Stabilizing Bit-Rates in Quantized Event Triggered Control Systems

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ABSTRACT

Event triggered systems are feedback systems that sample the state when the novelty in that state exceeds a threshold. Prior work has demonstrated that event-triggered feedback may have inter-sampling intervals that are, on average, greater than the sampling periods found in comparably performing periodic sampled data systems. This fact has been used to justify the claim that event-triggered systems are more efficient in their use of communication or computational resources than periodic sampled data systems. If, however, one accounts for quantization effects and maximum acceptable delays, then it is quite possible that the actual bit rates generated by event triggered systems may be greater than that of periodically triggered systems. This paper examines the bit-rates required to asymptotically stabilize nonlinear event triggered systems. In particular, this paper uses the scaling relationships between maximum delay, inter-sampling interval, and quantization error to bound stabilizing bit-rates for quantized event triggered control systems. Conditions are presented under which the stabilizing bit rates asymptotically go to a constant. In some cases, it is possible to show that this bit rate asymptotically goes to zero as the system approaches its equilibrium.

Keywords

Event triggered control, quantization, sampled-data systems

1. INTRODUCTION

State-dependent event-triggered control systems are systems that transmit the system state over the feedback channel when the difference between the current state and last sampled-state exceeds a state-dependent threshold. These systems were originally viewed as embedded computational systems [10]. In this case, one was interested in reducing how often the system state was sampled, as a means of reducing processor utilization. The concept of event-triggering can be easily extended to networked control systems [6] and wireless sensor-actuator networks [1], in which case the sampled state is *transmitted* over a communication channel.

Early interest in event-triggered control was driven by experimental results suggesting that these systems could have longer inter-sampling intervals than comparably performing periodic sampled-data systems [8, 9, 12]. In extending this idea to networked control systems, one might suppose that event-triggering can also reduce the system's usage of the communication channel since it might reduce the frequency at which feedback states are transported across the channel. This extension, however, is complicated by the fact that the communication channel is discrete in nature. Sampled states must first be quantized into a finite number of bits before being transmitted across the channel. Moreover, the transmitted bits must be delivered with a delay that does not de-stabilize the system. So an accurate measure of channel usage is the bit rate as defined by the number of bits per sampled state divided by the acceptable delay in message delivery. It means that the system's *stabilizing bit rate* (i.e., the bit rate assuring closed-loop stability) rather than the inter-transmission interval (i.e. the time between consecutive transmissions of the sampled state) provides a more realistic measure of channel usage in event-triggered networked control systems.

Prior work in state-dependent event-triggered control has used two different techniques to bound the inter-transmission times and acceptable delays. The method used in [9] bounds the minimum inter-transmission delay as a function of the open-loop system's Lipschitz constant. This work goes on to show that system stability is preserved for sufficiently small delays. More accurate measures of inter-transmission intervals were obtained in [3] using scaling properties of homogeneous systems. Quantitative bounds on both the inter-transmission time and maximum acceptable delay were obtained for self-triggered $\mathcal{L}_2$ systems in [13] and networked control systems [15]. The results in [13, 15] are significant because they show how the delay and inter-transmission time scale as a function of the last sampled state. These scaling properties led to the characterization [14] of event-triggered systems whose inter-transmission times exhibited *efficient attentiveness* (i.e. the inter-transmission intervals asymptotically approach infinity as the state approaches its equilibrium point). The approach used in this paper builds upon the techniques used in [14] to characterize how stabilizing bit rates scale as the system state approaches the equilibrium point.

This paper's bounds on stabilizing bit rates is reminiscent of earlier work on dynamic quantization. Prior work showed that static quantization maps required an infinite number of bits to achieve asymptotic stability [5]. With a finite number of bits, the best one can achieve is ultimate boundedness [16] when using static maps. This led to the development of dynamic quantization maps [4] in which the quantization map is dynamically varied to track state uncertainty. For linear systems, one was able to obtain bounds on the bit rate that were necessary and sufficient for stability, assuming a single sample delay [11]. In the case of nonlinear systems, lower bounds on the quantization rate were obtained [7]. The quantization maps developed in this paper are dynamic maps, similar to those used in [7]. This paper shows that the bit rates sufficient for stabilizing a nonlinear system asymptotically approach a constant for many cases. Under certain conditions, we recover the same bounds reported in [7]. Remarkably, we can sometimes find event-triggered systems whose stabilizing bit rates possess the efficient attentiveness property described in [14]; i.e., the bit rate asymptotically goes to zero as the system state approaches its equilibrium point.

The remainder of this paper is organized as follows. The notational conventions used throughout the paper are described in section 2. Section 3 describes the system model. Results on the asymptotic stability of quantized event triggered systems will be found in section 4. The paper's main results characterizing the asymptotic properties of the stabilizing bit rates will be found in section 5. Section 6 describes simulation results supporting the paper's main findings. Conclusions are stated in section 7.

2. MATHEMATICAL PRELIMINARIES

Throughout this paper the linear space of real n-vectors will be denoted as $\mathbb{R}^n$ and the set of non-negative reals will be denoted as $\mathbb{R}^+$. The norm of a vector $x \in \mathbb{R}^n$ will be denoted as $\|x\|$. Given the real-valued function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, we let $x(t)$ denote the value x takes at time $t \in \mathbb{R}^+$. The $\mathcal{L}$ infinity norm of a function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as $\|x\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t\geq 0} \|x(t)\|$. This function is said to be essentially bounded if $\|x\|_{\mathcal{L}_\infty} = M < \infty$ and the linear space of all essentially bounded real-valued functions will be denoted as $\mathcal{L}_\infty$. A subset $\Omega \subset \mathbb{R}^n$ is said to be compact if it is closed and bounded. We say a function g has the minimum order ζ, where ζ satisfies

$$\zeta = \min\{\eta \in \mathbb{R} : \lim_{s\to 0} \frac{g(s)}{s^\eta} \neq 0\}.$$

The minimum order of g is indicated by $\mathcal{O}_g$.

A given real valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(x) > 0$ for all $x \neq 0$. The function V is said to be radially unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$. A function $\alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is class $\mathcal{KL}$ if $\beta(\cdot, t)$ is class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Let Ω be a closed and bounded subset of $\mathbb{R}^n$. We say $f(\cdot) : \Omega \to \mathbb{R}^n$ is Lipschitz on Ω if for any $x, y \in \Omega$, we know there exists a constant $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Consider a system whose state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the initial value problem,

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

A point $\overline{x} \in \mathbb{R}^n$ is an *equilibrium point* of f if $f(\overline{x}) = 0$. We say that the equilibrium point is *stable* if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x_0 - \overline{x}\| < \delta$ implies $\|x(t) - \overline{x}\| < \epsilon$ for all $t \geq 0$. We say the equilibrium point is *asymptotically stable* if it is stable and $x(t) \to 0$ as $t \to \infty$.

Consider a system whose state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the initial value problem,

$$\dot{x}(t) = f(x(t), w(t)), \quad x(0) = x_0 \tag{1}$$

where $w(\cdot) : [0, \infty) \to \mathbb{R}^m$ is an essentially bounded real function.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$\underline{\alpha}(\|x\|) \leq V \leq \overline{\alpha}(\|x\|), \tag{2}$$

$$\frac{\partial V}{\partial x} f(x, w) \leq -\alpha(\|x\|) + \gamma(\|w\|_{\mathcal{L}_\infty}), \tag{3}$$

for all $(x, w) \in \mathbb{R}^n \times \mathbb{R}^m$, where $\underline{\alpha}$, $\overline{\alpha}$ are class $\mathcal{K}_\infty$ functions, and α, γ are class $\mathcal{K}$ functions, then the system (1) is input-to-state stable (ISS). The function V is called *ISS-Lyapunov* function.

3. PROBLEM STATEMENT

The system under study is a networked event-triggered control system with quantization. Figure 1 is a block diagram showing the components of this system.

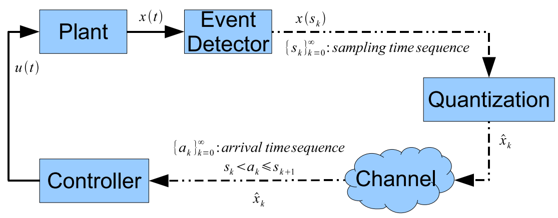

Figure 1: Networked Event-Triggered Control System with Quantization

The *plant*'s state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is an absolutely continuous function satisfying the initial value problem,

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{4}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0, 0) = 0$. The control signal $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^m$ is generated by the *controller* in figure 1. The vector $x_0 \in \mathbb{R}^n$ is the plant's initial condition.

The system state, $x(t)$, at time t is measured by the *event detector*. The event detector decides when to hand over the system state to the *quantizer*. The sequence of *sampling times* is denoted as $\{s_k\}_{k=0}^{\infty}$. For notational convenience, the kth consecutively sampled state $x(s_k)$ will be denoted as x_k. The kth *inter-sampling* interval is defined as $T_k = s_{k+1} - s_k$.

Upon receiving the sampled state, $x_k \in \mathbb{R}^n$, the *quantizer* converts this real vector into a finite bit representation. This quantized state is denoted as $\hat{x}_k \in \mathbb{R}^n$. The finite nature of the representation is modeled as a *quantization error*

$$\overline{e}_q(\|x_k\|) \geq \|x_k - \hat{x}_k\| \tag{5}$$

where $\overline{e}_q(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is actually a class $\mathcal{K}$ function of the norm of the last sampled state. By representing the quantization error in this manner, we obtain a dynamic quantizer similar to that used in previous papers on dynamic quantization [11, 7].

We define the *gap* between the current state and quantized state as $e_k(t) = x(t) - \hat{x}_k$. We assume that quantization is done instantaneously and that the quantizer transmits the quantized sampled state, $\hat{x}_k$, across the *channel*. The transmission times are therefore equivalent to the sampling times generated by the event trigger. The rest of the paper uses the terms transmission and sampling in an interchangeable way. The sampling times $\{s_k\}$ are generated by the event trigger so that the gap is always less than a state-dependent threshold function

$$\|e_k(t)\| \leq \theta(\|\hat{x}_k\|) \tag{6}$$

for all $t \in [s_k, s_{k+1}]$ where $k = 0, 1, \ldots, \infty$. The function $\theta(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function called the *threshold function*.

We assume that the quantized state, $\hat{x}_k$, is always successfully delivered to the controller. The channel, however, is assumed to introduce a finite delay into the delivery time. In particular, the arrival time of the kth sampled state $\hat{x}_k$ at the controller is denoted as $a_k \in \mathbb{R}^+$. This time is strictly greater than s_k. The delay of the kth message is $D_k = a_k - s_k$. We need to assume some orderliness to the transmission and delivery of such messages. In particular, we require that the transmission times, s_k, and arrival times, a_k, satisfy the following order $s_k < a_k \leq s_{k+1}$ for $k = 0, 1, \ldots, \infty$. Such a sequence of transmissions and arrivals will be said to be *admissible*.

Upon the arrival of the kth quantized state, $\hat{x}_k$, at the controller, a control input is computed and then held until the next quantized state is received. In other words, the control signal takes the form

$$u(t) = u_k = K(\hat{x}_k) \tag{7}$$

for $t \in [a_k, a_{k+1})$. The function $K(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $K(0) = 0$. As has been done in previous papers [9], this paper assumes that K is chosen so the system

$$\dot{z}(t) = f(z(t), K(z(t) + e(t))) \tag{8}$$

is input-to-state stable with respect to the signal $e \in \mathcal{L}_\infty$. This means, of course, that there exists a function $V(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions in equations (2-3). Note that this can be a very restrictive assumption since such K may not always exist [2].

4. ASYMPTOTIC STABILITY

This section characterizes a threshold function eq. (6) and a quantization error function eq. (5) such that the event-triggered system described in section 3 is asymptotically stable. As noted in the preceding section, the control function K eq. (7) is assumed to leave the system in equation (8) input-to-state stable with respect to the error e that enters through the controller. In particular, we assume there exists an ISS-Lyapunov function for system equation (8) with class $\mathcal{K}$ functions α and γ that satisfy inequality (3). The functions, α and γ, may be viewed as *specifications* on the transient and steady-state behavior, respectively, of the closed-loop system. With this view of these two functions, we will find it convenient to introduce a function $\xi(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ that takes values

$$\xi(s) = \gamma^{-1}(\varsigma\alpha(s)) \tag{9}$$

for some constant $\varsigma \in (0, 1)$ and $s \in \mathbb{R}^+$.

Recall that the plant for this system is characterized by the function f on the right hand side of equation (4). We assume that this function is Lipschitz on *compacts*. In particular, this means if we let Ω_k be a compact set containing the origin and all possible trajectories of $e_k(t)$ for any $t \in [a_k, a_{k+1})$, then

$$\|f(x, K(\hat{x}_k)))\| \leq \psi(\hat{x}_k, K(\hat{x}_k)) + L_{\Omega_k}\|e_k\| \tag{10}$$

where $\psi_k(\hat{x}_k, K(\hat{x}_k)) = \|f(\hat{x}_k, K(\hat{x}_k))\|$,and $e_k(t) = x(t) - \hat{x}_k$ is the gap function defined earlier. We can think of L_{Ω_k} as the Lipschitz constant over compact set Ω_k.

With these preliminaries we now state the main theorem of this section.

THEOREM 4.1. *Consider the system described in section 3. Assume that the threshold function θ and quantization error $\overline{e}_q$ satisfy*

$$\begin{aligned} \theta(\|\hat{x}_k\|) &< \underline{\xi}(\|\hat{x}_k\|) && (11)\\ \overline{e}_q(\|x_k\|) &< \theta(\|\hat{x}_k\|), \text{ and } \overline{e}_q(\|x_k\|) \leq \|x_k\| && (12) \end{aligned}$$

for all $\hat{x}_k, x_k \in \Gamma$, with $\underline{\xi}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ taking values

$$\underline{\xi}(s) = \sup\left\{\epsilon \in \mathbb{R}^+ : \epsilon \leq \xi(s - \epsilon), \epsilon \leq s, \forall s \in \mathbb{R}^+\right\}, \tag{13}$$

and $\Gamma = \{x \in \mathbb{R}^n : \|x\| \leq 2\underline{\alpha}^{-1} \circ \overline{\alpha}(\|x_0\|)\}$.

If the actual channel delay $D_k = a_k - s_k$ is always less than $\overline{D}_k = \min\{\underline{T}_k, \hat{D}_k\}$ for all $k = 0, 1, \ldots, \infty$, where

$$\begin{aligned} \underline{T}_k &= \frac{1}{L_{\Omega_k}}\left(\ln\left(1 + \frac{L_{\Omega_k}\theta(\|\hat{x}_k\|)}{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1})}\right)\right.\\ &\quad \left. - \ln\left(1 + \frac{L_{\Omega_k}\overline{e}_q(\|x_k\|)}{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1})}\right)\right) && (14)\\ \hat{D}_k &= \frac{1}{L_{\Omega_{k-1}}}\left(\ln\left(1 + L_{\Omega_{k-1}}\frac{\underline{\xi}(\|\hat{x}_{k-1})\|}{\psi(\hat{x}_{k-1}, u_{k-1})}\right)\right.\\ &\quad \left. - \ln\left(1 + L_{\Omega_{k-1}}\frac{\theta(\|\hat{x}_{k-1}\|)}{\psi(\hat{x}_{k-1}, u_{k-1})}\right)\right), && (15) \end{aligned}$$

and

$$\begin{aligned} \Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1}) =& (1 - \rho_k)|\psi(\hat{x}_k, K(\hat{x}_k)) - \psi(\hat{x}_k, K(\hat{x}_{k-1}))|\\ &+ \psi(\hat{x}_k, K(\hat{x}_{k-1})), && (16) \end{aligned}$$

$\Omega_k = \{x \in \mathbb{R}^n : \|x\| \leq \|\hat{x}_k\| + \underline{\xi}(\|\hat{x}_k\|)\}$ and $\rho_k = D_k/T_k$,

Then the closed-loop event triggered system is asymptotically stable and the inter-sampling interval T_k is always bounded below by $\underline{T}_k$ which is strictly greater than zero.

PROOF. We first show that $\hat{x}_k$ is always bounded by some class $\mathcal{KL}$ function. Once this is shown, it's easy to see that Ω_k is always bounded, and hence we have finite L_{Ω_k} for all $k = 0, 1, \cdots, \infty$. Based on the fact that L_{Ω_k} is always finite, we, then, prove that $\dot{V} < 0$ for all $t \geq 0$, and the asymptotic stability can be shown.

We first show that x_k is always bounded by some class $\mathcal{KL}$ function. If this is true, then $\hat{x}_k$ must be bounded. We know that there exists an ISS-Lyapunov function V such that (2) and (3) hold. If this V also satisfies

$$\dot{V} = \frac{\partial V}{\partial x} f(x, K(\hat{x}_k)) \leq -(1 - \varsigma)\alpha(\|x\|),$$

for all $x \in \mathbb{R}^n$, all $t \in [a_k, a_{k+1})$, all $k = 0, 1, \cdots, \bar{k}$ and some $\varsigma \in (0, 1)$, then there must exist a class $\mathcal{KL}$ function β such that $\|x(t)\| \leq \beta(\|x_0\|, t)$ for all $t \in [0, a_{\bar{k}+1})$.

Now, let's assume that there exists an integer k' such that

$$\|x_k\| \le \beta(\|x_0\|, s_k), \forall k = 0, 1, \cdots, k'$$
$$\|x_{k'+1}\| > \beta(\|x_0\|, s_{k'+1}). \quad (17)$$

Next, we show that $\dot{V} \le -(1-\varsigma)\alpha(\|x\|)$ for all $t \in [0, a_{k'+1})$. Therefore, $\|x(t)\| \le \beta(\|x_0\|, t)$ should hold for all $t \in [0, a_{k'+1})$, which contradicts our assumption that $\|x_{k'+1}\| > \beta(\|x_0\|, s_{k'+1})$.

To do so, we have to establish that $\dot{V} \le -(1-\varsigma)\alpha(\|x\|)$ for all $t \in [a_k, a_{k+1})$, and all $k = 0, 1, \cdots, k'$. Since $\|x(t)\| \le \beta(\|x_0\|, t)$ for all $t \in [0, s_{k'}]$, we know that $\|x_k\|$ are bounded for all $k = 0, 1, \cdots, k'$, and hence Ω_k is also bounded. It means that we can always find a finite Lipschitz constant L_{Ω_k} for all $x, \hat{x} \in \Omega_k$.

We first show that $e_k(t) < \underline{\xi}(\|\hat{x}_k\|)$ during the interval $[s_k, s_{k+1}]$, and then show that the inequality still holds during interval $[s_{k+1}, a_{k+1})$ with the maximum delay $\overline{D}_k$ given by (15). Finally, we can derive that $\dot{V} \le -(1-\varsigma)\alpha(\|x\|)$ by showing that $\underline{\xi}(\|\hat{x}_k\|) \le \xi(x(t))$ for all $t \in [a_k, a_{k+1})$ and all $k = 0, 1, \cdots, k'$.

During interval $[s_k, s_{k+1}]$, from inequality (11), it's easy to show that $\|e_k(t)\| \le \theta(\|\hat{x}_k\|) \le \underline{\xi}(\|\hat{x}_k\|)$. Besides, from the dynamic behavior of the gap $e_k(t)$ in this interval, we can also find the minimum inter-sampling interval which will be used in the next step.

The dynamic behavior of $e_k(t)$ satisfies

$$\dot{e}_k(t) = f(\hat{x}_k + e_k(t), K(\hat{x}_k)), \forall t \in [a_k, a_{k+1}).$$

Since we've shown that $\|e_k(t)\| < \underline{\xi}(\|\hat{x}_k\|)$ for all $t \in [s_k, s_{k+1}]$, it's easy to see that $x(t), \hat{x}_k \in \Omega_k$ for all $t \in [s_k, s_{k+1}]$ and all $k = 0, 1, \cdots, k'$. So, from (10) the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \le \|\dot{e}_k(t)\| \le \psi(\hat{x}_k, K(\hat{x}_{k-1})) + L_{\Omega_k}\|e_k(t)\|, \quad (18)$$

for all $t \in [s_k, a_k)$. Solving the dynamic inequality, we have

$$\|e_k(a_k)\| \le \bar{e}_q(\|x_k\|)e^{L_{\Omega_k} D_k} + \frac{\psi(\hat{x}_k, K(\hat{x}_{k-1}))}{L_{\Omega_k}}(e^{L_{\Omega_k} D_k} - 1).$$

For interval $[a_k, a_{k+1})$, the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \le \|\dot{e}_k(t)\| \le \psi(\hat{x}_k, K(\hat{x}_k)) + L_{\Omega_k}\|e_k(t)\|. \quad (19)$$

With $\|e_k(a_k)\|$ as the initial condition, we have

$$\|e_k(s_{k+1})\| \le \frac{\Psi_{k,k-1}(\hat{x}_k, \hat{x}_{k-1})}{L_{\Omega_k}}(e^{L_{\Omega_k} T_k} - 1) + \bar{e}_q(\|x_k\|)e^{L_{\Omega_k} T_k}. \quad (20)$$

We know that at s_{k+1}, $\|e_k(s_{k+1})\| = \theta(\|\hat{x}_k\|)$. Together with (20), we derive that $T_k \ge \underline{T}_k$, where $\underline{T}_k$ is given by (14). Moreover, with (12), it's easy to see that $T_k \ge \underline{T}_k > 0$. So we know that during time interval $[a_k, a_k + \underline{T}_k]$, there is no sampling. Since $D_k \le \overline{D}_k \le \underline{T}_k$, we can make sure that no sampling occurs during interval $[s_k, a_k]$.

Now, let's show that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|)$ for $t \in [s_{k+1}, a_{k+1})$ for $k = 0, 1, \cdots, k'$. We assume that it is not true. In this case, since $\overline{D}_{k+1}$ is the maximum delay, there must exists $D'_{k+1} < \overline{D}_{k+1} < \hat{D}_{k+1}$ such that at time $s_{k+1} + D'_{k+1}$, $\|e_k(s_{k+1} + D'_{k+1})\| = \underline{\xi}(\|\hat{x}_k\|)$, and $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|)$ for all $t \in [s_{k+1}, s_{k+1} + D'_{k+1}]$. So, we can use the same technology to show that

$$\|e_k(s_{k+1} + D'_{k+1})\| \le \theta(\|\hat{x}_k\|)e^{L_{\Omega_k} D'_{k+1}} + \frac{\psi(\hat{x}_k, K(\hat{x}_k))}{L_{\Omega_k}}(e^{L_{\Omega_k} D'_{k+1}} - 1). \quad (21)$$

Since $\|e_k(s_{k+1} + D'_{k+1})\| = \underline{\xi}(\|\hat{x}_k\|)$, we can derive that $D'_{k+1} \ge \hat{D}_{k+1}$, which contradicts the assertion that $D'_{k+1} < \hat{D}_{k+1}$. So the assumption which says that there exist some $t \in [s_{k+1}, a_{k+1})$ such that $\|e_k(t)\| > \underline{\xi}(\|\hat{x}_k\|)$ is not true. So, we can conclude that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|)$ for all $t \in [s_{k+1}, a_{k+1})$.

By now, we have shown that $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|)$ for all $t \in [s_k, a_{k+1})$, and $k = 0, 1, \cdots, k'$. The next, we will show that $\underline{\xi}(\|\hat{x}_k\|) < \xi(\|x(t)\|)$ for all $t \in [0, a_{k'+1})$.

From (13), we can derive that

$$\underline{\xi}(\|\hat{x}_k\|) \le \xi(\|\hat{x}_k\| - \underline{\xi}(\|\hat{x}_k\|)) \le \xi(\|\hat{x}_k\| - \|e_k(t)\|) \le \xi(\|x(t)\|), \quad (22)$$

for all $t \in [s_k, a_{k+1})$, and all $k = 0, 1, \cdots, k'$. Since $\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|) \le \xi(\|\hat{x}_k\|)$ for all $t \in [s_k, a_{k+1})$, and $k = 0, 1, \cdots, k'$, together with (3) and (9), we can show that

$$\dot{V} \le -(1-\varsigma)\alpha(\|x(t)\|), \quad (23)$$

for some constant $\varsigma \in (0, 1)$, all $t \in [s_k, a_{k+1})$, and all $k = 0, 1, \cdots, k'$, i.e. all $t \in [0, a_{k'+1})$.

Therefore, $\|x(t)\| \le \beta(\|x_0\|, t)$ for all $t \in [0, a_{k'+1})$, and hence $\|x_{k'+1}\| \le \beta(\|x_0\|, s_{k'+1})$. It contradicts our assumption in (17), which demonstrates that $\|x_k\| \le \beta(\|x_0\|, s_k)$, for all $k = 0, 1, \cdots, \infty$. If $\|x_k\|$ is bounded, $\hat{x}_k$ is also bounded, and hence so is Ω_k. From the fact that the system locally Lipschitz, we can conclude that L_{Ω_k} is always bounded.

Since we have proven L_{Ω_k} is always bounded, we can follow the same idea of proving that x_k is bounded to show that (23) is true for all $t \ge 0$, and hence the asymptotic stability of the system is shown. □

REMARK 4.2. *Inequality (11) and (12) assure that $\hat{D}_k$ and $\underline{T}_k$ are positive, and hence guarantee that $\overline{D}_k$ is always positive. We know that at time s_{k+1}, $\|e_k\| = \theta(\|\hat{x}_k\|)$. Meanwhile, $\underline{\xi}(\|\hat{x}_k\|)$ is an upper bound on $\|e_k\|$ to assure asymptotic stability. Since (11) holds, it means that we can accept some positive amount of delay such that $\|e_k\|$ can go beyond $\theta(\|\hat{x}_k\|)$ but within $\underline{\xi}(\|\hat{x}_k\|)$. (12) assures that the minimum inter-sampling interval $\underline{T}_k$ is always positive. We know that at s_k, $\|e_k\|$ is bounded from above by the quantization error $\bar{e}_q(\|x_k\|)$. If (12) holds, the next sampling time should be some time in the future. Since $\overline{D}_k$ takes the minimum of $\hat{D}_k$ and $\underline{T}_k$ which are both positive, $\overline{D}_k$ is positive, too.*

REMARK 4.3. *$\bar{e}_q$ can be chosen such that $\bar{e}_q(s) < \theta(s - \bar{e}_q(s))$ for all $s \in \Gamma$. We notice that in (12), θ and $\bar{e}_q$ are based on different variables. To make the design of quantization error easier, we can chose $\bar{e}_q(s)$ such that $\bar{e}_q(s) < \theta(s - \bar{e}_q(s))$ for all $\bar{e}_q(s) < s$. This inequality implies (12), since $\|\hat{x}_k\| \ge \|x_k\| - \bar{e}_q(\|x_k\|) \ge 0$.*

REMARK 4.4. *L_{Ω_k} converges to a finite constant as the state approaches the origin. In the proof of theorem 4.1, we show that $\|\hat{x}_k\|$ is always bounded. Since the radius of Ω_k (which is defined as a ball centered at the origin) is a class $\mathcal{K}$ function of $\hat{x}_k$, we can see that as time goes by, Ω_k converges to the origin, and L_{Ω_k} converges to a finite constant.*

Remark 4.5. *If we set the quantization error, network delay and noise to be 0, then the minimum inter-sampling intervals $\underline{T}_k$ in our work and Wang's work in [14] are the same. We both study the event triggered nonlinear systems. While Wang's work in [14] considered system noise without quantization error and delay, we consider quantization error and delay without system noise. If we set the quantization error, network delay and noise to be 0, we can see that the minimum inter-sampling intervals in both works are the same.*

If we set the quantization error and system noise to be 0, the minimum inter-sampling intervals and the maximum delays in both works are in similar forms. Wang's work in [13] studied the minimum inter-sampling interval and the maximum delay for event triggered systems. While Wang focused on linear systems which have system noise but no quantization error, we focus on nonlinear systems which have quantization error but no system noise. If we set the quantization error and system noise to be 0, we can find that the minimum inter-sampling intervals and the maximum delays are in similar forms.

Remark 4.6. *No sampling occurs during the interval $[s_k, s_k + \overline{D}_k)$, so we guarantee that $s_k < a_k \le s_{k+1}$. With the maximum delay $\overline{D}_k$ defined as the minimum of $\hat{D}_k$ and $\underline{T}_k$, we know that $\overline{D}_k \le \underline{T}_k$. $\underline{T}_k$ is the minimum inter-sampling interval, which means that no sampling occurs during time $[s_k, s_k + \underline{T}_k)$. Therefore, we can guarantee the admissible time sequence, i.e. $s_k < a_k \le s_{k+1}$.*

5. STABILIZING BIT-RATE

Stabilizing bit-rate is the bit-rate which is sufficient to guarantee the asymptotic stability of the system. This section studies three conditions under which the stabilizing bit-rate is 0, finite, and infinite.

Before talking about the stabilizing bit-rate, we first give a quantization map for the system given quantization error $\overline{e}_q(\|x_k\|)$. Since at sampling time s_k, both sensor and controller understand that $\|e_{k-1}(s_k)\| = \theta(\|\hat{x}_{k-1}\|)$, we only need to quantize the surface of the box $\|e_{k-1}(s_k)\| \le \theta(\|\hat{x}_{k-1}\|)$ (for convenience, we use infinity norm here). First, we use $\lceil \log_2 2n \rceil$ bits to identify which side e_{k-1} lies on, and then we cut this side uniformly into $\left\lceil \frac{\theta(\|\hat{x}_{k-1}\|)}{\overline{e}_q(\|x_k\|)} \right\rceil^{n-1}$ parts. If $e_{k-1}(s_k)$ lies on one of the small parts, then $e_{k-1}(s_k)$ will be quantized as the center of this part, and $\hat{x}_k$ can be calculated to be the sum of $\hat{x}_{k-1}$ and the quantized $e_{k-1}(s_k)$. In all, the number of bits used at the kth sampling is

$$N_k = \lceil \log_2 2n \rceil + (n-1) \left\lceil \log_2 \left\lceil \frac{\theta\|\hat{x}_{k-1}\|}{\overline{e}_q(\|x_k\|)} \right\rceil \right\rceil \tag{24}$$

We should notice that the number of bits transmitted at each time can be different, since we fix the quantization error instead of the number of bits. Also because the quantization error is fixed, uniform quantization minimizes the number of bits used to quantize the uncertainty set $\|e_{k-1}(s_k)\| = \theta(\|\hat{x}_{k-1}\|)$.

Now, let's define the stabilizing bit-rate as

$$\underline{r}_k = \frac{N_k}{\overline{D}_k}. \tag{25}$$

For convenience of the rest of this paper, we define $\phi_c(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$\psi(\hat{x}_k, K(\hat{x}_k)) \le \phi_c(\|\hat{x}_k\|) \tag{26}$$

$$\lim_{\hat{x}_k \to 0} \frac{\psi(\hat{x}_k, K(\hat{x}_k))}{\phi_c(\|\hat{x}_k\|)} > 0, \tag{27}$$

and $\phi_u(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$u_k = \|K(\hat{x}_k)\| \le \phi_u(\|\hat{x}_k\|). \tag{28}$$

$$\lim_{\hat{x}_k \to 0} \frac{\|K(\hat{x}_k)\|}{\phi_u(\|\hat{x}_k\|)} > 0 \tag{29}$$

The three conditions under which the stabilizing bit-rate is 0, finite and infinite are studied in the next three subsections.

5.1 Zero stabilizing bit-rate

Theorem 5.1. *We assume that all the assumptions in theorem 4.1 hold, if*

$$\lim_{s \to 0} \frac{\theta(s)}{\overline{e}_q(s)} < \infty, \tag{30}$$

$$\lim_{s \to 0} \frac{\phi_c(s)}{\theta(s)} = \lim_{s \to 0} \frac{\phi_u(s)}{\theta(s)} = \lim_{x \to 0} L_{\Omega_k} = 0, \tag{31}$$

then

$$\lim_{x \to 0} \underline{r}_k = 0.$$

Proof. We first show that $\lim_{x \to 0} N_k$ is bounded with (30), and then prove that $\overline{D}_k$ goes to infinity if (31) holds. With the stabilizing bit-rate defined as $\underline{r}_k = \frac{N_k}{\overline{D}_k}$, we then demonstrate that $\lim_{x \to 0} \underline{r}_k = 0$.

Let's first show that $\lim_{x \to 0} N_k < \infty$. It is easy to see that

$$1 \le \mathcal{O}_\xi = \mathcal{O}_{\underline{\xi}} \le \mathcal{O}_\theta, 1 \le \mathcal{O}_{\overline{e}_q} \tag{32}$$

The first inequality and the first equality can be derived from (13), the second inequality can be found from (11), and the third inequality can be shown from (11-13).

From (24), we know that if $\lim_{x \to 0} \frac{\theta\|\hat{x}_{k-1}\|}{\overline{e}_q(\|x_k\|)} < \infty$, then $\lim_{x \to 0} N_k < \infty$. From the fact that $\|x_k - \hat{x}_{k-1}\| = \theta(\|\hat{x}_{k-1}\|)$, we know that $\|x_k\| \ge |\|\hat{x}_{k-1}\| - \theta(\|\hat{x}_{k-1}\|)|$. So, we have

$$\begin{aligned} \lim_{x \to 0} \frac{\theta(\|\hat{x}_{k-1}\|)}{\overline{e}_q(\|x_k\|)} &< \lim_{\hat{x}_{k-1} \to 0} \frac{\theta(\|\hat{x}_{k-1}\|)}{\overline{e}_q(|\|\hat{x}_{k-1}\| - \theta(\|\hat{x}_{k-1}\|)|)} \\ &= \lim_{\hat{x}_{k-1} \to 0} \frac{\theta(\|\hat{x}_{k-1}\|)}{\overline{e}_q(\|\hat{x}_{k-1}\|)} \\ &< \infty \end{aligned} \tag{33}$$

The first equality holds because (32) is true, and the second inequality is derived from (30).

Next, we show that $\overline{D}_k$ goes to infinity as x approaches the origin. It is done by showing that both $\hat{D}_k$ and $\underline{T}_k$ converge to infinity as x approaches the origin.

For $\hat{D}_k$, it can be rewritten as

$$\hat{D}_k = \frac{1}{L_{\Omega_{k-1}}} \ln\left(1 + L_{\Omega_{k-1}} \tilde{D}_{k-1}\right),$$

where $\tilde{D}_{k-1} = \frac{\underline{\xi}(\|\hat{x}_{k-1}\|) - \theta(\|\hat{x}_{k-1}\|)}{\psi(\hat{x}_{k-1}, K(\hat{x}_{k-1})) + L_{\Omega_{k-1}} \theta(\|\hat{x}_{k-1}\|)}$.

If

$$\lim_{x \to 0} L_{\Omega_{k-1}} \tilde{D}_{k-1} > 0,$$

from (31), we know that $\hat{D}_k$ goes to infinity.

If

$$\lim_{x\to 0} L_{\Omega_{k-1}}\tilde{D}_{k-1} = 0,$$

then

$$\begin{aligned}\lim_{x\to 0}\hat{D}_k &= \lim_{\hat{x}_{k-1}\to 0}\tilde{D}_{k-1}\\ &\geq \lim_{\hat{x}_{k-1}\to 0}\frac{1-\frac{\theta(\|\hat{x}_{k-1}\|)}{\underline{\xi}(\|\hat{x}_{k-1}\|)}}{\frac{\phi_c(\|\hat{x}_{k-1}\|)}{\underline{\xi}(\|\hat{x}_{k-1}\|)}+L_{\Omega_{k-1}}\frac{\theta(\|\hat{x}_{k-1}\|)}{\underline{\xi}(\|\hat{x}_{k-1}\|)}} = \infty\end{aligned}$$

The first inequality holds because of inequality (26), and the second equality holds because (31) and (32) hold. Therefore, $\lim_{x\to 0}\hat{D}_k = \infty$.

For $\underline{T}_k$, it can be rewritten as

$$\underline{T}_k = \frac{1}{L_{\Omega_k}}\ln(1+L_{\Omega_k}\tilde{T}_k),$$

where $\tilde{T}_k = \frac{\theta(\|\hat{x}_k\|)-\bar{e}_q(\|x_k\|)}{\Psi_{k,k-1}(\hat{x}_k,\hat{x}_{k-1})+L_{\Omega_k}\bar{e}_q(\|x_k\|)}$.

If

$$\lim_{x\to 0} L_{\Omega_k}\tilde{T}_k > 0,$$

from (31), we know that $\underline{T}_k$ goes to infinity.

If

$$\lim_{x\to 0} L_{\Omega_k}\tilde{T}_k = 0,$$

then

$$\begin{aligned}\lim_{x\to 0}\underline{T}_k &= \lim_{x\to 0}\tilde{T}_k\\ &\geq \lim_{\hat{x}_k\to 0}\frac{1-\frac{\bar{e}_q(\|x_k\|)}{\theta(\|\hat{x}_k\|)}}{\frac{\phi_c(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|)}+2\frac{\psi(\hat{x}_k,K(\hat{x}_{k-1}))}{\theta(\|\hat{x}_k\|)}+\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\|\hat{x}_k\|)}}\end{aligned}$$

We know from (31) that $\lim_{\hat{x}_k\to 0}\frac{\phi_c(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|)} = 0$. If we can show that $\frac{\psi(\hat{x}_k,K(\hat{x}_{k-1}))}{\theta(\|\hat{x}_k\|)}$ and $\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\|\hat{x}_k\|)}$ converge to 0, then $\lim_{x\to 0}\underline{T}_k = \infty$.

$$\begin{aligned}&\lim_{x\to 0}\frac{\psi(\hat{x}_k,K(\hat{x}_{k-1}))}{\theta(\|\hat{x}_k\|)}\\ &\leq \lim_{\hat{x}_k\to 0}\left(\frac{\phi_c(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|)}+L'_{\Pi_{k,k-1}}\frac{\|u_k-u_{k-1}\|}{\theta(\|\hat{x}_k\|)}\right)\\ &\leq \lim_{\hat{x}_k\to 0}L'_{\Pi_{k,k-1}}\left(\frac{\phi_u(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|)}+\frac{\phi_u(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|)}\right)\\ &= \lim_{\hat{x}_k\to 0}L'_{\Pi_{k,k-1}}\frac{\phi_u(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|)},\end{aligned}$$

where $\Pi_{k,k-1} = \{u\in\mathbb{R}^m : \|u\|\leq\phi_u(\max\{\|\hat{x}_k\|,\|\hat{x}_{k-1}\|\})\}$. Since we've shown in theorem 4.1 that $\hat{x}_k$ is always bounded, $\Pi_{k,k-1}$ is bounded, and so is $L'_{\Pi_{k,k-1}}$ (the Lipschitz constant of f with respect to u). The first inequality holds because f is locally Lipschitz with respect to u, i.e. $\frac{|\psi(\hat{x}_k,u_{k-1}))-\psi(\hat{x}_k,u_k)|}{\|u_k-u_{k-1}\|}$ $\leq L'_{\Pi_{k,k-1}}$, the second inequality holds because $L'_{\Pi_{k,k-1}} < 0$, and inequality (28), (31) hold.

From $\|x_k-\hat{x}_{k-1}\| = \theta(\|\hat{x}_{k-1}\|)$, we can see that $\|\hat{x}_{k-1}\|\leq\|x_k\|+\theta(\|\hat{x}_{k-1}\|)\leq\|x_k\|+\xi(\|x_k\|)$. From $\|x_k-\hat{x}_k\|\leq\bar{e}_q(\|x_k\|)$, it can be show that $\|\hat{x}_k\|\geq\|x_k\|-\bar{e}_q(\|x_k\|)>0$ (the last inequality can be derived from (12)). So,

$$\begin{aligned}\lim_{x\to 0}\frac{\psi(\hat{x}_k,K(\hat{x}_{k-1}))}{\theta(\|\hat{x}_k\|)} &\leq \lim_{x_k\to 0}L'_{\Pi_{k,k-1}}\frac{\phi_u(\|x_k\|+\xi(\|x_k\|))}{\theta(\|x_k\|-\bar{e}_q(\|x_k\|))}\\ &= \lim_{x_k\to 0}L'_{\Pi_{k,k-1}}\frac{\phi_u(\zeta_1\|x_k\|)}{\theta(\zeta_2\|x_k\|)}\\ &= 0.\end{aligned}$$

The inequality, $\mathcal{O}_{\bar{e}_q}\geq 1$ in (32), implies the first equality. The second equality holds because of (31) and (30).

By now, we've shown that $\lim_{x\to 0}\frac{\psi(\hat{x}_k,K(\hat{x}_{k-1}))}{\theta(\|\hat{x}_k\|)} = 0$. Nest, we show that $\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\|\hat{x}_k\|)}$ converges to 0, too.

$$\begin{aligned}\lim_{x\to 0}\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\|\hat{x}_k\|)} &\leq \lim_{x\to 0}\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\|x_k\|-\bar{e}_q(\|x_k\|))}\\ &= \lim_{x\to 0}\frac{L_{\Omega_k}\bar{e}_q(\|x_k\|)}{\theta(\zeta_2\|x_k\|)}\\ &= 0.\end{aligned}$$

Therefore, $\lim_{x\to 0}\underline{T}_k = \lim_{x\to 0}\hat{D}_k = \infty$, which implies that $\lim_{x\to 0}\overline{D}_k = \infty$. Together with the fact that N_k converges to a finite constant, $\underline{r}_k$ goes to 0 as the state approaches the origin. □

REMARK 5.2. *Equalities in (31) indicate that $\|e_k(t)\|$ decreases faster than $\theta(\|\hat{x}_k\|)$ and $\underline{\xi}(\|\hat{x}_k\|)$, which leads to infinite minimum inter-sampling interval and infinite maximum delay. From (20), we can see that L_{Ω_k}, $\phi_c(s)$ and $\phi_u(r)$ (these two function consist $\Psi_{k,k-1}(s,r)$) reflect how fast $\|e_k(t)\|$ changes, and $\theta(s)$ is the threshold function. (31) indicates that $\phi_c(s)$ and $\phi_u(s)$ decrease faster than $\theta(s)$. With the fact that L_{Ω_k} goes to 0, we can conclude that $\|e_k(t)\|$ decreases faster than θ if (31) holds. Therefore, as the state approaches the origin, the inter-sampling interval becomes longer and longer, and finally goes to infinity. Since $\underline{\xi}$ decreases slower than θ as x goes to the origin (from (32)), $\underline{\xi}$ must decrease slower than $\|e_k(t)\|$. So, as x goes to the origin, $\hat{D}_k$ converges to infinity. Since both $\underline{T}_k$ and $\hat{D}_k$ go to infinity, the maximum delay $\overline{D}_k$ also goes to infinity.*

REMARK 5.3. *We don't consider the case when N_k goes to infinity as the state approaches the origin, because it is not practical to transmit infinite bits for one sampling. But if N_k goes to infinity, which is implied by the violation of (30), we can show that the stabilizing bit-rate still converges to 0 if (31) holds.*

5.2 Finite stabilizing bit-rate

THEOREM 5.4. *We assume that all the assumptions in theorem 4.1 hold. If (30) is true, and*

$$\lim_{s\to 0}\frac{\phi_c(s)}{\theta(s)} < \infty, \lim_{s\to 0}\frac{\phi_u(s)}{\theta(s)} < \infty, \lim_{x\to 0}L_{\Omega_k} = L < \infty \quad (34)$$

then there must exist a finite non-negative constant a and b such that

$$\lim_{x\to 0}\underline{r}_k = \frac{L}{\ln 2}(a(n-1)+b). \quad (35)$$

PROOF. Following the same idea in theorem 5.1, we show that N_k goes to finite. If the maximum delay $\overline{D}_k$ also converges to a non-zero constant, then the stabilizing bit-rate, $\underline{r}_k$, is finite. Taking a close look at $\underline{r}_k$, we see that $\underline{r}_k$ takes the form of $\underline{r}_k = \frac{L_{\Omega_k}}{\ln 2}(a_k(n-1)+b_k)$. If we can show that $\underline{r}_k$ converges to a finite number, then there must exist a finite constant a and b such that (35) holds.

Following the same idea in theorem 5.1, since (34), we can show that $\lim_{x\to 0}\hat{D}_k > 0$ and $\lim_{x\to 0}\underline{T}_k > 0$, and hence $\lim_{x\to 0}\overline{D}_k > 0$. □

REMARK 5.5. *Compared with theorem 5.1, theorem 5.4 is a more general case. We can see that if all the conditions in theorem 5.1 hold, then the conditions in theorem 5.4 must be satisfied. It indicates that 5.1 is a special case of 5.4, or in other words, 5.4 is a more general case of 5.1.*

REMARK 5.6. *Inequalities in(34) indicate that $\|e_k(t)\|$ decreases faster than or comparable to $\theta(\|\hat{x}_k\|)$ and $\underline{\xi}(\|\hat{x}_k\|)$ as x goes to the origin. Therefore, the minimum inter-sampling interval and the maximum delay converge to a finite number.*

REMARK 5.7. *The stabilizing bit-rates in Liberzon's work [7] and our work have the similar form. The difference is that the dimension in our case is $n-1$, and we have a bias term b. When we do the quantization, we only quantize the surface of the box $\|e_k(s_{k+1})\| \leq \theta(\|\hat{x}_k\|)$. So our quantization dimension is 1 less than the quantization dimension in [7]. Meanwhile, we need $\lceil \log_2(2n)\rceil$ bits to indicate which side $e_k(s_{k+1})$ lies on, which gives rise to the bias term b.*

5.3 Infinite stabilizing bit-rate

THEOREM 5.8. *If*

$$\lim_{s\to 0}\frac{\phi_c(s)}{\theta(s)} = \infty, \text{ or } \lim_{s\to 0}\frac{\phi_u(s)}{\theta(s)} = \infty, \tag{36}$$

then the stabilizing bit-rate goes to infinity, i.e.

$$\lim_{x\to 0}\underline{r}_k = \infty.$$

PROOF. First of all, from (24), we can see that the N_k is always positive.

Following the same steps in the proof of theorem 5.1, we can show that if (36) holds, then either $\hat{D}_k$ or $\underline{T}_k$ converges to 0, and hence $\overline{D}_k$ goes to 0.

Therefore, $\underline{r}_k$ goes to infinity. □

REMARK 5.9. *If (36) holds, $\|e_k(t)\|$ decreases slower than $\theta(\|\hat{x}_k\|)$ and $\underline{\xi}(\|\hat{x}_k\|)$ as the state goes to the origin. Therefore, the minimum inter-sampling interval and the maximum delay become shorter and shorter as x approaches the origin, and finally go to 0.*

6. SIMULATION RESULTS

In this section, we first design a threshold function θ and quantization error $\overline{e}_q$ for a nonlinear system to demonstrate theorem 4.1 and 5.1. Meanwhile, for the nonlinear system, we also compare our results with the results in Liberzon's work [7] to show that event triggered quantization can achieve better performance than the periodic quantization while using lower bit-rate than the periodic one. Then, theorem 4.1 and 5.4 are tested in a scalar linear case, and demonstrated to be true.

Now, let's consider a nonlinear system

$$\dot{x}_1 = x_1^3 + 2x_2^3 + u$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

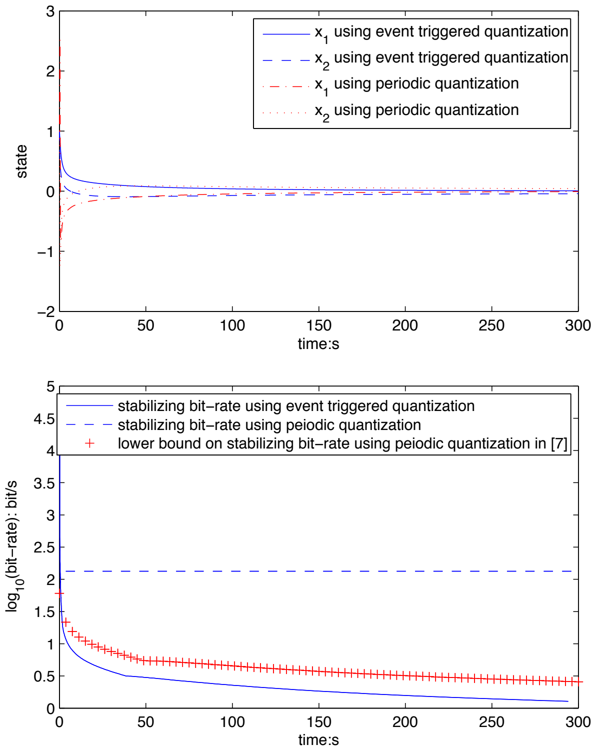

Figure 2: Zero stabilizing bit-rate

with $u = -2\hat{x}_1^3 - \hat{x}_2^3$ and $x(0) = [1; 1]$. We give the Lyapnov function as $V = x_1^4 + x_2^4$. It can be shown that $\dot{V} < 0$, if we set $\xi(s) = 0.2796s$, $\underline{\xi}(s) = 0.2185s$, $\phi_c(s) = 2s^3$, $\phi_u(s) = 3s^3$, $\theta(s) = 0.015s$, $\overline{e}_q(s) = 0.005s$. In this experiment, we set $L_{\Omega_k} = 21\|\hat{x}_k\|$.

We ran the system for 300 seconds, and always used $\overline{D}_k$ as the delay in the communication network. The state trajectories (solid line for x_1, and dashed line for x_2) are shown in the top plot in Figure 2 with x-axis the time axis and y-axis the state. They show that as time goes by, the two elements of the state go to 0 asymptotically. The bottom plot of Figure 2 shows the stabilizing bit-rate calculated from the simulation (solid line). Its x-axis indicates time, and the y-axis is $\log_{10}(\underline{r}_k)$. We can see that the stabilizing bit-rate gradually decreases to 0 as x approaches the origin. This behavior demonstrates theorem 5.1.

Moreover, we are also interested in comparing our results with Liberzon's work in [7]. To make these two works comparable, we first find the longest period to stabilize the system using periodic quantization (uniform quantization), and then we set the delay of each transmission to be one period. The longest period we found to stabilize the system is $T = 0.015s$, and the least number of bits of each transmission is $N = 2\log_2(\lceil e^{L_{\Omega_k}T}\rceil)$ if $\lceil e^{L_{\Omega_k}T}\rceil$ is odd, or $2\log_2(\lceil e^{L_{\Omega_k}T}\rceil + 1)$, otherwise (In [7], N is required to be odd). In this experiment, we still use $L_{\Omega_k} = 21\|\hat{x}_k\|$.

The state strategy (dot dashed line for x_1, and dotted line for x_2) is shown in the top plot of Figure 2. We see that it is asymptotically stable, but has very big overshoot in the transient process. Compared with the performance incurred by the periodic quantization, the event triggered quantiza-

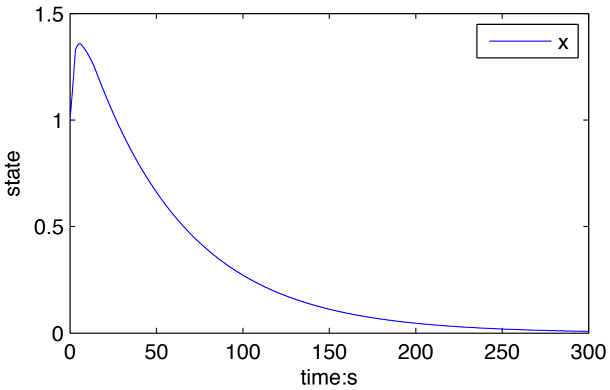

Figure 3: Finite stabilizing bit-rate

tion has smoother transient process, and almost the same convergence rate. Now, let's look at the stabilizing bit-rate using the periodic quantization. The lower bound on the stabilizing bit-rate using periodic transmission is indicated by crosses. When we look at the actual stabilizing bit-rate calculated from simulation, we find that it is more than 10 times higher than the theoretical minimum stabilizing bit-rate in the worst case, though we use the longest period and the least number of bits. That's because when we calculate N, we first compute $e^{L_{\Omega_k}T}$ which is always only a little bit above 1, and then take the ceil function of it which becomes 2, finally since it is not odd, we add 1 to it to make it odd. So we start from $e^{L_{\Omega_k}T}$ which is only a little bit above 1, and end with 3. Hence, the number of bits transmitted is always greater than the theoretical one, which results in higher stabilizing bit-rate than the theoretical lower bound on stabilizing bit-rate. This is especially true when L_{Ω_k} and T are small. If we compared the stabilizing bit-rate using periodic transmission (dashed line) with the stabilizing bit-rate using event triggered quantization (solid line), we can see that except at the very beginning of the time, our stabilizing bit-rate is always less than the stabilizing bit-rate using periodic transmission. Moreover, our stabilizing bit-rate is even less than the theoretical lower bound on the stabilizing bit-rate presented in [7]. In all, we conclude that event triggered quantization achieves better performance than the periodic quantization while using lower bit-rate than the periodic one.

Now, let's look at a linear case, we consider a system as

$$\dot{x} = 0.09x + u$$

with $u = -0.1\hat{x}_k$ and $x(0) = 1$. We choose the Lyapunov function as $V = x^2$. To achieve as small bit-rate as we can, we choose $\xi(s) = 0.1s$, $\underline{\xi}(s) = \frac{1}{11}s$, $\theta(s) = \frac{1}{25}s$ and $\overline{e}_q(s) = 0$ (in scalar case, we only need 1 bit to specify whether $e_{k-1}(s_k)$ is positive or negative). According to the dynamic of the system, $\phi_c(s) = 0.01s$, $\phi_u(s) = 0.1s$ and $L = 0.09$.

The system was run for 300 seconds. The top plot of Figure 3 shows the state trajectory of the system, which converges to the origin asymptotically but slowly. The slow behavior is mainly because when we design the threshold, our main purpose is to reduce the bit-rate as long as the asymptotic stability is guaranteed. As our bit-rate is 0.3335 bit/second, x took about 200 seconds to go below 0.05. The stabilizing bit-rate remains the same at 0.3335 bit/second for all time. That's because since every thing is linear, $\hat{D}_k$ is a constant which is always smaller than $\underline{T}_k$, and hence $\overline{D}_k$ is a constant. Since we transmit only 1 bit for every sampling, the bit-rate of the system remains the same. The upper bound on stabilizing bit-rate calculated from theorem 5.4 is 0.8142 bit/second. This is greater than the stabilizing bit-rate calculated from our experiment, which demonstrates theorem 5.4.

7. DISCUSSION AND FUTURE WORK

This paper explores a way to design the threshold function and quantization error to achieve zero or finite stabilizing bit-rate. First, a controller needs to be designed such that the closed loop system is ISS stable with respect to the local state gap $e_k = x - \hat{x}_k$. After the controller is fixed, the performance function ξ and $\underline{\xi}$, an upper bound on the closed loop dynamic behavior ϕ_c, and an upper bound on control input ϕ_u are determined. Based on $\underline{\xi}$, ϕ_c and ϕ_u, the threshold function are chosen such that the conditions in theorem 5.1 or 5.4, and (11) are all satisfied. Once the threshold function is chosen, the quantization error is decided to make sure that (12) and (30) hold.

In this paper, we give a quantization map without fully discussing whether this is the best choice such that the number of bits N_k is minimized at each sampling time. The quantization map that we present in this paper only quantizes the surface of an uncertainty set. We are not sure that this quantization map always uses smallest number of bits than any other quantization maps to achieve the same quantization level. The fully discussion of how to choose the optimal quantization map with event triggering law such that N_k is minimized while the maximum quantization error $\bar{e}_q$ is guaranteed will be in our future work.

In future work, we will also study the case when there is system noise. Some interesting questions arise. These questions include how the noise influences the stabilizing bit-rate, whether the same conditions assure finite stabilizing bit-rate, how the minimum inter-sampling interval and the maximum delay changes, and so on.

The work in this paper can be used as a basis to study the scheduling problem when there are several controllers sharing the same communication network. For one controller, we know the steady stabilizing bit-rate, the number of bits to be transmitted, and the maximum delay of the package. When there are several controllers sharing the same communication network, it's important to schedule the transmissions of the controllers such that all the controllers have enough information to stabilize their plants while the communication limit of the network is not exceeded.

8. ACKNOWLEDGEMENT

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