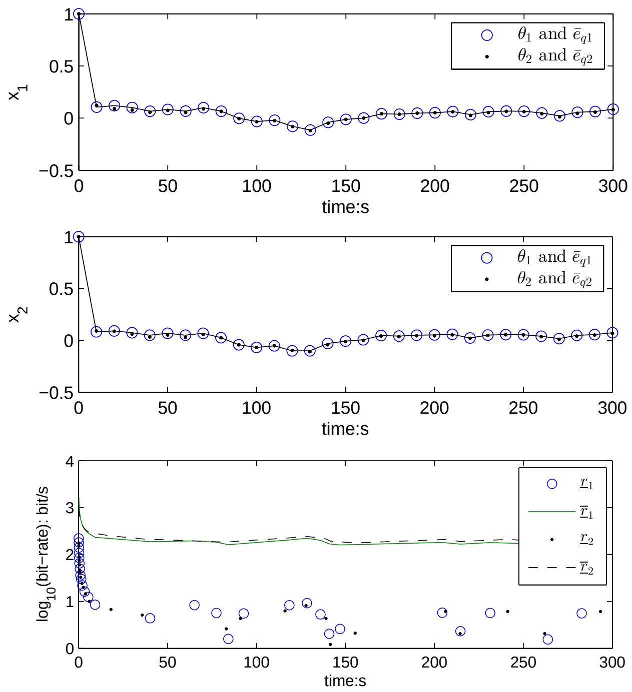


Fig. 2. State trajectories and stabilizing bit-rates with disturbances for the nonlinear system

We ran the system with disturbance for 300 seconds, and always used $\Delta_k$ as the delay in the communication network. The state trajectories and stabilizing bit-rates given by the two pairs of threshold functions and quantization errors are shown in figure 2. The top two plots show the state trajectories given by $\theta_1$ and $\bar{e}_{q1}$ (circles), $\theta_2$ and $\bar{e}_{q2}$ (dots). The $x$-axes of the top two plots indicate time, and the $y$-axes of the top two plots indicate $x_1$ and $x_2$, respectively. We can see that their state trajectories are very close to each other, and stay in a neighborhood of the origin. The bottom plot shows the stabilizing bit-rates of the two pairs with $x$-axis indicating time, and $y$-axis indicating $\log_{10} \underline{r}_k$. The stabilizing bit-rate of the first pair $\underline{r}_1$ and its upper bound $\overline{r}_1$ are expressed by circles and solid line, respectively. The stabilizing bit-rate of the second pair $\underline{r}_2$ and its upper bound $\overline{r}_2$ are expressed by dots and dashed line, respectively. We can see that for both pairs, the stabilizing bit-rates are bounded from above by the upper bounds given by theorem 10. We notice that the stabilizing bit-rates for both pair are close to each other. It is because as $x$ goes to the origin, the constant parts in threshold function and quantization error, which are the same in both pairs, dominate the stabilizing bit-rates.

We, then, ran the system without disturbance for 300 seconds to see whether there are differences between the stabilizing bit-rates of the two pairs of threshold functions and quantization errors. Again, the delay in the communication network is set to be $\Delta_k$. The system trajectories and stabilizing bit-rates are shown in figure 3. The top two plots in this figure give the state trajectories of the

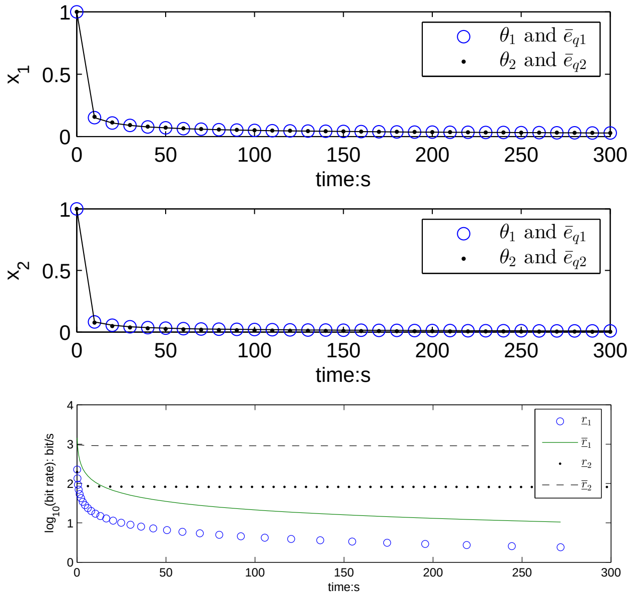


Fig. 3. State trajectories and stabilizing bit-rates without disturbances for the nonlinear system

two pairs of threshold functions and quantization errors (expressed by circles and dots, respectively) with $x$-axes indicating time and $y$-axes indicating state. We can see that they are all asymptotically stable. The bottom plot shows the stabilizing bit-rates of the two pairs of threshold functions and quantization errors with $x$-axis indicating time and $y$-axis indicating $\log_{10} \underline{r}_k$. The stabilizing bit-rate of the first pair $\underline{r}_1$ and its upper bound $\overline{r}_1$ are indicated by circles and solid line, respectively. We see that $\overline{r}_1$ is always above $\underline{r}_1$, and is about 5 times greater than $\underline{r}_1$ in the worst case. Moreover, as $x$ goes to the origin, both $\underline{r}_1$ and $\overline{r}_1$ converges to 0, which is expected by our results in Li et al. [2012]. The upper bound on the stabilizing bit-rate $\overline{r}_2$ given by the second pair (dashed line) is always above the stabilizing bit-rate $\underline{r}_2$. Each of them converges to a constant in less than 10 seconds. The constant that $\overline{r}_2$ converges to is about 10 times as the constant that $\underline{r}_2$ converges to.

## 7. FUTURE WORK

The results in this paper can be used as a foundation to study the scheduling problem in networked control systems. This paper provides the maximum delay $\Delta_k$ and the stabilizing bit-rate $\underline{r}_k$. With this information, communication channel can assign the communication resource to different control systems. Interesting topics includes the necessary and/or sufficient bandwidth to stabilize all control systems in the network, and the scheduling policy to achieve the necessary and/or sufficient bandwidth.

## REFERENCES


I.F. Akyildiz and I.H. Kasimoglu. Wireless sensor and actor networks: research challenges. *Ad hoc networks*, 2 (4):351–367, 2004.

D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, 45(6):1082–1097, 2000. ISSN 0018-9286.

A. Anta and P. Tabuada. Isochronous manifolds in self-triggered control. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 3194–3199. IEEE, 2009.

R.W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *Automatic Control, IEEE Transactions on*, 45(7):1279–1289, 2000.

D.F. Delchamps. Stabilizing a linear system with quantized state feedback. *Automatic Control, IEEE Transactions on*, 35(8):916–924, 1990.

J.P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. In *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*. Citeseer, 2002.

J.P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, 2007.

H.K. Khalil and JW Grizzle. *Nonlinear systems*, volume 3. Prentice hall, 1992.

Lichun Li, Michael Lemmon, and Xiaofeng Wang. Stabilizing bit-rates in quantized event triggered control systems. In *submitted to Hybrid Systems: Computation and Control*, 2012.

D. Liberzon and J.P. Hespanha. Stabilization of nonlinear systems with limited information feedback. *Automatic Control, IEEE Transactions on*, 50(6):910–915, 2005.

J. Sandee, W. Heemels, and P. van den Bosch. Case studies in event-driven control. In *Hybrid Systems: computation and control*, pages 762–765. Springer, 2007.

P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. In *Automatic Control, IEEE Transactions on*, volume 52, pages 1680–1685. IEEE, 2007.

P. Tabuada and X. Wang. Preliminary results on state-trigered scheduling of stabilizing control tasks. In *Decision and Control, 2006 45th IEEE Conference on*, pages 282–287. IEEE, 2006. ISBN 1424401712.

S. Tatikonda and S. Mitter. Control under communication constraints. *Automatic Control, IEEE Transactions on*, 49(7):1056–1068, 2004.

X. Wang and M.D. Lemmon. Finite-gain l2 stability in distributed event-triggered networked control systems with data dropouts. In *the European Control Conference*, 2009a.

X. Wang and M.D. Lemmon. Self-Triggered Feedback Control Systems With Finite-Gain $L_2$ Stability. In *Automatic Control, IEEE Transactions on*, volume 54, pages 452–467. IEEE, 2009b.

X. Wang and M.D. Lemmon. Attentively efficient controllers for event-triggered feedback systems. In *IEEE Conference on Decsion and Control*, 2011a.

Xiaofeng Wang and M.D. Lemmon. Event-triggering in distributed networked control systems. *Automatic Control, IEEE Transactions on*, 56(3):586 –601, march 2011b. ISSN 0018-9286. doi: 10.1109/TAC.2010.2057951.

W.S. Wong and R.W. Brockett. Systems with finite communication bandwidth constraints. ii. stabilization with limited information feedback. *Automatic Control, IEEE Transactions on*, 44(5):1049–1053, 1999.