Distributed Switching Control to Achieve Almost Sure Safety for Leader-Follower Nonholonomic Systems

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***Abstract*—Leader-follower formation control is a widely used distributed control strategy that needs systems to exchange their information over a wireless radio communication network to attain and maintain formations. These wireless networks are often subject to *deep fades*, where a severe drop in the quality of the communication link occurs. Such deep fades inevitably inject a great deal of stochastic uncertainties into the system, which significantly impacts the system's performance and stability, and causes unexpected safety problems in application like smart transportation system. Assuming an exponentially bursty channel that varies as a function of the vehicular states, this paper proposes a distributed switching control scheme under which the local controller is reconfigured in response to the changes of channel state, to assure *almost sure safety* for a chain of leader follower nonholonomic system. Here *almost sure safety* means that the likelihood of vehicular states entering a safe region asymptotically goes to one as time goes to infinity. Sufficient conditions are provided for each local vehicle to decide which controller is placed in the feedback loop to assure *almost sure safety* in the presence of *deep fades*. The simulation results of a chain of leader follower formation are used to illustrate the findings.**

I. Introduction

In the past decade, *leader follower formation control* has found extensive applications in industry and academia [1], [5], [9]–[11]. In formation control, agents coordinate with each other to form and maintain a specified formation. The coordination is often conducted distributedly over a wireless radio communication network. It is well known that such communication networks are subject to deep fading, which causes a severe drop in the network's quality-of-service (QoS). These deep fades induce a great amount of stochastic uncertainties into the system, and negatively impact the formation's performance and stability by interfering with the coordination between agents. The loss of coordination may cause serious safety issues in applications like smart transportation system [16], unmanned aerial vehicles system [14] and underwater autonomous vehicles [13]. These issues can be addressed by developing a distributed switching control system that detects such deep fades and adaptively reconfigures its controller to enforce a minimum safety requirement.

In real application, the safety issue is often examined in a stochastic setting by discussing the likelihood of a system state entering a forbidden or unsafe region. Traditionally, this has been done using mean square concepts in which the variance of some important system state, such as inter-vehicle distance, remains bounded. Such a concept is also analogous to the notion of *stochastic safety in probability* [12]. The common feature of above work is that they bound the likelihood of unsafe action occurring with a nonzero value, which still allows a finite probability for the system to be unsafe. This mean square safety or *stochastic safety in probability* criterion is simply not appropriate for many safety-critical systems such as smart transportation system where a small probability of danger can incur catastrophic failure. This paper suggests using a stronger notion of *almost sure safety* to assure the system state asymptotically goes to a safe equilibrium or a bounded safe set with probability one as time goes to infinity. In particular, *almost sure safety* in this paper refers to two strong notions of stochastic stability: *almost sure asymptotic stability* and *almost sure practical stability* [8].

The channel model that is used to attain *almost sure stability* must be carefully specified. Traditionally, this has been done by modeling channel fading as an *independent and identical distributed (i.i.d)* random process having either a Rayleigh or Rician distribution. This characterization might be reasonable for most stationary wireless networks, the use of i.i.d model is questionable in vehicular communication since the channel state is functionally dependent on the vehicle's physical state [2], [4]. A more realistic fading channel model was examined in [6], in which the channel is exponentially bursty and is dependent on the norm of the physical system's states. Such model is often referred to *exponentially bounded burstiness* (EBB) [18], and is more general in the sense that it can characterize the i.i.d channels as well as bursty channels that are often modeled as a two state Markov chain [17].

By using the EBB model that is functionally dependent on the vehicular state, this paper develops a distributed switching control scheme to assure *almost sure safety* for a chain of leader follower nonholonomic systems. The leader follower chain consists of a collection of leader follower pairs that require each follower to manipulate its linear and angular velocity to achieve and maintain a desired separation and relative bearing. The information of leader's bearing angle is transmitted over an exponentially bursty channel, which is accessed by a directional antenna mounted on each leading vehicle in the chain [7]. The stochastic uncertainty resulting from *deep fades* prevents each leader follower subsystem from maintaining formation safely. The cascaded structure of the leader follower chain exacerbates such uncertainty from upper system to the lower system, and therefore leads to catastrophic failure for the entire system. This paper proposes two switching rules to recover the safe behavior of the leader follower chain by adaptively selecting local controller in response to the changes of channel state, and by enforcing the upper systems to constrain their control actions as a function of the lower system's states. Sufficient conditions are provided for each vehicle to decide which controller is placed in the feedback loop to assure *almost sure asymptotic stability* and *almost sure practical stability* for the entire leader follower chain.

The layout of the paper is as follows. Section II introduces mathematical notations. Section III provides a system description and problem formulation. After that, Section IV discusses the main results. Then, Section V presents the simulation results of a leader follower chain with four vehicles. Finally, Section VI concludes the paper.

II. Mathematical Preliminaries

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and real numbers, respectively. Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and real numbers, respectively. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space. The ∞-norm on the vector $x \in \mathbb{R}^n$ is $|x| = \max |x_i| : 1 \leq i \leq n$, and the corresponding induced matrix norm is $\|A\| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |A_{ij}|$. Let $f(t) \in \mathbb{R}^n$ denote the value that function f takes at time $t \in \mathbb{R}$. Let $\{\tau_k\}_{k=0}^{\infty}$ denote a strictly monotone increasing sequence with $\tau_k \in \mathbb{R}_+$ for all $k \in \mathbb{Z}_+$ and $\tau_k < \tau_{k+1}$. Then, $f(\tau_k)$ denotes the value of function f at time τ_k. For simplicity, we

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let $f(k)$ denote $f(\tau_k)$ if its meaning is clear in the context. The left-hand limit at $\tau_k \in \mathbb{R}_+$ of a function $f(\cdot): \mathbb{R} \to \mathbb{R}^n$ is denoted by $f(k^-)$. Similarly, the right-hand limit of the function $f(k)$ is denoted by $f(k^+)$.

Consider a continuous-time random process $\{x(t) \in \mathbb{R}^n : t \in \mathbb{R}_+\}$ whose sample paths are right-continuous and satisfy the following differential equation,

$$\dot{x}(t) = f(x(t), u(t), w(t), d(t)) \tag{1}$$

where $u(\cdot): \mathbb{R}_+ \to \mathbb{R}^m$ is a control input, $d(t)$ is an external $\mathscr{L}_\infty$ disturbance with $|d(t)|_{\mathscr{L}_\infty} = D$ and $w(t)$ is a jump process

$$w(t) = \sum_{\ell=1}^{\infty} w_\ell \delta(t - \tau_\ell) \tag{2}$$

in which $\{w_\ell, \ell \in \mathbb{Z}_+\}$ is a Markov process describing the ℓth jump's size at jump instants $\{\tau_\ell\}_{\ell=1}^{\infty}$. The expectation of this stochastic process at time t will be denoted as $\mathbb{E}(x(t))$.

Let x^* be the equilibrium of system (1) with $f(x^*, 0, 0, 0) = 0$. The system in equations (1-2) is said to be *almost-surely asymptotically stable* with respect to x^*, if

$$\lim_{t\to\infty} \Pr\left\{ \sup_t |x(t)| \to x^* \right\} = 1$$

Given a constant positive $\Delta^* \in \mathbb{R}_+$, let $\Omega(\Delta^*)$ be a bounded set defined as $\Omega(\Delta^*) = \{x \in \mathbb{R}^n | |x - x^*| \le \Delta^*\}$. The system in equations (1-2) is said to be *almost-surely practical stable* with respect to $\Omega(\Delta^*)$, if there exists $\Delta > 0$ with $\Delta^* > \Delta$ such that if $|x(0) - x^*| \le \Delta$, then

$$\lim_{t\to\infty} \Pr\left\{ \sup_t |x(t)| \in \Omega(\Delta^*) \right\} = 1$$

The system in equations (1-2) is *almost sure safe* if it is *almost surely asymptotically stable* with respect to equilibrium x^* or *almost surely practical stable* with respect to set $\Omega(\Delta^*)$. x^* is called safe equilibrium, and the states in set $\Omega(\Delta^*)$ are safe states.

III. System Description

A. System Model

Figure 1 shows a string formation of N mobile robots. For each mobile robot, we consider the following kinematic model,

$$\dot{x}_i = v_i \cos(\theta_i),\ \dot{y}_i = v_i \sin(\theta_i),\ \dot{\theta}_i = \omega_i,\ i = 0, 1, \ldots, N-1 \tag{3}$$

where $(x_i(t), y_i(t))$ denotes the vehicle i's position at time $t \in \mathbb{R}_+$, $\theta_i(t)$ is the orientation of the vehicle relative to the x axis at time t. v_i and ω_i are the vehicle's speed and angular velocity that represent the control input.

As shown in Figure 1, the cascaded formation with N mobile robots consists of $N-1$ leader-follower pairs. In each leader-follower pair j, we assume that the leader can directly measure its relative bearing angle α_j to the follower. Similarly, the follower can measure its bearing angle ϕ_j to the leader. Both of the vehicles are able to measure the relative distance L_j. What is not directly known to the follower is the relative bearing angle α_j. In this paper, we consider the case when information about leader's bearing angle α_j is transmitted over a wireless channel. The channel is accessed through a directional antenna whose radiation pattern is shown in Figure 2.

The control objective of the cascaded formation is to have the follower in each leader-follower pair to regulate its speed and angular velocity to achieve and maintain a desired distance and bearing angle. Let L_{d_j} and α_{d_j} denote the desired inter-vehicle distance and relative bearing angle, respectively, in the jth leader-follower pair. It will therefore be convenient to characterize the time rate of change of the

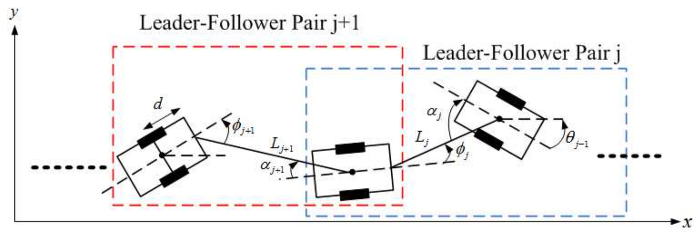

Fig. 1. A cascaded formation of nonholonomic vehicular system

relative distance L_j and leader's relative bearing angle α_j as follows [5]

$$\begin{aligned} \dot{L}_j &= v_{j-1}\cos\alpha_j - v_j \cos\phi_j - d\omega_j \sin\phi_j \\ \dot{\alpha}_j &= \frac{1}{L_j}\left(-v_{j-1}\sin\alpha_j - v_j \sin\phi_j + d\omega_j\cos\phi_j\right) + \omega_{j-1} \end{aligned} \tag{4}$$

where d is the distance from the vehicle's center to its front.

B. Information Structure

As discussed in the previous section, the leader's bearing angle α_j in each leader-follower pair must be transmitted to the follower over a wireless channel. In this regard, the information about α_j that is available to the follower is limited by the following two constraints,

- The state measurement $\alpha_j(t)$ is only taken at a sequence of time instants $\{\tau_k\}_{k=0}^{\infty}$ that satisfies $\tau_k < \tau_{k+1}, k = 1, 2, \ldots, \infty$.
- The sampled data $\alpha_j(\tau_k)$ is quantized with a finite number of bits $\bar{R}_j$, and is transmitted over an unreliable wireless channel with only first $R_j(k)$ bits ($R_j(k) \le \bar{R}_j$) received at the follower.

At kth sampling time instant, the triple $\{\hat{\alpha}_j(k^-), U_j(k), c_j(k)\}$ characterizes the information structure of the leader's bearing angle $\alpha_j(\tau_k)$ at the leader side. Assume that the measurement $\alpha_j(\tau_k)$ lies in an interval $[-U_j(k) + \hat{\alpha}_j(k^-), U_j(k) + \hat{\alpha}_j(k^-)]$ with $\hat{\alpha}_j(k^-)$ representing the "center" of the interval and $U_j(k)$ representing the length of the interval. The codeword $c_j(k) = \{b_{jl}(k)\}_{l=1}^{\bar{R}_j}$ consists of bits $b_{jl}(k) \in \{-1, 1\}$, and is constructed by truncating the first $\bar{R}_j$ bits of the following infinity bit sequence

$$\{\{b_{jl}(k)\}_{l=1}^{\infty} \in \{-1,1\}^\infty | \alpha_j(\tau_k) = \hat{\alpha}_j(k^-) + U_j(k) \sum_{l=1}^{\infty} \frac{1}{2^j} b_{jl}(k)\}$$

This corresponds to a uniform quantization of the sampled state within the interval $[-U_j(k) + \hat{\alpha}_j(k^-), U_j(k) + \hat{\alpha}_j(k^-)]$ with $\bar{R}_j$ number of bits.

We assume that the follower only successfully receives the first $R_j(k)$ bits in the codeword $c_j(k)$. The information structure at the follower side is another triple $\{\hat{\alpha}_j(k), U_j(k), \hat{c}_j(k)\}$ with $\hat{c}_j(k) = \{b_{jl}\}_{l=1}^{R_j(k)}$ and $\hat{\alpha}_j(k)$ being constructed as follows

$$\hat{\alpha}_j(k) = \hat{\alpha}_j(k^-) + U_j(k) \sum_{l=1}^{R_j(k)} \frac{1}{2^j} b_{jl}(k). \tag{5}$$

$\hat{\alpha}_j(k)$ is an estimate of the leader's bearing angle $\alpha_j(k)$ at time instant τ_k.

In order to reconstruct the estimate $\hat{\alpha}_j(k)$, it is necessary to synchronize the leader and follower in the sense that they have the same information structure. We assume a noiseless feedback channel, with each successfully received bit being acknowledged to the leader. This allows one to ensure that the information structures are synchronized between the leader and follower. The follower then uses the estimated bearing angle $\hat{\alpha}_j(k)$, and the measured inter-vehicle distance L_j, to select its speed, v_j, and angular velocity ω_j to achieve the control objective.

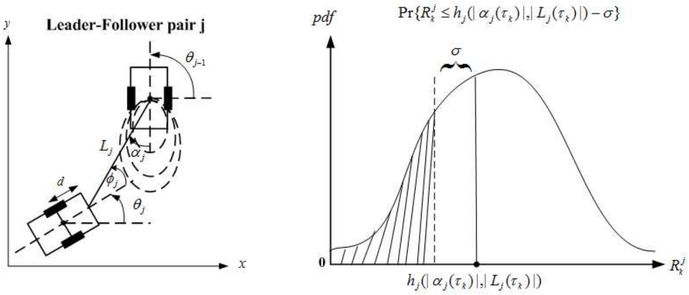

Fig. 2. Exponential Bounded Burstiness (EBB) Model for directional wireless channel

C. Wireless Channel

As shown in Figure 2, the leading vehicle in each pair uses a directional antenna to access the wireless channel. We assume the channels are free of interference from other leader-follower pairs, but the channel does exhibit deep fading. Deep fades occur when the channel gain drops below a threshold and stays below that threshold level for a random interval of time. Such fades are often modeled using two-state Markov chains [17].

We adopt an exponentially bounded burstiness (EBB) characterization of the fading channel. In particular, let $h(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ denote continuous, positive and monotone decreasing functions from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$. Assume the probability of successfully decoding $R_j(k)$ bits at each sampling time τ_k satisfies

$$\Pr\left\{R_j(k) \leq h(|\alpha_j(\tau_k)|, |L_j(\tau_k)|) - \sigma\right\} \leq e^{-\gamma(|\alpha_j(\tau_k)|,|L_j(\tau_k)|)\sigma} \quad (6)$$

for $|\alpha_j(\tau_k)| \leq \pi/2$ and $\sigma \in [0, h(|\alpha_j(\tau_k)|, |L_j(\tau_k))|]$ with

$$\Pr\left\{R_j(k) = 0\right\} = 1 \quad (7)$$

for $|\alpha_j(\tau_k)| > \pi/2, \forall k \in \mathbb{Z}_+$. We say such channels exhibit exponentially bounded burstiness (EBB). EBB characterizations can be used to describe a wide range of Markov channel models that include traditional i.i.d models as well as two-state Markov chain models. The analysis methods in this paper apply to a wide range of realistic channel conditions.

The equations (6) and (7) characterize the fact that if the follower vehicle is out of the antenna's radiation scope, i.e. $|\alpha_j(\tau_k)| > \pi/2$, then the communication link between the vehicles is broken. If the vehicle is within the scope, i.e. $|\alpha_j(\tau_k)| \leq \pi/2$, the probability of having a low bit rate is exponentially bounded.

As shown in Figure 2, the function $h(|\alpha_j|, |L_j|)$ in EBB model may be seen as a threshold characterizing the low bit rate region as a function of current formation's state. The exponent associated with exponential decrease is represented by a similar function $\gamma(|\alpha_j|, |L_j|)$. The two functions play different roles in the EBB model. Function $h(|\alpha_j|, |L_j|)$ characterizes the fact that as the absolute value of the formation's state L and α increase, the low bit rate threshold shrinks and moves toward the origin. Such activity can be induced due to path loss that is widely considered in the wireless communication community. On the other hand, the function $\gamma(|\alpha_j|, |L_j|)$ in the exponential bound models the fact that the likelihood of exhibiting a low bit rate increases as the formation state is away from the origin.

What should be apparent from the EBB model is that we are explicitly accounting for the relationship between channel state ($R_j(k)$) and formation configuration. A major goal of this paper is to exploit that relationship in deciding how to switch between different controllers to assure almost sure performance.

D. Distributed Switching Control

In this paper, the control objective is to steer the cascaded vehicular system shown in Figure 1 to a sequence of desired distances $\{L_{d_j}\}_{j=1}^{N-1}$ and bearing angles $\{\alpha_{d_j}\}_{j=1}^{N-1}$ in a distributed fashion, and then maintain around those set-points.

At each time instant $\{\tau_k\}_{k=0}^{\infty}$, the follower of each leader-follower pair switches among a group of controller gains to regulate its velocity and angular velocity to achieve the control objective. Let $K(k) := \{K_{\alpha_j}(k), K_{L_j}(k)\}$ denote the controller gain pair used for leader-follower pair j at time instant τ_k. These controller gains are selected from one pair of a collection of values $\mathscr{K}_j = \{K_{j_1}, K_{j_2}, \ldots, K_{j_M}\}$. Recall that the dynamics of formation configuration is equation (4), we use standard input to state feedback linearization to generate the control input

$$\begin{bmatrix} v_j \\ \omega_j \end{bmatrix} = \begin{bmatrix} -\cos\phi_j & -L_j \sin\phi_j \\ -\frac{\sin\phi_j}{d} & \frac{L_j}{d}\cos\phi_j \end{bmatrix} \begin{bmatrix} K_{L_j}(k)(L_{d_j} - L_j) \\ K_{\alpha_j}(k)(\alpha_{d_j} - \hat{\alpha}_j) \end{bmatrix} \quad (8)$$

over the time interval $[\tau_k, \tau_{k+1})$. The variable $\hat{\alpha}_j(t)$ is a continuous function over $[\tau_k, \tau_{k+1})$, and satisfies the following initial value problem,

$$\dot{\hat{\alpha}}_j = K_{\alpha_j}(k)(\alpha_{d_j} - \hat{\alpha}_j), \hat{\alpha}_j(\tau_k) = \hat{\alpha}_j(k) \quad (9)$$

where the estimate $\hat{\alpha}_j(k)$ is obtained from equation (5). With this control, the inter-vehicle distance L_j and bearing angle α_j satisfy the following differential equations over $[\tau_k, \tau_{k+1})$,

$$\begin{bmatrix} \dot{L}_j \\ \dot{\alpha}_j \end{bmatrix} = \begin{bmatrix} \cos\alpha_j & 0 \\ \frac{-\sin\alpha_j}{L_j} & 1 \end{bmatrix} \begin{bmatrix} v_{j-1} \\ \omega_{j-1} \end{bmatrix} + \begin{bmatrix} K_{L_j}(k)(L_{d_j} - L_j) \\ K_{\alpha_j}(k)(\alpha_{d_j} - \hat{\alpha}_j) \end{bmatrix} \quad (10)$$

for all $k = 1, 2, \ldots, \infty$.

The equations (9-10) represent the closed-loop system for the leader-follower pair j and can be viewed as an example of a jump nonlinear system given in equations (1-2). The $\mathscr{L}_\infty$ disturbance in the jth leader-follower system is $[v_{j-1}, \omega_{j-1}]$. The estimate of the bearing angle $\hat{\alpha}_j$ forms a jump process with jumps occurring at discrete time instants $\{\tau_k\}_{k=1}^{\infty}$. As shown in equation (5), the magnitude of the jump at each time instant is stochastically governed by the length of the uncertainty interval $U_j(k)$ and the number of received bits $R_j(k)$. Such jump process significantly impacts the formation performance of the cascaded system by pushing the formation state away from the equilibrium, which in turn leads to deep fades with a high probability. In the next section, we will show how to reconfigure the local controller gain in response to the changes of $U_j(k)$ and $R_j(k)$ such that almost sure performance is assured.

It is apparent from Figure 1 that vehicle j for $j = 1, 2, \ldots, N-2$ plays a leader in leader-follower pair $j+1$ as well as a follower in leader-follower pair j. In this regard, vehicle j could observe the full state α_{j+1} of the leader-follower subsystem $j+1$ because it serves the leadership in that system. By observing the behavior of the following vehicle, vehicle j for $j = 1, 2, \ldots, N-1$ can adjust its controller gain to overcome large overshoots in the following system. Such cooperative control strategy lessens the amplification on the disturbance from the upper leader-follower systems to the lower systems.

IV. Main Results

This paper's main results consist of two parts regarding the *safe* behavior of inter-vehicle distance L_j and bearing angle α_j for each leader-follower pair. Specifically, "safe" means that the vehicle does not collide with each other and the bearing angle is regulated to within the communication range almost surely. The first part of the results provides a sufficient condition under which the inter-vehicle distance L_j for $j=1,2,\dots,N-1$ is almost surely convergent to a compact invariant set regardless of the changes on channel state. Furthermore, we show that the inter-vehicle distance is almost surely convergent to the desired separation $L_{d_j}, j=1,2,\dots,N-1$ if the bearing angle $\alpha_j, j=1,2,\dots,N-1$ is almost surely convergent. The second part of the results derive sufficient conditions for the almost sure asymptotic stability and practical stability for the bearing angle $\alpha_j, j=1,2,\dots,N-1$.

In the main results, we use the fact that the leader's action in each leader-follower pair can be constrained as a function of the following system's state to assure the stability for the whole leader-follower system. Proposition IV.1 provides an explicit characterization of the bound on the leader's action, as well as a distributed way to achieve that bound. Using the results from Proposition IV.1, one can easily prove the first main result in this paper (Lemma IV.6), i.e. the convergence of inter-vehicle distance since the distance is measurable to both leader and follower. The more challenging and interesting part of the results is to guarantee the almost sure stability for the bearing angle α_j, which is presented in section IV-B.

The following Proposition is provided to assure the control input from upper leader-follower subsystem is bounded as a function of state estimates of the bottom system. The proof is provided in Appendix.

Proposition IV.1. *Consider the closed-loop system in equations (9-10), let $d\geq 1$, if there exists a sequence of controller gains $\{K_j(k)\}_{k=0}^{\infty}$, $K_j(k)=\{K_{L_j}(k),K_{\alpha_j}(k)\}\in\mathscr{K}_j$ such that for given monotone increasing functions $W_j(\cdot):\mathbb{R}_+\to\mathbb{R}_+, j=1,2,\dots,N-1$, the following inequality holds for all $k=0,1,\dots,\infty$*

$$\max\left\{\tilde{L}_{j,max},K_{\alpha_j}(k)|\tilde{\alpha}_j(k)|\right\}\leq\frac{W_j(|\tilde{\alpha}_{j+1}(k)|)}{(1+M_{L_j}(k))}\tag{11}$$

Where

$$\begin{aligned}\tilde{L}_{j,max}&=K_{L_j}(k)|\tilde{L}_j(k)|e^{K_{L_j}(k)T_k}+W_{j-1}(|\tilde{\alpha}_j(k)|)\left(e^{K_{L_j}(k)T_k}-1\right)\\ M_{L_j}(k)&=\max\left\{\overline{L}_j(\tau_k),\overline{L}_j(\tau_{k+1})\right\}\\ \overline{L}_j(t)&=\left(L_{d_j}+\frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}\right)\left(1-e^{-K_{L_j}(k)(t-\tau_k)}\right)\\ &\quad+L_j(k)e^{-K_{L_j}(k)(t-\tau_k)}\\ \tilde{\alpha}_j(k)&=\alpha_{d_j}-\hat{\alpha}_j(k),\quad\tilde{L}_j(k)=L_{d_j}-L_j(k)\end{aligned}$$

then

$$\sup_t\left|\left[\begin{array}{c}v_j(t)\\ \omega_j(t)\end{array}\right]\right|\leq W_j(|\tilde{\alpha}_{j+1}(k)|),t\in[\tau_k,\tau_{k+1})\tag{12}$$

Because of inequality (12), each leader-follower subsystem j in equation (10) can bound the external disturbance $[v_{j-1},\omega_{j-1}]$ by observing its local state estimate $\tilde{\alpha}_j$ at each time instant τ_k. Meanwhile, the subsystem $j-1$ can select its controller gain so that the control input $[v_{j-1},\omega_{j-1}]$ satisfies the bound in inequality (12) because the estimate of bearing angle $\tilde{\alpha}_j$ is always available to subsystem $j-1$. Such property provides a basis to design a distributed and cooperative switching law to assure the stability for the whole formation system.

Remark IV.2. *Functions $W_j(\cdot)$ are upper bounds on the control inputs of upper leader-follower system and the values of $W_j(\cdot)$ at each time instant τ_k can also be seen as feedback signals from the bottom system. Such feedback signals directly constrain the magnitude of control input from upper system, so that the disturbances are not amplified from upper system to bottom system.*

Remark IV.3. *The inequality (11) could be viewed as a switching rule for the leader-follower pair j to react to the changes on system $j+1$'s bearing angle. The switching rule applied over each time interval $[\tau_k,\tau_{k+1})$ is feasible because it is only based on the information that is available at time τ_k.*

With the validity of Proposition IV.1, the following corollary characterizes the propagated bound on the external inputs of the leader follower chain as a function of the bearing angle's estimate in each leader follower pair.

Corollary IV.4. *Suppose the hypothesis of Proposition IV.1 holds then,*

$$\max\{|v_0(k)|,|\omega_0(k)|\}\leq\tilde{W}_0\circ\tilde{W}_1^k\circ\cdots\circ\tilde{W}_j^k(|\tilde{\alpha}_{j+1}(k)|)\tag{13}$$

where $v_0(k)$ and $\omega_0(k)$ are the speed and angular velocity of the first vehicle in the chain, and

$$\tilde{W}_j^k(\cdot):=\frac{1}{(1+M_{L_j}(k))K_{\alpha_j}(k)}W_j(\cdot),j=1,\dots,N-2\tag{14}$$

Proof: Consider the first leader follower pair, the Proposition IV.1 implies

$$\max\{|v_0(k)|,|\omega_0(k)|\}\leq W_0(|\tilde{\alpha}_1(k)|)$$

Since

$$(1+M_{L_1}(k))K_{\alpha_1}(k)|\tilde{\alpha}_1(k)|\leq W_1(|\tilde{\alpha}_2(k)|)$$

by inequality (11), then

$$\max\{|v_0(k)|,|\omega_0(k)|\}\leq W_0\circ\tilde{W}_1^k(|\tilde{\alpha}_2(k)|)$$

Repeating above procedure leads to the final conclusion (13). ∎

Remark IV.5. *Inequality (13) implies: 1) The amplitude of inputs $(v_0(k),\omega_0(k))$ to the leader follower chain must approach to zero when the length of chain N increases to infinity, which is necessary to avoid string instability. 2) In order to assure almost sure stability for bearing angle α_j, the velocities of the first vehicle $(v_0(k),\omega_0(k))$ must varnish as time goes to infinity. However, this is not necessary for almost sure practical stability to hold.*

A. Almost Sure Convergence of Inter-vehicle Distance L_j

In this section, we present the first main results of this paper involving the almost sure convergence of inter-vehicle distance. First, the following lemma provides a sufficient condition on the controller gain K_{L_j}, under which one can show $L_j(t)$ converges at an exponential rate to an invariant set $\Omega_{\text{inv,j}}$ centered at the desired inter-vehicle distance L_{d_j}, for $j=1,2,\dots,N-1$ regardless of the change on channel state.

Lemma IV.6. *Let the hypothesis of proposition IV.1 hold, consider the system (9-10) with the selected controller gain $\{K_{L_j}(k),K_{\alpha_j}(k)\}\in\mathscr{K}_j$. If $K_{L_j}(k)>\frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho(L_{d_j}-d)}$ and $L_j(0)>d$, then for any sample path, $L_j(t)\geq d$ for all $t\in\mathbb{R}_+$ and there exists a finite time $\overline{T}>0$ such that $L_j(t)$ enters and remains in the set*

$$\Omega_{\text{inv,j}}\equiv\left\{L_j\in\mathbb{R}_+\,|\,|L_j-L_{d_j}|\leq\frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho K_{L_j}(k)}\right\}$$

for all $t\geq\overline{T}$ and any $\rho\in(0,1]$.

Proof: Consider the function $V(L_j)=\frac{1}{2}(L_j-L_{d_j})^2$ and closed-loop state equation (10). Taking the directional derivative of V over time interval $[\tau_k,\tau_{k+1})$ one obtains

$$\begin{aligned}\dot{V}(L_j)&=-K_{L_j}(L_j-L_{d_j})^2+(L_j-L_{d_j})\cdot v_{j-1}\cos\alpha_j\\ &\leq-K_{L_j}(1-\rho)(L_j-L_{d_j})^2-\rho\cdot K_{L_j}(L_j-L_{d_j})^2+|L_j-L_{d_j}|W_j(|\tilde{\alpha}_j(k)|)\end{aligned}$$

for any $\rho \in (0,1]$. The last inequality holds because of proposition IV.1. When $|L_j - L_{d_j}| \geq \frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho K_{L_j}}$, the following dissipative inequality holds,

$$\begin{aligned} \dot{V}(L_j) &\leq -K_{L_j}(1-\rho)(L_j - L_{d_j})^2 \\ &= -2K_{L_j}(1-\rho)V(L_j) \end{aligned} \tag{15}$$

This is sufficient to imply that $V(L_j(t))$ is an exponentially decreasing function of time that enters the set $\Omega_{\text{inv,j}}$ in finite time. $L_j(t) > d$ for all time since all L_j in $\Omega_{\text{inv,j}}$ satisfy

$$L_j \geq -\frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho K_{L_j}} + L_{d_j} > d \tag{16}$$

Since the time interval $[\tau_k, \tau_{k+1})$ is selected arbitrarily, the conclusion holds for any $k \in \mathbb{Z}_+$. ■

Remark IV.7. *Note that d is the distance from the center of the vehicle to the front of the vehicle. As shown in Figure 1, $L_j(t) > d$ simply means that the two vehicles do not collide with each other.*

Corollary IV.8. *Consider closed-loop system equations (9-10), let the hypotheses of Proposition IV.1 and Lemma IV.6 hold. If the bearing angle α_j is almost surely convergent to α_{d_j} with $W_{j-1}(0) = 0, j = 1,2,\ldots,N-1$, then the separation distance L_j almost surely converges to L_{d_j}.*

Proof: From Lemma IV.6, one knows that the inter-vehicle separation converges to a invariant set with size of $\frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho K_{L_j}(k)}$. With $W_{j-1}(0) = 0, j = 1,2,\ldots,N-1$, and $\lim_{k\to\infty} \Pr\{\alpha_j(k) \to \alpha_{d_j}\} = 1$, it is easy to show that the event $\lim_{k\to\infty} \frac{W_j(|\tilde{\alpha}_j(k)|)}{\rho K_{L_j}(k)} = 0$ occurs with probability one as time goes infinity, i.e. the separation $L_j(t)$ almost surely converges to L_{d_j}. ■

B. Almost Sure Asymptotic Stability and Practical Stability for Bearing Angle α_j

This section provides the second main result of this paper that assures almost sure asymptotic stability and almost sure practical stability for the bearing angle α_j. Figure 3 is used to interpret the basic idea and results. Two types of sets are depicted in Figure 3 with one enclosed by the blue curve, and the other one enclosed by the red curve. The blue curve enclosed set represents the partition generated by inequality $G(|\alpha_j|,|L_j|) \leq \eta_j$ with associated threshold $\eta_j \in (0,1)$, which is shown in Lemma IV.10. The red-curve enclosed area characterizes the target set where the system trajectory will converge to almost surely. The size of the target set is characterized by Δ_j^*. The almost sure asymptotic stability result is interpreted as a special case when the target set contains only origin.

The main result states that the bearing angle α_j will almost surely converge to the target set if the system trajectory enters and remains in the set enclosed by the blue curve. To assure the invariance of the blue curve enclosed set, we adopt a switching control strategy to reconfigure the control gain for each leader-follower pair. Figure 3 shows one possible evolution of the system trajectory α_j and L_j with the switching strategy. We use black dots to represent the estimates of the bearing angle $\hat{\alpha}_j(\tau_k)$ at each sampling time τ_k. A bar is used to characterize the uncertainty interval with the estimate $\hat{\alpha}_j(\tau_k)$ as its center. The length of bar can be viewed as an upper bound of the quantization error $|\alpha_j(\tau_k) - \hat{\alpha}_j(\tau_k)|$, and increases as the channel condition decreases. Therefore, the basic idea for switching is that when the system trajectory approaches the blue set's boundary with an increasing uncertainty length, an appropriate controller is re-selected to assure the stochastic variation on the uncertainty length satisfies a super-martingale inequality, which guarantees the system states converge to the target set with probability one.

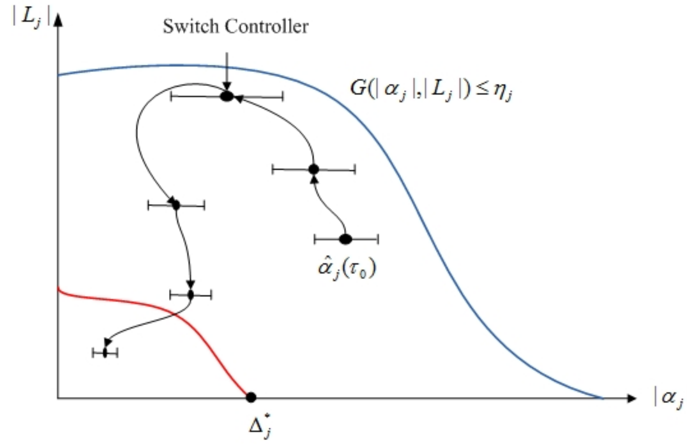

Fig. 3. Partition of formation state space.

To be more specific about the main result, first, a dynamic quantization method is used to show that the quantization error $|\alpha_j(\tau_k) - \hat{\alpha}_j(\tau_k)|$ can be bounded by a sequence that is recursively constructed (Lemma IV.9). Then, a sufficient condition is presented to select controllers, under which the sequence (Lemma IV.10) and bearing angle estimate (Lemma IV.12) satisfy supermartingale like inequalities. Finally, the super-martingale inequality condition leads to the proof of almost sure asymptotic stability (Theorem IV.13) and practical stability (Theorem IV.15) for the bearing angle α_j.

Recall that $\{\alpha_j(k^-), U_j(k)\}_{k=0}^{\infty}$ characterizes the quantizer's state at each time instance τ_k. The following lemma gives a recursive construction for this sequence such that the quantization error remains bounded by some function of $U_j(k)$ for all $k \geq 0$. Such predictable bound is used to switch controllers to assure almost sure performance. Note that the technique used to prove the Lemma follows the pattern in traditional dynamical quantization [3], [15].

Lemma IV.9. *Consider the closed-loop system (9-10), given the transmission time sequence $\{\tau_k\}_{k=0}^{\infty}$, and controller pairs $\{K_{L_j}(k), K_{\alpha_j}(k)\}_{k=0}^{\infty}$. Let $T_k = \tau_{k+1} - \tau_k$, let the hypothesis of proposition IV.1 and Lemma IV.6 hold, the initial ordered pair $\{\hat{\alpha}_j(0), U_j(0)\}$ is known to both leader and follower, and the initial state $\alpha_j(0) \in [-U_j(0), U_j(0)]$, $U_j(0) \leq \frac{\pi}{2}$. If the sequence $\{\alpha_j(k^-), U_j(k)\}_{k=0}^{\infty}$ is constructed by the following recursive equation,*

$$\begin{aligned} U_j(k+1) &= B_j(k)T_k + 2^{-R_j(k)}U_j(k) \quad (17) \\ \hat{\alpha}_j(k+1^-) &= (\hat{\alpha}_j(k^+) - \alpha_{d_j})e^{-K_{\alpha_j}(k)T_k} + \alpha_{d_j} \quad (18) \end{aligned}$$

where

$$B_j(k) = \max\left\{\frac{1}{\min\{L_{jmin}, L_j(k)\}}, 1\right\} W_{j-1}(|\tilde{\alpha}_j(k)|)$$

$$L_{jmin} = \left[-\tilde{L}_j(k) + \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}\right] e^{-K_{L_j}(k)T_k} + L_{d_j} - \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}$$

$$\tilde{L}_j(k) = L_{d_j} - L_j(k)$$

then the bearing angle $\alpha_j(k)$ for all $j = 1,2,\ldots,N-1$ generated by system equations (9-10) can be bounded as

$$|\alpha_j(k) - \hat{\alpha}_j(k^+)| \leq \overline{U}_j(k) \tag{19}$$

where $\overline{U}_j(k) = 2^{-R_j(k)}U_j(k)$ and $R_j(k)$ is the number of bits received over the time interval $[\tau_k, \tau_{k+1})$.

Proof: Let $e_j(t) = \alpha_j(t) - \hat{\alpha}_j(t)$ denote the estimation error. By inequality $\frac{d|e_j|}{dt} \leq \left|\frac{de_j}{dt}\right|$, the dynamic of $e_j(t)$ over time interval

$[\tau_k, \tau_{k+1})$ is bounded by

$$\begin{aligned}\frac{d|e_j|}{dt} &\leq \left|\left[\begin{array}{cc} -\frac{\sin\alpha_j}{L_j} & 1 \end{array}\right]\left[\begin{array}{c} v_{j-1} \\ \omega_{j-1} \end{array}\right]\right| \\ &\leq \left(\frac{1}{|L_j|}+1\right)\left|\left[\begin{array}{c} v_{j-1} \\ \omega_{j-1} \end{array}\right]\right| \\ &\leq \left(\frac{1}{|L_j|}+1\right)W_{j-1}(|\tilde{\alpha}_j(k)|) \end{aligned} \tag{20}$$

The last inequality holds because of Proposition IV.1. The explicit bound on $|L_j|$ over time interval $[\tau_k, \tau_{k+1})$ is $\dot{L}_j \geq K_{L_j}(k)(L_{d_j} - L_j) - |v_{j-1}| \geq K_{L_j}(k)(L_{d_j} - L_j) - W_{j-1}(|\tilde{\alpha}_j(k)|)$. Using Gronwall-Bellman inequality over $[\tau_k, \tau_{k+1})$, one has

$$\begin{aligned} L_j(t) &\geq \left[L_j(\tau_k) - \left(L_{d_j} - \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}\right)\right]e^{-K_{L_j}(k)(t-\tau_k)} \\ &+ L_{d_j} - \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}) \end{aligned}$$

Since $L_{d_j} \geq \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}$ and $L_j(t) > d$ from Lemma IV.6, we know $\inf_{\tau_k \leq t < \tau_{k+1}} L_j(t)$ is obtained at either $t = \tau_k$ or $t = \tau_{k+1}$,

$$L_j(t) \geq \inf_{\tau_k \leq t < \tau_{k+1}} L_j(t) = \min\left\{L_{jmin}, L_j(\tau_k)\right\} \tag{21}$$

where $L_{jmin} = [-\tilde{L}_j(k) + \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}]e^{-K_{L_j}(k)T_k} + (L_{d_j} - \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)})$. By inequality (21), (20) is rewritten as

$$\frac{d|e_j|}{dt} \leq \left(\frac{1}{\min\left\{L_{jmin}, L_j(\tau_k)\right\}}+1\right)W_{j-1}(|\tilde{\alpha}_j(k)|)$$

Solving above differential inequality, we have

$$|e_j(t)| \leq \underbrace{\left(\frac{1}{\min\left\{L_{jmin}, L_j(\tau_k)\right\}}+1\right)W_{j-1}(|\tilde{\alpha}_j(k)|)}_{B_j(k)}(t-\tau_k) + |e_j(\tau_k)|$$

For $t \to \tau_{k+1}$, one can get $|e(k+1^-)| \leq B_j(k)T_k + |e_j(k)|$. And assume that $|e_j(k)| \leq \overline{U}_j(k)$, then $|e(k+1^-)| \leq B_j(k)T_k + \overline{U}_j(k)$. We know that

$$|e(k+1^+)| \leq 2^{-R_j(k+1)}|e(k+1^-)| \leq 2^{-R_j(k+1)}\left(B_j(k)T_k + \overline{U}_j(k)\right)$$

From equation (17) and $\overline{U}_j(k+1) = 2^{-R_j(k+1)}U_j(k+1)$, we have $|e(k+1^+)| \leq \overline{U}_j(k+1)$. The equation (18) holds by simply considering the solution to the ODE $\dot{\tilde{\alpha}}_j = -K_{\alpha_j}\tilde{\alpha}_j$ with initial value $\tilde{\alpha}_j = \alpha_{d_j} - \hat{\alpha}_j(k^+)$. ■

With Lemma IV.9, the following lemma provides a sufficient condition on the selection of controller gains that leads to almost-surely practical stability for the bearing angle $\alpha_j, j = 1,2,\ldots,N-1$.

Lemma IV.10. *Consider the closed loop system in equations (9-10). Let*

$$G(|\alpha_j|,|L_j|) = e^{-h(|\alpha_j|,|L_j|)\gamma(|\alpha_j|,|L_j|)}(1 + h(|\alpha_j|,|L_j|)\gamma(|\alpha_j|,|L_j|))$$

be non-negative, monotone increasing function with respect to $|\alpha_j|$ and $|L_j|$ respectively. If there exists a sequence of controller gains $\{K_{L_j}(k), K_{\alpha_j}(k)\}_{k=0}^{\infty}$ with $K_j(k) = \{K_{L_j}(k), K_{\alpha_j}(k)\} \in \mathscr{K}_j$ for all $k \in \mathbb{Z}$ such that the Proposition IV.1 and following inequality hold for any $\eta_j \in (0,1)$

$$G(\overline{\alpha}_j(k+1), \overline{L}_j(k+1)) \leq \eta_j \tag{22}$$

$$\overline{\alpha}_j(k+1) = |-\tilde{\alpha}_j(k)e^{-K_{\alpha_j}(k)T_k} + \alpha_{d_j}| + B_j(k)T_k + \overline{U}_j(k)$$

$$\overline{L}_j(k+1) = L_{d_j} + \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)} - \left[\tilde{L}_j(k) + \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)}\right]e^{-K_{L_j}(k)T_k}$$

then

$$\mathbb{E}\left[\overline{U}_j(k+1)|\overline{U}_j(k)\right] \leq \eta_j\overline{U}_j(k) + \eta_jB_j(k)T_k, \forall k \in \mathbb{Z}_+ \tag{23}$$

Proof: Consider the sequence $\{\overline{U}_j(k)\}_{k=0}^{\infty}$ that satisfies equation (17) in Lemma IV.9, using the argument in [6], one has $\mathbb{E}[\overline{U}_j(k+1)|\overline{U}_j(k)] \leq G(|\alpha_j(k+1)|,|L_j(k+1)|)(B_j(k)T_k + \overline{U}_j(k))$. Let $G(|\alpha_j(k+1)|,|L_j(k+1)|) \leq \eta_j$, we have final conclusion (23) hold. In order to select the controller gain $\{K_{L_j}(k), K_{\alpha_j}(k)\}$ for the time interval $[\tau_k, \tau_{k+1})$, the selection decision is made based only on the information at time instant τ_k. Thus, we further bound the state $|\alpha_j(k+1)|$ and $|L_j(k+1)|$ by considering $|e_j(k+1^-)| = |\alpha_j(k+1^-) - \hat{\alpha}_j(k+1^-)| \leq B_j(k)T_k + \overline{U}_j(k)$. Since $\alpha_j(k+1) = \alpha_j(k+1^-)$, we have

$$\begin{aligned} |\alpha_j(k+1)| &\leq |\hat{\alpha}_j(k+1^-)| + B_j(k)T_k + \overline{U}_j(k) \\ &\leq |\alpha_{d_j} - \left(\alpha_{d_j} - \hat{\alpha}_j(k)\right)e^{-K_{L_j}(k)T_k}| + B_j(k)T_j + \overline{U}_j(k) \\ &\triangleq \overline{\alpha}_j(k+1) \end{aligned}$$

Similarly, one also has $|L_j(k+1)| \leq \overline{L}_j(k+1) = (L_{d_j} + \frac{W_{j-1}(|\tilde{\alpha}_j(k)|)}{K_{L_j}(k)})(1 - e^{-K_{L_j}(k)T_k}) + L_j(k)e^{-K_{L_j}(k)T_k}$ that is shown in Proposition IV.1. Since the function $G(|\alpha_j(k+1)|,|L_j(k+1)|)$ is monotone increasing function w.r.t $|\alpha_j(k+1)|$ and $|L_j(k+1)|$, then if $G(\overline{\alpha}_j(k+1), \overline{L}_j(k+1)) \leq \eta_j$, we have $G(|\alpha_j(k+1)|,|L_j(k+1)|) \leq \eta_j$, then the final conclusion holds. ■

Remark IV.11. *Function $G(\alpha_j, L_j)$ in condition (22) is directly related to the EBB model, and it generates a partition of the formation state space as shown in Figure 3. Each partition associates with a threshold η_j that characterizes the convergent rate for the uncertainty set. The aim of switching control strategy is to guarantee the condition (22) holds with a selected η_j.*

Similar to Lemma IV.10, the following lemma shows that the sequence of the estimate of bearing angle $\{\tilde{\alpha}_j(k)\}_{k=0}^{\infty}$ for $j = 1,2,\ldots,N-1$ satisfies super-martingale like property as sequence $\{\overline{U}_j(k)\}_{k=0}^{\infty}$ does.

Lemma IV.12. *Consider cascaded formation system in equations (9-10), given a sequence of controller pair $\{K_{L_i}(k), K_{\alpha_i}(k)\}_{k=0}^{\infty}$ with each $\{K_{L_i}(k), K_{\alpha_i}(k)\}$ selected at time instants $\{\tau_k\}_{k=0}^{\infty}$ and $\{K_{L_i}(k), K_{\alpha_i}(k)\} \in \mathscr{K}_i$. Let $K_{\alpha_i}^* = \min\{K_{\alpha_i}|K_{\alpha_i} \in \mathscr{K}_i\}$ and let $\mathscr{I}_k$ denote the information available at time instant τ_k, then we have*

$$\mathbb{E}\left[|\tilde{\alpha}_i(k+1)||\mathscr{I}_k\right] \leq e^{-K_{\alpha_i}^*T_k}|\tilde{\alpha}_i(k)| + \left(B_i(k)T_k + \overline{U}_i(k)\right)(1 - 2^{-\bar{R}_i})$$

Proof: Consider the time interval $[\tau_k, \tau_{k+1})$, by equation (9), we know that $\dot{\hat{\alpha}}_j = K_{\alpha_j}(k)(\alpha_{d_j} - \hat{\alpha}_j(t))$ with initial value $\hat{\alpha}_j(\tau_k)$. Therefore, let $\tilde{\alpha}_j(k) = \alpha_{d_j} - \hat{\alpha}_j(k)$, we have $\tilde{\alpha}_j(k+1^-) = e^{-K_{\alpha_j}(k)T_k}\tilde{\alpha}_j(k)$. Let $E_j(k+1) = \tilde{\alpha}_j(k+1) - \tilde{\alpha}_j(k+1^-)$, then $\tilde{\alpha}_j(k+1) = e^{-K_{\alpha_j}(k)T_k}\tilde{\alpha}_j(k) + E_j(k+1)$. Let $K_{\alpha_j}^* = \min\{K_{\alpha_j}|K_{\alpha_j} \in \mathscr{K}_j\}$, then

$$|\tilde{\alpha}_j(k+1)| \leq e^{-K_{\alpha_j}^*T_k}|\tilde{\alpha}_j(k)| + |E_j(k+1)| \tag{24}$$

The term $|E_j(k+1)|$ can be bounded as $|E_j(k+1)| \leq (B_j(k)T_k + \overline{U}_j(k))(1 - 2^{-R_j(k+1)})$. Taking the conditional expectation on both sides of inequality (24) with respect to the information $\mathscr{I}_k$ available at time instant τ_k and using above bound on $|E(k+1)|$ yield $\mathbb{E}\left[|\tilde{\alpha}_j(k+1)||\mathscr{I}_k\right] \leq e^{-K_{\alpha_j}^*T_k}|\tilde{\alpha}_j(k)| + \left(B_j(k)T_k + \overline{U}_j(k)\right)(1 - 2^{-R_j(k+1)})$. Since $R_j(k) \leq \bar{R}_j$ for all $k \in \mathbb{Z}_+$, the final conclusion holds. ■

With Lemma IV.10 and IV.12, we proceed to state the main theorem of almost sure asymptotic stability as follows,

Theorem IV.13. *Consider closed-loop system in equations (9-10). Let the hypothesis of Lemma IV.10 hold, suppose the coupling* *between the leader-follower subsystems is sufficiently weak and there exists a positive constant value ε_j such that*

$$B_j(k) = \max\left\{\frac{1}{\min\{L_{jmin}, L_j(k)\}}, 1\right\} W_{j-1}(|\tilde{\alpha}_j(k)|) \le \varepsilon_j |\tilde{\alpha}_j(k)|$$

for all $k \in \mathbb{Z}_+$, if

$$\max\{\eta_j + 1 - 2^{-\bar{R}_j}, (\eta_j + 1 - 2^{-\bar{R}_j})\varepsilon_j T_k + e^{-K^*_{\alpha_j} T_k}\} \le \delta \qquad (25)$$

where $\delta \in (0,1)$. Then the system state of bearing angle α_j almost surely asymptotically converges to α_{d_j} for $j = 1,2,\ldots,N-1$

Proof: We prove the almost sure convergence of α_i by proving $\lim_{k\to\infty} \mathbb{E}[\overline{U}_i(k) + \tilde{\alpha}_i(k)] \to 0$. Since $\alpha_i = \hat{\alpha}_i + e_i$, then $|\alpha_i(k) - \alpha_{d_i}(k)| \le |\hat{\alpha}_i(k) - \alpha_{d_i}| + \overline{U}_i(k)$. By Lemma IV.10 and Lemma IV.12, we have $\mathbb{E}[\overline{U}_i(k+1) + \tilde{\alpha}_i(k+1)] \le \delta_i \mathbb{E}[\overline{U}_i(k) + \tilde{\alpha}_i(k)]$ with $\delta_i \in (0,1)$ if inequality (25) holds. Then, it is clear that $\lim_{k\to\infty} \mathbb{E}[|\alpha_i(k) - \alpha_{d_i}(k)|] \to 0$. Using Markov inequality, we have $|\alpha_i(k) - \alpha_{d_i}(k)| \to 0$ almost surely, i.e. the bearing angle sequence $\{\alpha_i(k)\}$ almost surely converges to α_{d_i}. Because the state trajectory has no finite escape within each time interval $[\tau_k, \tau_{k+1})$, $\forall k \in \mathbb{Z}_+$. Then, the system state of bearing angle $\alpha_i(t)$ is almost surely convergent to α_{d_i}. ∎

Remark IV.14. *The weak coupling condition $B_j(k) \le \varepsilon_j |\tilde{\alpha}_j(k)|$ is equivalent to $W_{j-1}(\tilde{\alpha}_i(k)) \le \varepsilon_i |\tilde{\alpha}_j(k)|$ since $L_j(t) > d > 1$ for $t \in \mathbb{R}_+$.*

By Corollary IV.4, the first vehicle in the chain must eventually stop in order to guarantee almost sure asymptotic stability for the bearing angles. Although almost sure asymptotic stability is considered as a desired safety specification for safe-critical systems and has its own theoretical interest, it is clearly too restrictive to implement in the real application when non-varnishing disturbance exists. Almost sure practical stability is a weaker safety notion than almost sure asymptotic stability, and it allows the bearing angles to fluctuate within a reasonable safe set, and also the velocity of the vehicles to be nonzero. Theorem IV.15 provides a sufficient condition to assure almost sure practical stability for bearing angle $\alpha_j(t), j = 1,2,\ldots,N-1$.

Theorem IV.15. *Consider closed-loop system in equations (9-10). Let the hypothesis of Lemma IV.10 hold, for given positive values Δ^*_j, $j = 1,2,\ldots,N-1$, if there exists a controller pair $\{K_{L_j}(k), K_{\alpha_j}(k)\}$ with $\eta_j(k)$ such that*

$$B_j(k) \le \frac{1-r_j}{J_j} \min\{\Delta^*_j, |\tilde{\alpha}_j(k)| + \overline{U}(k)\}, j = 1,2,\ldots,N-1 \qquad (26)$$

with $r_j < 1$ where

$$r_j = \max\{\eta_j + 1 - 2^{-\bar{R}_j}, e^{-K^*_{\alpha_j} T_k}\} \qquad (27)$$

$$J_j = (\eta_j + 1 - 2^{-\bar{R}_j}) T_k \qquad (28)$$

Then the bearing angle α_j of leader follower pair i almost surely converges to a compact set defined by $\Omega_j = \{\alpha_j(t) : |\alpha_j(t) - \alpha_{d_j}| \le \Delta^*_j\}$.

Proof: By Lemma IV.10 and Lemma IV.12, one has

$$\begin{aligned}
&\mathbb{E}\left[|\tilde{\alpha}_j(k+1)| + \overline{U}_j(k+1) | \mathscr{I}_k\right] \\
&\le \max\{\eta_j + 1 - 2^{-\bar{R}_j}, e^{-K^*_{\alpha_j} T_k}\}\left(|\tilde{\alpha}_j(k)| + \overline{U}_j(k)\right) \\
&+ (\eta_j + 1 - 2^{-\bar{R}_j}) T_k B_j(k)
\end{aligned} \qquad (29)$$

Let $V_j(k) = |\tilde{\alpha}_j(k)| + \overline{U}_j(k)$, and consider function $V_j(k)$ as a candidate Lyapunov function. It is clear that $V_j(k) \ge 0$ for any $k \in \mathbb{Z}_+$. Then, we can rewrite inequality (29) into $\mathbb{E}[V_j(k+1)|V_j(k)] \le \mathbb{E}[V_j(k+1)|\mathscr{I}_j(k)] \le r_j V_j(k) + J_j B_j(k)$ Furthermore, if the controller gains $\{K_{L_j}(k), K_{\alpha_j}(k)\}$ are selected to assure $r_j < 1$, we have $\mathbb{E}[V_j(k+1)|V_j(k)] \le V_j(k) - [(1-r_j)V_j(k) - J_j B_j(k)]$. By condition (26), one can obtain

$$\begin{aligned}
\mathbb{E}[V_j(k+1)|V_j(k)] &\le V_j(k) + (1-r_j)\min\{\Delta^*_j - V_j(k), 0\} \\
&= V_j(k) - (1-r_j)\max\{V_j(k) - \Delta^*_j, 0\}
\end{aligned} \qquad (30)$$

From inequality (30), one can prove the bounded set $\hat{\Omega}_j = \{V_j(k) : V_j(k) \le \Delta^*_j\}$ is invariant with respect to system in equations (9) and (10) almost surely by considering

1 : when $V_j(k) \le \Delta^*_j$, inequality (30) is reduced to $\mathbb{E}[V_j(k+1)|V_j(k)] \le V_j(k)$, which implies that sequence $\{V_j(k)\}$ is a super-martingale and remains in the set $\hat{\Omega}_j$ almost surely.

2 : when $V_j(k) > \Delta^*_j$, $\exists \varepsilon > 0$ such that $\mathbb{E}[V_j(k+1)|V_j(k)] \le V_j(k) - \varepsilon$. Clearly, the trajectory of $V_j(k)$ will asymptotically decrease until reaching the set $\hat{\Omega}_j$ almost surely.

This condition can be viewed as a stochastic version of the LaSalle Theorem in discrete time system. With condition (30), one can easily attain the following almost sure convergence property for $V_j(k)$ with respect to set $\hat{\Omega}_j$ $\lim_{k\to+\infty} \Pr\{\sup_k V_j(k) \le \Delta^*_j\} \to 1$ Since $|\alpha_j(k) - \alpha_{d_j}| \le |\tilde{\alpha}_j(k)| + \overline{U}_j(k) = V_j(k)$, the almost sure convergence property for $V_j(k)$ leads to almost sure convergence for $|\alpha_j(k) - \alpha_{d_j}|$ with respect to set Ω_j. Since the state trajectories remains bounded within each transmission time interval $[\tau_k, \tau_{k+1})$ for all $k \in \mathbb{Z}_+$. Therefore, we have $\lim_{t\to+\infty} \Pr\{\sup_t |\alpha_j(t) - \alpha_{d_j}| \le \Delta^*_j\} \to 1$. ∎

Remark IV.16. *Inequality (26) characterizes an upper bound on the propagated disturbance $B_j(k)$ under which the leader follower pair j is almost sure practically stable. This upper bound is a increasing function of the size of target set Δ^*_j and the worst-case of bearing angle $|\tilde{\alpha}_j(k)| + \overline{U}(k)$, and a decreasing function of the ratio η_j.*

Remark IV.17. *Inequality (26) can be viewed as a distributed rule to select $\eta_j(k)$ to assure almost sure practical stability for each leader follower pair. The selected $\eta_j(k)$ is used in Lemma IV.10 to switch controller.*

The following Corollary shows an explicit bound on the bearing angle under which it is almost surely convergent to a "safe" set $\Omega_i(\Delta^*)$. Such bound is a function of η and Δ^*.

Corollary IV.18. *In Theorem IV.15, let the weak coupling assumption $W_j(\tilde{\alpha}_j(k)) \le \varepsilon_j |\tilde{\alpha}_j(k)|$ holds with $g_j(\eta_j) := \frac{1-r_j}{\varepsilon_j J_j} \ge 1$ and $r_j < 1$ where r_j and J_j are defined in equation (27). If*

$$|\tilde{\alpha}_j(k)| + \overline{U}_j(k) \le g_j(\eta_j)\Delta^* \qquad (31)$$

then the bearing angle α_j almost surely converges to a bounded set $\Omega_j = \{\alpha_j(t) : |\alpha_j(t) - \alpha_{d_j}| \le \Delta^*_j\}$.

Proof: From Theorem IV.15, we know that the sufficient condition to assure almost sure practical stability with set Ω_j is $B_j(k) \le \frac{1-r_j}{J_j} \min\{\Delta^*_j, |\tilde{\alpha}_j(k)| + \overline{U}(k)\}$. By weak coupling assumption $W_j(\tilde{\alpha}_j(k)) \le \varepsilon_j |\tilde{\alpha}_j(k)|$, the above sufficient condition holds, if

$$\begin{aligned}
|\tilde{\alpha}_j(k)| + \overline{U}_j(k) &\le \frac{1-r_j}{\varepsilon_j J_j} \min\{\Delta^*_j, |\tilde{\alpha}_j(k)| + \overline{U}(k)\} \\
&= g_j(\eta_j)\Delta^*
\end{aligned}$$

holds. The equality holds because $g_j(\eta_j) := \frac{1-r_j}{\varepsilon_j J_j} \ge 1$. Therefore, the conclusion holds. ∎

Remark IV.19. *$g_j(\eta_j)$ is a monotone decreasing function with respect to η_j, and it characterizes the size of the region from which the state almost surely converges to the set Ω_j with size Δ^*. The inequality (31) may be viewed as a partition of the physical state in the sense that small η_j gives rise to large contraction set.*

V. Simulation Experiments

This section presents simulation experiments examining the resilience of our proposed switched controller to deep fades, and also demonstrates the benefits of using almost sure practical stability as a safety measurement over the traditional mean square stability.

A. Simulation Setup

In the simulation, we consider $N = 4$ vehicles that is cascaded in a string as shown in Figure 1. Each leader-follower pair uses a two-state Markov chain model to simulate the fading channel between the leader and follower. The two-state Markov chain has two states with one representing the good channel condition and the other one representing the bad channel condition. Here, the "good channel state" simply means the transmitted bit is successfully received, while the "bad channel state" means the failure of receiving the bit.

Following the characterization of Makov chain model in [17], one can find that the conditional probability for good channel state is a monotone decreasing function of $\frac{L_j(t)}{\cos\alpha_j(t)}$, while the conditional probability for bad channel state is a monotone decreasing function of $\frac{\cos\alpha_j(t)}{L_j(t)}$. The explicit function form depends on the distribution of the channel gain. In this simulation, we therefore use $p_{11} = e^{-3\times10^{-3}(\frac{L_j(t)}{\cos\alpha_j(t)})^2}$ to denote the conditional probability for the good channel state. Let $p_{22} = e^{-6\times10^{2}(\frac{\cos\alpha_j(t)}{L_j(t)})^2}$ represent the conditional probability for the bad channel condition. Hence, the corresponding transition probabilities between these states are $1-p_{11}$ and $1-p_{22}$. Then, we utilize the EBB model in equation (6) to characterize the low bit region generated by the two-state Markov chain model. The corresponding functions in EBB model (6) are $h(\alpha_j, L_j) = \bar{R}_j e^{-3\times10^{-4}(\frac{L_j(t)}{\cos\alpha_j(t)})^2}, \gamma(\alpha_j, L_j) = e^{-4.5\times10^{-3}(\frac{L_j(t)}{\cos\alpha_j(t)})^2}$ with $\bar{R}_j = 2$ representing two bits are transmitted at each sampling period.

The 100 ms sampling time that is widely used in mobile robot system, is selected for each leader-follower pair ($j = 1,2,3$), i,e, $T_k = 0.1$ sec for all $k \in \mathbb{Z}_+$. The functions $W_{j-1}(\cdot)$ in Proposition IV.1 are selected to be linear functions $W_{j-1}(|\tilde{\alpha}_j(t)|) = a_j|\tilde{\alpha}_j(t)| + b_j$ with parameters selected as $a_1 = 0.1, b_1 = 0.01; a_2 = 0.8, b_2 = 2; a_3 = 1, b_3 = 4$. The value of the parameter sets are chosen to be increasing with respect to j to guarantee the feasibility of the controller selection for each leader-follower system.

In this simulation, we consider an interesting and realistic scenario that the fourth vehicle from far distance intends to merge into the other three closed-clustered vehicles. Hence, the initial states for three leader-follower pairs ($j = 1,2,3$) are set as $\alpha_1(\tau_0) = \frac{\pi}{3}, \alpha_2(\tau_0) = \frac{\pi}{4}, \alpha_3(\tau_0) = \frac{\pi}{6}$. with initial uncertainty length $U_j(\tau_0) = \frac{\pi}{6}$, and $L_1(\tau_0) = 7.1, L_2(\tau_0) = 7.1, L_3(\tau_0) = 99$. By switching controller pairs from the following pool $\mathscr{K}_j = \{(K_{L_j}, K_{\alpha_j}) : 0 < K_{L_j} \le 100, 0 < K_{\alpha_j} \le 100\}$. each leader-follower pair is required to achieve and maintain around desired setpoints $\alpha_{d_j} = 0, L_{d_j} = 2, j = 1,2,3$.

B. Simulation Results

A Monte Carlo method was used to verify that the system has almost surely practical stability when Proposition IV.1 and Theorem IV.15 hold. Each simulation example is run 100 times over a time interval from 0 to 10 seconds.

In the first simulation, we select the controllers for each leader-follower pair from $\mathscr{K}_j, j = 1,2,3$ so that Proposition IV.1 and Theorem IV.15 hold at each time instant τ_k. Figure 4 show the maximum and minimum values of the system states L_j and α_j, $j = 1,2,3$ evaluated over all the 100 runs. The maximum value is marked by red lines and the minimum value is marked by blue lines. The two dashed lines in Figure 4 represent the upper and lower bound for the relative bearing α, i.e. $|\alpha_j| \le \pi/2$, which characterizes the safety region. We can see from Figure 4 that the maximum and minimum values of the system states asymptotically converge to a bounded set containing the desired set-points $\alpha_{d_j} = 0$ and $L_{d_j} = 2$. This is precisely the behavior that one would expect if the system is almost sure practical stable. These results therefore, seem to confirm our statement in Theorem IV.15. Figures 5-6 show one sample of

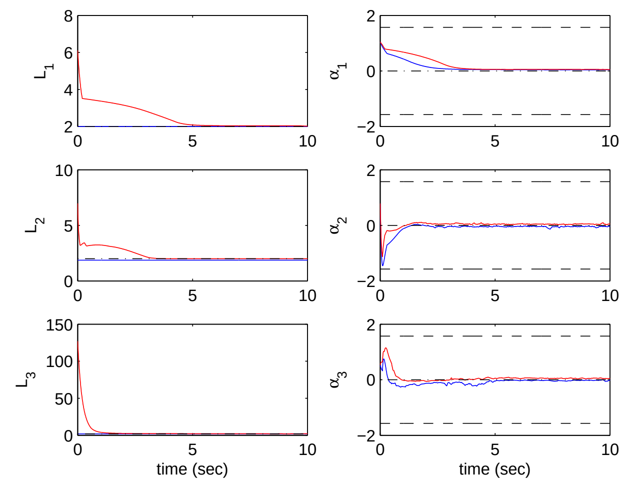

Fig. 4. The maximum and minimum value of separation L_j and bearing angle α_j for leader-follower pair, $j = 1,2,3$.

switching controller profile and channel state for each leader-follower pair. The top plot in Figure 5 shows the switching controller profile for the leader-follower pair 1 with red line marked as controller gain K_{α_1} and blue line as controller gain K_{L_1}. The bottom one is the switching controller profile for leader-follower pair 2 with the same marking rule. These plots show that the controller gains stay low at the first two seconds to avoid causing large disturbance to the bottom system, then switch from low to high when the systems approach the equilibrium and are confident that the channel state will always stay good. The top plot in Figure 6 is the switching controller profile for the leader-follower system 3 with same marking rule, and the bottom plot is the channel state $R_3(k)$ that characterizes the number of successfully received bits at each time interval. We can clearly see from the plots that the controller for system 3 starts with low gains to compensate the effect caused by a short string of zero bits at the beginning, and then switches from low gain to high gain when channel condition stays good. These results demonstrate that channel state indeed is used as a feedback signal to switch the controller. In the second simulation, we studied the benefits of almost sure practical stability as a safety measurement over the traditional mean square stability. Traditional mean square stability requires the second moment of the system state converges to a positive constant value, but it does not put any constraint on the sample path which might potentially cause safety issues. For a fair comparison, the same simulation setup and parameters are applied in this simulation with the only difference on the controllers. One type of controller used in this simulation is a mean square stabilizing controller, which is selected to guarantee mean square stability for each leader-follower pair. The other type of controller is the switching controller proposed in this paper to guarantee almost sure practical stability for each leader-follower pair. The switching control strategy uses the mean square stabilizing controller as its initial controller.

Figure 7 shows a comparison of the maximum and minimum

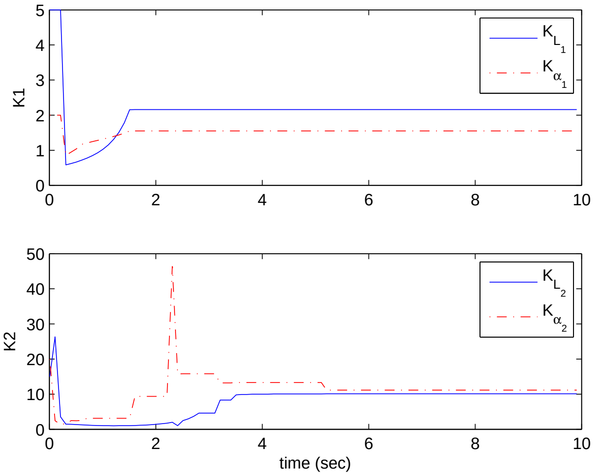

Fig. 5. One sample of switching controller profile for leader-follower pair 1 and 2

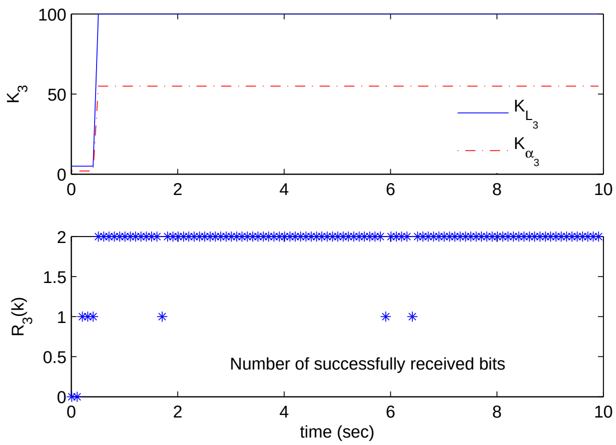

Fig. 6. One sample of switching profile and channel state for leader-follower pair 3

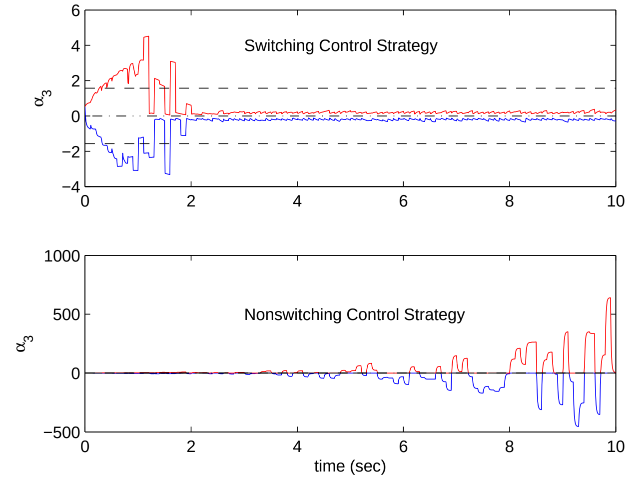

Fig. 7. The maximum and minimum system trajectory for leader-follower pair 3 with switching controller (Top) and non-switching controller pair(Bottom) $K_{L_3} = 2$ and $K_{\alpha_3} = 50$

values of the bearing angle α_3 for leader-follower pair 3 with the switching controller case in the top plot and the mean square controller $K_1 = (5, 0.5); K_2 = (5, 0.5); K_3 = (2, 50)$ in the bottom plot. It is worth noting that (K_1, K_2, K_3) is just one of the many selections in our simulation. Because of the space limitation, we only use (K_1, K_2, K_3) as an example to demonstrate the results. It is clear from Figure 7 that the system's sample path goes unbounded as time increases by using a mean square stabilizing controller, but it converges asymptotically to a bounded set by using a switching controller. These results suggest that the composition of mean square stable systems does not guarantee mean square stability for the whole system, while the composition of almost sure stable systems may still guarantee almost sure stability for the whole system.

VI. Conclusions

This paper studies the almost sure safety property for a chain of leader-follower nonholonomic system in the presence of deep fades that exhibits exponentially bounded burstiness and varies as a function of vehicular state. The concept of almost sure safety is examined in terms of almost sure asymptotic stability and practical stability, which specifically requires the vehicular system to maintain a desired formation with absence of collision and communication loss. Switching strategy is adopted to assure almost sure safety for the leader follower system by adaptively reconfiguring local controller gains to the changes of channel state as well as the lower system's state. Sufficient conditions are provided to decide which controller is placed in the feedback loop at each transmission time. As a result of the correlation between channel state and physical vehicular state, sufficient conditions turn out to be partition rules for the system state. Each region is associated with corresponding controller sets to achieve almost sure safety. The simulation results of a four-vehicle leader follower formation control are provided to support our theoretical analysis and also illustrate the benefit of using almost surely practical stability as a safety measurement over traditional mean square stability. In this paper, each communication link is assumed to only subject to deep fades. However, the real wireless communication on transportation system inevitably invokes significant interference, which is an interesting and critical problem in large scale system. it turns out that interference is highly related to the physical position of the user as well as geometry of the network. The results of this paper provide potential benefits to address the interference problem in multi-agent wireless networked system.

Acknowledgments

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VII. APPENDIX

Proof of Proposition IV.1:

Proof: Consider the infinite norm of the control input given in equation (8),

$$\left|\left[\begin{array}{c} v_j(t) \\ \omega_j(t) \end{array}\right]\right| \leq \left\|\left[\begin{array}{cc} -\cos\phi_j & -L_j\sin\phi_j \\ -\frac{\sin\phi_j}{d} & \frac{L_j}{d}\cos\phi_j \end{array}\right]\right\| \left|\left[\begin{array}{c} K_{L_j}(k)(L_{d_j}-L_j) \\ K_{\alpha_j}(k)(\alpha_{d_j}-\hat{\alpha}_j) \end{array}\right]\right|$$
$$\leq \left(1+|L_j(t)|\right)\max\{K_{L_j}(k)|\tilde{L}_j(t)|, K_{\alpha_j}(k)|\tilde{\alpha}_j(t)|\} \quad (32)$$

with $\tilde{L}_j(t) = L_{d_j} - L_j(t)$. The supreme of $|L_j(t)|$ over time interval $[\tau_k, \tau_{k+1})$ can be obtained by considering $\dot{L}_j(t) \leq K_{L_j}(k)(L_{d_j} - L_j)(t) + W_{j-1}(|\tilde{\alpha}_j(k)|)$. Using Gronwall Bellman theorem to solve above inequality and yield,

$$\begin{aligned} &L_j(t) \\ &\leq L_j(k)e^{-K_{L_j}(k)(t-\tau_k)} + \left(L_{d_j} + \frac{W_{j-1}(|\alpha_j(k)|)}{K_{L_j}(k)}\right)\left(1-e^{-K_{L_j}(k)(t-\tau_k)}\right) \\ &\triangleq \overline{L}_j(t) \end{aligned}$$

Assume $L_j(t) > 0$ (In Lemma IV.9, we prove that if controller gain $K_{L_j}(k)$ is selected sufficiently large, $L_j(t) > d > 0$ holds for all $t \geq 0$), and because $\frac{d\overline{L}_j}{dt} \geq 0$ or $\frac{d\overline{L}_j}{dt} < 0$ over interval $[\tau_k, \tau_{k+1})$. In other words, $\overline{L}_j(t)$ is a monotone function over $[\tau_k, \tau_{k+1})$. Thus $\sup_{\tau_k \leq t < \tau_{k+1}} L_j(t)$ is obtained when $t = \tau_k$ or $t \to \tau_{k+1}$, i.e.

$$L_j(t) = \max\left\{\overline{L}_j(\tau_k), \overline{L}_j(\tau_{k+1})\right\} \triangleq M_{L_j}(k) \quad (33)$$

Note that over time interval $[\tau_k, \tau_{k+1})$, one has $\frac{d|\tilde{L}_j(t)|}{dt} \leq K_{L_j}(k)|\tilde{L}_j(t)| + W_{j-1}(\tilde{\alpha}_j(k))$ thus

$$\begin{aligned} &\sup_{\tau_k \leq t \leq \tau_{k+1}} K_{L_j}(k)|\tilde{L}_j(t)| \\ &= K_{L_j}(k)|\tilde{L}_j(k)|e^{K_{L_j}(k)T_k} + W_{j-1}(|\tilde{\alpha}_j(k)|)\left(e^{K_{L_j}(k)T_k}-1\right) \\ &\triangleq \tilde{L}_{j,max}(k) \end{aligned} \quad (34)$$

By inequalities (33-34), (32) can be further bounded

$$\left|\left[\begin{array}{c} v_j(t) \\ \omega_j(t) \end{array}\right]\right|_\infty \leq (1+M_{L_j}(k))\max\left\{\tilde{L}_{j,max}(k), K_{\alpha_j}(k)|\tilde{\alpha}_j(t)|\right\} \quad (35)$$

with $\tilde{\alpha}_j(t) = \alpha_{d_j} - \hat{\alpha}_j(t)$ satisfying $\dot{\tilde{\alpha}}_j = -K_{\alpha_j}(k)\tilde{\alpha}_j, t \in [\tau_k, \tau_{k+1})$ with initial value $\tilde{\alpha}_j(\tau_k)$. From the solution of the above ODE, it is obvious that $|\tilde{\alpha}_j(t)| < |\tilde{\alpha}_j(\tau_k)|$, then it is straightforward to show that if the condition (11) is satisfied, the inequality (12) holds. ∎